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ARTICLE

Beyond the broken tetrahedron

August Y. Chen[1] and Bjarne Schülke[2]

[1]Cornell University, Ithaca, NY, USA and [2]Extremal Combinatorics and Probability Group, Institute for Basic Science, Daejeon, South Korea
Corresponding author: Bjarne Schülke; Email: schuelke@ibs.re.kr

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Abstract

Here we consider the hypergraph Turán problem in uniformly dense hypergraphs as was suggested by Erdős and Sós. Given a 3-graph F, the uniform Turán density $\pi_{\square}(F)$ of F is defined as the supremum over all $d \in [0,1]$ for which there is an F-free uniformly d-dense 3-graph, where uniformly d-dense means that every linearly sized subhypergraph has density at least d. Recently, Glebov, Král', and Volec and, independently, Reiher, Rödl, and Schacht proved that $\pi_{\square}(K_4^{(3)-}) = \frac{1}{4}$, solving a conjecture by Erdős and Sós. Despite substantial attention, the uniform Turán density is still only known for very few hypergraphs. In particular, the problem due to Erdős and Sós to determine $\pi_{\square}(K_4^{(3)})$ remains wide open.

In this work, we determine the uniform Turán density of the 3-graph on five vertices that is obtained from $K_4^{(3)-}$ by adding an additional vertex whose link forms a matching on the vertices of $K_4^{(3)-}$. Further, we point to two natural intermediate problems on the way to determining $\pi_{\square}(K_4^{(3)})$, and solve the first of these.

Keywords: Hypergraphs; Turán density

2020 MSC Codes: Primary: 05C65; Secondary: 05D99

1. Introduction

Turán's paper [21] in which he shows that complete, balanced, $(r-1)$-partite graphs maximise the number of edges among all K_r-free graphs is often seen as the starting point of extremal combinatorics. Already in this paper, he asked for similar results for hypergraphs. More than 80 years later, still little is known in this regard.

A k-uniform hypergraph (or k-graph) $H = (V, E)$ consists of a vertex set V and an edge set $E \subseteq V^{(k)} = \{e \subseteq V : |e| = k\}$. The extremal number for $n \in \mathbb{N}$ and a k-uniform hypergraph F is the maximum number of edges in an F-free k-uniform hypergraph on n vertices and it is denoted by $\mathrm{ex}(n, F)$. The Turán density of F is $\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{k}}$ (this limit was shown to exist by monotonicity in [13]). The aforementioned result by Turán was later generalised to the Erdős-Stone-Simonovits theorem [7, 9], which determines the Turán density of any graph F to be $\frac{\chi(F)-2}{\chi(F)-1}$, where $\chi(F)$ is the chromatic number of F.

For hypergraphs, only few (non-trivial) Turán densities are know, one prominent example being the Fano plane [2, 5, 10, 15]. The Turán density of the complete 3-uniform hypergraph on four vertices, $K_4^{(3)}$, is still not known, although it is widely conjectured to be 5/9. In fact, the problem is even open for the 3-uniform hypergraph that consists of three edges on four vertices, denoted by $K_4^{(3)-}$. For more on the hypergraph Turán problem, we refer to the survey by Keevash [14].

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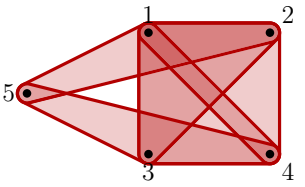


Figure 1. The hypergraph $F_\star$.

In this work, we consider a variant of the hypergraph Turán problem that was suggested by Erdős and Sós [6, 8] who asked for the maximum density of F-free hypergraphs H if we require H to be uniformly dense. More precisely, given $d \in [0,1]$ and $\eta > 0$, a 3-uniform hypergraph $H = (V, E)$ is called *uniformly* (d, η)*-dense* if for every $U \subseteq V$, we have $|U^{(3)} \cap E| \geqslant d\binom{|U|}{3} - \eta|V|^3$. Then the uniform Turán density of a 3-uniform hypergraph F is defined as

$$\pi_{\cdot\cdot}(F) = \sup\{d \in [0,1] : \text{ for every } \eta > 0 \text{ and } n \in \mathbb{N}, \text{ there is an } F\text{-free, uniformly } (d,\eta)\text{-dense 3-graph } H \text{ with } |V(H)| \geqslant n\}.$$

Erdős and Sós specifically asked to determine $\pi_{\cdot\cdot}(K_4^{(3)})$ and $\pi_{\cdot\cdot}(K_4^{(3)-})$. Similarly as with the original Turán density, these problems turned out to be very difficult. Only recently, Glebov, Král', and Volec [12] and Reiher, Rödl, and Schacht [19] independently solved the latter, showing that $\pi_{\cdot\cdot}(K_4^{(3)-}) = 1/4$ and thus confirming a conjecture by Erdős and Sós. We refer to Reiher's survey [18] for a full description of the landscape of extremal problems in uniformly dense hypergraphs. Further, Reiher, Rödl, and Schacht [20] *characterised* all 3-uniform hypergraphs F with $\pi_{\cdot\cdot}(F) = 0$ and showed that the minimum positive uniform Turán density of any 3-uniform hypergraph is at least $\frac{1}{27}$. Very recently, Garbe, Král', and Lamaison [11] indeed found 3-uniform hypergraphs whose uniform Turán density is $\frac{1}{27}$. Also very recently, Bucić, Cooper, Král', Mohr, and Correia [4] determined the uniform Turán density of all tight 3-uniform cycles of length at least five. Despite significant attention (see [1, 3, 18]), there are few hypergraphs F for which $\pi_{\cdot\cdot}(F)$ is known.

In this work, we add another example, namely $F_\star$, which we define as the hypergraph on five vertices with edge set $\{123, 124, 134, 125, 345\}$, see Fig. 1. In other words, $F_\star$ is the 3-uniform hypergraph obtained from a $K_4^{(3)-}$ by adding an additional vertex whose link is a matching.

Theorem 1.1. *We have* $\pi_{\cdot\cdot}(F_\star) = 1/4$.

We establish this result by developing a novel method to establish the existence of a hypergraph F in a uniformly (d, η)-dense hypergraph H, building off the techniques of Reiher, Rödl, and Schacht [19]. We provide a brief sketch of our arguments at the end of this section. Since this article was first made available online, our methods have been used by Li, Lin, Wang, and Zhou [16] to describe a large family of 3-uniform hypergraphs all of which have uniform Turán density 1/4. Surprisingly, an adaptation of our method has also been used by Piga, Sales, and the second author [17] to determine the *codegree* Turán density of any tight cycle minus an edge.

One potential difficulty in moving from determining $\pi_{\cdot\cdot}(K_4^{(3)-})$ to $\pi_{\cdot\cdot}(K_4^{(3)})$, which has been described as perhaps the most urgent problem in this area [18], is that there are no 3-uniform hypergraphs F with $K_4^{(3)-} \subsetneq F \subsetneq K_4^{(3)}$ that could serve as intermediate steps. However, Reiher, Rödl, and Schacht reduced the problem of determining the uniform Turán density of any hypergraph to a problem in *reduced hypergraphs* (see Theorem 2.4). The terminology of reduced hypergraphs gives rise to two natural problems 'between' determining $\pi_{\cdot\cdot}(K_4^{(3)-})$ and $\pi_{\cdot\cdot}(K_4^{(3)})$. Our second result solves the first of these. Since the notation needed is not the lightest, we postpone the introduction of the notation to Section 2 and ask the reader unfamiliar with it to read it first before continuing here.

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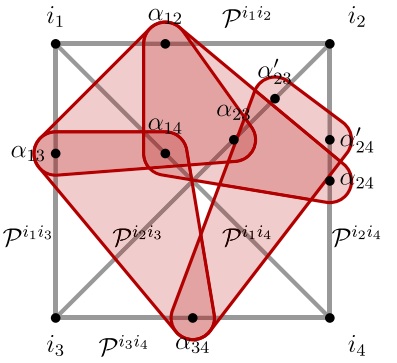


Figure 2. The reduced image of $K_4^{(3)-}$ with an additional edge and one vertex 'glued' (α_{34}).

Suppose $\mathcal{A} = (I, \mathcal{P}^{ij}, \mathcal{A}^{ijk})$ is a $(d, \therefore)$-dense reduced hypergraph and (λ, φ) is a reduced map from $K_4^{(3)-}$ (with vertex set [4] and apex at vertex 1) to $\mathcal{A}$. Let $\lambda(j) = i_j$ and $\varphi(jk) = \alpha_{jk}$ for $jk \in [4]^{(2)}$. Now note that if $\alpha_{23}\alpha_{24}\alpha_{34} \in E(\mathcal{A}^{i_2i_3i_4})$ (that is, if there is an edge in $\mathcal{A}^{i_2i_3i_4}$ which intersects $\{\alpha_{23}, \alpha_{24}, \alpha_{34}\}$ in three vertices) then (λ, φ) would actually be a reduced map from $K_4^{(3)}$ to $\mathcal{A}$. If $\mathcal{A}$ is $(d, \therefore)$-dense for any $d > 0$, then $\mathcal{A}^{i_2i_3i_4}$ does contain edges (which intersect $\{\alpha_{23}, \alpha_{24}, \alpha_{34}\}$ in zero or more vertices). This leads to the question which density d we need to guarantee that there is an edge in $\mathcal{A}^{i_2i_3i_4}$ which intersects $\{\alpha_{23}, \alpha_{24}, \alpha_{34}\}$ in at least one (or in at least two) vertices. In this sense, moving from the problem for $K_4^{(3)-}$ to the problem for $K_4^{(3)}$ can be described as 'gluing' first one, then two, and eventually three of the vertices in the constituent $\mathcal{A}^{i_2i_3i_4}$. As mentioned earlier, upon gluing three vertices, then one obtains the emergence of the desired $K_4^{(3)}$. The setting in $\mathcal{A}$ with one vertex 'glued' is shown in Fig. 2.

Our second result states that an intersection in at least one vertex (or the first gluing) is already guaranteed if the density is larger than 1/4 (which is clearly best possible). More precisely, we prove the following.

Theorem 1.2. *For $\varepsilon > 0$, there is some $M \in \mathbb{N}$ such that in every $(\frac{1}{4} + \varepsilon, \therefore)$-dense reduced hypergraph $\mathcal{A} = ([M], \mathcal{P}^{ij}, \mathcal{A}^{ijk})$, there are indices $i_1, i_2, i_3, i_4 \in [M]$ and vertices $\alpha_{jk} \in \mathcal{P}^{i_ji_k}$ for $jk \in [4]^{(2)}$, as well as vertices $\alpha'_{23} \in \mathcal{P}^{i_2i_3}$ and $\alpha'_{24} \in \mathcal{P}^{i_2i_4}$, such that $\alpha_{13}\alpha_{14}\alpha_{34} \in E(\mathcal{A}^{i_1i_3i_4})$, $\alpha_{12}\alpha_{13}\alpha_{23} \in E(\mathcal{A}^{i_1i_2i_3})$, $\alpha_{12}\alpha_{14}\alpha_{24} \in E(\mathcal{A}^{i_1i_2i_4})$, and $\alpha'_{23}\alpha'_{24}\alpha_{34} \in E(\mathcal{A}^{i_2i_3i_4})$.*

Here, α_{34} corresponds to the vertex resulting from gluing (see Fig. 2).

To determine $\pi_{\therefore}(K_4^{(3)})$, an interesting next step is to determine the optimal density d such that we find such a structure with two vertices glued, i.e., only requiring one of the vertices α'_{23} and α'_{24}.

The proofs of Theorem 1.1 and Theorem 1.2 both utilise our novel method and could be formulated as one. For both results, we need to 'glue' one vertex in the reduced hypergraph (for an illustration, see Fig. 3. for Theorem 1.1 and Fig. 2 for Theorem 1.2 respectively). For clarity however, we give the key steps separately in Section 3 (from Lemma 3.3 onward) and Section 4, respectively.

Our approach. At a high level, our method proceeds by iteratively preparing 'rows' in the reduced hypergraph in a mutually compatible fashion, and subsequently applying the pigeonhole principle across many rows. This entails the emergence of substructures in the reduced hypergraph which intersect in one vertex, which is precisely the 'glued' vertex. To prepare a single row we utilise ideas from the work of Reiher, Rödl, and Schacht [19]. The crucial new step, however, is to combine the structures from several rows by the pigeonhole principle. In order to make this possible, the preparation of one row has to be done more carefully than what a simple application of [19] would give.

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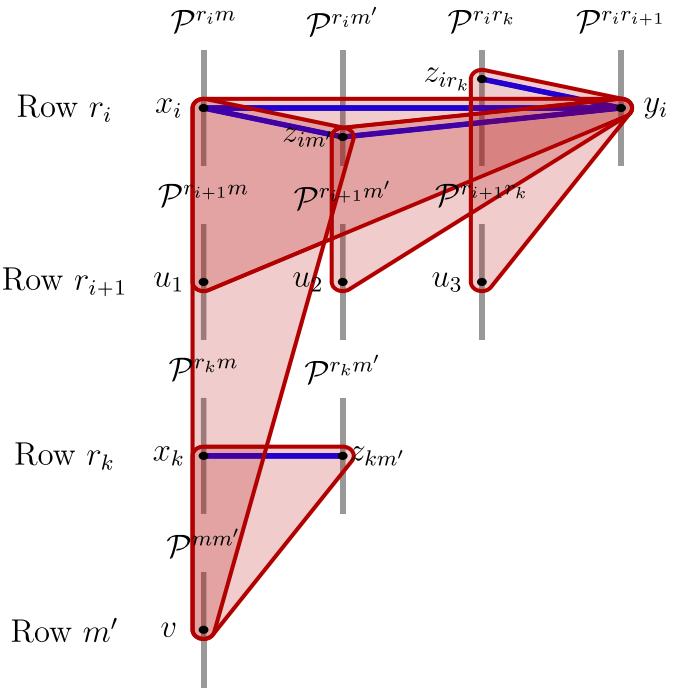


Figure 3. The reduced image of $F_\star$ in the reduced hypergraph $\mathcal{A}$. As before, the grey lines between two indices i and j represent the vertex class $\mathcal{P}^{ij}$.

It is natural to expect that this method can be used to determine the uniform Turán density of other 3-uniform hypergraphs which arise similarly by 'gluing' a vertex in the reduced hypergraph. Indeed, since the first publication of this article, Li, Lin, Wang, and Zhou [16] used our method to describe a family of 3-uniform hypergraphs all of which have uniform Turán density $\frac{1}{4}$. The aforementioned application of our method to the codegree Turán problem by Piga, Sales, and the second author from [17] is somewhat surprising, though, since usually different variants of the hypergraph Turán density behave quite differently and require separate approaches.

2. Preliminaries

When dealing with singleton sets, we may omit the parentheses, for instance, given a set I and $r \in I$, we may simply write $I \setminus r$ etc.

When proving $\pi_{\cdot\cdot}(K_4^{(3)-}) = \frac{1}{4}$, Reiher, Rödl, and Schacht [19] used the hypergraph regularity lemma. After 'regularising' a hypergraph, the problem reduces to an extremal problem in the reduced hypergraph. We will follow this approach and use the terminology introduced in [19] and laid out systematically by Reiher in his survey [18].

For a hypergraph H, we denote the number of its edges by $e(H) = |E(H)|$ and the number of its vertices by $v(H) = |V(H)|$. Given a set I, we denote by $I^{(\ell)}$ the set of all ℓ-element subsets of I. When considering such subsets, we often omit the parentheses and write for instance ijk for an element of $I^{(3)}$, instead of $\{i, j, k\}$. For a hypergraph H, we define the shadow of H as $\partial H = \{e \setminus x : x \in e \in E(H)\}$.

Definition 2.1. *Let I be a set of indices, for all $ij \in I^{(2)}$, let $\mathcal{P}^{ij}$ be a set of vertices which are disjoint for $ij \neq i'j'$, and for $ijk \in I^{(3)}$, let $\mathcal{A}^{ijk}$ be a 3-partite 3-graph with partition classes $\mathcal{P}^{ij}$, $\mathcal{P}^{ik}$, and $\mathcal{P}^{jk}$. Then the $\binom{|I|}{2}$-partite 3-graph $\mathcal{A}$ with vertex set $V(\mathcal{A}) = \bigcup_{ij \in I^{(2)}} \mathcal{P}^{ij}$ and edge set $E(\mathcal{A}) = \bigcup_{ijk \in I^{(3)}} E(\mathcal{A}^{ijk})$ is called a reduced hypergraph with index set I, vertex classes $\mathcal{P}^{ij}$ and constituents $\mathcal{A}^{ijk}$.*

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For brevity, we often simply write 'let $\mathcal{A} = (I, \mathcal{P}^{ij}, \mathcal{A}^{ijk})$ be a reduced hypergraph' instead of 'let $\mathcal{A}$ be a reduced hypergraph with index set I, vertex classes $\mathcal{P}^{ij}$ and constituents $\mathcal{A}^{ijk}$'.

Next, let us take a look at how the density condition in our host hypergraphs translates to a density condition in the respective reduced hypergraphs.

Definition 2.2. *Let $\mathcal{A} = (I, \mathcal{P}^{ij}, \mathcal{A}^{ijk})$ be a reduced hypergraph and let $d \in [0,1]$ be a real number. If $e(\mathcal{A}^{ijk}) \geqslant d|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|$ holds for all $ijk \in I^{(3)}$, we say that $\mathcal{A}$ is $(d, \therefore)$-dense.*

The following definition of a reduced map describes the setting in which a hypergraph F can be embedded into a host hypergraph H (by using the hypergraph regularity method) in terms of the reduced hypergraph of H.

Definition 2.3. *A reduced map from a 3-graph F to a reduced hypergraph $\mathcal{A} = (I, \mathcal{P}^{ij}, \mathcal{A}^{ijk})$ is a pair (λ, φ) such that*

(1) *$\lambda : V(F) \to I$ and $\varphi : \partial F \to V(\mathcal{A})$;*

(2) *for $uv \in \partial F$, we have $\lambda(u) \neq \lambda(v)$ and $\varphi(uv) \in \mathcal{P}^{\lambda(u)\lambda(v)}$;*

(3) *if $uvw \in E(F)$, then $\varphi(uv)\varphi(uw)\varphi(vw) \in E(\mathcal{A}^{\lambda(u)\lambda(v)\lambda(w)})$.*

If there is a reduced map from F to $\mathcal{A}$, we say that $\mathcal{A}$ contains a reduced image of F, otherwise $\mathcal{A}$ is called F-free.

Now the following theorem due to Reiher, Rödl, and Schacht (and appearing in this formulation in Reiher's survey [18]) reduces the problem of determining $\pi_{\therefore}(F)$ to a problem about reduced hypergraphs.

Theorem 2.4. *For a 3-graph F, we have*

$$\pi_{\therefore}(F) = \sup\{d \in [0,1] : \text{For every } M \in \mathbb{N}\text{, there is a } (d, \therefore)\text{−dense, } F\text{−free reduced hypergraph } \mathcal{A} \text{ with index set } [M]\}.$$

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Since $K_4^{(3)-} \subseteq F_\star$, we have $1/4 = \pi_{\therefore}(K_4^{(3)-}) \leqslant \pi_{\therefore}(F_\star)$.

Due to Theorem 2.4, it is thus enough to show that for every $\varepsilon > 0$, there is some large $M \in \mathbb{N}$ such that every $(\frac{1}{4} + \varepsilon, \therefore)$-dense reduced hypergraph with index set $[M]$ contains a reduced image of $F_\star$. Let thus $\varepsilon > 0$ be given, without loss of generality $\varepsilon \ll 1$, and choose $M, M_*, m \in \mathbb{N}$, and $\delta > 0$ such that

$$M^{-1} \ll M_*^{-1} \ll m^{-1} \ll \delta \ll \varepsilon \ll 1\,. \tag{3.1}$$

Then let $\mathcal{A}$ be a $(\frac{1}{4} + \varepsilon, \therefore)$-dense reduced hypergraph with index set $[M]$, vertex classes $\mathcal{P}^{ij}$ and constituents $\mathcal{A}^{ijk}$. We will show that $\mathcal{A}$ contains a reduced image of $F_\star$.

We begin with a 'cleaning process' (until Lemma 3.2) similar to the one in [19]. However, we will allow for a bit more 'robustness'. Consider any $ijk \in [M]^{(3)}$ with $i < j < k$. We define the bipartite graph Q^i_{jk} on the vertex set $\mathcal{P}^{ij} \cup \mathcal{P}^{ik}$ by placing an edge between $w \in \mathcal{P}^{ij}$ and $v \in \mathcal{P}^{ik}$ if there are at least $\varepsilon^2|\mathcal{P}^{jk}|$ vertices u in $\mathcal{P}^{jk}$ such that $wvu \in E(\mathcal{A}^{ijk})$. Similarly, we define the bipartite graph Q^k_{ij} on the vertex set $\mathcal{P}^{ik} \cup \mathcal{P}^{jk}$ by placing an edge between $v \in \mathcal{P}^{ik}$ and $w \in \mathcal{P}^{jk}$ if there are at least $\varepsilon^2|\mathcal{P}^{ij}|$ vertices u in $\mathcal{P}^{ij}$ such that $uvw \in E(\mathcal{A}^{ijk})$. We write $N_{Q^i_{jk}}(x)$ and $d_{Q^i_{jk}}(x)$ for the neighbourhood and the degree of a vertex x in the graph Q^i_{jk}, respectively, and similarly we define $N_{Q^k_{ij}}(x)$ and $d_{Q^k_{ij}}(x)$. □

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Lemma 3.1. *For any $ijk \in [M]^{(3)}$ with $i < j < k$, we have*

$$\sum_{v \in \mathcal{P}^{ik}} d_{Q^i_{jk}}(v) d_{Q^k_{ij}}(v) \geqslant \big(\frac{1}{4} + \frac{\varepsilon}{2}\big)|\mathcal{P}^{ij}||\mathcal{P}^{jk}||\mathcal{P}^{ik}|\,.$$

Proof. We use a double-counting argument. For $v \in \mathcal{P}^{ik}$, set $T(v) = \{w_1 v w_2 \in E(\mathcal{A}^{ijk})\}$ and $S(v) = \{w_1 v w_2 \in E(\mathcal{A}^{ijk}) : vw_1 \in E(Q^i_{jk}), vw_2 \in E(Q^k_{ij})\}$,

$$S_1(v) = \{w_1 v w_2 \in E(\mathcal{A}^{ijk}) : w_1 \in \mathcal{P}^{ij}, vw_1 \notin E(Q^i_{jk})\}, \text{ and}$$
$$S_2(v) = \{w_1 v w_2 \in E(\mathcal{A}^{ijk}) : w_2 \in \mathcal{P}^{jk}, vw_2 \notin E(Q^k_{ij})\}\,.$$

Note that by double-counting, we have the following.

$$\begin{aligned}\big(\frac{1}{4} + \varepsilon\big)|\mathcal{P}^{ij}||\mathcal{P}^{jk}||\mathcal{P}^{ik}| &\leqslant \sum_{v \in \mathcal{P}^{ik}} |T(v)| \leqslant \sum_{v \in \mathcal{P}^{ik}} |S(v)| + |S_1(v)| + |S_2(v)| \\ &\leqslant \sum_{v \in \mathcal{P}^{ik}} d_{Q^i_{jk}}(v) d_{Q^k_{ij}}(v) + 2\varepsilon^2 |\mathcal{P}^{ij}||\mathcal{P}^{jk}||\mathcal{P}^{ik}|\,.\end{aligned}$$

Since $\varepsilon \ll 1$, the assertion follows. □

Given $ijk \in [M]^{(3)}$ with $i < j < k$, by Lemma 3.1 and the Cauchy-Schwartz inequality, we have either

$$\sum_{v \in \mathcal{P}^{ik}} d_{Q^i_{jk}}(v)^2 \geqslant \big(\frac{1}{4} + \frac{\varepsilon}{2}\big)|\mathcal{P}^{ij}|^2|\mathcal{P}^{ik}|\,, \tag{3.2}$$

in which case we colour ijk blue, and if this is not true, we must have

$$\sum_{v \in \mathcal{P}^{ik}} d_{Q^k_{ij}}(v)^2 \geqslant \big(\frac{1}{4} + \frac{\varepsilon}{2}\big)|\mathcal{P}^{jk}|^2|\mathcal{P}^{ik}|\,,$$

in which case we colour ijk red.

By (3.1) and by Ramsey's theorem, there is a subset of M_* indices, without loss of generality $[M_*]$, such that all 3-subsets of this set are coloured with the same colour, say blue.

There is one more step in the 'cleaning process', for which we apply Ramsey's theorem to a different colouring of the 3-subsets of $[M_*]$. For $ijk \in [M_*]^{(3)}$ with $i < j < k$ and $r \in \mathbb{N}$, let

$$S^i_{jk}(r) = \{x \in \mathcal{P}^{ik} : d_{Q^i_{jk}}(x) \geqslant \big(\frac{1}{2} + r\delta\big)|\mathcal{P}^{ij}|\}\,.$$

Since ijk was coloured in blue, (3.2) entails

$$\big(\frac{1}{4} + \frac{\varepsilon}{2}\big)|\mathcal{P}^{ij}|^2|\mathcal{P}^{ik}| \leqslant \sum_{v \in \mathcal{P}^{ik}} d_{Q^i_{jk}}(v)^2 \leqslant \big(\frac{1}{2} + \delta\big)^2|\mathcal{P}^{ij}|^2|\mathcal{P}^{ik} \setminus S^i_{jk}(1)| + |\mathcal{P}^{ij}|^2|S^i_{jk}(1)|\,.$$

Because of (3.1), we infer that $|S^i_{jk}(1)| \geqslant \delta|\mathcal{P}^{ik}|$.

Clearly the $S^i_{jk}(r)$'s are decreasing and $S^i_{jk}(r) = \varnothing$ for $r > \frac{1}{2\delta}$, so we may colour ijk by the maximum $r \in [\frac{1}{2\delta}]$ (w.l.o.g., we assume that $\frac{1}{2\delta}$ is an integer) for which $|S^i_{jk}(r)| \geqslant \delta|\mathcal{P}^{ik}|$.

Again by Ramsey's theorem, as $m \ll M_*$ and after potentially reordering the indices, we can suppose that $[m]$ is a blue clique in the first colouring and an r_*-coloured clique in the second colouring, for some $r_* \in [\frac{1}{2\delta}]$. The following is the conclusion of our cleaning.

(★) Every $ijk \in [m]^{(3)}$ with $i < j < k$ satisfies (3.2) and $|S^i_{jk}(r_*)| \geqslant \delta|\mathcal{P}^{ik}| > |S^i_{jk}(r_* + 1)|\,.$

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Now, for all $i \in [m]$, define the graph $Q^i = \bigcup_{1 \leqslant i < j < k \leqslant m} Q^i_{jk}$. We need the following auxiliary result, which is a robust version of a result in [19].

Lemma 3.2. *For $i, j_1, j_2, k \in [m]$ with $i < j_1 < j_2 < k$, and any $x \in S^i_{j_1k}(r_*) \cap S^i_{j_2k}(r_*)$, there are at least $\delta|\mathcal{P}^{ij_1}||\mathcal{P}^{ij_2}|$ triangles xyz in Q^i with $y \in \mathcal{P}^{ij_2}$, $z \in \mathcal{P}^{ij_1}$.*

Proof. Let A_{j_1} be the neighbours of x in $\mathcal{P}^{ij_1}$, and let A_{j_2} be the neighbours of x in $\mathcal{P}^{ij_2}$. By definition of x (and S^i_{jk}), we have that $|A_{j_1}| \geqslant (\frac{1}{2} + r_*\delta)|\mathcal{P}^{ij_1}|$, $|A_{j_2}| \geqslant (\frac{1}{2} + r_*\delta)|\mathcal{P}^{ij_2}|$.

Also, denote $B_{j_1} = \mathcal{P}^{ij_1} \setminus A_{j_1}$ and $B_{j_2} = \mathcal{P}^{ij_2} \setminus A_{j_2}$. Now let $C = B_{j_2} \cap S^i_{j_1j_2}(r_* + 1)$. By ($\star$), we know $|C| < \delta|\mathcal{P}^{ij_2}|$, and by definition of $S^i_{j_1j_2}(r_* + 1)$, we have $d_{Q^i_{j_1j_2}}(y) < (\frac{1}{2} + (r_* + 1)\delta)|\mathcal{P}^{ij_1}|$ for all $y \in B_{j_2} \setminus C$.

Note that any edge between A_{j_1} and A_{j_2} gives rise to a triangle of the kind the statement asks for, so suppose for the sake of contradiction that there are less than $\delta|\mathcal{P}^{ij_1}||\mathcal{P}^{ij_2}|$ edges between A_{j_1} and A_{j_2}.

Then, when setting $d'_{Q^i_{j_1j_2}}(y) = |N_{Q^i_{j_1j_2}}(y) \cap A_{j_1}|$ for all $y \in A_{j_2}$, we have $\sum_{y \in A_{j_2}} d'_{Q^i_{j_1j_2}}(y) < \delta|\mathcal{P}^{ij_1}||\mathcal{P}^{ij_2}|$. Since each $d'_{Q^i_{j_1j_2}}(y) \leqslant |\mathcal{P}^{ij_1}|$, we also know $\sum_{y \in A_{j_2}} \big(d'_{Q^i_{j_1j_2}}(y)\big)^2 \leqslant \delta|\mathcal{P}^{ij_1}|^2|\mathcal{P}^{ij_2}|$. By our conditions on $|A_{j_1}|$, we have $d_{Q^i_{j_1j_2}}(y) - d'_{Q^i_{j_1j_2}}(y) \leqslant |B_{j_1}| \leqslant (\frac{1}{2} - r_*\delta)|\mathcal{P}^{ij_1}|$. Hence

$$\begin{aligned}
\sum_{y \in A_{j_2}} d_{Q^i_{j_1j_2}}(y)^2 &= \sum_{y \in A_{j_2}} \big(d_{Q^i_{j_1j_2}}(y) - d'_{Q^i_{j_1j_2}}(y) + d'_{Q^i_{j_1j_2}}(y)\big)^2 \\
&\leqslant |A_{j_2}|(\tfrac{1}{2} - r_*\delta)^2|\mathcal{P}^{ij_1}|^2 + \delta|\mathcal{P}^{ij_1}|^2|\mathcal{P}^{ij_2}| + 2(\tfrac{1}{2} - r_*\delta)\delta|\mathcal{P}^{ij_1}|^2|\mathcal{P}^{ij_2}| \\
&< |A_{j_2}|(\tfrac{1}{2} - r_*\delta)^2|\mathcal{P}^{ij_1}|^2 + 2\delta|\mathcal{P}^{ij_1}|^2|\mathcal{P}^{ij_2}|\,.
\end{aligned}$$

Now recalling ($\star$) and splitting the sum yields

$$\begin{aligned}
&\big(\tfrac{1}{4} + \tfrac{\varepsilon}{2}\big)|\mathcal{P}^{ij_1}|^2|\mathcal{P}^{ij_2}| \leqslant \sum_{x \in \mathcal{P}^{ij_2}} d_{Q^i_{j_1j_2}}(x)^2 \\
&\leqslant \sum_{x \in B_{j_2} \setminus C} d_{Q^i_{j_1j_2}}(x)^2 + \sum_{x \in A_{j_2}} d_{Q^i_{j_1j_2}}(x)^2 + \sum_{x \in C} d_{Q^i_{j_1j_2}}(x)^2 \\
&\leqslant |B_{j_2}|\big(\tfrac{1}{2} + (r_* + 1)\delta\big)^2|\mathcal{P}^{ij_1}|^2 + |A_{j_2}|\big(\tfrac{1}{2} - r_*\delta\big)^2|\mathcal{P}^{ij_1}|^2 + 3\delta|\mathcal{P}^{ij_1}|^2|\mathcal{P}^{ij_2}|\,.
\end{aligned}$$

Since $|A_{j_2}| + |B_{j_2}| = |\mathcal{P}^{ij_2}|$ and $|B_{j_2}| \leqslant (\frac{1}{2} - r_\star\delta)|\mathcal{P}^{ij_2}|$, the right-hand side is maximised when $|B_{j_2}| = (\frac{1}{2} - r_\star\delta)|\mathcal{P}^{ij_2}|$ and $|A_{j_2}| = (\frac{1}{2} + r_\star\delta)|\mathcal{P}^{ij_2}|$. Cancelling $|\mathcal{P}^{ij_1}|^2|\mathcal{P}^{ij_2}|$ entails

$$\begin{aligned}
\frac{1}{4} + \frac{\varepsilon}{2} &\leqslant \big(\tfrac{1}{2} - r_*\delta\big)\big(\tfrac{1}{2} + (r_* + 1)\delta\big)^2 + \big(\tfrac{1}{2} + r_*\delta\big)\big(\tfrac{1}{2} - r_*\delta\big)^2 + 3\delta \\
&\leqslant \big(\tfrac{1}{2} - r_*\delta\big)\big(\tfrac{1}{2} + r_*\delta\big)^2 + \big(\tfrac{1}{2} + r_*\delta\big)\big(\tfrac{1}{2} - r_*\delta\big)^2 + 6\delta \\
&< \frac{1}{4} + 6\delta\,.
\end{aligned}$$

Since $\delta \ll \varepsilon$, this yields a contradiction. □

In our proof, we will iteratively prepare a number of 'rows', each of which is a subgraph of a Q^r, by finding many triangles that are 'compatible' with each other. By definition of Q^r, for each edge in $e \in E(Q^r)$, there exists a small subset X in some vertex class of $Q^{r'}$ for some $r' > r$, such that every $x \in X$ together with e forms an edge in $\mathcal{A}$. This subset X can be thought of as the 'projection' of

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the row r onto the row r'. After preparing many rows, we will show that some of these projections have to intersect. From this, the reduced image of $F_\star$ will arise (see Fig. 4).

The following lemma encapsulates the preparation of one row which is 'paid for' by shrinking the index set.

Lemma 3.3. *Let $I \subseteq [m]$ with $|I| > \frac{2}{\delta^2}$, $m \in I$, and $r = \min I$. Then there are $J \subseteq I \setminus r$, with $m \in J$ and $|J| \geqslant \delta^2|I|$, $x \in \mathcal{P}^{rm}$, $y \in \mathcal{P}^{rr'}$, where $r' = \min J$, and $z_j \in \mathcal{P}^{rj}$ for all $j \in J \setminus \{r', m\}$ such xz_jy forms a triangle in Q^r.*

Proof. First, we find $x \in \mathcal{P}^{rm}$. As $|S^r_{jm}(r_*)| \geqslant \delta|\mathcal{P}^{rm}|$ for all $j \in I \setminus \{m, r\}$, it follows that there is some set $I' \subseteq I \setminus \{r, m\}$ such that $|I'| \geqslant \delta(|I| - 2)$ and some $x \in \bigcap_{j\in I'} S^r_{jm}(r_*)$.

Let $r' = \min I'$; next, we find $y \in \mathcal{P}^{rr'}$. For any $j \in I'$, let A_j be the set of neighbours of x in $\mathcal{P}^{rj}$. Since $x \in S^r_{jm}(r_*) \cap S^r_{r'm}(r_*)$, by Lemma 3.2, there are at least $\delta|\mathcal{P}^{rj}||\mathcal{P}^{rr'}|$ edges in Q^r between A_j and $A_{r'}$. For $j \in I' \setminus \{r'\}$, let D_j be the set of vertices in $A_{r'}$ that are connected by some edge in Q^r to a vertex in A_j. Then, $|D_j| \geqslant \frac{\delta|\mathcal{P}^{rj}||\mathcal{P}^{rr'}|}{|\mathcal{P}^{rj}|} = \delta|\mathcal{P}^{rr'}|$ and double-counting yields that there is some vertex $y \in A_{r'}$ that lies in D_j for at least $\frac{\delta|\mathcal{P}^{rr'}|(|I'|-1)}{|\mathcal{P}^{rr'}|} = \delta(|I'| - 1)$ such $j \in I'$. Denote the set of such $j \in I'$ by I'' and let $J = I'' \cup \{m, r'\}$.

Now $r' = \min J$ and for all $j \in I'' = J \setminus \{r', m\}$, we have that $y \in D_j$. Hence, there is some vertex $z_j \in A_j$ such that z_jy is an edge in Q^r. Since $z_j \in A_j$ and $y \in A_{r'}$, we know that xz_j and xy are also edges in Q^r. Thus, for all $j \in J \setminus \{r', m\}$, xz_jy is a triangle in Q^r. The lower bound on $|J|$ follows as $|J| \geqslant \delta(|I'| - 1) + 2 \geqslant \delta^2(|I| - 2) - \delta + 2 \geqslant \delta^2|I|$. □

Now, we apply Lemma 3.3 iteratively, in each step preparing a row and shrinking the index set. First, let $I_0 = [m]$, and $r_1 = \min I_0$. Lemma 3.3 provides some $I_1 \subseteq I_0 \setminus r_1$ with $m \in I_1$ and $|I_1| \geqslant \delta^2|I_0|$, $x_1 \in \mathcal{P}^{r_1m}$, $y_1 \in \mathcal{P}^{r_1r_2}$, where $r_2 = \min I_1$, and $z_{1j} \in \mathcal{P}^{r_1j}$ for all $j \in I_1 \setminus \{r_2, m\}$ such that $x_1z_{1j}y_1$ forms a triangle in Q^{r_1}.

Next, we apply Lemma 3.3 again, but this time to I_1. Generally, assume that for some $i \in [\frac{2}{\varepsilon^2}]$, we have defined sets $I_0, \dots, I_i \subseteq [m]$, indices $r_1, \dots r_{i+1} \in [m]$, and for all $j \in [i]$ and $k \in I_j \setminus \{r_{j+1}, m\}$, vertices $x_j \in \mathcal{P}^{r_jm}$, $y_j \in \mathcal{P}^{r_jr_{j+1}}$, and $z_{jk} \in \mathcal{P}^{r_jk}$ such that the following holds. For all $j \in [i]$, we have $I_j \subseteq I_{j-1} \setminus r_j$, $m \in I_j$, $r_{j+1} = \min I_j$, and $|I_j| \geqslant \delta^2|I_{j-1}|$, and further for all $k \in I_j \setminus \{r_{j+1}, m\}$, $x_jz_{jk}y_j$ forms a triangle in Q^{r_j}. Then applying Lemma 3.3 to I_i yields $I_{i+1} \subseteq I_i \setminus r_{i+1}$ with $m \in I_{i+1}$ and $|I_{i+1}| \geqslant \delta^2|I_i|$, $x_{i+1} \in \mathcal{P}^{r_{i+1}m}$, $y_{i+1} \in \mathcal{P}^{r_{i+1}r_{i+2}}$, where $r_{i+2} = \min I_{i+1}$, and $z_{i+1j} \in \mathcal{P}^{r_{i+1}j}$ for all $j \in I_{i+1} \setminus \{r_{i+2}, m\}$ such that $x_{i+1}z_{i+1j}y_{i+1}$ forms a triangle in $Q^{r_{i+1}}$.

Note that we can perform this process $\frac{2}{\varepsilon^2} + 1$ times since in the last step (and thus in all steps prior) we have that $|I_{\frac{2}{\varepsilon^2}}| \geqslant (\delta^2)^{\frac{2}{\varepsilon^2}} m > \frac{2}{\delta^2}$ (using the hierarchy (3.1)) and thus we can still construct the set $I_{\frac{2}{\varepsilon^2}+1}$ with $|I_{\frac{2}{\varepsilon^2}+1}| \geqslant 3$.

After this iteration procedure, let the maximum element of $I_{\frac{2}{\varepsilon^2}+1} \setminus m$ be m'. For all $i \in [\frac{2}{\varepsilon^2} + 1]$, since $I_{\frac{2}{\varepsilon^2}+1} \subseteq I_i$, there is some triangle $x_iz_{im'}y_i$ in Q^{r_i} and we set $e_i = x_iz_{im'}$. Note that e_i is between $\mathcal{P}^{r_im}$ and $\mathcal{P}^{r_im'}$. Since $e_i \in E(Q^{r_i})$, we know that there is a set $H(e_i) \subseteq \mathcal{P}^{m'm}$ with $|H(e_i)| \geqslant \varepsilon^2|\mathcal{P}^{m'm}|$ such that $x_iz_{im'}v \in E(\mathcal{A}^{r_im'm})$ for every $v \in H(e_i)$. Thus, there are at least three indices $i, j, k \in [\frac{2}{\varepsilon^2} + 1]$ such that $H(e_i) \cap H(e_j) \cap H(e_k) \neq \varnothing$. Without loss of generality, assume $i < j < k$.

Let $v \in H(e_i) \cap H(e_j) \cap H(e_k)$ and note that $r_k > r_j \geqslant r_{i+1}$ and that since $|I_{\frac{2}{\varepsilon^2}+1}| \geqslant 3$ and $m = \max I_{\frac{2}{\varepsilon^2}+1}$, $m' = \max I_{\frac{2}{\varepsilon^2}+1} \setminus m$, we know $r_k < m'$. Thus, there are $z_{im'} \in \mathcal{P}^{r_im'}$ and $z_{ir_k} \in \mathcal{P}^{r_ir_k}$ such that $x_iz_{im'}y_i$ is a triangle in Q^{r_i} and $z_{ir_k}y_i \in E(Q^{r_i})$ (we do not need the other edges in the respective triangle). Now these edges imply the existence of vertices $u_1 \in \mathcal{P}^{r_{i+1}m}$, $u_2 \in \mathcal{P}^{r_{i+1}m'}$, $u_3 \in$

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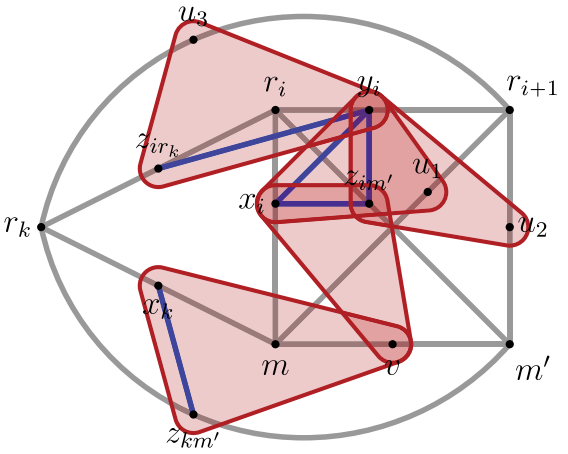


Figure 4. $F_\star$ embedded across several prepared rows in the reduced hypergraph $\mathcal{A}$. Blue lines represent edges in Q^r.

$\mathcal{P}^{r_{i+1}r_k}$ such that $x_iy_iu_1 \in E(\mathcal{A}^{r_ir_{i+1}m})$, $z_{im'}y_iu_2 \in E(\mathcal{A}^{r_ir_{i+1}m'})$, and $z_{ir_k}y_iu_3 \in E(\mathcal{A}^{r_ir_{i+1}r_k})$. Together with the edges $x_iz_{im'}v \in E(\mathcal{A}^{r_im'm})$ and $x_kz_{km'}v \in E(\mathcal{A}^{r_km'm})$, this yields a reduced embedding of $F_\star$. This iteration process and the resulting embedding is illustrated in Fig. 4.

Indeed, as shown in Fig. 3, the edges mentioned show that when setting $\lambda(1) = r_i$, $\lambda(2) = r_{i+1}$, $\lambda(3) = m$, $\lambda(4) = m'$, and $\lambda(5) = r_k$ and $\varphi(12) = y_i$, $\varphi(13) = x_i$, $\varphi(14) = z_{im'}$, $\varphi(15) = z_{ir_k}$, $\varphi(23) = u_1$, $\varphi(24) = u_2$, $\varphi(25) = u_3$, $\varphi(34) = v$, $\varphi(35) = x_k$, and $\varphi(45) = z_{km'}$, then (λ, φ) is a reduced embedding of $F_\star$ into $\mathcal{A}$. □

4. Proof of Theorem 1.2

Proof. Let $\varepsilon > 0$ be given, without loss of generality, $\varepsilon \ll 1$, and choose $M, M_*, m \in \mathbb{N}$ and $\delta > 0$ such that

$$M^{-1} \ll M_*^{-1} \ll m^{-1} \ll \delta \ll \varepsilon \ll 1\,. \tag{4.1}$$

Then let $\mathcal{A}$ be a $(\frac{1}{4} + \varepsilon)$-dense reduced hypergraph with index set $[M]$, vertex classes $\mathcal{P}^{ij}$ and constituents $\mathcal{A}^{ijk}$. We need to show that there are four indices $i_1, i_2, i_3, i_4 \in [M]$ and vertices $x_{\alpha\beta} \in \mathcal{P}^{i_\alpha i_\beta}$ for $\alpha\beta \in [4]^{(2)}$ as well as $x'_{23} \in \mathcal{P}^{i_2i_3}$ and $x'_{24} \in \mathcal{P}^{i_2i_4}$ such that $x_{12}x_{13}x_{23} \in E(\mathcal{A}^{i_1i_2i_3})$, $x_{12}x_{14}x_{24} \in E(\mathcal{A}^{i_1i_2i_4})$, $x_{13}x_{14}x_{34} \in E(\mathcal{A}^{i_1i_3i_4})$, and $x'_{23}x'_{24}x_{34} \in E(\mathcal{A}^{i_2i_3i_4})$.

We proceed as in the last section until Lemma 3.3. Instead of this lemma, we need the following variant. □

Lemma 4.1. *Let $m_1, m_2 \in \mathbb{N}$ with $m_1 \geqslant \frac{m_2+1}{\delta^{m_2}}$ and let $I \subseteq [m]$ with $|I| \geqslant m_1$, $m \in I$ and $r = \min I$. Then there are $J \subseteq I \setminus r$ with $m \in J$ and $|J| \geqslant m_2$, $x \in \mathcal{P}^{rm}$ and $z_k \in \mathcal{P}^{rk}$ for all $k \in J \setminus \{r', m\}$, where $r' = \min J$, such that for all $jk \in (J \setminus m)^{(2)}$ with $j < k$, there is some $y \in \mathcal{P}^{rj}$ such that xz_ky forms a triangle in Q^r.*

Proof. We will choose the z_k iteratively and with each choice the set of considered indices will shrink. As $|S^r_{jm}(r_*)| \geqslant \delta|\mathcal{P}^{rm}|$ for all $j \in I \setminus \{m, r\}$, it follows that there is some set $I' \subseteq I \setminus \{r, m\}$ such that $|I'| \geqslant \delta(|I| - 2)$ and some $x \in \bigcap_{j\in I'} S^r_{jm}(r_*)$. For any $k \in I'$, let A_k be the set of neighbours of x in $\mathcal{P}^{rk}$. Since for all $jk \in I'^{(2)}$ with $j < k$, we have $x \in S^r_{km}(r_*) \cap S^r_{jm}(r_*)$, by Lemma 3.2, there are at least $\delta|\mathcal{P}^{rk}||\mathcal{P}^{rj}|$ edges in Q^r between A_k and A_j. Let now $k_1 = \max I'$ and let D_j be the set of vertices in $\mathcal{P}^{rk_1}$ which lie in a triangle between A_{k_1} and A_j. Note that $|D_j| \geqslant \delta|\mathcal{P}^{rk_1}|$. Therefore, there is some set $I'_1 \subseteq I' \setminus k_1$ with $|I'_1| \geqslant \delta(|I'| - 1)$ and some $z_{k_1} \in \bigcap_{j\in I'_1} D_j$. Set $I_1 = I'_1 \cup \{k_1, m\}$

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and note that $|I_1| \geqslant \delta^2|I|$ and that for all $j \in I_1$ with $j < k_1$, there is some $y \in \mathcal{P}^{rj}$ such that $xz_{k_1}y$ forms a triangle in Q^r.

Now suppose that we have found indices $m > k_1 > \ldots > k_i$ in I for some $i < m_2$, a set $I_i \subseteq I \setminus r$ with $\{m, k_1, \ldots, k_i\} \subseteq I_i$ and $|I_i| \geqslant \delta^{i+1}|I|$, and vertices $z_k \in \mathcal{P}^{rk}$ for all $k \in \{k_1, \ldots, k_i\}$ such that for all $k \in \{k_1, \ldots, k_i\}$ and $j \in I_i$ with $j < k$, there is some $y \in \mathcal{P}^{rj}$ such that xz_ky forms a triangle in Q^r. Then let $I'_{i+1} = I_i \setminus \{m, k_1, \ldots, k_i\}$ and note that $|I'_{i+1}| \geqslant \delta^{i+1}m_1 - (i+1) \geqslant 1$ (this is where we use the lower bound on m_1; we need $|I'_{i+1}| \geqslant 1$ for the following maximum to be well defined). Let $k_{i+1} = \max I'_{i+1}$ and for $j \in I'_{i+1} \setminus k_{i+1}$, let D'_j be the set of vertices in $\mathcal{P}^{rk_{i+1}}$ which lie in a triangle between A_{k+1} and A_j. As before, we have that $|D'_j| \geqslant \delta|\mathcal{P}^{rk_{i+1}}|$ and thus, there is some set $I''_{i+1} \subseteq I'_{i+1} \setminus k_{i+1}$ and some $z_{k_{i+1}} \in \bigcap_{j \in I''_{i+1}} D'_j$ (if $I''_{i+1} = \varnothing$, pick $z_{k_{i+1}} \in \mathcal{P}^{rk_{i+1}}$ arbitrarily). Set $I_{i+1} = I''_{i+1} \cup \{m, k_1, \ldots k_{i+1}\}$ and note that

$$|I_{i+1}| \geqslant \delta(|I_i| - (i+2)) + i + 2 \geqslant \delta|I_i| \geqslant \delta^{i+2}|I|\,,$$

and that for all $k \in \{k_1, \ldots k_{i+1}\}$ and $j \in I_{i+1}$ with $j < k$, there is some $y \in \mathcal{P}^{rj}$ such that xz_ky forms a triangle in Q^r. We cease this process once I_{m_2} is chosen (corresponding to $i = m_2 - 1$ in the above). Note that $\{m, k_1, \ldots, k_{m_2}\} \subseteq I_{m_2}$.

Eventually, we set $J = \{m, k_1, \ldots, k_{m_2}\}$ and note that J, x, and z_k for $k \in J$ have the asserted properties. □

Similarly as in the last section, we will now conclude the proof by an iteration process, this time repeatedly applying Lemma 4.1. Let $m_1, \ldots m_{\frac{1}{\varepsilon^2}+1} \in \mathbb{N}$ with

$$m^{-1} = m_1^{-1} \ll m_2^{-1} \ll \cdots \ll m_{\frac{1}{\varepsilon^2}+1} \ll \delta\,, \tag{4.2}$$

and note that this is compatible with the hierarchy (4.1). Set $I_0 = [m]$, $r_1 = 1$ (note that we will use I_i here in a different sense than in the proof of Lemma 4.1 but essentially in the same sense as in the proof of Theorem 1.1). Generally, assume that we have defined, for some $i \in [\frac{1}{\varepsilon^2}]$, sets $I_0, \ldots, I_i \subseteq [m]$, indices $r_1, \ldots, r_{i+1} \in [m]$, and for each $i' \in [i]$ and $k \in I_{i'} \setminus \{r_{i'+1}, m\}$, vertices $x_{i'} \in \mathcal{P}^{r_{i'}m}$ and $z_{i'k} \in \mathcal{P}^{r_{i'}k}$ such that the following holds. For all $i' \in [i]$, we have $I_{i'} \subseteq I_{i'-1} \setminus r_{i'}$, $m \in I_{i'}$, $r_{i'+1} = \min I_{i'}$, and $|I_{i'}| \geqslant m_{i'}$, and further for all $jk \in (I_{i'} \setminus m)^{(2)}$ with $j < k$, there is some $y \in \mathcal{P}^{r_{i'}j}$ such that $x_{i'}z_{i'k}y$ forms a triangle in $Q^{r_{i'}}$.

Then we apply Lemma 4.1 with I_i instead of I, and m_i and m_{i+1} in place of m_1 and m_2, respectively. This yields a set $I_{i+1} \subseteq I_i \setminus r_{i+1}$ with $m \in I_{i+1}$ and $|I_{i+1}| \geqslant m_{i+1}$, a vertex $x_{i+1} \in \mathcal{P}^{r_{i+1}m}$, and for all $k \in I_{i+1} \setminus \{r_{i+2}, m\}$, where $r_{i+2} = \min I_{i+1}$, vertices $z_{i+1k} \in \mathcal{P}^{r_{i+1}k}$ such that for all $jk \in (I_{i+1} \setminus m)^{(2)}$ with $j < k$, there is some $y \in \mathcal{P}^{r_{i+1}j}$ such that $x_{i+1}z_{i+1k}y$ forms a triangle in $Q^{r_{i+1}}$.

Due to (4.2), we can proceed in this manner until $i = \frac{1}{\varepsilon^2} + 1$, that is, we can construct a set $I_{\frac{1}{\varepsilon^2}+1}$ with $|I_{\frac{1}{\varepsilon^2}+1}| \geqslant m_{\frac{1}{\varepsilon^2}+1}$ in this manner. Let $m' = \max I_{\frac{1}{\varepsilon^2}+1} \setminus m$. During the iteration, we found for each $i \in [\frac{1}{\varepsilon^2} + 1]$ vertices x_i and $z_{im'}$ such that, in particular, $x_iz_{im'} \in E(Q^{r_i})$. We set $e_i = x_iz_{im'}$. By definition of Q^{r_i}, there is a set $H(e_i) \subseteq \mathcal{P}^{m'm}$ with $|H(e_i)| \geqslant \varepsilon^2|\mathcal{P}^{m'm}|$ such that $x_iz_{im'}v \in E(\mathcal{A}^{r_im'm})$ for every $v \in H(e_i)$. Thus there are two indices $i, j \in [\frac{1}{\varepsilon^2} + 1]$ such that $H(e_i) \cap H(e_j) \neq \varnothing$. W.l.o.g., assume $i < j$.

Let $v \in H(e_i) \cap H(e_j)$ and let $y \in \mathcal{P}^{r_ir_j}$ such that $x_iz_{im'}y$ forms a triangle in Q_{r_i} (such a y exists due to the choice of x_i and $z_{im'}$). The definition of Q^{r_i} implies that there are vertices $u_1 \in \mathcal{P}^{r_jm}$ and $u_2 \in \mathcal{P}^{r_jm'}$ such that $x_iyu_1 \in E(\mathcal{A}^{r_ir_jm})$ and $z_{im'}yu_2 \in E(\mathcal{A}^{r_ir_jm'})$. Since we also have the edges $x_iz_{im'}v \in E(\mathcal{A}^{r_im'm})$ and $x_jz_{jm'}v \in E(\mathcal{A}^{r_jm'm})$, the indices r_i, r_j, m', and m and the vertices x_i, $z_{im'}$, y, x_j, u_1, $z_{jm'}$, u_2, and v form the desired configuration. This is illustrated in Fig. 5. □

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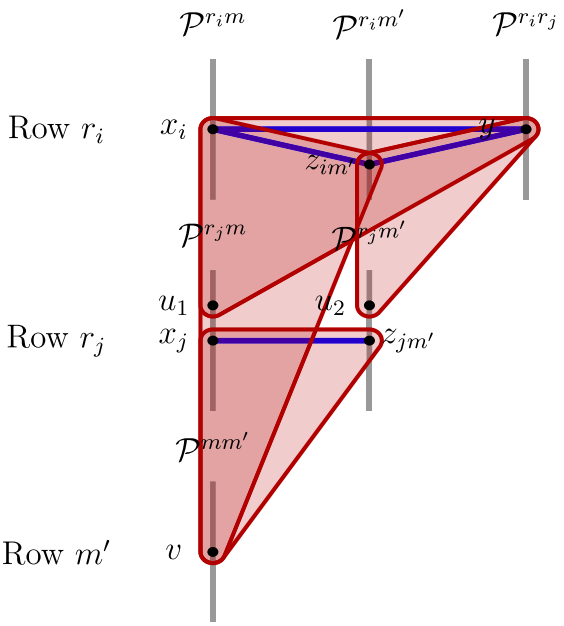


Figure 5. $K_4^{(3)-}$ with one vertex glued embedded in the reduced hypergraph $\mathcal{A}$ that we obtain, following the procedure above.

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References

[1] Balogh, J., Clemen, F. C. and Lidický, B. (2022) *Hypergraph Turán Problems in, ell$_2$-Norm, Surveys in Combina- Torics 2022*, Vol. 481 London Math. Soc. Lecture Note Ser., Cambridge: Cambridge Univ. Press, pp. 21–63. MR4421399

[2] Bellmann, L. and Reiher, C. (2019) Turán's theorem for the Fano plane. *Combinatorica* **39**(5) 961–982. DOI: 10.1007/s00493-019-3981-8. MR4039597.

[3] Berger, S., Piga, S, Reiher, C., Rödl, V. and Schacht, M. (2021) Turán density of cliques of order five in 3-uniform hypergraphs with quasirandom links. *Procedia. Comput. Sci.* **195** 412–418. DOI: 10.48550/ARXIV.2206.07354. MR4742567.

[4] Bucić, M., Cooper, J. W., KráI, D., Mohr, S. and Correia, D. M. (2023) Uniform Turán density of cycles. *Trans. Amer. Math. Soc.* **376**(7) 4765–4809. DOI: 10.1090/tran/8873. MR4608432.

[5] De Caen, D. and Füredi, Z. (2000) The maximum size of 3-uniform hypergraphs not containing a Fano plane. *J. Combin. Theory Ser. B* **78**(2) 274–276. DOI: 10.1006/jctb.1999.1938. MR1750899.

[6] Erdős, P. (1990) Problems and results on graphs and hypergraphs: similarities and differences. In *Mathematics of Ramsey Theory*, Vol. 5 Algorithms Combin., Berlin: Springer, pp. 12–28. DOI: 10.1007/978-3-642-72905-8_2. MR 1083590

[7] Erdős, P. and Simonovits, M. (1966) A limit theorem in graph theory. *Studia Sci. Math. Hungar.* **1** 51–57. MR 205876.

[8] Erdős, P. and Sós, V. T. (1982) On Ramsey—Turán type theorems for hypergraphs. *Combinatorica* **2**(3) 289–295. DOI: 10.1007/BF02579235. MR 698654

[9] Erdős, P. and Stone, A. H. (1946) On the structure of linear graphs. *Bull. Amer. Math. Soc.* **52** 1087–1091.

[10] Füredi, Z. and Simonovits, M. (2005) Triple systems not containing a Fano configuration. *Comb. Prob. Comp.* **14**(4) 467–484, DOI: 10.1017/S0963548305006784. MR 2160414.

[11] Garbe, F., Král', D. and Lamaison, A. (2024) Hypergraphs with minimum positive uniform Turán density. *Israel J. Math.* **259**(2) 701–726. DOI: 10.1007/s11856-023-2554-0. MR4732978.

[12] Glebov, R., Král', D. and Volec, J. (2016) A problem of Erdős and Sós on 3-graphs. *Israel J. Math.* **211**(1) 349–366. DOI: 10.1007/s11856-015-1267-4. MR 3474967.

[13] Katona, G., Nemetz, T. and Simonovits, M. (1964) On a problem of Turán in the theory of graphs. *Mat. Lapok* **15** 228–238, (Hungarian, with English and Russian summaries). MR 172263.

https://doi.org/10.1017/S0963548325100199 Published online by Cambridge University Press

[14] Keevash, P. (2011) Hypergraph Turán problems. In *Surveys in Combinatorics 2011*, Vol. 392 London Math. Soc. Lecture Note Ser., Cambridge: Cambridge Univ. Press, pp. 83–139. MR 2866732
[15] Keevash, P. and Sudakov, B. (2005) The Turán number of the Fano plane. *Combinatorica* **25**(5) 561–574. DOI: 10.1007/s00493-005-0034-2. MR 2176425.
[16] Li, H., Lin, H., Wang, G.H. and Zhou, W.L. (2025) Hypergraphs with a quarter uniform Turán density. *J. Oper. Res. Soc. China*. DOI: 10.1007/s40305-025-00619-7.
[17] Piga, S., Sales, M. and Schülke, B. (2023) The codegree Turán density of tight cycles minus one edge. *Comb. Prob. Comp.* **32**(6) 881–884. DOI: 10.1017/S0963548323000196.
[18] Reiher, C. (2020) Extremal problems in uniformly dense hypergraphs. *European J. Combin.* **88** 103117. DOI: 10.1016/j.ejc.2020.103117. MR 4111729,
[19] Reiher, C., Rödl, V., Schacht, M. (2018a) On a Turán problem in weakly quasirandom 3-uniform hypergraphs. *J. Eur. Math. Soc. (JEMS)* **20**(5) 1139–1159, DOI: 10.4171/JEMS/784. MR 3790065.
[20] Reiher, C., Rödl, V. and Schacht, M. (2018b) Hypergraphs with vanishing Turán density in uniformly dense hypergraphs. *J. Lond. Math. Soc.* **97**(1) 77–97, DOI: 10.1112/jlms.12095. MR 3764068.
[21] Turán, P. (1941) Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok* **48** 436–452, (Hungarian, with German summary). MR 18405.

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