HARMONIC INTEGRALS ON ALGEBRAIC VARIETIES

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In my book *The Theory and Applications of Harmonic Integrals* (hereinafter referred to as *H.I.*), Chapter IV is devoted to the application of the theory to algebraic varieties. In order to introduce harmonic integrals on an algebraic variety I have to assign a metric to the variety; this metric is not related to the variety in any invariant sense, and indeed its introduction is extremely artificial. Nevertheless it turns out that we can deduce from the properties of the harmonic integrals a number of properties of the variety which are birationally invariant, that is, do not depend on the choice of metric. This remarkable fact seems to indicate either that the invariant properties in question should be obtainable directly without the introduction of the harmonic integrals, or that the metric introduced is not so artificial as it first appeared to be. This note is intended to examine this question.

We begin with a commentary on the analysis of *H.I.* Chapter IV. The theorems proved there fall into two classes, those which depend only on local properties of the metric and properties common to all Riemannian manifolds, and those which depend on the special topological properties of the algebraic manifold in question. We adopt the notation of *H.I.*, denoting the variety by V_m, the section of it by $m-p$ generally situated primes by V_p, the Riemannian of V_m by $M(V_m)$, and the image of V_p on $M(V_m)$ by $M(V_p)$. $M(V_p)$ is a $2p$-cycle of the $2m$-dimensional manifold $M(V_m)$. Local parameters are introduced on $M(V_m)$ by taking any set of complex parameters $z^1, \ldots, z^m$ in the corresponding neighbourhood of V_m, and writing $z^j = x^j + ix^{m+j}$; $x^1, \ldots, x^{2m}$ are then permissible local parameters on $M(V_m)$, but sometimes it is more convenient to use $z^1, \ldots, z^m$ and their conjugate imaginaries $\bar{z}^1, \ldots, \bar{z}^m$ as local parameters. When the real local parameters $x^1, \ldots, x^{2m}$ are used it is convenient to define components of tensors $T^{j_1 \ldots j_q}_{i_1 \ldots i_p}$ for all integral values of the indices by the conventions (cf. *H.I.* p. 162)

$$T^{j_1 \ldots j_{r-1}\, 2m+j_r\, j_{r+1} \ldots j_q}_{i_1 \ldots \ldots \ldots \ldots \ldots i_p} = - T^{j_1 \ldots j_{r-1}\, j_r\, j_{r+1} \ldots j_q}_{i_1 \ldots \ldots \ldots \ldots \ldots i_p},$$

$$T^{j_1 \ldots \ldots \ldots \ldots \ldots j_q}_{i_1 \ldots i_{r-1}\, 2m+i_r\, i_{r+1} \ldots i_p} = - T^{j_1 \ldots \ldots \ldots \ldots \ldots j_q}_{i_1 \ldots i_{r-1}\, i_r\, i_{r+1} \ldots i_p}.$$

The metric $g_{ij}\, dx^i dx^j$ introduced in *H.I.* has the property:

$$g_{ij} = \frac{\partial^2 \psi}{\partial x^i \partial x^j} + \frac{\partial^2 \psi}{\partial x^{m+i} \partial x^{m+j}}, \tag{1}$$

where ψ is a function of $x^1, \ldots, x^{2m}$ which is analytic in the neighbourhood considered. From this fact, several properties of the Christoffel symbols and of the Riemann curvature tensor are deduced (*H.I.* § 39·1). The first point which I wish to make is that these properties follow from the expression given for g_{ij}, but do not depend on the particular function ψ which appears. By means of these properties Theorems I, II, III are proved (§§ 40 and 42), and it thus follows that these theorems depend on

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the form of the metric, but not on the particular function ψ. Further, the fact that the 2-form

$$\omega = \tfrac{1}{2} g_{m+ij}\, dx^i dx^j,$$

and that the forms

$$\omega_1 = \omega, \quad \omega_2 = \omega_1 \times \omega, \ldots, \quad \omega_k = \omega_{k-1} \times \omega, \ldots$$

are harmonic depends only on the form of g_{ij} and not on ψ. On the other hand, the periodic properties of ω_k proved in §§ 41·1, 41·2 depend on special topological properties of $M(V_m)$. The analysis of the harmonic forms given in the following sections (42·1–43·3) does not depend on these periodic properties, and hence is valid for any metric with the property (1), irrespective of the function ψ chosen. It should be observed, further, that we only need to have property (1) locally, in the neighbourhood of any point; we do not require to have the same function ψ at all points.

In particular, we can deduce from the fact that the metric can be expressed in the form (1), locally, that any harmonic p-form P $(0 \leqslant p \leqslant m)$ can be written in the form

$$P = P_p + P_{p-2} \times \omega_1 + P_{p-4} \times \omega_2 + \ldots,$$

where P_k is an *effective* k-form, and that there are exactly $R_p - R_{p-2}$ linearly independent effective p-forms.

The sections which follow deal with the periodic properties of the effective p-forms, and of the various sets of ineffective p-forms. These properties do depend on particular topological properties of $M(V_m)$. The first point to notice is that the classification of the cycles of $M(V_m)$ is not absolute, but is relative to the system of prime sections of V_m. Let $|\, U_{m-1} \,|$ be a linear system of varieties on V_m which satisfies the conditions necessary in order that it can be used in Lefschetz's analysis of the cycles on V_m. We can classify the cycles of V_m into sets of effective cycles and ineffective cycles of various classes with respect to $|\, U_{m-1} \,|$. The number of p-cycles in each class will be the same as the number of p-cycles in the same class when they are classified with respect to $|\, V_{m-1} \,|$, but the actual cycles in the classes will not in general be the same, except when U_{m-1} and V_{m-1} are algebraically related.

In *H.I.* § 45·2, it is shown that the ineffective harmonic integrals of multiplicity p and class k have non-zero periods only on the ineffective p-cycles of class k, *when the cycles are classified with respect to the prime sections* V_{m-1} *of* V_m. This implies that the metric $g_{ij}\, dx^i dx^j$ chosen is related in some way to the prime sections of V_m. An examination of the arguments used shows that the relation between the classification of the harmonic integrals into effective integrals and ineffective integrals of the various classes, and the classification of the cycles of $M(V_m)$, with respect to $|\, V_{m-1} \,|$ follows from the following properties of the form ω:

$$\int_{M(V_1)} \omega \neq 0; \tag{2}$$

$$\int_{\Gamma_2} \omega = 0, \tag{3}$$

whenever Γ_2 is a 2-cycle whose intersection number $(\Gamma_2 . M(V_{m-1}))$ with $M(V_{m-1})$ is zero.

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Now suppose that in any way we find another metric $\bar{g}_{ij}\,dx^i dx^j$ with the properties

$$\bar{g}_{ij} = \frac{\partial^2 \overline{\psi}}{\partial x^i \partial x^j} + \frac{\partial^2 \overline{\psi}}{\partial x^{m+i} \partial x^{m+j}}, \tag{1}'$$

locally, for some function $\overline{\psi}$; if $\nu = \frac{1}{2!}\bar{g}_{m+ij}\,dx^i dx^j$,

$$\int_{M(V_1)} \nu \neq 0, \tag{2}'$$

and

$$\int_{\Gamma_2} \nu = 0, \tag{3}'$$

whenever $(\Gamma_2 . M(V_{m-1})) = 0$. Then we can construct harmonic integrals in this metric, and classify them as before, showing that the ineffective integrals of multiplicity p and class k have non-zero periods only on the ineffective p-cycles of class k, when the cycles are classified with respect to $|V_{m-1}|$. From this it follows at once that any ineffective harmonic p-form of class k, constructed for the metric $\bar{g}_{ij}\,dx^i dx^j$, is homologous to an ineffective harmonic p-form of class k, constructed for the metric $g_{ij}\,dx^i dx^j$, and conversely. Thus, if the two sets of ineffective harmonic p-forms of class k are suitably arranged, they have the same period matrix. We can indeed go further. For reasons explained in *H.I.*, we may confine our attention to ineffective integrals of class 0, that is, effective integrals. When we use $z^1, \ldots, z^m, \bar{z}^1, \ldots, \bar{z}^m$ as parameters the $R_p - R_{p-2}$ effective p-forms ($p \leqslant m$) can be divided into sets of ρ_p^h forms of type h ($h = 0, \ldots, p$), a p-form being of type h if it can be written as

$$\sum_{i_1 \ldots i_{p-h}=1}^{m} \sum_{j_1 \ldots j_h}^{m} A_{i_1 \ldots i_{p-h} j_1 \ldots j_h}\, dz^{i_1} \ldots dz^{i_{p-h}}\, d\bar{z}^{j_1} \ldots d\bar{z}^{j_h}$$

in which each term involves exactly h differentials $d\bar{z}^i$. We have, of course,

$$\rho_p^h = \rho_p^{p-h}, \quad \sum_{h=0}^{p} \rho_p^h = R_p - R_{p-2}.$$

The lemma of *H.I.* § 47·3 shows that every effective p-form of type h constructed for the metric $\bar{g}_{ij}\,dx^i dx^j$ is homologous to an effective p-form of type h constructed for the metric $g_{ij}\,dx^i dx^j$, and conversely, and hence, if the two sets of effective p-forms of type h are suitably arranged they have the same period matrix.

These period matrices, and the number ρ_p^h, are thus determined for the system $|V_{m-1}|$ by any metric $g_{ij}\,dx^i dx^j$ satisfying the conditions expressed in equations (1), (2), (3). We now show that the matrices having these properties form a set determined naturally by $M(V_{m-1})$. In a recent paper (*Proc. Cambridge Phil. Soc.* 43 (1947), 577–580) I proved a theorem which, when applied to $M(V_1)$, tells us that there exists a real 2-form

$$\nu = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\, dz^i d\bar{z}^j$$

with the properties:

(*a*) $\nu \to 0$;

(*b*) $\int_{M(V_1)} \nu \neq 0$;

(*c*) $\int_{\Gamma_2} \nu = 0$,

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whenever $(\Gamma_2 . M(V_{m-1})) = 0$. This is a perfectly general theorem concerning $M(V_m)$ and does not depend on the existence of harmonic integrals; the proof given did in fact use harmonic integrals, but this was merely for convenience, and a direct proof can easily be constructed. The form ν is not, of course, uniquely determined by these properties, and we are concerned with the set of forms satisfying the conditions. We note that the form ω is one such form.

The condition (a) implies that

$$\frac{\partial a_{ji}}{\partial z^k} = \frac{\partial a_{ki}}{\partial z^j}, \quad \frac{\partial a_{ij}}{\partial \bar{z}^k} = \frac{\partial a_{ik}}{\partial \bar{z}^j}$$

(for $i, j, k = 1, \ldots, m$). Hence it follows that, locally, a_{ij} can be written as

$$a_{ij} = 2\sqrt{-1}\,\frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j},$$

for some function ϕ. Now use the real parameters $x^1, \ldots, x^{2m}$. We find that

$$\nu = \frac{1}{2!} h_{ij}\, dx^i dx^j,$$

where

$$h_{ij} = \frac{\partial^2 \phi}{\partial x^{m+i} \partial x^j} - \frac{\partial^2 \phi}{\partial x^i \partial x^{m+j}}.$$

We now define a tensor $\bar{g}_{ij}$ by the conditions

$$\bar{g}_{m+ij} = h_{ij}$$

(where, as usual $g_{2m+ij} = -g_{ij}$, etc.). Then

$$\bar{g}_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} + \frac{\partial^2 \phi}{\partial x^{m+i} \partial x^{m+j}}$$

and is a symmetric tensor. Hence, provided $k\bar{g}_{ij}\, dx^i dx^j$ is positive definite, where k is some constant, it can be used as a metric to construct a system of harmonic integrals related as above to the prime section of V_m, and conversely all the metrics $\bar{g}_{ij}\, dx^i dx^j$ which satisfy $(1)', (2)', (3)'$ can be obtained in this way. Thus the class of metrics which lead to harmonic integrals which can be classified in the manner described to correspond to the prime sections of V_m can be obtained from the forms ν which satisfy the conditions (a), (b), (c), and hence as a set they are determined by intrinsic properties of $M(V_m)$.

The condition that $\bar{g}_{ij}\, dx^i dx^j$ be positive definite can be expressed as follows. Let

$$\nu_1 = \nu, \quad \nu_2 = \nu_1 \times \nu, \ldots,$$

and let

$$\nu_k = \nu^k_{i_1 \ldots i_k j_1 \ldots j_k} dz^{i_1} \ldots dz^{i_k} d\bar{z}^{j_1} \ldots d\bar{z}^{j_k}.$$

Let (b_{ij}) be any $m \times k$ matrix of rank k. Then if

$$(-\sqrt{-1})^k \nu^k_{i_1 \ldots i_k j_1 \ldots j_k} \begin{vmatrix} b_{i_1 1} & \ldots & b_{i_k 1} \\ \vdots & & \\ b_{i_1 k} & \ldots & b_{i_k k} \end{vmatrix} \begin{vmatrix} \bar{b}_{j_1 1} & \ldots & \bar{b}_{j_k 1} \\ \vdots & & \\ \bar{b}_{j_1 k} & \ldots & \bar{b}_{j_k k} \end{vmatrix} > 0,$$

for all admissible b_{ij} for $k = 1, \ldots, m$, and at all points of $M(V_m)$. $\bar{g}_{ij}\, dx^i dx^j$ is positive definite. The condition merely expresses the fact that the 'measure' of any analytic element of k dimensions on V_m, defined by $\bar{g}_{ij}\, dx^i dx^j$, is positive.

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The theorem quoted also tells us that, given any system $|U_{m-1}|$ of varieties of dimension $m-1$ on V_m which satisfy the conditions necessary for Lefschetz's theory of cycles, there exist forms ν having the properties (a), (b), (c), but with U_1, U_{m-1} substituted for V_1, V_{m-1}. If we use one of these to construct a metric and then classify the corresponding harmonic integrals, the classification will correspond to the classification of the cycles relative to the system $|U_{m-1}|$. In general, the period matrix of the effective p-fold integrals of type k for this metric will not, however, be the same as the period matrix of the effective p-fold integrals of type k for the metric $g_{ij}\,dx^i dx^j$, which is related to the system $|V_{m-1}|$, since the effective p-cycles are not the same in the two classifications. Thus the period matrices are relative invariants of the linear systems $|U_{m-1}|$, $|V_{m-1}|$, and are determined by the set of forms ν satisfying (a), (b), (c) for the systems $|U_{m-1}|$, $|V_{m-1}|$.

We can, however, show that the numbers ρ_p^h do not depend on whether we consider the system $|U_{m-1}|$ or $|V_{m-1}|$, that is, they are relative invariants of V_m, unaltered by any birational transformation of V_m which is $1-1$ without exception. It is clear that if P is a $(p-2k)$-form of type $k-h$, $P\times\omega_h$ is a p-form of type k. Hence the R_p harmonic p-form, constructed for the admissible metric $g_{ij}\,dx^i dx^j$ can be divided into forms of type $0, 1, 2, \ldots$, the number of forms of type k being $\rho_p^k+\rho_{p-2}^{k-1}+\rho_{p-4}^{k-2}+\ldots$. Similarly, if $\tilde{g}_{ij}\,dx^i dx^j$ is another admissible metric associated with a linear system $|U_{m-1}|$, and σ_p^h is the number of effective harmonic p-forms for this metric which are of type k, the total number of harmonic p-forms of type k is $\sigma_p^k+\sigma_{p-2}^{k-1}+\sigma_{p-4}^{k-2}+\ldots$. But an argument exactly similar to that given in *H.I.* § 47·3 shows that any harmonic p-form of type k constructed for the one metric is homologous to a harmonic p-form of type k constructed for the other metric, and hence we have

$$\rho_p^k+\rho_{p-2}^{k-1}+\rho_{p-4}^{k-2}+\ldots=\sigma_p^k+\sigma_{p-2}^{k-1}+\sigma_{p-4}^{k-2}+\ldots.$$

If we next consider the harmonic $(p-2)$-forms of type $k-1$, we obtain, similarly

$$\rho_{p-2}^{k-1}+\rho_{p-4}^{k-2}+\ldots=\sigma_{p-2}^{k-1}+\sigma_{p-4}^{k-2}+\ldots,$$

and, subtracting the two equations, we have

$$\rho_p^k=\sigma_p^k.$$

The number ρ_p^0 is, of course, the pth geometric genus of V_m.

In conclusion, I should like to refer to a problem which is suggested by the properties of the form ν, satisfying (a), (b), (c), discussed above. The function ϕ defined by

$$a_{ij}=2\sqrt{-1}\,\frac{\partial^2\phi}{\partial z^i\partial\bar{z}^j}$$

cannot be defined as finite at all points of $M(V_m)$. For then, ν would be the exterior derivative of $\sum_{i=1}^{m}\frac{\partial\phi}{\partial z^i}dz^i$, and hence would have all its periods zero, which would contradict (b). In the case when ν is the original 2-form ω, it is known that ϕ can be taken as a function which is finite at all points of $M(V_m)$, except on a certain $M(V_{m-1})$, where it

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becomes logarithmically infinite. It would be interesting to know whether this result could be generalized as follows:

Let γ_{2m-2} be a $(2m-2)$-cycle of $M(V_m)$ on which every $(2m-2)$-fold integral of type $(m-1)$ has zero periods. There exists a closed 2-form ν, which is not a null form,

$$\nu = a_{ij}\, dz^i\, d\bar{z}^j$$

on $M(V_m)$ such that ν has zero period on every 2-cycle having zero intersection number with γ_{2m-2}. Can the differential equation

$$2\sqrt{-1}\,\frac{\partial^2\phi}{\partial z^i\,\partial\bar{z}^j} = a_{ij}$$

be solved so that ϕ has singularities only on an *algebraic* locus of $m-1$ complex dimensions homologous to γ_{2m-2}? Here γ_{2m-2} takes the place of $M(V_{m-1})$ and the condition involving the $(2m-2)$-fold integrals is just the condition that γ_{2m-2} is algebraic. By using the full resources of harmonic integrals the result can be deduced provided that the algebraic system of varieties homologous to γ_{2m-2} satisfy certain simple conditions, but one would like a direct proof; for such a proof might well lead to a generalization which would yield a solution of the problem of finding necessary and sufficient conditions that a $2k$-cycle of $M(V_m)$ be algebraic.

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