A triangular exploration

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In this Article we study the following problem:

Let $\triangle ABC$ be an acute-angled triangle. Let the points D, E, F on the sides BC, CA and AB, respectively, be such that AD is the median from A, BE is the internal angle bisector of $\angle ABC$, and CF is the altitude from C. This is shown in Figure 1.

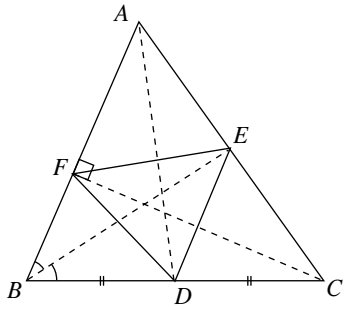


FIGURE 1

If $\triangle ABC$ is similar to $\triangle XYZ$ where X, Y, Z are distinct elements of the set $\{D, E, F\}$ then what can be said about $\triangle ABC$?

It is clear that if $\triangle ABC$ is equilateral then $\triangle DEF$, being the medial triangle of $\triangle ABC$, is similar to $\triangle ABC$. But in the problem under consideration we have six triangles, namely $\triangle DEF$, $\triangle EFD$, $\triangle FDE$, $\triangle EDF$, $\triangle DFE$ and $\triangle FED$, at least one of which is similar to $\triangle ABC$ and we wish to ascertain if $\triangle ABC$ is equilateral in all cases.

In the rest of the Article we shall denote the angles of $\triangle ABC$ by $\angle A$, $\angle B$ and $\angle C$ respectively, and those of $\triangle DEF$ by $\angle D$, $\angle E$ and $\angle F$ respectively. The lengths of BC, CA and AB are denoted by a, b and c respectively.

Case 1: $\triangle ABC$ is similar to $\triangle DEF$ (see Figure 2).

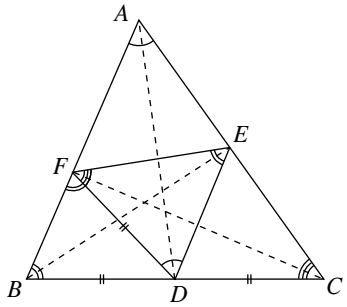


FIGURE 2

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Observe that

$$\angle AFE = 180° - \angle B - \angle C = \angle A$$

implying $EA = EF$. Since $\angle AFC = 90°$ we have

$$\angle EFC = 90° - \angle AFE = 90° - \angle A = \angle ECF.$$

Therefore $EF = EC = EA$. Hence E is the midpoint of AC. Using the angle bisector theorem it follows that

$$BA = BC.$$

Thus $\angle A = \angle C$. As D is the midpoint of BC and E is the midpoint of AC, $DE \,//\, AB$, and so by alternate angles

$$\angle E = \angle AFE$$

and this is the same as $\angle B = \angle A$. Hence $\angle A = \angle B = \angle C$ and ΔABC is equilateral.

Case 2: ΔABC is similar to ΔDFE (see Figure 3).

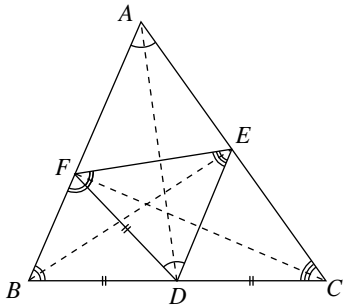


FIGURE 3

We will show that under this hypothesis the angles of ΔABC cannot be distinct. In order to accomplish this we shall rule out all the six possibilities, namely $\angle A > \angle B > \angle C$, $\angle A > \angle C > \angle B$, $\angle B > \angle A > \angle C$, $\angle B > \angle C > \angle A$, $\angle C > \angle A > \angle B$ and $\angle C > \angle B > \angle A$, one by one. Before we set out, for ease of reference, let us state that

$$DE = \frac{ab}{2c};\ EF = \frac{a^2}{2c};\ FD = \frac{a}{2}.$$

2.1 ($\angle A > \angle B > \angle C$)

Observe that $\angle CED = 2\angle A - \angle B > \angle C$. Therefore $DC > DE$ and so $\frac{a}{2} > \frac{ab}{2c}$, implying $c > b$, which is a contradiction.

2.2 ($\angle A > \angle C > \angle B$)

Here $180° - 2\angle B > \angle A$, so from ΔAEF, $AE > EF$ and so $\frac{bc}{c+a} > \frac{a^2}{2c} > \frac{c}{2}$, implying $2b > c + a$, which is absurd because $a > c > b$.

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2.3 ($\angle B > \angle A > \angle C$)

Here $\angle EDC = 2\angle B - \angle A > \angle C$, so $CE > DE$ and so $\dfrac{ab}{c+a} > \dfrac{ab}{2c}$, yielding $c > a$, a contradiction.

2.4 ($\angle B > \angle C > \angle A$)

In this case $180° - 2\angle B < \angle A$ implies $AE < EF$, and so $\dfrac{bc}{c+a} < \dfrac{a^2}{2c} < \dfrac{c}{2}$, implying $2b < c + a$, a contradiction since $b > c > a$.

2.5 ($\angle C > \angle A > \angle B$)

Observe that $\angle C > 2\angle B - \angle A = \angle EDC$ implies $DE > CE$ and so $\dfrac{ab}{2c} > \dfrac{ab}{c+a}$, showing $a > c$, a contradiction.

2.6 ($\angle C > \angle B > \angle A$)

Here $\angle C > 2\angle A - \angle B = \angle CED$ implies $DE > DC$ and so $\dfrac{ab}{2c} > \dfrac{a}{2}$, implying $b > c$, a contradiction.

Therefore at least two angles of ΔABC are equal and there are three possibilities:

(i) $\angle A = \angle B$; (ii) $\angle B = \angle C$; (iii) $\angle C = \angle A$.

We deal with these three cases now.

(i) ($\angle A = \angle B$)

Since $\angle D = \angle DFB$, $DE \parallel AB$. As D is the midpoint of BC and $DE \parallel AB$, E is the midpoint of AC. Therefore, by the angle bisector theorem, $BA = BC$ i.e. $\angle A = \angle C$. Hence ΔABC is equilateral.

(ii) ($\angle B = \angle C$)

In this case, $DB = DF = DE = DC$. Thus $BCEF$ is a cyclic quadrilateral. Therefore

$$\angle AEF = 2\angle B - \angle A = \angle B,$$

whence $\angle A = \angle B$. Thus ΔABC is equilateral.

(iii) ($\angle C = \angle A$)

Here $BA = BC$ and since BE is the internal bisector of B, $BE \perp AC$. Thus

$$\angle BFC = \angle BEC = 90°$$

whence $BCEF$ is cyclic. Therefore

$$2\angle B + \angle C = 180° = \angle B + 2\angle C$$

whence $\angle B = \angle C$. Hence ΔABC is equilateral.

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Case 3: ΔABC is similar to ΔFDE (see Figure 4).

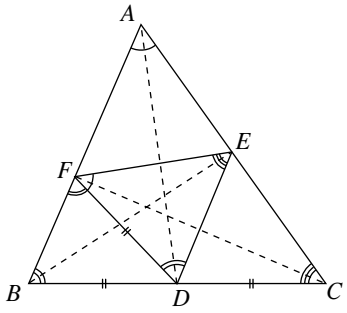


FIGURE 4

Since $DF = DB = \frac{1}{2}a$, $\angle DFB = \angle B$, it follows that

$$\angle AFE = 180^\circ - \angle A - \angle B = \angle C,$$

whence quadrilateral $BCEF$ is cyclic and since D is the centre of the circle passing through B, C, E and F, we have $DE = DF$. This implies $\angle E = \angle F$. But $\angle F = \angle A$ and $\angle E = \angle C$. Hence $\angle A = \angle C$, which implies $BC = AB$. Thus ΔABC is isosceles. But it need not be equilateral.

Case 4: ΔABC is similar to ΔEDF (see Figure 5).

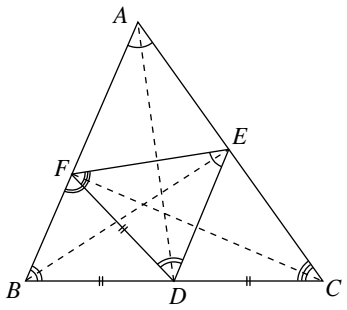


FIGURE 5

In this case

$$\angle AFE = 180^\circ - \angle C - \angle B = \angle A.$$

But then $\angle AFE = \angle A = \angle E$ whence $DE \,//\, AB$ and consequently E is the midpoint of AC. As the internal angle bisector BE of $\angle ABC$ bisects CA, from Case 1 we obtain $c = a$ and so $AB = BC$. In both cases (Case 3 and Case 4) one can construct an isosceles triangle with the desired properties.

Case 5: ΔABC is similar to ΔEFD (see Figure 6).

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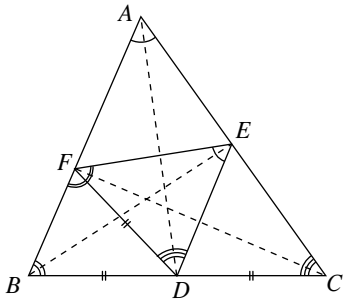


FIGURE 6

Since $FD = \frac{1}{2}a$ and

$$\frac{FD}{a} = \frac{EF}{c} = \frac{DE}{b}$$

we have $EF = \frac{1}{2}c$ and $DE = \frac{1}{2}b$. Also $\angle AFE = 180° - 2\angle B$. Therefore, by the sine rule applied to ΔAFE we obtain

$$\frac{EF}{\sin \angle A} = \frac{AE}{\sin(180° - 2\angle B)}.$$

On using $EF = \frac{1}{2}c$, $AE = \dfrac{bc}{c + a}$, $a = 2R \sin \angle A$, $b = 2R \sin \angle B$ and $\sin 2\angle B = 2 \sin \angle B \cos \angle B$ this reduces to

$$a = (c + a) \cos \angle B,$$

whence

$$b^2 = c^2 + a^2 - \frac{2ca^2}{c + a}.$$

Angle-chasing yields $\angle DEC = 180° - 2\angle B$ and applying the sine rule to ΔEDC we obtain

$$\frac{DC}{\sin(180° - 2\angle B)} = \frac{DE}{\sin \angle C}.$$

Recalling that $DC = \frac{1}{2}a$ and $DE = \frac{1}{2}b$ we obtain, after a bit of simplification,

$$\cos \angle B = \frac{ac}{2b^2},$$

which, when combined with $\cos \angle B = \dfrac{a}{c + a}$, yields

$$2b^2 = c^2 + ac.$$

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Eliminating b from the two expressions of b^2 we obtain

$$(c - a)^2 (c + 2a) = 0,$$

whence $c = a$ and $\cos\angle B = \dfrac{a}{c + a} = \dfrac{1}{2}$, implying $\angle B = 60°$. Thus $\angle A = \angle B = \angle C$ and ΔABC is equilateral.

Case 6: ΔABC is similar to ΔFED (see Figure 7).

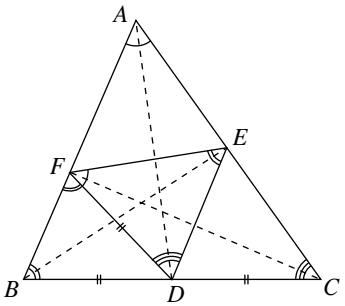


FIGURE 7

We observe that

$$\angle AFE = 180° - \angle A - \angle B = \angle C,$$

whence $BCEF$ is a cyclic quadrilateral. Therefore $\angle BEC = \angle BFC = 90°$ and since D is the midpoint of BC we have $DE = DF$ which implies $\angle A = \angle B$. Thus $CA = CB$, and since CF is the altitude on AB, it bisects $\angle C$. Therefore

$$\tfrac{1}{2}\angle B = \angle EBF = \angle ECF = \tfrac{1}{2}\angle C,$$

showing $\angle B = \angle C$. Hence ΔABC is equilateral.

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