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On Dirichlet spaces with zero sets containing all Blaschke sequences

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Abstract. We consider the family of infinite positive Borel measures μ in the unit disc, with finite degree of contact at the unit circle, and the set E_μ of points in the unit circle for which every neighborhood contains infinite mass of μ. Assuming that E_μ is a Carleson set, we show that all Blaschke sequences are zero sets for the corresponding Dirichlet space D_μ.

1 Introduction

Let $w \in L^1(\mathbb{D})$ be a nonnegative superharmonic function in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The weighted Dirichlet space D_w is the set of holomorphic functions f in $\mathbb{D}$, for which

$$\int_{\mathbb{D}} |f'(z)|^2 w(z) dA(z) < +\infty,$$

where A denotes the area measure in $\mathbb{C}$. For $w \equiv 1$, we obtain the classical Dirichlet space $\mathcal{D}$ and for $w_p(z) = (1-|z|^2)^p$, $p \in (0,1)$, we obtain the well-known Dirichlet spaces D_p having radial weights. Another well-studied type of weighted Dirichlet spaces was initiated by Richter [12] and corresponds to the case that w is a positive harmonic function on $\mathbb{D}$. The superharmonically weighted Dirichlet spaces D_w have been introduced and studied by Aleman [1]. We refer the reader to [2, 9] and the references therein, for more information on Dirichlet spaces.

A sequence $(z_n) \subset \mathbb{D}$ is said to be a zero set for D_w if there exists $f \in D_w$ such that f is not identically zero and $f(z) = 0$ if and only if $z = z_n$ for some $n \in \mathbb{N}$. It is well known (see, e.g., [1, 5]) that D_w is contained in the Hardy–Hilbert space H^2 of holomorphic functions in $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z) < +\infty.$$

It follows (see, e.g., [8]) that every zero set (z_n) for D_w is a Blaschke sequence; that is, (z_n) satisfies the Blaschke condition

$$\sum_{n=1}^{\infty} (1-|z_n|) < +\infty.$$

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It is well known that there exist nonnegative superharmonic weights $w \in L^1(\mathbb{D})$, for which the family of zero sets of D_w is strictly contained in the family of Blaschke sequences. For example (see [7]), for the classical Dirichlet space $\mathcal{D}$, there exists a Blaschke sequence $(z_n) \subset \mathbb{D}$, which converges "very tangentially" at some point in the unit circle, that is not a zero set for $\mathcal{D}$. The same result holds for the Dirichlet spaces D_p, $p \in (0,1)$ (see [11, p. 1984] and the references therein).

In [4], Bahajji-El Idrissi and El-Fallah posed the following problem.

Problem 1.1 Determine all superharmonically weighted Dirichlet spaces for which every Blaschke sequence is a zero set.

Earlier, in [10], Guillot studied the above problem when w is a positive harmonic function. In particular, he considered weights that are Poisson integrals

$$P_\nu(z) = \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2} d\nu(\zeta),$$

where ν is a finite positive Borel measure on the unit circle $\partial\mathbb{D}$ having the form

$$\nu = \sum_{n=1}^{\infty} c_n \delta_{\zeta_n},$$

where $c_n > 0$, $(\zeta_n) \subset \partial\mathbb{D}$, and δ_a is the Dirac measure at $a \in \mathbb{C}$. The Dirichlet space corresponding to $w = P_\nu$ is denoted by D_ν. Guillot proved that every Blaschke sequence is a zero set for D_ν if and only if

$$\int_{\partial\mathbb{D}} \log\left(\int_{\partial\mathbb{D}} \frac{1}{|\zeta - \eta|^2} d\nu(\zeta) \right) ds(\eta) < +\infty, \tag{1.1}$$

where s is the normalized Lebesgue measure on $\partial\mathbb{D}$. Recently, in [6], Bao, Mashreghi, Wang, and Wulan completely solved Problem 1.1 for harmonic weights, proving that the above equivalence holds for every finite positive Borel measure ν. Moreover, they considered weights that are Green potentials

$$U_\mu(z) = \int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{z-w}\right| d\mu(w), \qquad z \in \mathbb{D},$$

where μ is a positive Borel measure on the unit disc $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} (1-|z|^2) d\mu(z) < +\infty. \tag{1.2}$$

The Green potential $U_\mu \not\equiv +\infty$ exactly when (1.2) holds (see, e.g., [3, p. 98]). The Dirichlet space corresponding to $w = U_\mu$ is denoted by D_μ. From the Riesz representation theorem (see, e.g., [3, pp. 105–106]), every positive superharmonic function w in $\mathbb{D}$ is equal to the sum $U_\mu + P_\nu$, for a positive Borel measure μ in $\mathbb{D}$ satisfying (1.2) and a finite positive Borel measure on $\partial\mathbb{D}$. Therefore, the examination of the weights U_μ and P_ν covers all positive superharmonic functions.

The case of a finite positive Borel measure μ in $\mathbb{D}$ is trivial. If $\mu(\mathbb{D}) < +\infty$, then all bounded holomorphic functions are contained in D_μ (see, e.g., [1, Proposition 3.1, p. 83]); in particular, all Blaschke products are contained in D_μ and therefore all Blaschke

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sequences are zero sets for D_μ. In what follows, we will consider infinite measures μ. In [6], the authors showed that a sufficient condition for all Blaschke sequences to be zero sets for D_μ is that

$$\int_{\partial\mathbb{D}} \log(B_\mu(\zeta))ds(\zeta) < +\infty, \tag{1.3}$$

where

$$B_\mu(\zeta) = \frac{1}{2\pi}\int_{\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(z), \qquad \zeta \in \partial\mathbb{D},$$

is the balayage of μ. Also, assuming that one of the conditions

C1) there exists $C > 0$, such that

$$\operatorname{supp}(\mu) \subset \left\{ z \in \mathbb{D} : \frac{|1-z|^4}{1-|z|} < C \right\},$$

C2) there exist $r \in (0,1)$ and an ordered Blaschke sequence $(a_n) \subset \mathbb{D}$, such that

$$\operatorname{supp}(\mu) \subset \bigcup_{n=1}^{+\infty} \left\{ z \in \mathbb{D} : \left| \frac{z - a_n}{1 - \overline{a_n} z} \right| < r \right\},$$

for the support of the measure μ holds, they proved that (1.3) is a necessary and sufficient condition for every Blaschke sequence to be a zero set for D_μ.

In this article, our purpose is to enlarge the class of infinite Borel measures μ on $\mathbb{D}$ for which every Blaschke sequence is a zero set for D_μ. This will be done by considering a class of measures satisfying a condition weaker than (C1) and show that, for these measures, condition (1.3) holds.

Let $k \in \mathbb{N}$ and $E \subset \partial\mathbb{D}$. Following [7, p. 315], a set $S \subset \mathbb{D}$ is said to have finite degree of contact k at E if there exists a constant $M > 0$ such that

$$\operatorname{dist}(z, E)^k \leq M(1 - |z|^2),$$

for all $z \in S$. In this terminology, condition (C1) says that the support of the measure μ is contained in an open set that has finite degree of contact 4 at $\{1\}$.

Let μ be an infinite positive Borel measure in $\mathbb{D}$, satisfying condition (1.2). From (1.2), it follows that μ is finite on each compact subset of $\mathbb{D}$. We will denote by $D(z, r)$ the open disc in $\mathbb{C}$ centered at $z \in \mathbb{C}$ with radius $r > 0$. Let

$$E_\mu = \{\zeta \in \partial\mathbb{D} : \mu(D(\zeta, \varepsilon)) = +\infty,\ \forall \varepsilon > 0\}.$$

We note that E_μ is not empty. Suppose to the contrary that $E_\mu = \varnothing$. Then, for every $\zeta \in \partial\mathbb{D}$, there exists $\varepsilon_\zeta > 0$, such that $\mu(D(\zeta, \varepsilon_\zeta)) < +\infty$. Since $\partial\mathbb{D}$ is compact, it will be covered by finitely many discs having finite μ-measure. Since the remaining of $\mathbb{D}$ is compact, we conclude that μ is finite, contradiction. An infinite Borel measure μ in $\mathbb{D}$ is said to have a finite degree of contact at $\partial\mathbb{D}$ if there exist $k \in \mathbb{N}$ and an open set $G \subset \mathbb{D}$, such that

1) $\mu(\mathbb{D}\backslash G) < +\infty$, and
2) G has finite degree of contact k at E_μ.

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We will need one more definition before stating the main result. A closed subset $E \subset \partial\mathbb{D}$ is said to be a Carleson set if

$$\int_{\partial\mathbb{D}} \log\left(\frac{1}{\operatorname{dist}(\zeta, E)}\right) ds(\zeta) < +\infty.$$

Theorem 1.2 *Let μ be a positive infinite Borel measure in $\mathbb{D}$ satisfying condition (1.2). Suppose that μ has a finite degree of contact at $\partial\mathbb{D}$ and E_μ is a Carleson set. Then,*

$$\int_{\partial\mathbb{D}} \log(B_\mu(\zeta)) ds(\zeta) < +\infty$$

and all Blaschke sequences are zero sets for D_μ.

As Guillot has pointed out in [10, p. 270], the proof of Theorem 1.2 will be based on the existence of outer functions vanishing in a certain order on the set E_μ. We refer the reader to [8] for the definition and information about outer functions. Theorem 1.2 will be proved in Section 2. In Section 3, we show that for measures μ having finite degree of contact 2 at $\partial\mathbb{D}$, the set E_μ has zero length. The final section contains some remarks and questions.

2 Proof of Theorem 1.2

In this section we will prove Theorem 1.2. For a bounded holomorphic function $f \in H^\infty$ in $\mathbb{D}$, we will write

$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|.$$

From the hypothesis, we have that there exist $k \in \mathbb{N}$, $M > 0$ and an open set $G \subset \mathbb{D}$, such that

$$\mu(\mathbb{D}\backslash G) < +\infty \tag{2.1}$$

and

$$\operatorname{dist}(z, E_\mu)^k \le M(1 - |z|^2), \tag{2.2}$$

for all $z \in G$. Also, since E_μ is assumed to be a Carleson set, there exists an outer function g, whose derivative g' is continuous in the closed unit disc $\overline{\mathbb{D}}$, such that $g(\zeta) = 0$ for every $\zeta \in E_\mu$ (see, e.g., [9, Theorem 4.4.3, p. 65]). It follows that g satisfies a Lipschitz condition in $\overline{\mathbb{D}}$; that is, there exists $L > 0$ such that

$$|g(z) - g(w)| \le L|z - w|,$$

for all $z, w \in \overline{\mathbb{D}}$. In particular, for all $z \in \mathbb{D}$ and $\zeta \in E_\mu$,

$$|g(z)| \le L|z - \zeta|$$

and

$$|g(z)| \le L \inf_{\zeta\in E_\mu} |z - \zeta| = L \operatorname{dist}(z, E_\mu). \tag{2.3}$$

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Let $F = g^k$ and note that F is an outer function. Then, since g and g' are bounded in $\overline{\mathbb{D}}$,

$$\|F'\|_{H^\infty} \le k\|g^{k-1}\|_{H^\infty}\|g'\|_{H^\infty} < +\infty$$

and

$$\int_{\mathbb{D}} |F'(z)|^2 U_\mu(z)dA(z) \le k^2\|g^{k-1}\|^2_{H^\infty}\|g'\|^2_{H^\infty}\int_{\mathbb{D}} U_\mu(z)dA(z). \tag{2.4}$$

From (1.2), we have that $U_\mu \not\equiv +\infty$. Let $a_0 \in \mathbb{D}$, such that $U_\mu(a_0) < +\infty$. Let

$$\sigma(z) = \frac{a_0 - z}{1 - \overline{a_0}z}, \quad z \in \mathbb{D},$$

and note that $U_\mu \circ \sigma$ is superharmonic in $\mathbb{D}$. Then,

$$\begin{aligned}\int_{\mathbb{D}} U_\mu(z)dA(z) &= \int_{\mathbb{D}} U_\mu(\sigma(w))|\sigma'(w)|^2 dA(w)\\ &= \int_{\mathbb{D}} U_\mu(\sigma(w))\frac{(1-|a_0|^2)^2}{|1-\overline{a_0}w|^4}dA(w)\\ &\le \left(\frac{1+|a_0|}{1-|a_0|}\right)^2\int_{\mathbb{D}} U_\mu(\sigma(w))dA(w)\\ &\le \left(\frac{1+|a_0|}{1-|a_0|}\right)^2 U_\mu(\sigma(0))\\ &= \left(\frac{1+|a_0|}{1-|a_0|}\right)^2 U_\mu(a_0) < +\infty. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we obtain that

$$\int_{\mathbb{D}} |F'(z)|^2 U_\mu(z)dA(z) < +\infty$$

and $F \in D_\mu$.

We will show that $F \in L^2(\mu)$. From (2.2) and (2.3), we obtain that

$$|F(z)| = |g(z)|^k \le L^k \operatorname{dist}(z, E_\mu)^k \le L^k M(1-|z|^2), \tag{2.6}$$

for every $z \in G$. Therefore,

$$\begin{aligned}\int_G |F(z)|^2 d\mu(z) &\le L^{2k}M^2\int_G (1-|z|^2)^2 d\mu(z)\\ &\le L^{2k}M^2\int_{\mathbb{D}}(1-|z|^2)d\mu(z) < +\infty. \end{aligned} \tag{2.7}$$

Also, from (2.1),

$$\int_{\mathbb{D}\setminus G} |F(z)|^2 d\mu(z) \le \|g\|^k_{H^\infty}\mu(\mathbb{D}\setminus G) < +\infty. \tag{2.8}$$

From (2.7) and (2.8), we conclude that $F \in L^2(\mu)$.

Next, we will show that all Blaschke sequences are zero sets for D_μ. Let $(z_n) \subset \mathbb{D}$ be a Blaschke sequence and let B denote the corresponding Blaschke product

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(see, e.g., [8]). Since F is an outer function in $D_\mu \cap L^2(\mu)$,

$$\int_{\mathbb{D}} |F(z)|^2(1-|B(z)|^2)d\mu(z) \leq \int_{\mathbb{D}} |F(z)|^2 d\mu(z) < +\infty$$

and from the inner–outer factorization criterion for the membership of holomorphic functions in D_μ (see [5, Theorem 4.3, p. 732]) we obtain that $BF \in D_\mu$. Since F is outer, B and BF have the same zeros in $\mathbb{D}$. Therefore, (z_n) is a zero set for D_μ.

Finally (see [5, Theorem 3.4, p. 728]), we will use the identification of $D_\mu \cap L^2(\mu)$ as the weighted Hardy space H^2_μ of the functions $f \in H^2$ whose radial limit function f^*, defined for almost every $\zeta \in \partial\mathbb{D}$ by

$$f^*(\zeta) = \lim_{r\to 1} f(r\zeta),$$

satisfies the following condition:

$$\int_{\partial\mathbb{D}} |f^*(\zeta)|^2 B_\mu(\zeta) ds(\zeta) < +\infty.$$

We have shown that F is an outer function in H^2_μ, therefore

$$\int_{\partial\mathbb{D}} |F^*(\zeta)|^2 B_\mu(\zeta) ds(\zeta) < +\infty.$$

By Jensen's inequality (see, e.g., [13, Chapter 3]),

$$\begin{aligned} +\infty &> \log\left(\int_{\partial\mathbb{D}} |F^*(\zeta)|^2 B_\mu(\zeta) ds(\zeta)\right) \\ &\geq \int_{\partial\mathbb{D}} \log\left(|F^*(\zeta)|^2 B_\mu(\zeta)\right) ds(\zeta) \\ &= \int_{\partial\mathbb{D}} \log|F^*(\zeta)|^2 ds(\zeta) + \int_{\partial\mathbb{D}} \log(B_\mu(\zeta)) ds(\zeta). \end{aligned}$$

Since F is outer, $\log|F^*| \in L^1(\partial\mathbb{D})$. It follows that

$$\int_{\partial\mathbb{D}} \log(B_\mu(\zeta)) ds(\zeta) < +\infty,$$

and the proof is complete.

3 On the set E_μ

In this section, we will show that for an infinite Borel measure μ in $\mathbb{D}$, which has a finite degree of contact 2 at $\partial\mathbb{D}$ and satisfies condition (1.2), the set E_μ has zero length. We start with an auxiliary lemma. For $\zeta \in \partial\mathbb{D}$ and $A > 0$, let

$$\Omega(\zeta, A) = \left\{z \in \mathbb{D} : |\zeta - z|^2 < A(1-|z|^2)\right\}$$

be a horodisc in $\mathbb{D}$.

Lemma 3.1 *Let μ be a Borel measure in $\mathbb{D}$ and suppose that $\mu(\Omega(\zeta, A)) = +\infty$, for some $\zeta \in \partial\mathbb{D}$ and $A > 0$. Then, $B_\mu(\zeta) = +\infty$.*

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Proof From the hypothesis, we have

$$B_\mu(\zeta) \geq \frac{1}{2\pi}\int_{\Omega(\zeta,A)} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(z) \geq \frac{1}{2\pi}\int_{\Omega(\zeta,A)} (1/A) d\mu(z) = +\infty.$$

■

We proceed to state and the main result of this section.

Theorem 3.2 *Let μ be an infinite Borel measure in $\mathbb{D}$ satisfying condition (1.2). Suppose that there exists an open set $G \subset \mathbb{D}$, such that $\mu(\mathbb{D}\backslash G) < +\infty$ and G has finite degree of contact 2 at E_μ. Then, $s(E_\mu) = 0$.*

Proof We begin by showing that $B_\mu = +\infty$ on E_μ. Let $\zeta \in E_\mu$. From the assumption, it follows that there exists $A > 0$ such that $\mu(\Omega(\zeta, A)) = +\infty$. Let λ denote the restriction of μ to $\Omega(\zeta, A)$. From Lemma 3.1,

$$B_\mu(\zeta) \geq B_\lambda(\zeta) = +\infty. \tag{3.1}$$

Also, from Theorem 1.2, we obtain that $\log(B_\mu) \in L^1(\partial\mathbb{D})$. Since $B_\mu = +\infty$ on E_μ, the conclusion follows. ■

4 Remarks and questions

Let μ be an infinite Borel measure in $\mathbb{D}$, which has a finite degree of contact at $\partial\mathbb{D}$ and satisfies condition (1.2). From Theorem 1.2, condition (1.3) holds and the outer function O_μ, satisfying $|O_\mu^*|^2 = 1/B_\mu$ on $\partial\mathbb{D}$, is well defined. From [6, inequality (2.1), p. 4], $O_\mu \in H^\infty$. Also,

$$\int_{\partial\mathbb{D}} |(BO_\mu)^*(\zeta)|^2 B_\mu(\zeta) ds(\zeta) = 1,$$

so $BO_\mu \in H^2_\mu = D_\mu \cap L^2(\mu)$ whenever B is a Blaschke product, or a general inner function in $\mathbb{D}$. In particular, the auxiliary outer function F in the proof of Theorem 1.2 can be replaced by O_μ.

It seems natural to ask whether the conclusions in our main results are true, without the assumption that the measure has finite degree of contact in the unit circle. We now state these questions formally.

Question 4.1 *Let μ be an infinite Borel measure in $\mathbb{D}$ satisfying condition (1.2). Suppose that all Blaschke sequences are zero sets for D_μ. Does it follow that $s(E_\mu) = 0$ and/or $B_\mu = +\infty$ on E_μ?*

Note that all Carleson sets in $\partial\mathbb{D}$ have zero length. A negative answer to Question 4.1 would imply that (1.3) is not a necessary condition for all Blaschke sequences to be zero sets for D_μ, while a positive answer to Question 4.1 would lead to a second question about the equivalence of the two properties. If $\log(B_\mu) \in L^1(\partial\mathbb{D})$, then the outer function O_μ is well defined, $BO_\mu \in D_\mu$ for all Blaschke products B and all Blaschke sequences would be zero sets for D_μ. The next question is about the converse of this implication.

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Question 4.2 *Let μ be a positive infinite Borel measure in $\mathbb{D}$ satisfying condition (1.2). Is it true that, all Blaschke sequences are zero sets for D_μ if and only if condition (1.3) holds?*

A positive answer on Question 4.2 would imply a positive answer to Question 4.1 and would determine all purely superharmonic Dirichlet spaces D_μ for which every Blaschke sequence is a zero set; thus, it would provide a complete solution to Problem 1.1.

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