J. K-Theory **7** (2011), 499–525
doi:10.1017/is011003012jkt149
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# Model structures for pro-simplicial presheaves

by

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**Abstract**

This paper displays model structures for the category of pro-objects in simplicial presheaves on an arbitrary small Grothendieck site. The first of these is an analogue of the Edwards-Hastings model structure for pro-simplicial sets, in which the cofibrations are monomorphisms and the weak equivalences are specified by comparisons of function complexes. Other model structures are built from the Edwards-Hastings structure by using Bousfield-Friedlander localization techniques. There is, in particular, an $n$-type structure for pro-simplicial presheaves, and also a model structure in which the map from a pro-object to its Postnikov tower is formally inverted.

*Key Words:* simplicial presheaves, pro-objects, $n$-types, Postnikov tower.

*Mathematics Subject Classification 2010:* Primary 14F35; Secondary 18F20, 55U35.

## Introduction

This paper describes various model structures for the category of pro-objects in simplicial presheaves on an arbitrary small Grothendieck site.

The most fundamental of these structures is a generalization of the Edwards-Hastings model structure for pro-simplicial sets [5], in which a cofibration is a monomorphism in the category of pro-objects, and a weak equivalence $f : X \to Y$ is a map which induces a weak equivalence

$$\varinjlim_j \mathbf{hom}(Y_j, Z) \xrightarrow{f^*} \varinjlim_i \mathbf{hom}(X_i, Z)$$

of (homotopy) colimit function complexes for all injective fibrant simplicial presheaves $Z$. This model structure gives the category of pro-simplicial presheaves the structure of a proper closed simplicial model category. The relevant statements are Theorem 14 and Theorem 18, which are proved in the third section of this paper. The first two sections contain preliminary results which are used in their proofs.

*This research was supported by the Natural Sciences and Engineering Research Council of Canada

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In applications, a good model structure on a category of pro-objects is one in which an object is equivalent to its Postnikov tower, and one can find such a model structure for the category of pro-simplicial presheaves by a Bousfield-Friedlander localization technique. Explicitly, one shows that the derived Postnikov tower functor $X \mapsto \mathbf{P}_* X$ satisfies a standard list of conditions, and then a theorem of Bousfield implies that there is a model structure on the pro-simplicial presheaves category for which the weak equivalences (here called pro-equivalences) are those maps $f : X \to Y$ which induce weak equivalences $\mathbf{P}_* X \to \mathbf{P}_* Y$ in the Edwards-Hastings model structure.

The construction of this pro-equivalence model structure mimics the construction of the stable category from the levelwise equivalence model structure on spectra. The general technique was recently used in Biedermann's construction of a model structure for $n$-types of simplicial presheaves [2]. Biedermann's model structure is described in Section 4 of this paper, and then promoted to an $n$-type model structure for pro-simplicial presheaves in Theorem 29. The pro-equivalence model structure is derived essentially simultaneously, and is proved in Theorem 31. Theorem 29 and Theorem 31 appear in the final section, and are the main results of this paper.

The results of this paper were designed to give a common framework for traditional étale homotopy theory and the homotopy theory of simplicial presheaves, for the purpose of properly comparing the theories. That framework is now given by categories of pro-simplicial presheaves on various étale sites, but the consequences of the main results of this paper are far from having been fully worked out.

We do know that the "pro-attitude" enabled by Theorem 31 has some relevance for cohomological descent questions, in that it provides ways of avoiding some convergence problems for descent spectral sequences arising from Postnikov towers of simplicial presheaves. There is some evidence for this claim in [11].

The referee says that the Edwards-Hastings model structure of Theorem 14 coincides with the strict model structure for pro-simplicial presheaves which is defined by Isaksen in [7] — one needs Lemma 19 below to see this. Isaksen uses a definition of weak equivalence which is much closer to the original Edwards-Hastings definition, and the central technical device of [7] is his Lemma 3.2. This result asserts, in the context of simplicial presheaves, that a natural transformation which is a pro-isomorphism can be refined up to cofinality by a transformation which has a factorization in the diagram category as a cofibration followed by a fibration, both of which are pro-isomorphisms.

The referee also points out that the model structure of Theorem 31 can be derived from the model structures of Theorem 23 by using the filtered model category techniques of Fausk and Isaksen [6]. In contrast, the proof of Theorem 31

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which is given here uses standard techniques, and is quite simple. Biedermann's idea that the Postnikov section functors behave like the classical $\Omega$-spectrum construction is really quite lovely.

I would like to thank the referee for his or her remarks.

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## 1. Pro-objects and cofinal functors

Recall that a *pro-object* in a category $\mathbf{E}$ is a functor

$$X : I \to \mathbf{E},$$

where $I$ is a small left filtered category.

If $Y$ is an object of $\mathbf{E}$, write

$$\hom(X,Y) = \varinjlim_{i \in I} \hom(X_i, Y).$$

An element of this set $\hom(X,Y)$ is a morphism $X \to Y$ in the pro-category.

More generally, if $Y : J \to \mathbf{E}$ is an arbitrary pro-object, then

$$\hom(X,Y) = \varprojlim_{j} \hom(X, Y_j)$$

specifies the set of morphisms $X \to Y$ in the *pro-category* $pro-\mathbf{E}$. Morphisms in the pro-category are often called *pro-maps*.

A pro-map $X \to Y$ can equivalently be viewed as a natural transformation

$$\hom(Y, \ ) \to \hom(X, \ )$$

of the functors $\mathbf{E} \to \mathbf{Set}$ which are pro-represented by $X$ and $Y$.

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**Example 1** 1) All natural transformations $X \to X'$ of functors defined on a left filtered category $I$ represent pro-maps in a canonical way.

2) Suppose that $\phi : I \to J$ is a functor between left filtered categories and that $X : J \to \mathbf{E}$ is a pro-object in $\mathbf{E}$, and write $\phi_* X$ for the composite pro-object $X\phi$. The identity maps $X_{\phi(i)} \to X_{\phi(i)}$ define a pro-map

$$X \to \phi_* X.$$

An isomorphism in the category $pro - \mathbf{E}$ is usually called a *pro-isomorphism*. The standard identification of pro-maps with natural transformations of pro-representable functors leads to the observation that a pro-map $\phi : X \to Y$ is a pro-isomorphism if and only if precomposition with $\phi$ induces bijections

$$\hom(Y, Z) \xrightarrow{\phi^*} \hom(X, Z)$$

for all objects $Z$ of $\mathbf{E}$.

**Lemma 2** *Suppose given a pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow[\alpha_*]{} & D \end{array}$$

*of natural transformations of set-valued functors defined on a left filtered category $I$, and suppose that $\alpha$ is a pro-isomorphism. Then the map $\alpha_*$ is a pro-isomorphism.*

*Proof:* The induced diagram

$$\begin{array}{ccc} \hom(D, Z) & \longrightarrow & \hom(C, Z) \\ \downarrow & & \downarrow \\ \hom(B, Z) & \xrightarrow[\alpha^*]{} & \hom(A, Z) \end{array}$$

is a pullback for all objects $Z$, and the map $\alpha^*$ is a bijection since $\alpha$ is a pro-isomorphism. It follows that the induced map

$$\hom(D, Z) \to \hom(C, Z)$$

is a bijection for all objects $Z$, so that $\alpha_*$ is a pro-isomorphism. □

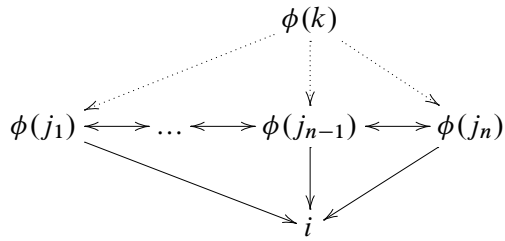
A functor $\phi : J \to I$ between left filtered categories is *cofinal* if all slice categories $\phi / i$ are path-connected — one finds this definition, for example, in [12].

**Lemma 3** *A functor $\phi : J \to I$ between left filtered categories is cofinal if and only if all slice categories $\phi / i$ are left filtered.*

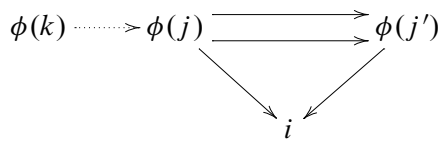
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*Proof:* Every left filtered category is path connected.

Thus, assume that all slice categories $\phi/i$ are connected. Every (solid arrow) path




has a dotted arrow lower bound in $\phi/i$, since $J$ is left filtered. Thus, any two objects of $\phi/i$ have a common lower bound. Any two morphisms




have a dotted arrow equalizer since $J$ is left filtered. □

If the functor $\phi$ is cofinal and $X$ is a pro-object defined on $I$, then the canonical map $X \to \phi_* X$ is a pro-isomorphism. One proves this by showing that the induced map

$$\hom(\phi_* X, Z) \to \hom(X, Z)$$

is an isomorphism for all objects $Z$ of **E**.

**Example 4** Suppose that $I$ is a small left filtered category. Let $\mathrm{Ar}(I)$ be the category whose objects are the arrows $\alpha : i \to j$ of $I$. Then the category $\mathrm{Ar}(I)$ is left filtered, and the source and target functors

$$s,t : \mathrm{Ar}(I) \to I$$

are cofinal.

Suppose that $X$ and $Y$ are pro-objects, defined on the categories $I$ and $J$, respectively, and let $f : X \to Y$ be a pro-map. The category $R(f)$ of representatives of $f$ has for objects the morphisms

$$\phi : X(i) \to Y(j)$$

such that $\phi$ represents a component $f_j : X \to Y(j)$ of $f$. A morphism $(\alpha,\beta) : \phi \to \psi$ of $R(f)$ consists of morphisms $\alpha : i \to i'$ of $I$ and $\beta : j \to j'$ of $J$ such that the

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diagram

$$\begin{array}{ccc} X(i) & \xrightarrow{\phi} & Y(j) \\ {\scriptstyle X(\alpha)}\downarrow & & \downarrow{\scriptstyle Y(\beta)} \\ X(i') & \xrightarrow[\psi]{} & Y(j') \end{array}$$

commutes. The category $R(f)$ is left filtered. The following result appears in [1] — its proof is an illuminating exercise.

**Lemma 5** *Both the source $s : R(f) \to I$ and target $t : R(f) \to J$ functors are cofinal, the objects $\phi : X(i) \to Y(j)$ define a natural transformation*

$$\gamma_f : s_* X \to t_* Y,$$

*and the diagram of pro-maps*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ s_* X & \xrightarrow[\gamma_f]{} & t_* Y \end{array}$$

*commutes.*

The collection of opposite categories $\alpha^{op}$ of ordinal number posets $\alpha$ is a natural source of left filtered categories. More generally, ordinal numbers are special cases of *strongly directed sets* — these are posets in which any two elements have an upper bound — and the opposite $\alpha^{op}$ of a strongly directed set $\alpha$ is a left filtered category.

Suppose that $I$ is a left filtered category, and (following [5]) let $M(I)$ be the set of all finite subcategories $d : D \subset I$ such that $D$ has an initial object $e$. A morphism of $M(I)$ is an inclusion of subcategories.

Suppose that $E$ is a finite subcategory of $I$ and that $I$ is infinite. Then there is an object $e \in I$ outside of $E$ and morphisms $\alpha_v : e \to v$ for every $v \in E$ such that the diagrams

$$\begin{array}{ccc} & e & \\ {\scriptstyle \alpha_v}\swarrow & & \searrow{\scriptstyle \alpha_{v'}} \\ v & \longrightarrow & v' \end{array}$$

commute for every morphism $v \to v'$ of $E$. Write $CE = C_e E$ for the image in $I$ of the resulting functor $E \times \mathbf{1} \to I$. Then $CE$ is a finite subcategory of $I$ and $e$ is initial in $CE$. Say that $e$ is a *cone object* for $E$.

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Suppose that $D$ and $D'$ are finite subcategories of $I$ (with initial objects), and write $D \cup D'$ for the smallest subcategory of $I$ which contains $D$ and $D'$. Every morphism of $D \cup D'$ can be written as a finite string of morphisms

$$i_0 \xrightarrow{\alpha_1} i_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} i_n$$

where each $\alpha_i$ is in $D$ or $D'$, and it follows that $D \cup D'$ is a finite subcategory of $I$. The category $C(D \cup D')$ is a finite subcategory of $I$ with initial object. Thus $C(D \cup D')$ is an upper bound for both $D$ and $D'$ in $M(I)$, and so $M(I)$ is a strongly directed set.

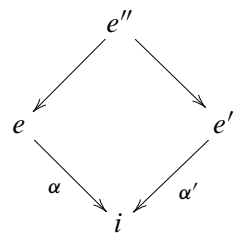
Note that an object $D$ of $M(I)$ has at most finitely many subobjects, so that the poset $M(I)$ is *cofinite* as well as strongly directed.

**Lemma 6** *Suppose that $I$ is an infinite left filtered category. Then the functor*

$$e : M(I)^{op} \to I$$

*which takes the finite subcategory $D$ to its initial object $e$ is cofinal.*

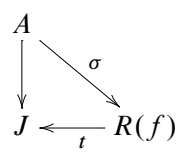
*Proof:* Suppose given finite subcategories $D, D'$ of $I$ with initial objects $e, e'$, respectively. Suppose given maps $\alpha : e \to i$ and $\alpha' : e' \to i$. Choose an object $e''$ outside of $D$ and $D'$ such that $e''$ is a cone object for $D \cup D'$ and the diagram




commutes. It follows that the category $e/i$ is connected. □

**Lemma 7** *Suppose that $f : X \to Y$ is a pro-map, where $X$ and $Y$ are defined on $I$ and $J$ respectively and $J^{op}$ is a cofinite strongly directed set. Then the target functor $t : R(f) \to J$ has a section.*

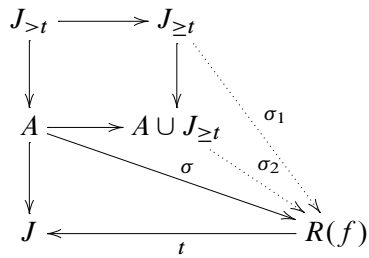
*Proof:* Consider the poset of partial sections




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which are defined on non-empty subcategories $A \subset J$ of $J$. This poset is non-empty, because for every $j \in J$ there is a representative $X(i_j) \to Y(j)$ of $f_j$, so that there is a partial section over the subcategory $\{j\}$.

Given a partial section as above, suppose that $A \neq J$. Then there is a maximal object $t$ of $J$ which is not in $A$ since $J^{op}$ is cofinite. Form the diagram




by choosing $\sigma_1$ which then induces $\sigma_2$. The map $\sigma_1$ is determined by finding a representative $\sigma_1(t) : X(i_t) \to Y(t)$ of $f_t$ such that the diagram

$$\begin{array}{ccc} X(i_t) & \xrightarrow{\sigma_1(t)} & Y(t) \\ \downarrow & & \downarrow \\ \lim\limits_{\longleftarrow s>t} X(i_s) & \xrightarrow{\sigma_*} & \lim\limits_{\longleftarrow s>t} Y(s) \end{array}$$

defined by the finite subcategory $J_{>s}$ commutes.

Finish the argument by using Zorn's lemma. □

Suppose that $f : X \to Y$ is a natural transformation of pro-objects $J \to \mathbf{E}$. Then $p$ can be identified with a pro-object $f : J \to \mathbf{E^1}$, where $\mathbf{E^1}$ is the category of morphisms in $\mathbf{E}$. If $\phi : A \to B$ is a natural transformation of pro-objects $I \to \mathbf{E}$, then a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\beta]{} & Y \end{array} \tag{1}$$

in the pro-category $pro-\mathbf{E}$ can be identified with a map $(\alpha,\beta) : \phi \to f$ in the pro-category $pro-\mathbf{E^1}$.

Suppose that $J^{op}$ is a cofinite strongly directed set. Then it follows from Lemma 7 that the target functor

$$t : R(\alpha,\beta) \to J$$

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has a section $\sigma : J \to R(\alpha, \beta)$. Write $\sigma_* = s \cdot \sigma$. It also follows that the commutative diagram (1) is defined by the commutative square of natural transformations of $J$-diagrams

$$\begin{array}{ccc} A(\sigma_*(j)) & \xrightarrow{\alpha_{\sigma(j)}} & X(j) \\ {\scriptstyle \phi(\sigma_*(j))}\downarrow & & \downarrow{\scriptstyle f(j)} \\ B(\sigma_*(j)) & \xrightarrow[\beta_{\sigma(j)}]{} & Y(j) \end{array}$$

## 2. Diagrams of simplicial presheaves

Suppose that $I$ is a small left filtered category. It is well known (see, for example, [8, 2.37] or [9]) that the category of $I$-diagrams in the category $s\mathrm{Pre}(\mathcal{C})$ of simplicial presheaves on a small site $\mathcal{C}$ has a model structure for which the weak equivalences and cofibrations are defined pointwise: a natural transformation $X \to Y$ of $I$-diagrams is a *pointwise weak equivalence* (respectively *cofibration*) if and only all component maps $X_i \to Y_i$ are local weak equivalences (respectively cofibrations) of simplicial presheaves, and the *injective fibrations* are defined by a right lifting property. This is the *injective model structure* for $I$-diagrams in simplicial presheaves.

Suppose that $J$ is a strongly directed set, and consider the corresponding category of $J^{op}$ diagrams in simplicial presheaves.

Take $s \in J$ and write $J_{\leq s}$ for the subset of $J$ consisting of all $t \leq s$. If $X : J_{\leq s}^{op} \to \mathbf{Set}$ is a functor and $L_s(X) : J^{op} \to \mathbf{Set}$ is its left Kan extension, then

$$\begin{aligned} L_s(X)(t) &= \varinjlim_{t \leq u \leq s} X(u), \\ &= \begin{cases} X(t) & \text{if } t \leq s, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Write $J_{<s}$ for the subset of $J$ consisting of all $t < s$, and let $L_{<s}Y$ be the left Kan extension of a functor $Y : J_{<s}^{op} \to \mathbf{Set}$. Then, similarly,

$$L_{<s}(X)(t) = \begin{cases} X(t) & \text{if } t < s, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

Suppose that $A \to B$ is a cofibration of $J^{op}$-diagrams, and form the diagram

$$\begin{array}{ccc} L_{<s}(A) & \longrightarrow & L_{<s}(B) \\ \downarrow & & \downarrow \\ L_{\leq s}(A) & \longrightarrow & L_{\leq s}(B) \end{array}$$

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Then

$$(L_{\leq s}(A) \cup_{L_{<s}(A)} L_{<s}(B))(t) = \begin{cases} B(t) & \text{if } t < s, \\ A(s) & \text{if } t = s, \text{ and} \\ \emptyset & \text{otherwise} \end{cases}$$

and it follows that the map

$$L_{\leq s}(A) \cup_{L_{<s}(A)} L_{<s}(B) \to L_{\leq s}(B)$$

is a cofibration which is trivial if $A \to B$ is trivial.

**Lemma 8** *Suppose that $J$ is a cofinite strongly directed set, and that $p : X \to Y$ is a morphism of $J^{op}$-diagrams in $s\mathrm{Pre}(\mathcal{C})$. Then $p$ is a fibration (respectively trivial fibration) if and only if it has the right lifting property with respect to all maps*

$$L_{\leq s}(A) \cup_{L_{<s}(A)} L_{<s}(B) \to L_{\leq s}(B)$$

*which are induced by trivial cofibrations (respectively cofibrations) $A \to B$.*

*Proof:* We'll prove the fibration statement only. The trivial fibration statement is similar.

Certainly, if $p$ is a fibration then it has the advertised lifting property by the observations above.

For the converse, it suffices to show that $p$ has the right lifting property with respect to all trivial cofibrations $D \to L_{\leq s}(B)$. In effect, consider the poset of partial lifts

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle\neq}\downarrow & \nearrow & \\ A_0 & & \downarrow{\scriptstyle p} \\ \downarrow & & \\ B & \longrightarrow & Y \end{array}$$

where $A \to B$ is a trivial cofibration. This poset is non-empty: if $x \in B(s) - A(s)$ then $x \in L_{\leq s}(B)$ , and $p$ has the right lifting property with respect to the trivial cofibration

$$L_{\leq s}(A) = L_{\leq s}(B) \cap A \to L_{\leq s}(B),$$

so there is a partial lift

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle\neq}\downarrow & \nearrow & \\ L_{\leq s}(B) \cup A & & \downarrow{\scriptstyle p} \\ \downarrow & & \\ B & \longrightarrow & Y \end{array} .$$

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The poset of partial lifts has maximal elements, by Zorn's Lemma, and a maximal partial lift must be a full lift $B \to X$, by the same argument as above.

Now consider a lifting problem

$$\begin{array}{ccc} D & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ L_{\leq s}(B) & \longrightarrow & Y \end{array}$$

such that $i$ is a trivial cofibration. The poset $J_{\leq s}$ is finite, so there is a minimal $t \leq s$ with $B(t) - D(t) \neq \emptyset$. Then

$$L_{\leq t}(B) \cap D = L_{\leq t}(D) = L_{\leq t}(D) \cup_{L_{<t}(D)} L_{<t}(B),$$

and so $p$ has the right lifting property with respect to $D \cap L_{\leq t}(B) \to L_{\leq t}(B)$. It follows that there is a non-trivial partial lift

$$\begin{array}{ccc} D & \longrightarrow & X \\ {\scriptstyle \neq}\downarrow & \nearrow & \\ L_{\leq t}(B) \cup D & & \downarrow{\scriptstyle p} \\ \downarrow & & \\ L_{\leq s}(B) & \longrightarrow & Y \end{array} .$$

Finish the proof with a Zorn's Lemma argument. □

**Corollary 9** *Suppose that $J$ is a cofinite strongly directed set. Then a map $p : X \to Y$ of $J^{op}$-diagrams in $s\mathrm{Pre}(\mathcal{C})$ is a fibration (respectively trivial fibration) if and only if all maps*

$$X_s \to \lim_{\overleftarrow{t<s}} X_t \times_{\lim_{\overleftarrow{t<s}} Y_t} Y_s$$

*are fibrations (respectively trivial fibrations).*

## 3. The Edwards-Hastings structure

Write $pro - s\mathrm{Pre}(\mathcal{C})$ for the category of pro-objects in the simplicial presheaf category $s\mathrm{Pre}(\mathcal{C})$.

A pro-map $j : A \to B$ is a *cofibration* if it is isomorphic in the pro-category to natural transformation which is a pointwise cofibration. A pro-map $p : X \to Y$ is a *fibration* if it is a retract of a fibration $Z \to W$ of $J^{op}$-diagrams, where $J$ is a cofinite strongly directed set.

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A pro-map $f : X \to Y$ is a *weak equivalence* if and only if it induces weak equivalences of simplicial sets

$$f^* : \mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$$

for all injective fibrant simplicial presheaves $Z$. Here, the filtered colimit of function complexes

$$\mathbf{hom}(X, Z) = \varinjlim_{i \in I} \mathbf{hom}(X_i, Z)$$

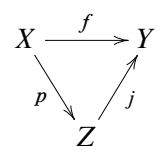
defines the object $\mathbf{hom}(X, Z)$ in the category of simplicial sets.

Observe that any natural transformation of $I$-diagrams which is a pointwise weak equivalence is a weak equivalence of pro-objects in the sense just described.

**Lemma 10** *A pro-map $j : A \to B$ is a cofibration if and only if it is a monomorphism of the pro-category.*

*Proof:* Suppose that $j : A \to B$ is a pointwise cofibration. Then $j$ is a natural map of functors $I \to s\mathrm{Pre}(\mathcal{C})$ such that each component map $j : A_i \to B_i$ is a monomorphism of simplicial presheaves. The map $j$ is a monomorphism in the pro-category, by an easy argument.

Suppose that $f : X \to Y$ is a natural transformation of pro-objects $I \to s\mathrm{Pre}(\mathcal{C})$ which represents a monomorphism. Then $f$ has an epi-monic factorization




such that $p$ is a pointwise epimorphism and $j$ is a pointwise monomorphism. The map $p$ is also an epimorphism in the pro-category.

But then $p$ is an isomorphism of the pro-category. In effect, the diagram

$$X \times_Z X \rightrightarrows X \xrightarrow{p} Z$$

is a coequalizer in the pro-category. The map $p$ is a monomorphism, and the composites $p \cdot pr_1$, $p \cdot pr_2$ with the two projections in the coequalizer picture are equal, so that $pr_1 = pr_2$ in the pro-category. The identity $1_X$ satisfies $1_X \cdot pr_1 = 1_X \cdot pr_2$, so there is a morphism $\sigma : Z \to X$ such that $\sigma \cdot p = 1_X$. Finally, $p \cdot \sigma \cdot p = p$ and $p$ is an epimorphism, so that $p \cdot \sigma = 1_Z$.

It then follows that the map $f$ is isomorphic to the pointwise cofibration $j$ in the pro-category. □

**Corollary 11** *The class of cofibrations is closed under retraction.*

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The following result is a consequence of the discussion following Lemma 7.

**Lemma 12** *1) Every fibration has the right lifting property with respect to all pointwise trivial cofibrations.*

*2) Suppose that $p : Z \to W$ is a pointwise trivial injective fibration of $J^{op}$-diagrams, where $J$ is a strongly directed cofinite set. Then $p$ has the right lifting property with respect to all cofibrations.*

*Proof:* For the proof of statement 1), suppose that $p : X \to Y$ is an injective fibration of $J^{op}$-diagrams, where $J$ is a cofinite strongly directed set, and suppose that the natural transformation $i : A \to B$ is a pointwise trivial cofibration of $I$-diagrams. Suppose given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \tag{2}$$

Then there is a functor $\sigma_* : J \to I$ such that the diagram (2) is defined by a commutative diagram of $J$-diagrams

$$\begin{array}{ccc} A(\sigma_*(j)) & \xrightarrow{\alpha_{\sigma(j)}} & X(j) \\ {\scriptstyle i(\sigma_*(j))}\downarrow & \nearrow & \downarrow{\scriptstyle f(j)} \\ B(\sigma_*(j)) & \xrightarrow[\beta_{\sigma(j)}]{} & Y(j) \end{array}$$

The transformation $i\sigma_*$ is a trivial cofibration of $J$-diagrams, so that the dotted arrow lifting exists in $J$-diagrams, and hence in the pro-category.

The proof of statement 2) is similar. □

**Corollary 13** *Every pro-map $f : X \to Y$ has factorizations*

$$\begin{array}{ccccc} & & U & & \\ & {\scriptstyle i}\nearrow & & \searrow{\scriptstyle p} & \\ X & & \xrightarrow{f} & & Y \\ & {\scriptstyle j}\searrow & & \nearrow{\scriptstyle q} & \\ & & V & & \end{array}$$

*where $i$ is a trivial cofibration and $p$ is a fibration, and $j$ is a cofibration and $q$ is a trivial fibration.*

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*Proof:* The map $f$ is pro-isomorphic to a natural transformation $f'$ defined on a cofinite strongly directed set $J$, by Lemma 5 and Lemma 6. The natural transformation $f'$ has the required factorizations in the injective model structure on the category $s\mathrm{Pre}(\mathcal{C})^J$ of $J$-diagrams. □

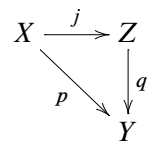
**Theorem 14** *The category $pro-s\mathrm{Pre}(\mathcal{C})$, with the definitions of cofibration, fibration and weak equivalence given above, has the structure of a closed model category.*

The model structure for the category $pro-s\mathrm{Pre}(\mathcal{C})$ of Theorem 14 specializes to the Edwards-Hastings model structure for pro-simplicial sets [5]. We shall say that the model structure of Theorem 14 is the *Edwards-Hastings model structure* for pro-simplicial presheaves.

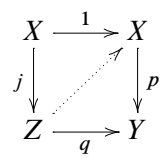
*Proof:* The completeness axiom **CM1**, the weak equivalence axiom **CM2** and the retract axiom **CM3** are trivial to verify. The factorization axiom **CM5** is a consequence of Corollary 13.

Lemma 16 below asserts that every fibration has the right lifting property with respect to all trivial cofibrations.

Subject to having Lemma 16, suppose that the map $p : X \to Y$ is a fibration and a weak equivalence, and suppose further that $p$ is an injective fibration of $J^{op}$-diagrams, where $J$ is a strongly directed cofinite set. Then $p$ has a factorization




in $J^{op}$-diagrams, such that $q$ is a trivial injective fibration and $j$ is a cofibration. But then $j$ is also a weak equivalence, so the lifting exists in the diagram




by Lemma 16, so that $p$ is a retract of $q$. It then follows from Lemma 12 that $p$ has the right lifting property with respect to all cofibrations. We have verified **CM4**. □

**Lemma 15** *Suppose that the cofibration $i : A \to B$ has the left lifting property with respect to all fibrations $p : X \to Y$ such that $X$ and $Y$ are fibrant. Then $i$ has the left lifting property with respect to all fibrations.*

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*Proof:* Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where $p$ is an injective fibration of $J$-diagrams and $J$ is a cofinite strongly directed set. Form the diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & LX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ Y & \xrightarrow[j]{} & LY \end{array}$$

in $J$-diagrams, where the horizontal maps are fibrant models. Find a factorization

$$\begin{array}{ccc} LX & \xrightarrow{\alpha} & Z \\ & {\scriptstyle p_*}\searrow & \downarrow{\scriptstyle q} \\ & & LY \end{array}$$

of $p_*$ in $J$-diagrams, where $q$ is a fibration and $\alpha$ is a trivial cofibration. Then $i : A \to B$ has the left lifting property with respect to the induced fibration $q_* : Y \times_{LY} Z \to Y$, and the map $\theta : X \to Y \times_{LY} Z$ is a weak equivalence of $J$-diagrams because the injective model structure on $J$-diagrams is proper.

Find a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & V \\ & {\scriptstyle \theta}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \times_{LY} Z \end{array}$$

of the weak equivalence $\theta$ in $J$-diagrams, where $j$ is a trivial cofibration and $\pi$ is a trivial fibration. Then the cofibration $i : A \to B$ has the left lifting property with respect to $\pi$ by Lemma 12, so that $i$ has the left lifting property with respect to the composite $q_* \cdot \pi$. The map $p$ is a retract of $q_* \cdot \pi$ by a standard argument, so that $i$ has the left lifting property with respect to $p$. □

**Lemma 16** *Suppose that the pro-map $j : A \to B$ is a cofibration and a weak equivalence. Then $j$ has the left lifting property with respect to all fibrations $p : X \to Y$.*

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*Proof:* On account of Lemma 15, we can assume that $p : X \to Y$ is an injective fibration of $J^{op}$ diagrams for some cofinite strongly directed set $J$, and that $X$ and $Y$ are injective fibrant.

Recall that a pro-map $\phi : A \to X$ consists of pro-maps $\phi_s : A \to X_s$, $s \in J$, such that the diagrams

$$\begin{array}{ccc} & \overset{\phi_s}{\nearrow} & X_s \\ A & & \downarrow \\ & \underset{\phi_t}{\searrow} & X_t \end{array} \tag{3}$$

commute for all $t < s$.

Consider the special case in which $p : X \to Y$ is an injective fibration of simplicial presheaves, and that the objects $X$ and $Y$ are injective fibrant. Suppose that the map $j : A \to B$ is a natural transformation of functors $I \to s\mathrm{Pre}(\mathcal{C})$, and that $j$ is a pointwise cofibration. In the diagram

$$\begin{array}{ccc} \mathbf{hom}(B,X) & \xrightarrow{p_*} & \mathbf{hom}(B,Y) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ \mathbf{hom}(A,X) & \xrightarrow[p_*]{} & \mathbf{hom}(A,Y) \end{array}$$

the instances of $j^*$ are weak equivalences (in fact, trivial fibrations) of simplicial sets, so that the fibration

$$(j^*, p_*) : \mathbf{hom}(B,X) \to \mathbf{hom}(A,X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B,Y)$$

(which is a filtered colimit of fibrations, indexed on $I$) is trivial. It follows that the dotted arrow lifting exists in the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

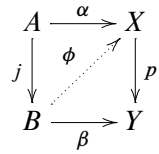
so that the map $p : X \to Y$ has the right lifting property with respect to $j$.

More generally, if $p : X \to Y$ is an injective fibration of simplicial presheaves with $Y$ not necessarily injective fibrant, then $p : X \to Y$ has the right lifting property with respect to $j$ — see the argument for Lemma 15.

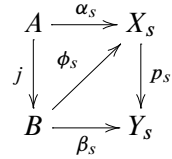
Suppose that $S \subset J$ is subposet of $J$ which is closed under subobjects in the

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sense that if $t \leq s$ and $s \in S$, then $t \in S$. Consider the lifting problem

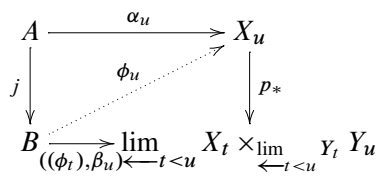



A partial lift $\phi$ on $S$ consists of a collection of pro-maps $\phi_s : B \to X_s$, $s \in S$, such that the diagrams




commute, and that $\phi$ is functorial on $S$ in the sense that the diagrams (3) commute for $t \leq s$ in $S$.

Suppose that the element $u$ of $J - S$ is minimal. If $t < u$ then $t \in S$, and there is a commutative diagram of pro-maps

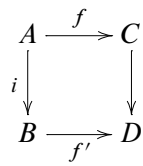



The map $p_*$ is an injective fibration of simplicial presheaves by Corollary 9, and so the dotted arrow $\phi_u$ exists, making the diagram commute.

Every partial lift $\phi$ defined on a subset $S$ of $J$ which is closed under subobjects can therefore be extended to a partial lift defined on $S \cup \{u\}$ where $u \in J - S$ is minimal. A Zorn's Lemma argument finishes the proof. □

**Lemma 17** *The Edwards-Hastings model structure on* $pro - s\mathrm{Pre}(\mathcal{C})$ *is proper.*

*Proof:* Suppose given a pushout diagram




such that $i$ is a cofibration and $f$ is a weak equivalence. Suppose that $Z$ is an

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injective fibrant simplicial presheaf. Then the diagram

$$\begin{array}{ccc} \mathbf{hom}(D,Z) & \xrightarrow{f'^*} & \mathbf{hom}(B,Z) \\ \downarrow & & \downarrow{\scriptstyle i^*} \\ \mathbf{hom}(C,Z) & \xrightarrow[f^*]{} & \mathbf{hom}(A,Z) \end{array}$$

is a pullback in simplicial sets, in which the map $i^*$ is a fibration and $f^*$ is a weak equivalence. It follows that $f'^*$ is a weak equivalence. This is true for all injective fibrant objects $Z$, so that $f'$ is a weak equivalence.

Suppose given a pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{f'} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[f]{} & Y \end{array}$$

in which $p$ is a fibration and $f$ is a weak equivalence. We can assume that this is a diagram of natural transformations on a strongly directed cofinite set $J$. In this case, $p$ must be an injective fibration of $J$-diagrams, because it has the right lifting property with respect to all sectionwise trivial cofibrations. The map $f$ has a factorization

$$\begin{array}{ccccc} B & & \xrightarrow{f} & & Y \\ & {\scriptstyle i}\searrow & & \nearrow{\scriptstyle q} & \\ & & U & & \end{array}$$

in the $J$-diagram category, where $i$ is a sectionwise trivial cofibration and $q$ is an injective fibration. Then $q$ is a fibration and a weak equivalence of the pro-category, so that it pulls back to a trivial fibration $q_*$ along $p$. In the diagram

$$\begin{array}{ccccc} A & \xrightarrow{i_*} & U \times_Y Y & \xrightarrow{q_*} & X \\ \downarrow & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[i]{} & U & \xrightarrow[q]{} & Y \end{array}$$

the map $p_*$ is an injective fibration and $i$ is a sectionwise weak equivalence, so that the map $i_*$ is a sectionwise weak equivalence. It follows that the map $f' = q_* i_*$ is a weak equivalence of the pro-category. □

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Suppose that $X$ and $Y$ are pro-simplicial presheaves. The function complex $\mathbf{hom}(X,Y)$ is the simplicial set whose $n$-simplices are the pro-maps $X \times \Delta^n \to Y$. There are isomorphisms

$$\hom(K, \mathbf{hom}(X,Y)) \cong \hom(X \times K, Y)$$

for pro-objects $X$ and $Y$. If $i : A \to B$ is a cofibration of pro-objects and $j : K \to L$ is a cofibration of simplicial sets, then the induced map

$$(B \times K) \cup_{(A \times K)} (A \times L) \to B \times L$$

is a cofibration, which is trivial if either $i$ or $j$ is trivial. The category $pro-s\mathrm{Pre}(\mathcal{C})$ therefore has the structure of a closed simplicial model category.

We therefore have the following:

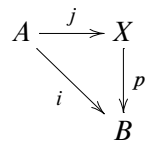
**Theorem 18** *The Edwards-Hastings model structure on $pro-s\mathrm{Pre}(\mathcal{C})$ satisfies the axioms for a proper closed simplicial model category.*

The following result will be important in applications:

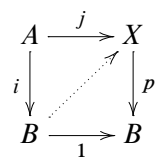
**Lemma 19** *1) Suppose that the pro-map $i : A \to B$ is a cofibration and a weak equivalence. Then $i$ is a retract of a natural transformation $j : A \to X$ which is a pointwise cofibration and a pointwise weak equivalence.*

*2) Suppose that $p : X \to Y$ is a fibration and a weak equivalence. Then $p$ is a retract of a natural transformation $q : Z \to Y$ which is an injective fibration and a pointwise weak equivalence.*

*Proof:* Suppose that $i : A \to B$ is a cofibration and a weak equivalence. We can assume that $i$ is a natural transformation of $J^{op}$-diagrams, where $J$ is a cofinite strongly directed set. The map $i$ has a factorization




in $J^{op}$-diagrams, where $j$ is a pointwise cofibration and a pointwise weak equivalence, and $p$ is an injective fibration. Then $p$ is a fibration for the Edwards-Hastings structure, so that the lift exists in the diagram




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and so $i$ is a retract of $j$ in the pro-category.

The proof of statement 2) is similar. □

**Remark 20** 1) Lemma 19 implies that the model structure of Theorem 14 and Theorem 18 specializes to the Edwards-Hastings model structure on pro-simplicial sets. It is a consequence of Lemma 19 that the two model structures have the same weak equivalences, while they have the same fibrations by definition.

2) Lemma 19 also implies that the Edwards-Hastings weak equivalences for pro-objects in simplicial presheaves coincide with the strict weak equivalences, as defined by Isaksen [7]. The Edwards-Hastings structure and Isaksen's strict model structure also have the same cofibrations, so these two model structures coincide.

## 4. $n$-types of simplicial presheaves

Suppose that $X$ is a simplicial presheaf, and write

$$\mathbf{P}_n(X) = LP_n \mathrm{Ex}^\infty X$$

and let $p_n : X \to \mathbf{P}_n X$ be the natural map. Here, $\mathrm{Ex}^\infty$ is the natural fibrant model for simplicial sets, and $Y \to LY$ is the natural injective fibrant model for simplicial presheaves. Following [2], a simplicial presheaf map $f : X \to Y$ is an *$n$-equivalence* if the induced map

$$\mathbf{P}_n X \to \mathbf{P}_n Y$$

is a local (hence sectionwise) weak equivalence. Equivalently, this means that the diagrams

$$\begin{array}{ccc} \tilde{\pi}_k X & \longrightarrow & \tilde{\pi}_k Y \\ \downarrow & & \downarrow \\ \tilde{X}_0 & \longrightarrow & \tilde{Y}_0 \end{array}$$

are pullbacks in the sheaf category for $1 \leq k \leq n$, and that the map

$$\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$$

is an isomorphism of sheaves. It follows that the functor $X \mapsto \mathbf{P}_n X$ preserves weak equivalences, and that the map $p_n : X \to \mathbf{P}_n X$ is an $n$-equivalence for $n \geq 0$.

**Remark 21** It is part of the canon of classical homotopy theory that an $n$-equivalence is a map $f : X \to Y$ whose homotopy fibres are $n$-connected. This definition is not quite applicable in simplicial presheaf categories, because the

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homotopy fibres of a map of simplicial presheaves are only locally defined and hence lose much of their usual meaning.

The $n^{th}$ derived Postnikov section functor $X \mapsto \mathbf{P}_n X$ is globally defined for simplicial presheaves $X$, and has the nice stability properties described below. The corresponding definition of $n$-equivalence which is given here is universal for simplicial presheaf and diagram categories, and is easily manipulated. From this point of view, it is the right definition of $n$-equivalence for simplicial presheaves.

Here is a list of axioms for a functor $Q : s\mathrm{Pre}(\mathcal{C}) \to s\mathrm{Pre}(\mathcal{C})$ equipped with a natural transformation $\eta : X \to QX$:

**A4** The functor $Q$ preserves weak equivalences.

**A5** The maps $\eta, Q(\eta) : QX \to QQX$ are weak equivalences.

**A6** Suppose given a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

with $p$ a fibration, such that the maps $\eta : Y \to QY$, $\eta : X \to QX$ and $QB \to QY$ are weak equivalences. Then $QA \to QX$ is a weak equivalence.

A simplicial presheaf $X$ for which the natural map $X \to \mathbf{P}_n X$ is a local weak equivalence is an *$n$-type*. This means precisely that the homotopy group sheaves $\tilde{\pi}_k X$ are trivial for $k > n$. It is an exercise to show that any simplicial presheaf $\mathbf{P}_n Y$ is an $n$-type.

**Lemma 22** *Suppose given a pullback diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

*with $p$ a fibration, such that $Y$ is an $n$-type and the map $B \to Y$ is an $n$-equivalence. Then $A \to X$ is an $n$-equivalence.*

Lemma 22 can be proved with a comparison of long exact sequences in sheaves of homotopy groups. It can also be proved with a Boolean localization argument which reduces the statement of the Lemma to its analogue for simplicial sets.

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Say that an *n-fibration* is a map which has the right lifting property with respect to all cofibrations which are $n$-equivalences.

It is a consequence of the constructions that the functor $Q = \mathbf{P}_n$ satisfies axioms **A4** and **A5**, and it follows from Lemma 22 that the axiom **A6** is satisfied. The following theorem of Biedermann [2] is then a formal consequence of a result of Bousfield [3], which is itself a refinement of the Bousfield-Friedlander localization technique [4].

**Theorem 23** *1) The category $s\mathrm{Pre}(\mathcal{C})$ of simplicial presheaves on $\mathcal{C}$, together with the classes of cofibrations, $n$-equivalences and $n$-fibrations, satisfies the axioms for a proper closed simplicial model category.*

*2) A map $p : X \to Y$ is an $n$-fibration if and only if it is an injective fibration and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathbf{P}_n X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ Y & \xrightarrow[\eta]{} & \mathbf{P}_n Y \end{array}$$

*is homotopy cartesian in the injective model structure for $s\mathrm{Pre}(\mathcal{C})$.*

**Remark 24** Biedermann shows [2], with a simple argument, that the model structure of Theorem 23 is cofibrantly generated. This result also follows from a bounded cofibration argument, as in [10].

## 5. The pro-equivalence model structure

Throughout this section, the cofibrations, weak equivalences and fibrations for the category $pro - s\mathrm{Pre}(\mathcal{C})$ are those of the Edwards-Hastings model structure of Theorem 14.

The Postnikov tower construction gives a natural pro-map $\eta : X \to \mathbf{P}_* X$ for all pro-simplicial presheaves $X$.

A pro-map $f : X \to Y$ is an *n-equivalence* if the induced map $\mathbf{P}_n X \to \mathbf{P}_n Y$ is a weak equivalence (ie. an Edwards-Hastings weak equivalence). The map $f : X \to Y$ is said to be a *pro-equivalence* if it is an $n$-equivalence for all $n \geq 0$.

**Lemma 25** *Every weak equivalence $f : X \to Y$ is an $n$-equivalence, for all $n \geq 0$.*

*Proof:* By Lemma 19, if the pro-map $i : A \to B$ is a cofibration and a weak equivalence in the Edwards-Hastings model structure, then it is a retract of a natural transformation $j : A \to X$ which is a pointwise trivial cofibration. The map $\mathbf{P}_n A \to \mathbf{P}_n X$ is a pointwise weak equivalence since the functor $X \mapsto \mathbf{P}_n X$ preserves weak

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equivalences of simplicial presheaves. It follows that the pro-map $i_* : \mathbf{P}_n A \to \mathbf{P}_n B$ is a weak equivalence.

Lemma 19 also implies that the functor $\mathbf{P}_n$ takes trivial fibrations to weak equivalences. □

**Lemma 26** *1) The natural pro-map $\eta : X \to \mathbf{P}_* X$ is a pro-equivalence.*

*2) The pro-map $\eta : \mathbf{P}_* X \to \mathbf{P}_* \mathbf{P}_* X$ is a weak equivalence.*

*Proof:* Suppose that $Z$ is an injective fibrant simplicial presheaf.

For a fixed $s$, the map

$$\varinjlim_k \mathbf{hom}(\mathbf{P}_n \mathbf{P}_k X_s, Z) \to \mathbf{hom}(\mathbf{P}_n X_s, Z)$$

is a weak equivalence, since the map $\mathbf{P}_n X_s \to \mathbf{P}_n \mathbf{P}_k X_s$ is a weak equivalence of simplicial presheaves for $k \geq n$. It follows that the induced map

$$\varinjlim_s \varinjlim_k \mathbf{hom}(\mathbf{P}_n \mathbf{P}_k X_s, Z) \to \varinjlim_s \mathbf{hom}(\mathbf{P}_n X_s, Z)$$

is a weak equivalence, giving statement 1).

Statement 2) follows from essentially the same observation: the map

$$\mathbf{P}_n X_s \to \mathbf{P}_k \mathbf{P}_n X_s$$

is a weak equivalence if $k \geq n$. □

**Lemma 27** *A pro-map $f : X \to Y$ is a pro-equivalence if and only if the induced pro-map $\mathbf{P}_* X \to \mathbf{P}_* Y$ is a weak equivalence.*

*Proof:* Suppose that the induced map $f_* : \mathbf{P}_* X \to \mathbf{P}_* Y$ is a weak equivalence. The induced map

$$\mathbf{P}_n \mathbf{P}_* X \to \mathbf{P}_n \mathbf{P}_* Y$$

is a weak equivalence by Lemma 25, and in the diagram

$$\begin{array}{ccc} \mathbf{P}_n X & \longrightarrow & \mathbf{P}_n Y \\ \downarrow & & \downarrow \\ \mathbf{P}_n \mathbf{P}_* X & \longrightarrow & \mathbf{P}_n \mathbf{P}_* Y \end{array}$$

the vertical maps are weak equivalences by Lemma 26. It follows that $f : X \to Y$ is an $n$-equivalence. This is true for all $n \geq 0$, so that $f$ is a pro-equivalence.

The converse is trivial. □

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Say that a pro-object $X$ is an *$n$-type* if the canonical natural transformation $X \to \mathbf{P}_n(X)$ is a weak equivalence.

We have the following analogue of Lemma 22:

**Lemma 28** *Suppose given a pullback diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array} \tag{4}$$

*in $pro-s\mathrm{Pre}(\mathcal{C})$ such that $p$ is a fibration, $Y$ is an $n$-type, and $B \to Y$ is an $n$-equivalence. Then the map $A \to X$ is an $n$-equivalence.*

*Proof:* The diagram (4) is isomorphic to a diagram of natural transformations of $J^{op}$-diagrams, where $J$ is a strongly directed cofinite set, and where the map $p$ is an injective fibration of $J^{op}$-diagrams. It follows that all component simplicial presheaf maps $p : X_s \to Y_s$ are injective fibrations of simplicial presheaves.

Form the diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \xrightarrow{i} & Z \\ \downarrow & & \downarrow & & \downarrow q \\ B & \longrightarrow & Y & \xrightarrow[\simeq]{} & \mathbf{P}_n Y \end{array}$$

so that $q$ is an injective fibration and $i$ is a pointwise trivial cofibration. The induced map

$$\mathbf{P}_n B \times_{\mathbf{P}_n Y} Z \to Z$$

is a weak equivalence, as is the map

$$A \to B \times_{\mathbf{P}_n Y} Z,$$

by properness of the Edwards-Hastings structure. In the pullback diagram

$$\begin{array}{ccc} B \times_{\mathbf{P}_n Y} Z & \xrightarrow{\theta} & \mathbf{P}_n B \times_{\mathbf{P}_n Y} Z \\ \downarrow & & \downarrow q_* \\ B & \longrightarrow & \mathbf{P}_n B \end{array}$$

the map $q_*$ is a pointwise fibration and the canonical map $B \to \mathbf{P}_n B$ is a pointwise $n$-equivalence, so the map $\theta$ is a pointwise $n$-equivalence by Lemma 22.

It follows that the composite

$$A \to X \xrightarrow{i} Z$$

is an $n$-equivalence, so that the map $A \to X$ is an $n$-equivalence. □

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An *n-fibration* is a map which has the right lifting property with respect to all cofibrations which are $n$-equivalences. We then have the following:

**Theorem 29** *1) The category $pro-s\mathrm{Pre}(\mathcal{C})$ of pro-simplicial presheaves on $\mathcal{C}$, together with the classes of cofibrations, $n$-equivalences and $n$-fibrations, satisfies the axioms for a proper closed model category.*

*2) A pro-map $p : X \to Y$ is an $n$-fibration if and only if it is a fibration and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathbf{P}_n X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ Y & \xrightarrow[\eta]{} & \mathbf{P}_n Y \end{array}$$

*is homotopy cartesian in the Edwards-Hastings model structure for $pro-s\mathrm{Pre}(\mathcal{C})$.*

*Proof:* Axiom **A4** is Lemma 25, axiom **A5** is trivial, and axiom **A6** is proved in Lemma 28. □

**Lemma 30** *Suppose given a pullback diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

*in $pro-s\mathrm{Pre}(\mathcal{C})$, with $p$ a fibration, and such that the maps $Y \to \mathbf{P}_* Y$ and $\mathbf{P}_* B \to \mathbf{P}_* X$ are weak equivalences. Then the map $A \to X$ is a pro-equivalence.*

*Proof:* We can assume that the pullback diagram is formed in $J^{op}$-diagrams, where $J$ is a strongly directed cofinite set, and that $p$ is an injective fibration of $J^{op}$-diagrams.

By an argument similar to that for Lemma 28, it suffices to show that, for all pullback diagrams

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & \mathbf{P}_* B \end{array}$$

in $(J \times \mathbb{N})^{op}$-diagrams with $p$ an injective fibration, the induced map $A \to X$ is an $n$-equivalence for all $n$.

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Pick a number $n \geq 0$. In all pullback diagrams

$$\begin{array}{ccc} A_{i,k} & \longrightarrow & X_{i.k} \\ \downarrow & & \downarrow p \\ B_i & \longrightarrow & \mathbf{P}_k B_i \end{array}$$

with $k \geq n$, the induced map $A_{i,k} \to X_{i,k}$ is an $n$-equivalence by Lemma 28. It follows by a cofinality argument that the pro-map $A \to X$ is an $n$-equivalence, as required. □

We have proved the following:

**Theorem 31** *1) The category $pro-s\mathrm{Pre}(\mathcal{C})$ of pro-simplicial presheaves on $\mathcal{C}$, together with the classes of cofibrations, pro-equivalences and pro-fibrations, satisfies the axioms for a proper closed model category.*

*2) A map $p : X \to Y$ is an pro-fibration if and only if it is a fibration and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathbf{P}_* X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ Y & \xrightarrow[\eta]{} & \mathbf{P}_* Y \end{array}$$

*is homotopy cartesian in the Edwards-Hastings model structure for $pro-s\mathrm{Pre}(\mathcal{C})$.*

*Proof:* Axiom **A4** follows from Lemma 25, axiom **A5** is Lemma 26, and axiom **A6** follows from Lemma 30. □

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Received: January 12, 2010

https://doi.org/10.1017/is011003012jkt149 Published online by Cambridge University Press