Math. Proc. Camb. Phil. Soc. (1996), **120**, 291
Printed in Great Britain

Lagrangian submanifolds satisfying a basic equality

BY BANG-YEN CHEN

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824-1027 *U.S.A.*
E-mail address: bychen@math.msu.edu

AND LUC VRANCKEN†

Departement Wiskunde, Celestijnenlaan 200 *B*, *B*-3001 *Leuven, Belgium*
e-mail address: luc.vrancken©wis.kuleuven.ac.be

(*Received* 6 *April* 1995; *revised* 7 *August* 1995)

Abstract

In [3], B. Y. Chen proved that, for any Lagrangian submanifold M in a complex space-form $M^n(4c)$ $(c = \pm 1)$, the squared mean curvature and the scalar curvature of M satisfy the following inequality:

$$H^2 \geqslant \frac{2(n+2)}{n^2(n-1)}\tau - \left(\frac{n+2}{n}\right)c.$$

He then introduced three families of Riemannian n-manifolds $P_a^n(a > 1)$, $C_a^n(a > 1)$, $D_a^n(0 < a < 1)$ and two exceptional n-spaces F^n, L^n and proved the existence of a Lagrangian isometric immersion p_a from P_a^n into $\mathbb{C}P^n(4)$ and the existence of Lagrangian isometric immersions f, l, c_a, d_a from F^n, L^n, C_a^n, D_a^n into $\mathbb{C}H^n(-4)$, respectively, which satisfy the equality case of the inequality. He also proved that, beside the totally geodesic ones, these are the only Lagrangian submanifolds in $\mathbb{C}P^n(4)$ and in $\mathbb{C}H^n(-4)$ which satisfy this basic equality. In this article, we obtain the explicit expressions of these Lagrangian immersions. As an application, we obtain new Lagrangian immersions of the topological n-sphere into $\mathbb{C}P^n(4)$ and $\mathbb{C}H^n(-4)$.

1. *Introduction*

If $f: M \to \tilde{M}^n$ is an isometric immersion from a Riemannian n-manifold M into a Kaehler n-manifold $\tilde{M}^n$, then M is called a Lagrangian submanifold if the almost complex structure J of $\tilde{M}^n$ carries each tangent space of M into its normal space. Lagrangian submanifolds appear naturally in the context of classical mechanics and mathematical physics.

Let M be a Riemannian n-manifold with sectional curvature function K. The scalar curvature τ of M at p is a main intrinsic invariant defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j), \tag{1·1}$$

† The second author is a Senior Research Assistant of the National Fund for Scientific Research (Belgium). Part of this work was done while he was visiting Michigan State University in February 1995. He would like to thank Professors D. Blair and B. Y. Chen for their hospitality.

https://doi.org/10.1017/S0305004100074867 Published online by Cambridge University Press

where $e_1, \ldots, e_n$ is an orthonormal basis of tangent space T_pM. For a Lagrangian immersion from M into $\tilde{M}^n$, the squared mean curvature function H^2 on M is the main extrinsic invariant.

Let $\mathrm{cn}(x,k)$ and $\mathrm{dn}(x,k)$ be the usual Jacobi's elliptic functions with modulus k (see, for instance, [6]). It is well known that $\mathrm{cn}(x,k), \mathrm{dn}(x,k)$ are doubly periodic functions. We put

$$\mu_a = \frac{\sqrt{(a^2-1)}}{\sqrt{2}} \mathrm{cn}\left(ax, \frac{\sqrt{(a^2-1)}}{\sqrt{2a}}\right), \quad a > 1, \tag{1·2}$$

$$\eta_a = \frac{\sqrt{(a^2+1)}}{\sqrt{2}} \mathrm{dn}\left(\frac{\sqrt{(a^2+1)}}{\sqrt{2}} x, \frac{\sqrt{2a}}{\sqrt{(a^2+1)}}\right), \quad 0 < a < 1, \tag{1·3}$$

and

$$\rho_a = \frac{\sqrt{(a^2+1)}}{\sqrt{2}} \mathrm{cn}\left(ax, \frac{\sqrt{(a^2+1)}}{\sqrt{2a}}\right), \quad a > 1. \tag{1·4}$$

Let $S^n(c)$ and $H^n(-c)$ denote the n-sphere with constant sectional curvature c and the real hyperbolic n-space with constant sectional curvature $-c$, respectively. For $n \geqslant 3$, we denote by P_a^n, D_a^n and C_a^n the warped products $I \times_{\mu_a} S^{n-1}(\frac{a^4-1}{4})$, $\mathbb{R} \times_{\eta_a} H^{n-1}(\frac{a^4-1}{4})$ and $I \times_{\rho_a} S^{n-1}(\frac{a^4-1}{4})$ with warped functions μ_a, η_a and ρ_a, respectively, and I are the maximal open intervals containing 0 on which the corresponding warped functions are positive. Moreover, we denote by F^n and L^n the warped products $\mathbb{R} \times_{1/\sqrt{2}} H^{n-1}(-\frac{1}{4})$ and $\mathbb{R} \times_{\mathrm{sech}(x)} \mathbb{R}^{n-1}$, respectively. For $n = 2$ we shall replace $S^{n-1}(\frac{a^4-1}{4})$ or $H^{n-1}(\frac{a^4-1}{4})$ by the real line $\mathbb{R}$ to define P_a^2, D_a^2, C_a^2, F^2 and L^2. It is easy to see that F^2 is a flat surface and D_a^n, F^n and L^n are complete Riemannian n-manifolds, but P_a^n and C_a^n are not complete. Furthermore, these Riemannian n-manifolds are conformally flat. Topologically, S^n is the two point compactification of both P_a^n and C_a^n (see [3] for details). We denote by $\mathbb{C}P^n(4)$ and $\mathbb{C}H^n(-4)$ the complex projective n-space and the complex hyperbolic n-space with constant holomorphic sectional curvatures 4 and -4, respectively.

In [3] B. Y. Chen proved the following theorems.

THEOREM A. (1) *If M is a Lagrangian submanifold of $\mathbb{C}P^n(4)$ $(n > 1)$, then the squared mean curvature H^2 and the scalar curvature τ of M satisfy*

$$H^2 \geqslant \frac{2(n+2)}{n^2(n-1)} \tau - \left(\frac{n+2}{n}\right). \tag{1·5}$$

(2) *For each P_a^n, there exists a Lagrangian isometric immersion*

$$p_a : P_a^n \to \mathbb{C}P^n(4)$$

which satisfies equality in (1·5) *identically.*

(3) *Conversely, suppose $\iota : M \to \mathbb{C}P^n(4)$ is a Lagrangian isometric immersion satisfying equality in* (1·5) *identically. Then either M is a totally geodesic submanifold and hence M is locally isometric to the real projective space $\mathbb{R}P^n(1)$ or the set U of non-totally geodesic points in M is a dense subset of M, U is an open portion of a P_a^n with $a > 1$ and up to rigid motions of $\mathbb{C}P^n(4)$, the immersion ι is given by p_a.*

THEOREM B. (1) *If M is a Lagrangian submanifold of $\mathbb{C}H^n(-4)$ $(n > 1)$, then we have*

$$H^2 \geqslant \frac{2(n+2)}{n^2(n-1)} \tau + \left(\frac{n+2}{n}\right). \tag{1·6}$$

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(2) *For each* C_a^n *(respectively,* D_a^n, F^n, *or* L^n*), there exists a Lagrangian isometric immersion* $c_a: C_a^n \to \mathbb{C}H^n(-4)$ *(respectively,* $d_a: D_a^n \to \mathbb{C}H^n(-4)$, $f: F^n \to \mathbb{C}H^n(-4)$, *or* $l: L^n \to \mathbb{C}H^n(-4)$*) which satisfies equality in* (1·6) *identically.*

(3) *Conversely, suppose* $\iota: M \to \mathbb{C}H^n(-4)$ *is a Lagrangian isometric immersion satisfying equality in* (1·6) *identically. Then either* (*a*) *M is a totally geodesic submanifold and hence M is locally isometric to the real hyperbolic space* $\mathbb{R}H^n(-1)$ *or* (*b*) *M is an open portion of a* $D_a^n (0 < a < 1)$, F^n *or* L^n, *and up to rigid motions of* $\mathbb{C}H^n(-4)$, *the immersion* ι *is given by* d_a, f, *or* l, *or* (*c*) *the set U of non-totally geodesic points in M is a dense subset of M, U is an open portion of* $C_a^n (a > 1)$ *and up to rigid motions of* $\mathbb{C}H^n(-4)$, *the immersion* ι *is given by* c_a.

In Section 2, we explain the method for obtaining the explicit expressions of the non-totally geodesic Lagrangian immersions which satisfy the equality case of the inequalities (1·5) or (1·6). In Sections 3–7, we obtain the explicit coordinate expressions of the Lagrangian immersions p_a, f, l, c_a and d_a. By applying the coordinate expressions of p_a and c_a, we prove that both p_a and c_a can be extended to their two points compactifications. As an application, we obtain Lagrangian immersions $\bar{p}_a$ and $\bar{c}_a$ of the topological n-sphere S^n into $\mathbb{C}P^n(4)$ and $\mathbb{C}H^n(-4)$ which satisfy the equality case of the inequality. Recall that, according to [1], the only minimal Lagrangian immersion of S^2 into $\mathbb{C}P^n(4)$ is the totally geodesic one.

Using immersions $\bar{p}_a$ and $\bar{c}_a$ we improve part (3) of Theorem A and part (3) of Theorem B as follows.

THEOREM A-(3′). *Suppose* $\iota: M \to \mathbb{C}P^n(4)$ *is a Lagrangian isometric immersion satisfying the equality case of* (1·5) *identically. Then either M is a totally geodesic submanifold or up to rigid motions of* $\mathbb{C}P^n(4)$, $\iota(M)$ *is an open portion of a* $\bar{p}_a(S^n)$ *with* $a > 1$.

THEOREM B-(3′). *Suppose* $\iota: M \to \mathbb{C}H^n(-4)$ *is a Lagrangian isometric immersion satisfying the equality case of* (1·6) *identically. Then either M is a totally geodesic submanifold or up to rigid motions of* $\mathbb{C}H^n(-4)$, $\iota(M)$ *is an open portion of a* $d_a(D_a^n)$ $(0 < a < 1)$, $f(F^n)$, $l(L^n)$, *or* $\bar{c}_a(S^n)$ $(a > 1)$.

2. *A general construction method*

In this section we discuss our method for obtaining the explicit expressions of the non-totally geodesic Lagrangian submanifolds in $\tilde{M}^n(4c)$ which satisfy the following equality:

$$H^2 = \frac{2(n+2)}{n^2(n-1)}\tau - \left(\frac{n+2}{n}\right)c. \tag{2·1}$$

For simplicity, we call (2·1) *the basic equality.*

Case 1. $M^n(4c) = \mathbb{C}P^n(4)$. In this case, we consider the Hopf fibration

$$\pi: S^{2n+1}(1) \to \mathbb{C}P^n(4).$$

On $S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ we consider the Sasakian structure ϕ (the projection of the complex structure J of $\mathbb{C}^{n+1}$ on the tangent bundle of $S^{2n+1}(1)$) and the structure vector field $\xi = Jx$, where x is the position vector. An isometric immersion $f: M \to S^{2n+1}(1)$ is called *C-totally real* if ξ is normal to $f_*(TM)$. Note that for a C-totally

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real submanifold, we have $\langle \phi(f_*(TM)), f_*(TM)\rangle = 0$, where $\langle\ ,\ \rangle$ denotes the inner product on $\mathbb{C}^{n+1}$. On $\mathbb{C}^{n+1}$ we consider the complex structure J induced by $i = \sqrt{-1}$. The main results of [7] can be specialized to our situation as follows.

Let $g: M \to \mathbb{C}P^n(4)$ be a Lagrangian isometric immersion. Then there exists an isometric covering map $\tau: \hat{M} \to M$, and a C-totally real isometric immersion $f: \hat{M} \to S^{2n+1}(1)$ such that $g(\tau) = \pi(f)$. Hence every Lagrangian immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a C-totally real immersion of the same Riemannian manifold. Conversely, let $f: \hat{M} \to S^{2n+1}(1)$ be a C-totally real isometric immersion. Then $g = \pi(f): M \to \mathbb{C}P^n(4)$ is again an isometric immersion, which is Lagrangian. Under this correspondence, the second fundamental forms h^f and h^g of f and g satisfy $\pi_* h^f = h^g$. Moreover, h^f is horizontal with respect to π. (We shall denote h^f and h^g simply by h.)

From the observation above, it is clear that in order to obtain an explicit expression of a desired Lagrangian submanifold in $\mathbb{C}P^n(4)$, it is sufficient to construct a corresponding C-totally real submanifold in $S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ which satisfies the same basic equality in $S^{2n+1}(1)$.

Case (2). $\tilde{M}^n(4c) = \mathbb{C}H^n(-1)$. In this case, we consider the complex number $(n+1)$-space $\mathbb{C}_1^{n+1}$ endowed with the pseudo-Euclidean metric g_0 given by (for the details, cf. [5])

$$g_0 = -dz_1\, d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j\, d\bar{z}_j. \tag{2·2}$$

Put
$$H_1^{2n+1}(-1) = \{z = (z_1, z_2, \ldots, z_{n+1}) : \langle z, z\rangle = -1\}, \tag{2·3}$$

where $\langle\ ,\ \rangle$ denotes the inner product on $\mathbb{C}_1^{n+1}$ induced from g_0. $H_1^{2n+1}(-1)$ is known as an anti-de Sitter space-time.

We put
$$T_z' = \{z \in \mathbb{C}^{n+1} : \operatorname{Re}\langle u, z\rangle = \operatorname{Re}\langle u, iz\rangle = 0\}$$

and
$$H_1^1 = \{\lambda \in \mathbb{C} : \lambda\bar{\lambda} = 1\}.$$

Then we have an H_1^1-action on $H_1^{2n+1}(-1)$, $z \mapsto \lambda z$. At each point $z \in H_1^{2n+1}(-1)$, the vector iz is tangent to the flow of the action. Since g_0 is Hermitian, we have $\operatorname{Re} g_0(iz, iz) = -1$. Note that the orbit is given by $x_t = (\cos t + i \sin t)\, z$ and $dx_t/dt = iz_t$. Thus the orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2n+1}/\sim$, under the identification induced from the action, is the complex hyperbolic space $\mathbb{C}H^n(-4)$ with constant holomorphic sectional curvature -4, with the complex structure J induced from the canonical complex structure J on $\mathbb{C}_1^{n+1}$ via the totally geodesic fibration:

$$\pi: H_1^{2n+1}(-1) \to \mathbb{C}H^n(-4). \tag{2·4}$$

Just as in Case (1), let $g: M \to \mathbb{C}H^n(-4)$ be a Lagrangian isometric immersion. Then there exists an isometric covering map $\tau: \hat{M} \to M$, and a C-totally real isometric immersion $f: \hat{M} \to H_1^{2n+1}(-1)$ such that $g(\tau) = \pi(f)$. Hence every totally real immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a C-totally real immersion. Conversely, let $f: \hat{M} \to H_1^{2n+1}(-1)$ be a C-totally real isometric immersion. Then $g = \pi(f): M \to \mathbb{C}H^n(-4)$ is again an isometric immersion, which is Lagrangian. Similarly, under this correspondence, the second fundamental forms h^f and h^g of f and g satisfy $\pi_* h^f = h^g$. Moreover, h^f is horizontal with respect to π. (We shall also denote h^f and h^g simply by h.)

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From the observation above, it is also clear that in order to obtain an explicit expression of a Lagrangian submanifold in $\mathbb{C}H^n(-4)$ satisfying the basic equality, it is sufficient to construct a corresponding C-totally real submanifold in $H_1^{2n+1}(-1) \subset \mathbb{C}_1^{n+1}$ which satisfies the same equality in $H_1^{2n+1}(-1)$.

The method of our construction of the desired C-totally real submanifolds works as follows.

Assume M is a submanifold of $S^{2n+1}(1)$ (or in $H_1^{2n+1}(-1)$). Denote by $\tilde{\nabla}$ and ∇ the Levi–Civita connections of $\mathbb{C}^{n+1}$ (or of $\mathbb{C}_1^{n+1}$) and of M, respectively. Let h denote the second fundamental form of M in $S^{2+1}(1)$ (or in $H_1^{2n+1}(-1)$). Since $S^{2n+1}(1)$ and $H_1^{2n+1}(-1)$ are totally umbilical with 1 as mean curvature in $\mathbb{C}^{n+1}$ and in $\mathbb{C}_1^{n+1}$, respectively, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) - \epsilon x, \tag{2·5}$$

where $\epsilon = 1$ whenever the ambient space is $\mathbb{C}^{n+1}$ and $\epsilon = -1$ whenever the ambient space is $\mathbb{C}_1^{n+1}$.

If M is one of the warped products $I \times N^{n-1}$ given by F^n, L^n, $P_a^n(a > 1)$, $C_a^n(a > 1)$, $D_a^n(0 < a < 1)$ mentioned in Section 1, then the warped product metric on M is given by

$$g = dx^2 + \lambda^2 g_0, \tag{2·6}$$

where g_0 is the canonical metric of the second component N^{n-1} and $\lambda = \lambda(x)$ is $1/\sqrt{2}$, $\operatorname{sech} x$, μ_a, η_a or ρ_a, respectively, where μ_a, η_a, ρ_a are defined by (1·2)–(1·4). When M is D_a^n, F^n or L^n, $I = \mathbb{R}$.

Assume M is isometrically immersed in $S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ (or in $H_1^{2n+1}(-1) \subset \mathbb{C}_1^{n+1}$) as a C-totally real submanifold satisfying the basic equality (2·1). Let $z = z(x, y_1, \ldots, y_{n-1})$ denote the immersions in $\mathbb{C}^{n+1}$ (or in $\mathbb{C}_1^{n+1}$). Then, by applying (2·5) and a result of [3], we have

$$z_{xx} = 3\lambda i z_x - \epsilon z, \quad z_x = \frac{\partial z}{\partial x}, \quad z_{xx} = \frac{\partial^2 z}{\partial x^2}, \tag{2·7}$$

$$\tilde{\nabla}_Y z_x = \left(i\lambda + \frac{\lambda'}{\lambda}\right) Y, \tag{2·8}$$

$$\tilde{\nabla}_Y \tilde{\nabla}_Z z = \langle Y, Z \rangle \left\{ \left(i\lambda - \frac{\lambda'}{\lambda}\right) z_x - \epsilon z \right\} + D_Y Z, \tag{2·9}$$

where $\langle\ ,\ \rangle$ is the inner product on $\mathbb{C}^{n+1}$ (or on $\mathbb{C}_1^{n+1}$), Y, Z are vector fields tangent to the second component N^{n-1} of the warped product, and $D_Y Z$ is the component of $\tilde{\nabla}_Y Z$ tangent to the second component.

Since M is isometrically immersed in $S^{2n+1}(1)$ or in $H_1^{2n+1}(-1)$ as a C-totally real submanifold, we have

$$\langle z_x, iz \rangle = \langle Y, iz \rangle = \langle z_x, iY \rangle = \langle z_x, Y \rangle = 0, \tag{2·10}$$

where Y, Z are tangent vectors perpendicular to z_x. Because M lies either in $S^{2n+1}(1)$ or in $H_1^{2n+1}(-1)$, we also have

$$\langle z, z \rangle = \epsilon. \tag{2·11}$$

In principle, we shall obtain the explicit expression of the desired isometric immersions by solving system (2·7)–(2·9) of partial differential equations under the conditions given by (2·10)–(2·11).

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3. *Lagrangian immersion* $p_a: P_a^n \to \mathbb{C}P^n(4)$

We divide into two cases.

Case (a). $n = 2$. Recall that for each $a > 1$, P_a^2 is the warped product surface $I \times_{\mu_a} \mathbb{R}$ whose warped product metric is given by $g = dx^2 + \mu_a^2 dy^2$, where μ_a is defined by (1·2). Put $e_1 = \partial/\partial x$ and $e_2 = 1/\mu_a \partial/\partial y$.

Let $p_a: P_a^2 \to \mathbb{C}P^2(4)$ be the Lagrangian isometric immersion of P_a^2 into $\mathbb{C}P^2(4)$ introduced in [3] which satisfies the basic equality. If $\hat{p}_a: P_a^2 \to S^5(1) \subset \mathbb{C}^3$ is a lift of p_a (cf. Section 2), then $\hat{p}_a$ is a C-totally real isometric immersion which satisfies the same basic equality in $S^5(1)$.

Assume that $\hat{p}_a$ is defined by $z = z(x, y)$. Then $z = z(x, y)$ must satisfy the differential system (2·7)–(2·9), which in this case reduces to

$$z_{xx} = 3\mu_a i z_x - z, \tag{3·1}$$

$$z_{xy} = \left(i\mu_a + \frac{\mu_a'}{\mu_a}\right) z_y, \tag{3·2}$$

$$z_{yy} = (i\mu_a^3 - \mu_a \mu_a') z_x - \mu_a^2 z, \tag{3·3}$$

where
$$\mu_a = ak\,\mathrm{cn}(ax, k), \quad k = \frac{\sqrt{(a^2-1)}}{\sqrt{2}\,a}, \quad a > 1. \tag{3·4}$$

Since the lift of $\hat{p}_a$ is a C-totally real immersion into $S^5(1) \subset \mathbb{C}^3$, $z = z(x, y)$ also satisfies conditions given in (2·10)–(2·11).

From (3·2) and (3·3) we have $z_{yyy} = -(\mu_a^4 + \mu_a'^2 + \mu_a^2) z_y$. On the other hand, from (3·4), we have $\mu_a^4 + \mu_a'^2 + \mu_a^2 = c^2$, $c = \sqrt{(a^4-1)}/2$. Thus we get $z_{yyy} + c^2 z_y = 0$. Therefore $z = z(x, y)$ takes the form

$$z(x, y) = A_1(x)\, e^{icy} + A_2\, e^{-icy} + B(x), \tag{3·5}$$

for some vector functions $A_1(x)$, $A_2(x)$, $B(x)$. By using (3·2) and (3·5), we find

$$A_i'(x) = \left(i\mu_a + \frac{\mu_a'}{\mu_a}\right) A_i(x), \quad i = 1, 2. \tag{3·6}$$

Therefore
$$A_i(x) = \frac{b_i}{ak} \mu_a \,\mathrm{Exp}\left(i \int \mu_a\, dx\right), \quad i = 1, 2, \tag{3·7}$$

for some vectors $b_1, b_2 \in \mathbb{C}^3$. Since $\int \mu_a\, dx = \int ak\,\mathrm{cn}(ax, k)\, dx = \cos^{-1}(\mathrm{dn}(ax, k))$, we obtain

$$A_i(x) = b_i\,\mathrm{cn}(ax, k)\,(\mathrm{dn}(ax, k) + ik\,\mathrm{sn}(ax, k)). \tag{3·8}$$

On the other hand, (3·1) and (3·5) imply that $A_1(x)$, $A_2(x)$ and $B(x)$ satisfy the second order homogeneous linear differential equation:

$$\phi''(x) = 3iak\,\mathrm{cn}(ax, k)\,\phi'(x) - \phi(x). \tag{3·9}$$

Since $\phi_1(x) = \mathrm{cn}(ax, k)\,(\mathrm{dn}(ax, k) + ik\,\mathrm{sn}(ax, k))$ is a solution of (3·9), the method of reduction of order implies that a second independent solution of (3·9) is given by

$$\phi_1(x) \int \mathrm{nc}^2(ax, k)\,\mathrm{Exp}\left(i \int^x ak\,\mathrm{cn}(at, k)\, dt\right) dx.$$

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Thus, a second independent solution takes the form

$$\phi_2(x) = a^3k^2(1-k^2)\,\phi_1(x)\int(\mathrm{dc}(ax,k)\,\mathrm{nc}(ax,k)+ik\,\mathrm{sc}(ax,k)\,\mathrm{nc}(ax,k))\,dx$$

$$= (\mathrm{dn}(ax,k)+ik\,\mathrm{sn}(ax,k))((1-k^2)\,\mathrm{sn}(ax,k)+ik\,\mathrm{dn}(ax,k)). \tag{3·10}$$

Consequently, $z(x,y)$ takes the form

$$z(x,y) = f_1(x)(C_1\cos(cy)+C_2\sin(cy)+C_7)$$
$$+f_2(x)(C_3\cos(cy)+C_4\sin(cy)+C_8)+C_5g_1(x)+C_6g_2(x), \tag{3·11}$$

where

$$\left.\begin{aligned} &f_1(x) = \mathrm{dn}(ax,k)\,\mathrm{cn}(ax,k), \quad f_2(x) = k\,\mathrm{sn}(ax,k)\,\mathrm{cn}(ax,k),\\ &g_1(x) = (1-2k^2)\,\mathrm{sn}(ax,k)\,\mathrm{dn}(ax,k),\\ &g_2(x) = k(1-k^2)\,\mathrm{sn}^2(ax,k)+k\,\mathrm{dn}^2(ax,k). \end{aligned}\right\} \tag{3·12}$$

From (3·12), we have

$$\left.\begin{aligned} &f_1(0)=1, \quad f_1'(0)=0, \quad f_1''(0)=-a^2(1+k^2), \quad f_1'''(0)=0,\\ &f_2(0)=0, \quad f_2'(0)=ak, \quad f_2''(0)=0, \quad f_2'''(0)=-a^3k(4+k^2),\\ &g_1(0)=0, \quad g_1'(0)=a(1-2k^2), \quad g_1''(0)=0, \quad g_1'''(0)=a^3(8k^4-2k^2-1),\\ &g_2(0)=k, \quad g_2'(0)=0, \quad g_2''(0)=2a^2k(1-2k^2), \quad g_2'''(0)=0. \end{aligned}\right\} \tag{3·13}$$

Because $\hat{p}_a$ is C-totally real in $S^5(1)$, up to rigid motions of $S^5(1)$, we may choose

$$\left.\begin{aligned} &e_1 = z(0,0), \quad e_2 = iz(0,0), \quad e_3 = z_x(0,0),\\ &e_4 = iz_x(0,0), \quad e_5 = \frac{1}{ak}z_y(0,0), \quad e_6 = \frac{1}{ak}iz_y(0,0). \end{aligned}\right\} \tag{3·14}$$

Then e_1, e_3, e_5 are the canonical basis of $\mathbb{C}^3$ (over $\mathbb{C}$). By using (3·11)–(3·14), we obtain

$$\left.\begin{aligned} &e_1 = C_1+kC_6+C_7, \quad e_5 = \frac{c}{ak}C_2,\\ &e_3 = akC_3+a(1-2k^2)\,C_5+akC_8. \end{aligned}\right\} \tag{3·15}$$

From (3·2) and (3·11)–(3·14), we also obtain

$$e_6 = \frac{1}{ak}iz_y(0,0) = \frac{1}{a^2k^2}z_{xy}(0,0) = \frac{c}{ak}C_4. \tag{3·16}$$

Hence, we obtain from (3·20) and (3·21) that

$$C_2 = \frac{ak}{c}e_5, \quad C_4 = \frac{ak}{c}e_6. \tag{3·17}$$

Now, by (3·1), (3·3) and (3·11)–(3·15), we may obtain

$$2e_1 = \frac{3c^2}{a^2k^2}C_1-a^2(1+k^2)\,e_1+3a^2k(1-k^2)\,C_6, \tag{3·18}$$

and

$$2ake_4 = \frac{c^2}{a^2k^2}C_1-a^2(1+k^2)\,e_1+3a^2k(1-k^2)\,C_6. \tag{3·19}$$

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From (3·15), (3·18) and (3·19), we get

$$\left.\begin{aligned} C_1 &= \frac{a^2k^2}{c^2}(e_1 - ake_4), \quad C_7 = 0, \\ C_6 &= \frac{1}{3a^2k(1-k^2)}((a^2+a^2k^2-1)e_1 + 3ake_4). \end{aligned}\right\} \tag{3·20}$$

From (3·1) we obtain

$$z_{xxx}(0,0) = -(9a^2k^2+1)z_x(0,0) - 3akiz(0,0). \tag{3·21}$$

Thus, by (3·11)–(3·15), we may obtain

$$3ake_2 = ak(4a^2-8a^2k^2-1)(C_3+C_8) + a(1-2k^2)(a^2-5a^2k^2-1)C_5. \tag{3·22}$$

Also, from (3·1) and (3·3), we have

$$z_{yyx}(0,0) = -a^4k^4z_x(0,0) - a^3k^3iz(0,0). \tag{3·23}$$

Therefore, by (3·11)–(3·15), (3·22) and (3·23), we get

$$\left.\begin{aligned} C_3 &= \frac{a^2k^2}{c^2}(e_2 + ake_3), \quad C_8 = 0, \\ C_5 &= -\frac{a^2k^3}{c^2(1-2k^2)}e_2 + \frac{c^2-a^4k^4}{c^2a(1-2k^2)}e_3. \end{aligned}\right\} \tag{3·24}$$

Since $k = \sqrt{(a^2-1)}/\sqrt{2}a$ and $c = \sqrt{(a^4-1)}/2$, (3·17), (3·20) and (3·24) imply

$$\left.\begin{aligned} &C_1 = \frac{2}{a^2+1}(e_1 - ake_4), \quad C_2 = \frac{\sqrt{2}}{\sqrt{(1+a^2)}}e_5, \qquad C_3 = \frac{2}{a^2+1}(e_2 + ake_3), \\ &C_4 = \frac{\sqrt{2}}{\sqrt{(1+a^2)}}e_6, \qquad C_5 = \frac{2a}{a^2+1}(-ake_2 + e_3), \quad C_6 = \frac{2a}{a^2+1}(ake_1 + e_4). \end{aligned}\right\} \tag{3·25}$$

Therefore, if we choose the complex Euclidean coordinate system on $\mathbb{C}^3$ in such a way that $e_1 - ake_4$ is in the direction of $(0,1,0)$, $-ake_2 + e_3$ is in the direction of $(1,0.0)$, and e_5 is $(0,0,1)$, then we obtain

$$\left.\begin{aligned} z(x,y) = \frac{\sqrt{2}}{\sqrt{(1+a^2)}}\Bigg(&\frac{1}{a}\operatorname{sn}(ax,k)\operatorname{dn}(ax,k) + iak\left(1 + \frac{1}{a^2}\operatorname{sn}^2(ax,k)\right), \\ &(\operatorname{dn}(ax,k) + ik\operatorname{sn}(ax,k))\operatorname{cn}(ax,k)\cos\left(\frac{\sqrt{(a^4-1)}}{2}y\right), \\ &(\operatorname{dn}(ax,k) + ik\operatorname{sn}(ax,k))\operatorname{cn}(ax,k)\sin\left(\frac{\sqrt{(a^4-1)}}{2}y\right)\Bigg), \quad k = \frac{\sqrt{(a^2-1)}}{\sqrt{2}a}, a > 1. \end{aligned}\right\} \tag{3·26}$$

By a direct computation, we know that (2·10) and (2·11) are satisfied and (3·26) defines a C-totally real isometric imbedding of P_a^2 into $S^5(1) \subset \mathbb{C}^e$ whose squared mean curvature function and whose scalar curvature satisfy the basic equality:

$$H^2 = 2(\tau - 1). \tag{3·27}$$

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The projection of this immersion gives rise to a Lagrangian isometric immersion of P_a^2 into $\mathbb{C}P_2(4)$. According to theorem 1 of [3], up to rigid motions of $\mathbb{C}P^2(4)$, (3·26) provides us the unique Lagrangian isometric immersion $p_a: P_a^2 \to \mathbb{C}P^2(4)$ defined in [3] which satisfies basic equality (3·27).

Notice that the function $z = z(x, y)$ given in (3·26) is a doubly periodic map from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{C}^3$. Now we claim that (3·27) defines a C-totally real imbedding of the topological 2-sphere S^2 into $S^5(1) \subset \mathbb{C}^3$. This can be seen as follows.

Put

$$\left.\begin{aligned} y_1 &= \mathrm{sn}(ax, k), \quad y_2 = \mathrm{cn}(ax, k)\cos\left(\frac{\sqrt{(a^4-1)}}{2}y\right), \\ y_3 &= \mathrm{cn}(ax, k)\sin\left(\frac{\sqrt{(a^4-1)}}{2}y\right). \end{aligned}\right\} \tag{3·28}$$

Since $\mathrm{dn}(ax, k) = \sqrt{[1-k^2\,\mathrm{sn}^2(ax, k)]}$, (3·28) yield

$$\begin{aligned} z(y_1, y_2, y_3) = \frac{\sqrt{2}}{\sqrt{(1+a^2)}}\Big(&\frac{y_1}{a}\sqrt{[(1-k^2)y_1^2+y_2^2+y_3^2]}+iak\left[\left(1+\frac{1}{a^2}\right)y_1^2+y_2^2+y_3^2\right], \\ &\{\sqrt{[(1-k^2)y_1^2+y_2^2+y_3^2]}+iky_1\}y_2, \quad \{\sqrt{[(1-k^2)y_1^2+y_2^2+y_3^2]}+iky_1\}y_3\Big), \end{aligned} \tag{3·29}$$

where $y_1^2+y_2^2+y_3^2 = 1$.

It is easy to verify that (3·29) defines an imbedding of S^2 into $S^5(1) \subset \mathbb{C}^3$. Let $\bar{P}_a^2$ denote the Riemannian 2-manifold given by S^2 endowed with the metric induced from $\mathbb{C}^3$ via (3·29). Then P_a^2 is a dense open subset of $\bar{P}_a^2$ with two points removed. Moreover, the isometric imbedding of $\bar{P}_a^2$ into $S^5(1) \subset \mathbb{C}^3$ defined by (3·29) coincides with the C-totally real isometric imbedding of P_a^2 into $S^5(1) \subset \mathbb{C}^3$ on P_a^2. Therefore, the Lagrangian isometric immersion $\bar{p}_a: \bar{P}_a^2 \to \mathbb{C}P^2(4)$ induced from (3·26) (or equivalently from (3·29)) can be regarded as the complete extension of the Lagrangian isometric immersion $p_a: P_a^2 \to \mathbb{C}P^2(4)$ introduced in [3]. According to the uniqueness of such Lagrangian immersion (theorem 3·1 of [4]), (3·26) (equivalently (3·29)) provides an explicit expression of $p_a: P_a^2 \to \mathbb{C}P^2(4)$.

Case (b). $n \geqslant 3$. In this case, P_a^n is the warped product $I \times_{\mu_a} S^{n-1}(a^4-1)/4)$ which is clearly isometric to $I \times_{\frac{2}{\sqrt{(a^4-1)}}\mu_a} S^{n-1}(1)$. Because of the existence theorem, we have an isometric immersion

$$P_a: I \times_{\frac{2}{\sqrt{(a^4-1)}}\mu_a} S^{n-1}(1) \to S^{2n+1}(1) \subset \mathbb{C}^{n+1}$$

satisfying (2·7)–(2·11). Denote by x the parameter on I and by $y = (y_1, \ldots, y_n)$ with $\sum_{j=1}^n y_j^2 = 1$ a point of $S^{n-1}(1)$. Then (2·7) implies that there exist complex valued functions α, β on I such that

$$p_a(x, y) = \alpha(x)\,p_a(0, y) + \beta(x)\,p_{a*}\left(\frac{\partial}{\partial x}\right)(0, y), \tag{3·30}$$

where α and β are the solutions of the differential equation:

$$f'' = 3\mu_a i f' - f$$

with initial conditions $f(0) = 1$, $f'(0) = 0$ for α, and $f(0) = 0$, $f'(0) = 1$ for β, respectively.

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Using the solutions obtained in Case (a), we obtain that $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ are given by

$$\alpha_1 = \frac{2}{1+a^2}(\mathrm{dn}(ax,k)\,\mathrm{cn}(ax,k) + a^2k^2\left(1+\frac{1}{a^2}\mathrm{sn}^2(ax,k)\right),$$

$$\alpha_2 = \frac{2}{1+a^2}k\,\mathrm{sn}(ax,k)(\mathrm{cn}(ax,k) - \mathrm{dn}^2(ax,k)),$$

$$\beta_2 = \frac{2}{1+a^2}\mathrm{sn}(ax,k)\left(ak^2\,\mathrm{cn}(ax,k) + \frac{1}{a}\mathrm{dn}^2(ax,k)\right),$$

$$\beta_2 = \frac{2}{1+a^2}\left[ak\left(1+\frac{1}{a^2}\mathrm{sn}^2(ax,k)\right) - ak\,\mathrm{dn}(ax,k)\,\mathrm{cn}(ax,k)\right].$$

Now, we look at the map

$$p: S^{n-1} \to S^{2n+1}(1) \subset \mathbb{C}^{n+1}: y \mapsto p_a(0,y)$$

which clearly is an immersion. Denote the induced metric on S^{n-1} by $\langle\langle\,,\,\rangle\rangle$. Since $\mu_a(0) = ak$ and $\mu_a'(0) = 0$, we deduce from (2·9) that for V and W tangent to S^{n-1}, we have

$$\tilde{\nabla}_V p_*(W) = p_*(D_V W) + \langle\langle V, W\rangle\rangle akip_{a^*}\left(\frac{\partial}{\partial x}\right)\Big|_{x=0},$$

$$\tilde{\nabla}_V ip_{a^*}\left(\frac{\partial}{\partial x}\right)\Big|_{x=0} = -akp_*(V),$$

showing that $p(S^{n-1})$ is contained as a totally umbilical hypersurface in a totally geodesic S^n in S^{2n+1}. Moreover, since $p_{a^*}(\partial/\partial x)|_{x=0}$, ip and $ip_*(V)$ are normal to this S^n, this S^n is also C-totally real in $S^{2n+1}(1)$. Hence, we can decompose p into 2 maps p_1 and p_2:

$$p_1: S^{n-1} \to S^n \subset \mathbb{R}^{n+1}, \quad p_2: S^n \to S^{2n+1}(1) \subset \mathbb{C}^{n+1},$$

where p_2 is totally geodesic, C-totally real immersion. Since $p_1(S^{n-1})$ is totally umbilical with constant curvature $1+a^2k^2$, we may assume that

$$p_1(y_1,\ldots,y_n) = \frac{\sqrt{2}}{\sqrt{(1+a^2)}}(ak, y_1,\ldots,y_n);$$

and since p_2 is totally geodesic and C-totally real, we may assume that

$$p_2(x_1,\ldots,x_{n+1}) = (ix_1, x_2,\ldots,x_{n+1}).$$

Hence
$$p_a(0,y) = p(y) = \frac{\sqrt{2}}{\sqrt{(1+a^2)}}(iak, y_1,\ldots,y_n). \tag{3.31}$$

Next, we look at $p_{a^*}(\partial/\partial x)(0,y)$. Since $\mu_a'(0) = 0$, we have

$$\tilde{\nabla}_V\left[p_{a^*}\left(\frac{\partial}{\partial x}\right)\Big|_{x=0} - akip_a|_{x=0}\right] = 0.$$

Hence there exist a constant vector C such that

$$p_{a^*}\left(\frac{\partial}{\partial x}\right)\Big|_{x=0} = akip_a|_{x=0} + C_1.$$

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Using (2·10), we see that there exist a real number δ such that $C_1 = \delta e_1$ and

$$0 = \left\langle p_{a*}\left(\frac{\partial}{\partial x}\right)\Big|_{x=0}, ip_a|_{x=0}\right\rangle = ak - \delta\langle ie_1, p_a|_{t=0}\rangle = ak\left(1 - \frac{\sqrt{2}}{\sqrt{(1+a^2)}}\delta\right).$$

Hence

$$p_{a*}\left(\frac{\partial}{\partial x}\right)\Big|_{x=0} = \frac{\sqrt{2}}{\sqrt{(1+a^2)}}\left(-a^2k^2 + \frac{1+a^2}{2}, aky_1, \ldots, aky_n\right)$$

$$= \frac{\sqrt{2}}{\sqrt{(1+a^2)}}(1, aky_1, \ldots, aky_n). \tag{3·32}$$

Combining this with (3·30) and (3·31), we obtain

$$p_a(x,y) = \frac{\sqrt{2}}{\sqrt{(1+a^2)}}\left(\frac{1}{a}\operatorname{sn}(ax,k)\operatorname{dn}(ax,k) + iak\left(1 + \frac{1}{a^2}\operatorname{sn}^2(ax,k)\right),\right.$$

$$\left.(\operatorname{dn}(ax,k) + ik\operatorname{sn}(ax,k))\operatorname{cn}(ax,k)\,y_1, \ldots, (\operatorname{dn}(ax,k) + ik\operatorname{sn}(ax,k))\operatorname{cn}(ax,k)\,y_n\right), \tag{3·33}$$

$$k = \frac{\sqrt{(a^2-1)}}{\sqrt{2}\,a}, \quad a > 1,$$

where $y_1^2 + \ldots + y_n^2 = 1$.

It is clear that (3·33) can also be expressed as follows:

$$z(x_1, \ldots, x_{n+1}) = \frac{\sqrt{2}}{\sqrt{(1+a^2)}}\left(\frac{x_1}{a}\sqrt{\left((1-k^2)x_1^2 + \sum_{j=2}^{n+1} x_j^2\right)} + iak\left[\left(1 + \frac{1}{a^2}\right)x_1^2 + \sum_{j=2}^{n+1} x_j^2\right],\right.$$

$$\left.\left[\sqrt{\left((1-k^2)x_1^2 + \sum_{j=2}^{n+1} x_j^2\right)} + ikx_1\right]x_2, \ldots, \left[\sqrt{\left((1-k^2)x_1^2 + \sum_{j=2}^{n+1} x_j^2\right)} + ikx_1\right]x_{n+1}\right), \tag{3·34}$$

where $\sum_{j=1}^{n+1} x_j^2 = 1$.

It is easy to verify that (3·34) defines an imbedding of S^n into $S^{2n+1}(1) \subset \mathbb{C}^{n+1}$. Let $\bar{P}_a^n$ denote the Riemannian n-manifold given by S^n endowed with the metric induced from $\mathbb{C}^{n+1}$ via (3·34). Then P_a^n is a dense open subset of $\bar{P}_a^n$ with two points removed. Moreover, the isometric imbedding of $\bar{P}_a^n$ into $S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ defined by (3·34) coincides with the C-totally real isometric imbedding of P_a^n into $S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ on P_a^n. Therefore, the Lagrangian isometric immersion $\bar{p}_a : \bar{P}_a^n \to \mathbb{C}P^n(4)$ induced from (3·34) can be regarded as the complete extension of the Lagrangian isometric immersion $p_a : P_a^n \to \mathbb{C}P^n(4)$ introduced in [3]. According to the uniqueness of such Lagrangian immersion (theorem 3·1 of [4]), (3·34) provides an explicit expression of $p_a : P_a^n \to \mathbb{C}P^n(4)$.

Consequently, we obtain

THEOREM 1.

(1) (3·33) (*equivalently*, (3·34)) *defines a C-totally real isometric imbedding from* $\bar{P}_a^n$ *into* $S^{2n+1}(1)$ *which satisfies the basic equality*; *hence, via the Hopf fibration*, (3·34) *gives rise to a Lagrangian isometric immersion* $\bar{p}_a$ *of* $\bar{P}_a^n$ *into* $\mathbb{C}P^n(4)$ *which also satisfies the basic equality*;

(2) *C-totally real isometric immersions of* $\bar{P}_a^n$ *into* $S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ *which satisfy the basic equality are congruent to the one defined by* (3·34); *and*

(3) *up to rigid motions of* $\mathbb{C}P^n(4)$, *the Lagrangian isometric immersion of* $\bar{P}_a^n$ *into* $\mathbb{C}P^n(4)$ *induced from* (3·34) *is the complete extension of the Lagrangian isometric immersion* $p_a : P_a^n \to \mathbb{C}P^n(4)$.

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4. *Lagrangian immersion* $f: F^n \to \mathbb{C}H^n(-4)$

We recall that F^2 is defined as the complete flat surface given by the warped product $\mathbb{R} \times_{1/\sqrt{2}} \mathbb{R}$. Put

$$e_1 = \frac{\partial}{\partial x} \quad \text{and} \quad e_2 = \sqrt{2}\frac{\partial}{\partial y}.$$

Let $f: F^2 \to \mathbb{C}H^2(-4)$ be the Lagrangian isometric immersion of F^2 into $\mathbb{C}H^2(-4)$ introduced in [3] which satisfies the basic equality. If $\hat{f}: F^2 \to H_1^5(1) \subset \mathbb{C}_1^3$ is a lift of f, then $\hat{f}$ is a C-totally real isometric immersion of F^2 into $H_1^5(-1)$ which satisfies the same basic equality in $H_1^5(-1)$.

Assume that the lift $\hat{f}$ is defined by $z = z(x, y)$. Then $z = z(x, y)$ satisfies the differential system (2·7)–(2·9), which in this case reduces to

$$z_{xx} = \frac{3i}{\sqrt{2}} z_x + z, \tag{4·1}$$

$$z_{xy} = \frac{i}{\sqrt{2}} z_y, \tag{4·2}$$

$$z_{yy} = \frac{i}{2\sqrt{2}} z_x + \frac{z}{2}. \tag{4·3}$$

Since F^2 is a C-totally real surface in $H_1^5(-1) \subset \mathbb{C}_1^3$, the immersion also satisfies the conditions given in (2·10)–(2·11).

By solving (4·1), we obtain

$$z(x, y) = A(y)\, e^{ix/\sqrt{2}} + B(y)\, e^{i\sqrt{2}x}, \tag{4·4}$$

where $A(y)$ and $B(y)$ are $\mathbb{C}_1^3$-valued functions in y. (4·2) and (4·4) imply that $B(y)$ is a constant vector which is simply denoted by b. Hence, from (4·3), we have $4A''(y) + A(y) = 0$. Therefore

$$z(x, y) = \left[c_1 \cosh\left(\frac{y}{2}\right) + c_2 \sinh\left(\frac{y}{2}\right)\right] e^{ix/\sqrt{2}} + b\, e^{i\sqrt{2}x} \tag{4·5}$$

for some vectors $c_1, c_2, b \in \mathbb{C}_1^3$.

Now, by applying (2·10), (2·11) and (4·5), we know that vectors c_1, c_2, b defined in (4·5) satisfy the following conditions:

$$\langle c_1, b\rangle = \langle c_2, b\rangle = \langle c_1, c_2\rangle = 0, \quad \langle b, b\rangle = 1, \quad \langle c_1, c_2\rangle = -\langle c_1, c_1\rangle = 2. \tag{4·6}$$

Thus, c_1, c_2, b are mutually orthogonal, b, c_2 are space-like vectors, and c_1 is a time-like vector. It is easy to see that $c_1 = (\sqrt{2}, 0, 0)$, $b = (0, 1, 0)$ and $c_2 = (0, 0, \sqrt{2})$ satisfy (4·6).

Now, we consider a map from F^2 into $H_1^5(-1) \subset \mathbb{C}_1^3$ given by

$$z(x, y) = \left(\sqrt{2}\cosh\left(\frac{y}{2}\right) e^{i\frac{x}{\sqrt{2}}}, \sqrt{2}\sinh\left(\frac{y}{2}\right) e^{i\frac{x}{\sqrt{2}}}, e^{i\sqrt{2}x}\right). \tag{4·7}$$

By a direct computation, we know that (4·7) defines a C-totally real isometric immersion of F^2 into $H_1^5(-1) \subset \mathbb{C}_1^3$ whose squared mean curvature function and whose scalar curvature satisfy the basic equality.

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For $n \geqslant 3$, F^n is the warped product $\mathbb{R} \times_{1/\sqrt{2}} H^{n-1}(-\frac{1}{4})$ which is isometric to the warped product $\mathbb{R} \times_{\sqrt{2}} H^{n-1}(-1)$. By applying an argument similar to case (b) of Section 3, we may obtain a C-totally real isometric immersion from F^n into $H_1^{2n+1}(-1) \subset \mathbb{C}_1^{n+1}$ given by

$$(\sqrt{2}\, y_1 e^{i\frac{x}{\sqrt{2}}}, \ldots, \sqrt{2}\, y_n e^{i\frac{x}{\sqrt{2}}}, e^{i\sqrt{2}\,x}), \tag{4·8}$$

where $y_1^2 - y_2^2 - \ldots - y_n^2 = 1$.

By direct computation, we know that the immersion defined by (4·8) satisfies the basic equality. The projection of this immersion gives rise to a Lagrangian isometric immersion of F^n into $\mathbb{C}H^n(-4)$ satisfying the basic equality. Hence, according to the uniqueness of such immersion (cf. theorem 4 of [3]), (4·8) provides an explicit expression of the Lagrangian immersion $f: F^n \to \mathbb{C}H^n(-4)$.

Consequently, we obtain

THEOREM 2. (1) (4·8) *defines a C-totally real isometric immersion from* F^n *into* $H_1^{2n+1}(-1)$ *which satisfies the basic equality; and hence the projection of* (4·8) *gives rise to a Lagrangian isometric immersion of* F^n into $\mathbb{C}H^n(-4)$ *which also satisfies the equality;*

(2) *C-totally real isometric immersions of* F^n *into* $H_1^{2n+1}(1) \subset \mathbb{C}_1^{n+1}$ *which satisfy the basic equality are congruent to the one given by* (4·8); *and*

(3) *up to rigid motions of* $\mathbb{C}H^n(-4)$, *the Lagrangian isometric immersion of* F^n *into* $\mathbb{C}H^n(-4)$ *induced from* (4·8) *is the Lagrangian isometric immersion* $f: F^n \to \mathbb{C}H^n(-4)$.

Remark 4·1. The totally real immersion of F^n into $\mathbb{C}_1^{n+1}$ defined by (4·7) is a 2-type immersion of order $[1, 2]$ associated with the first two nonzero eigenvalues of the Laplacian of F^n (cf. [2]).

5. *Lagrangian immersion* $l: L^n \to \mathbb{C}H^n(-4)$

L^n is the warped product space $\mathbb{R} \times_{\operatorname{sech}(x)} \mathbb{R}^{n-1}$ with metric

$$g = dx^2 + (\operatorname{sech}^2 x)(dy_1^2 + \ldots + dy_{n-1}^2).$$

Put $$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \cosh x \frac{\partial}{\partial y_1}, \ldots, e_n = \cosh x \frac{\partial}{\partial y_{n-1}}.$$

Let $l: L^n \to \mathbb{C}H^n(-4)$ be the Lagrangian isometric immersion from L^n into $\mathbb{C}H^n(-4)$ introduced in [3]. Assume $\hat{l}: L^n \to H_1^{2n+1}(-1) \subset \mathbb{C}_1^{n+1}$ is a lift of the immersion l (cf. Section 2). Then $\hat{l}$ is a C-totally real immersion of L^n into $H_1^{2n+1}(-1)$ which satisfies the same basic equality in $H_1^{2n+1}(-1)$. Assume $\hat{l}$ is defined by $z = z(x, y_1, \ldots, y_{n-1})$. Then z satisfies the differential system (2·7)–(2·9), which in this case reduces to

$$z_{xx} = 3i(\operatorname{sech} x)\, z_x + z, \tag{5·1}$$

$$z_{xy_j} = (i \operatorname{sech} x - \tanh x)\, z_{y_j}, \tag{5·2}$$

$$(\cosh^2 x)\, z_{y_j y_k} = \delta_{jk}((\tanh x + i \operatorname{sech} x)\, z_x + z), \quad j, k = 1, \ldots, n-1. \tag{5·3}$$

Since the lift $\hat{l}: L^n \to H_1^{2n+1}(-1) \subset \mathbb{C}_1^{n+1}$ is a C-totally real submanifold in $H^{2n+1}(-1)$, the immersion also satisfies the conditions given by (2·10)–(2·11).

Now, we proceed to solve the differential equation (5·1) as follows.

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First, let $u = \sinh x$. Equation (5·1) can be rewritten as

$$(1+u^2)\,z_{uu}+(u-3i)\,z_u-z=0. \tag{5·4}$$

It is easy to see that (5·4) has a linear solution given by $z = u-3i$. Therefore $z_1 = \sinh x-3i$ is a solution of (5·1).

Next, we use the Wronskian to obtain a second independent solution of (5·1). This is done in what follows.

Suppose $z_2 = z_2(x)$ is a second independent solution of (5·1). Let $W = W(x)$ denote the Wronskian of z_1, z_2, i.e. $W = z_1 z_2'-z_1' z_2$. Because z_1, z_2 are independent solutions of (5·2), the Wronskian W takes the following form:

$$\begin{aligned} W(x) &= C\,\mathrm{Exp}\left(3i\int^x \operatorname{sech} t\,dt\right) = C\,\mathrm{Exp}\,(3i\tan^{-1}(\sinh x)) \\ &= C\{\cos(3\tan^{-1}(\sinh x))+i\sin(3\tan^{-1}(\sinh x))\}, \end{aligned} \tag{5·5}$$

where C is a non-zero constant. Without loss of generality, we may assume $C = 1$. Thus, from (5·5), we have

$$W = \{(4\operatorname{sech}^2 x-3)\operatorname{sech} x+i(3-4\tanh^2 x)\tanh x\}. \tag{5·6}$$

Therefore, a second independent solution of (5·2) is given by

$$z_2 = (\sinh x-3i)\int^x \frac{\operatorname{sech} x(4\operatorname{sech}^2 x-3)+i\tanh x(3-4\tanh^2 x)}{(\sinh x-3i)^2}dx. \tag{5·7}$$

On the other hand, by a direct long computation, we have

$$\int^x \frac{(4\operatorname{sech}^2 x-3)\operatorname{sech} x+i(3-4\tanh^2 x)\tanh x}{(\sinh x-3i)^2}dx = \frac{1}{2(1-i\sinh x)(\sinh x-3i)}. \tag{5·8}$$

Thus, by (5·7) and (5·8), we get

$$z_2 = \frac{1}{2(1-i\sinh x)}, \tag{5·9}$$

which is a second independent solution of (5·1). Therefore, the general solution of (5·1) is given by

$$z = C(y_1,\ldots,y_{n-1})(\sinh x-3i)+\frac{D(y_1,\ldots,y_{n-1})}{1-i\sin x}. \tag{5·10}$$

By (5·2), (5·3), (5·10) and a direct computation, we have

$$z(x,y_1,\ldots,y_{n-1}) = \alpha(\sinh x-3i)+\frac{1}{1-i\sinh x}\left(\beta+\sum_{j=1}^{n-1}\gamma_j y_j-i\alpha\sum_{j=1}^{n-1}y_j^2\right), \tag{5·11}$$

for some vectors α, β, γ in $\mathbb{C}_1^{n+1}$. By choosing

$$\begin{gathered} e_1 = z(0,\ldots,0),\quad e_2 = iz(0,\ldots,0),\quad e_3 = z_x(0,\ldots,0), \\ e_4 = iz_x(0,\ldots,0),\quad e_5 = z_{y_1}(0,\ldots,0),\quad e_6 = iz_{y_1}(0,\ldots,0),\ldots, \\ e_{2n+1} = z_{y_{n-1}}(0,\ldots,0),\quad e_{2n+2} = iz_{y_{n-1}}(0,\ldots,0), \end{gathered}$$

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we obtain

$$z=\frac{1}{2}\Big(-\frac{1+i\sum_j y_j^2}{1-i\sinh x}+i(\sinh x-3i),-\frac{3i+\sum_j y_j^2}{1-i\sinh x}$$
$$-i(\sinh x-3i),\frac{2y_1}{1-i\sinh x},\ldots,\frac{2y_{n-1}}{1-i\sinh x}\Big). \quad (5{\cdot}12)$$

By a direct long computation, we know that (5·12) defines a C-totally real isometric immersion from L^n into $H_1^{2n+1}(-1)$ which satisfies the basic equality. Therefore, the projection of this immersion gives rise to a Lagrangian isometric immersion of L^n into $\mathbb{C}H^n(-4)$ which also satisfies the basic equality in $\mathbb{C}H^n(-4)$. Thus, according to theorem 2 of [3], (5·12) provides us an explicit expression of the Lagrangian immersion $l: L^n\to\mathbb{C}H^n(-4)$.

Consequently, we obtain

THEOREM 3. (1) (5·12) *defines a Lagrangian isometric immersion from* L^n *into* $H_1^{2n+1}(-1)$ *which satisfies the basic equality; and hence the projection of* (5·12) *gives rise to a Lagrangian isometric immersion of* L^n *into* $\mathbb{C}H^n(-4)$ *which also satisfies the basic equality;*

(2) *C-totally real isometric immersions of* L^n *into* $H_1^{2n+1}(1)\subset\mathbb{C}_1^{n+1}$ *which satisfy the basic equality are congruent to the one given by* (5·12); *and*

(3) *up to rigid motions of* $\mathbb{C}H^n(-4)$, *the Lagrangian isometric immersion of* L^n *into* $\mathbb{C}H^n(-4)$ *induced from* (5·12) *is the Lagrangian immersion* $l: L^n\to\mathbb{C}H^n(-4)$.

6. *Lagrangian immersions* c_a *and* d_a

For $n\geqslant 3$, C_a^n, $a>1$, is the warped product $I\times_{\rho_a}S^{n-1}(\frac{a^4-1}{4})$ which is isometric to $I\times_{\frac{2}{\sqrt{(a^4-1)}}\rho_a}S^{n-1}(1)$ and C_a^2 is the warped product $I\times_{\rho_a}\mathbb{R}$, where ρ_a is defined by (1·4). By applying a method similar to the ones given in Sections 3 and 4, we may obtain a C-totally real isometric immersion from C_a^n into $H_1^{2n+1}(1)\subset\mathbb{C}_1^{n+1}$ defined by

$$\frac{\sqrt{2}}{\sqrt{(a^2-1)}}\Big(\frac{y_1}{a}\sqrt{[(1-k^2)y_1^2+y_2^2+\ldots+y_{n+1}^2]}iak\Big[\Big(1-\frac{1}{a^2}\Big)y_1^2+y_2^2+\ldots+y_{n+1}^2\Big],$$
$$\{\sqrt{[(1-k^2)y_1^2+y_2^2+\ldots+y_{n+1}^2]}+iky_1\}y_2,\ldots,\{\sqrt{[(1-k^2)y_1^2+y_2^2}$$
$$+\ldots+y_{n+1}^2]}+iky_1\}y_{n+1}\Big), \quad (6{\cdot}1)$$

where $y_1^2+\ldots+y_{n+1}^2=1$.

By a direct long computation, we know that (6·1) defines a C-totally real isometric imbedding of C_a^n into $H_1^{2n+1}(-1)\subset\mathbb{C}_1^{n+1}$ which satisfies the basic equality. The projection of this immersion gives rise to a Lagrangian isometric immersion of C_a^n into $\mathbb{C}H^n(-4)$ satisfying the basic equality. Thus, according to theorem 2 of [3], the projection of (6·1) provides us an explicit expression of the Lagrangian immersion c_a of C_a^n in $\mathbb{C}H^n(-4)$ introduced in [3].

Notice that as in the case of P_a^n, (6·1) defines a C-totally real imbedding from the topological n-sphere S^n into $H_1^{2n+1}(1)\subset\mathbb{C}_1^{n+1}$.

Let $\bar{C}_a^n$ denote S^n endowed with the Riemannian metric induced from the pseudo-Euclidean metric of $\mathbb{C}_1^{n+1}$ via (6·1). Then C_a^n is a dense open subset of $\bar{C}_a^n$ with two points removed. Moreover, the isometric imbedding of $\bar{C}_a^n$ into $H_1^{2n+1}(-1)\subset\mathbb{C}_1^{n+1}$

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given by (6·1) coincides with the C-totally real isometric imbedding of C_a^n into $H_1^{2n+1}(-1) \subset \mathbb{C}_1^{n+1}$ on C_a^n. Therefore, the Lagrangian isometric immersion $\bar{c}_a$: $\bar{C}_a^n \to \mathbb{C}H^n(-4)$ induced from (6·1) can be considered as the complete extension of the Lagrangian isometric immersion c_a: $C_a^n \to \mathbb{C}H^n(-4)$ introduced in [3].

Consequently, by applying theorem 4 of [3], we obtain

THEOREM 4. (1) (6·1) *defines a Lagrangian isometric imbedding from* $\bar{C}_a^n$ *into* $H_1^{2n+1}(-1)$ *which satisfies the basic equality; hence* (6·1) *gives rise to a Lagrangian isometric immersion* $\bar{c}_a$ *of* $\bar{C}_a^n$ *into* $\mathbb{C}H^n(-4)$ *which satisfies the basic equality;*

(2) *C-totally real isometric immersions of* $\bar{C}_a^n$ *into* $H_1^{2n+1}(1) \subset \mathbb{C}_1^{n+1}$ *which satisfy the basic equality are congruent to the one given by* (6·1); *and,*

(3) *up to rigid motions of* $\mathbb{C}H^n(-4)$, *the Lagrangian isometric immersion of* $\bar{C}_a^n$ *into* $\mathbb{C}H^n(-4)$ *induced from* (6·1) *provides us the explicit expression of the Lagrangian immersion* c_a: $C_a^n \to \mathbb{C}H^n(-4)$.

Finally, we consider Lagrangian immersion d_a: $D_a^n \to \mathbb{C}H^n(-4)$.

For $n \geqslant 3$, D_a^n, $0 < a < 1$, is the warped product $I \times_{\rho_a} H^{n-1}(\frac{a^4-1}{4})$ which is isometric to $I \times_{\frac{2}{\sqrt{(a^2-1)}}\rho_a} H^{n-1}(-1)$ and C_a^2 is the warped product $I \times_{\eta_a} \mathbb{R}$, where ρ_a is defined by (1·3). By applying a method similar to the ones given in Sections 3 and 4, we may obtain a C-totally real isometric immersion from D_a^n into $H_1^{2n+1}(1) \subset \mathbb{C}^{n+1}$ given by

$$\frac{1}{\sqrt{(1-a^2)}}\left(\sqrt{2}\,\mathrm{dn}\left(\frac{ax}{k},k\right)\left[\mathrm{cn}\left(\frac{ax}{k},k\right)+\mathrm{i}\,\mathrm{sn}\left(\frac{ax}{k},k\right)\right]y_1 \ldots, \sqrt{2}\,\mathrm{dn}\left(\frac{ax}{k},k\right)\left[\mathrm{cn}\left(\frac{ax}{k},k\right)\right.\right.$$

$$\left.\left.+i\,\mathrm{sn}\left(\frac{ax}{k},k\right)\right]y_n, \frac{1}{\sqrt{(1+a^2)}}\left[2\,\mathrm{cn}\left(\frac{ax}{k},k\right)\mathrm{sn}\left(\frac{ax}{k},k\right)+i\left[1-a^2-2\,\mathrm{cn}^2\left(\frac{ax}{k},k\right)\right]\right)\right), \quad (6{\cdot}2)$$

$$k = \frac{\sqrt{2}\,a}{\sqrt{(1+a^2)}}, 0 < a < 1,$$

where $y_1^2 - y_2^2 - \ldots - y_n^2 = 1$. By a direct long computation, we know that (6·2) defines a C-totally real isometric immersion of D_a^n into $H_1^{2n+1}(1) \subset \mathbb{C}_1^{n+1}$ which satisfies the basic equality. The projection of this immersion gives rise to a Lagrangian isometric immersion of D_a^n into $\mathbb{C}H^n(-4)$ satisfying the basic equality. Thus, according to theorem 4 of [3], (6·2) provides an explicit expression of the Lagrangian immersion d_a: $C_a^n \to \mathbb{C}H^n(-4)$.

Consequently, we obtain

THEOREM 5. (1) (6·2) *defines a Lagrangian isometric immersion from* D_a^n *into* $H_1^{2n+1}(-1)$ *which satisfies basic equality; and hence* (6·2) *gives rise to a Lagrangian isometric immersion* d_a *of* D_a^n *into* $\mathbb{C}H^n(-4)$ *which also satisfies the basic equality;*

(2) *C-totally real isometric immersions of* D_a^n *into* $H_1^{2n+1}(1) \subset \mathbb{C}_1^{n+1}$ *which satisfy the basic equality are congruent to the one given by* (6·2); *and*

(3) *up to rigid motions of* $\mathbb{C}H^n(-4)$, *the Lagrangian isometric immersion of* D_a^n *into* $\mathbb{C}H^n(-4)$ *induced from* (6·2) *is the Lagrangian immersion* d_a: $D_a^n \to \mathbb{C}H^n(-4)$.

Remark 5·1. (Added 9 August 1995) After this paper was submitted for publication, I. Castro and F. Urbano kindly informed the first author that they had independently obtained the basic inequality of [3] for Lagrangian surfaces M^2 in $\mathbb{C}P^2$ and in $\mathbb{C}H^2$. Moreover, they had investigated the equality case from twistor theory points of view. In particular, they had proved that a Lagrangian surface M^2 in $\mathbb{C}P^2$ satisfies the basic

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equality if and only if it has holomorphic twistor lift. (For details see their joint article 'Twistor holomorphic Lagrangian surfaces in complex projective and hyperbolic planes', to appear in *Ann. Glob. Anal. Geom.*)

REFERENCES

[1] J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward. On conformal minimal immersions of S^2 into $\mathbb{C}P^n$. *Math. Ann.* **279** (1988), 599–620.

[2] B. Y. Chen. *Total mean curvature and submanifolds of finite type* (World Scientific, 1984).

[3] B. Y. Chen. Jacobi's elliptic functions and Lagrangian immersions. *Proc. Royal Soc. Edinburgh Sect.* A, *Math.* **126** (1996) (to appear).

[4] B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken. An exotic totally real minimal immersion of S^3 in $\mathbb{C}P^3$ and its characterization. *Proc. Royal Soc. Edinburgh Sect.* A, *Math.* **126** (1996) (in press).

[5] B. Y. Chen, G. D. Ludden and S. Montiel. Real submanifolds of a Kaehlerian manifold. *Algebra, Groups and Geometries* **1** (1984), 176–212.

[6] D. F. Lawden, *Elliptic functions and applications* (Springer-Verlag, 1989).

[7] H. Reckziegel. Horizontal lifts of isometric immersions into the bundle space of a pseudo-Riemannian submersion; in *Global differential geom and global analysis* (1984). Lecture Notes in Mathematics (Springer-Verlag, 1985), pp. 264–279.

https://doi.org/10.1017/S0305004100074867 Published online by Cambridge University Press