—Technical Note—

A NEW SOLUTION TO FŐPPL-HENCKY MEMBRANE EQUATION

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ABSTRACT

In this note, Föppl-Hencky membrane equation in the case of axisymmetric deformation was derived, and its power series solution was presented by using the displacement-based solution method. The result shows that both the displacement-based solution method and the stress-based solution method are effective for the solution to Föppl-Hencky equation. But in comparison with the latter, the former makes the solving process some more concise. In addition, some issues concerned were also discussed.

Keywords: Föppl-Hencky membrane, Axisymmetric deformation, Power series solution, Displacement.

1. INTRODUCTION

Föppl-Hencky membrane equation originates in classical Föppl-von Kármán equation dealing with the large deflection problem of thin plates [1-5]. In view of the fact that the plate is very thin, the bending stiffness in the plate may be negligible, thus such a plate problem is transformed into a flexible membrane one in which only the tensile effect is considered. By neglecting bending stiffness term in Föppl-von Kármán equation, we may obtain Föppl-Hencky membrane equation in rectangular coordinates (x, y) as follows

$$\frac{q}{h}+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \varphi}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 \varphi}{\partial x\partial y}=0, \tag{1a}$$

$$\frac{\partial^4 \varphi}{\partial x^4}+2\frac{\partial^4 \varphi}{\partial x^2\partial y^2}+\frac{\partial^4 \varphi}{\partial y^4}=E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right], \tag{1b}$$

where $w = w(x, y)$ is the out-plane deflection, $\varphi = \varphi(x, y)$ is the Airy stress function, E is Young's modulus of elasticity, h is the thickness of the membrane and q is the intensity of the external loads applied. Equation (1a) shows the relation of equilibrium between the external loads and the membrane response to tensile deformation, while Eq. (1b) indicates the consistency relation between the deflection and the stress. Föppl-Hencky membrane equation is considered to be an effective mathematical model to analyze flexible membrane problems.

During the past hundred years, researchers have applied themselves to the solution of Eqs. (1a, b) in different ways, including analytical and numerical study. The analytical method is often the most challenging due to the high-order partial differential equations Eqs. (1a, b). Based on a given stress and deflection pattern in the form of power series, Hencky [6] solved the problem of axisymmetric deformation of a uniformly loaded circular membrane fixed at the outer edge, usually called Föppl-Hencky problem for short. Later, Hencky [7] adopted finite difference method to solve this problem. A computational error in the solution presented by Hencky [6] was corrected by Chien [8] and Alekseev [9], respectively, and also by Campbell [10] where the problem was generalized to include arbitrary initial tension. Hencky solution is regarded as the first solution of circular membrane problems and is often cited with regard to this problem. Alekseev [11] investigated the problem of axisymmetric deformation of an annular circular membrane fixed at the outer edge and uniformly loaded along the inner edge, and presented only a partial solution of the problem. Sun *et al*. [12] presented the complete solution of Alekseev problem. The solution of the problem of axisymmetric deformation of a circular membrane under the action of uniformly-distributed loads in its central portion was presented by Chien *et al*. [13]. Chen and Zheng [14] dealt with the problem of axisymmetric deformation of a circular membrane fixed at the outer edge and loaded at the center, via the application of a so-called extended Hencky transformation. Jin [15] dealt with this problem by using the additional restriction condition that the stress component (radial or circumferential) is always nonnegative. Based on the complete solution of Alekseev problem, Sun *et al*. [16] also dealt with this problem via the approach to taking

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limit value. In fact, Hencky solution was obtained only for the case of uniform lateral loading (the case of gravity loading), as for the case of static pressure loading, the radial component of the pressure acting on the deformed membrane is obviously neglected by Hencky solution. But the investigating results for the two types of loading (gravity and pressure), presented by Fichter [17], have been shown to agree quite closely when the membrane is lightly loaded and to diverge slowly as the load intensity is increased. Recently, Lim [18] carried out an investigation on the effect of auxeticity toward the deflection of a circular membrane using an exact model and also introduced a semi-empirical model to facilitate convenient but accurate calculation of the membrane deflection, regardless of the membrane being conventional or auxetic. Due to the somewhat intractable nonlinear equations, closed-form solutions of membrane problems are available in a few cases, but usually a variety of methods is utilized to obtain numerical solutions for displacements, strains, and stresses [19].

Recall from the classical solution to Föppl-Hencky problem [13, 20] that, the problem was first formulated as a mathematical expression in terms of stress, strain and displacement (a set of differential equations containing stress, strain and displacement), a two-order differential equation containing only stress was then obtained by eliminating the displacement components from these differential equations and was solved by using the power series method, and further all solutions for stress, strain, displacement in the form of power series was derived. Roughly, this classical solution may be called the stress-based solution method. Here we shall reconsider the solution of Föppl-Hencky problem, with emphasis on the displacement-based solution method, so that we can see what is going on.

2 MEMBRANE EQUATION IN AN AXISYMMETRIC PROBLEM

In polar coordinates (r, θ), let $w = w(r, \theta)$ and $\varphi = \varphi(r, \theta)$, Föppl-Hencky membrane equation may be changed as, from Eqs. (1a, b),

$$\frac{q}{h} + L(w, \varphi) = 0, \tag{2a}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \varphi = -\frac{E}{2} L(w, w), \tag{2b}$$

where the operator

$$\begin{aligned} L(w, \varphi) = {} & \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right) \\ & + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\frac{\partial^2 \varphi}{\partial r^2} - 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right) \end{aligned} \tag{3}$$

and the another operator $L(w, w)$ may easily be obtained by substituting φ in Eq. (3) with w. Especially, for an axisymmetric problem, it is obvious that $w = w(r)$ and $\varphi = \varphi(r)$. The relation among radial membrane stress, circumferential membrane stress and stress function are

$$\sigma_r = \frac{1}{r}\frac{d\varphi}{dr} \text{ and } \sigma_t = \frac{d^2\varphi}{dr^2} = \frac{d}{dr}(r\sigma_r) \tag{4a}$$

and the axisymmetric conditions at the membrane center is

$$\frac{dw}{dr} = 0 \text{ and } \sigma_r \text{ is a finite value at } r = 0. \tag{4b}$$

Thus, Eqs. (2a, b) may be reduced into, via Eq. (4a),

$$\frac{q}{h} + \frac{1}{r}\frac{d}{dr}\left(r\sigma_r \frac{dw}{dr}\right) = 0, \tag{5a}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 \varphi + \frac{E}{r}\frac{dw}{dr}\frac{d^2 w}{dr^2} = 0. \tag{5b}$$

Integrating Eq. (5a) with respect to r, we obtain the equilibrium equation (which is perpendicular to the plane of the circular membrane)

$$h\sigma_r \frac{dw}{dr} = -\frac{1}{2} rq, \tag{6a}$$

where the second condition in Eq. (4b) is used. At the same time, note that the operator $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2$ may be written as the form $\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right]\right\}$, hence substituting it into Eq. (5b) and integrating with respect to r, we obtain the consistency equation

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r^2 h\sigma_r\right)\right] + \frac{Eh}{2}\left(\frac{dw}{dr}\right)^2 = 0, \tag{6b}$$

where the first condition in Eq. (4b) is used. Equations (6a, b) are the Föppl-Hencky membrane equation in the case of axisymmetric deformation, where there are one independent variable r and two dependant variables σ_r and w. For a circular membrane with radius a and fixed at its perimeter, the corresponding boundary conditions are

$$\frac{u}{r} = 0 \text{ and } w = 0 \text{ at } r = a, \tag{7a, b}$$

where u is the radial displacement and u/r is the circumferential strain e_t. If ν is Poisson's ratio, then u/r may be written as, via physical equations,

$$\frac{u}{r} = \frac{1}{Eh}(h\sigma_t - \nu h\sigma_r) = \frac{1}{Eh}\left[\frac{d}{dr}\left(rh\sigma_r\right) - \nu h\sigma_r\right]. \tag{8}$$

Here note an interesting phenomenon: in the problem of a circular plate fixed at its perimeter, the condition $dw/dr = 0$ at $r = a$ is usually used as a boundary condition, while in Eq. (7), such a condition seems to be van-

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ishing (in fact, in circular membrane problems $dw/dr \neq 0$ at $r = a$). From such a phenomenon, perhaps one may guess that the circular membrane problem can be solved, obviously, only by eliminating dw/dr (rather than σ_r) from Eqs. (6a, b). But in fact, eliminating σ_r (rather than dw/dr) is also effective for the problem solved, as demonstrated in what follows.

3 POWER SERIES SOLUTION

Let us introduce the following dimensionless variables

$$Q = \frac{a^4 q}{Eh^4}, W = \frac{w}{h}, S_r = \frac{a^2 \sigma_r}{Eh^2}, x = \frac{r^2}{a^2}, \tag{9}$$

transform Eqs. (6a, b) into

$$\frac{dW}{dx} S_r = -\frac{Q}{4}, \tag{10a}$$

$$\frac{d^2}{dx^2}\left(xS_r\right) + \frac{1}{2}\left(\frac{dW}{dx}\right)^2 = 0 \tag{10b}$$

and write the boundary conditions (7a, b) as

$$\frac{u}{r} = \frac{h^2}{a^2}\left[2x\frac{dS_r}{dx} + (1-\nu)S_r\right] = 0 \quad \text{and } W = 0 \text{ at } x = 1. \tag{11a, b}$$

Eliminating S_r from Eqs. (10a, b), we may obtain an equation which contains only W

$$2Q\frac{dW}{dx}\frac{d^2W}{dx^2} + Qx\frac{dW}{dx}\frac{d^3W}{dx^3} - 2Qx\left(\frac{d^2W}{dx^2}\right)^2 + 2\left(\frac{dW}{dx}\right)^5 = 0. \tag{12}$$

Expand W into the power series of the x, i.e., let

$$W(x) = Q^{1/3}\sum_{n=0}^{\infty} b_n x^n. \tag{13}$$

Substituting Eq. (13) and its derivatives into Eq. (12), the general solution of Eq. (12) may be written as

$$W(x) = Q^{1/3}\left(b_0 + b_1 x - \frac{1}{2}b_1^4 x^2 + \frac{5}{9}b_1^7 x^3 - \frac{55}{72}b_1^{10}x^4 + \frac{7}{6}b_1^{13}x^5 - \frac{205}{108}b_1^{16}x^6 + \frac{17051}{5292}b_1^{19}x^7 - \frac{2864485}{508032}b_1^{22}x^8 + \frac{103863265}{10287648}b_1^{25}x^9 - \cdots\right). \tag{14}$$

If letting $c = -2b_1^3$, Eq. (14) can easily be transformed into

$$W(x) = Q^{1/3}\left\{b_0 - (\frac{c}{2})^{1/3}x\left[1 + \frac{1}{4}cx + \frac{5}{36}(cx)^2 + \frac{55}{576}(cx)^3 + \frac{7}{96}(cx)^4 + \frac{205}{3456}(cx)^5 + \frac{17051}{338688}(cx)^6 + \frac{2864485}{65028096}(cx)^7 + \frac{103863265}{2633637888}(cx)^8 + \cdots\right]\right\}. \tag{15}$$

Further, we write Eq. (15) into

$$W(x) = Q^{1/3}\left[b_0 - \left(\frac{c}{2}\right)^{1/3} xg(cx)\right], \tag{16}$$

where

$$g(x) = 1 + \frac{1}{4}x + \frac{5}{36}x^2 + \frac{55}{576}x^3 + \frac{7}{96}x^4 + \frac{205}{3456}x^5 + \frac{17051}{338688}x^6 + \frac{2864485}{65028096}x^7 + \frac{103863265}{2633637888}x^8 + \cdots. \tag{17}$$

From Eqs. (16) and (17), Eq. (11b) gives

$$b_0 = \left(\frac{c}{2}\right)^{1/3} g(c). \tag{18}$$

Thus, from Eqs. (16) and (18), one has

$$W(x) = \left(\frac{Qc}{2}\right)^{1/3}[g(c) - xg(cx)]. \tag{19}$$

From Eqs. (10a) and (19), it is found that

$$S_r = \left(\frac{Qc}{2}\right)^{2/3}\frac{1}{2c}f(cx), \tag{20}$$

where

$$f(x) = 1 - \frac{1}{2}x - \frac{1}{6}x^2 - \frac{13}{144}x^3 - \frac{17}{288}x^4 - \frac{37}{864}x^5 - \frac{1205}{36288}x^6 - \frac{219241}{8128512}x^7 - \frac{6634069}{292626432}x^8 - \frac{51523763}{2633637888}x^9 - \cdots. \tag{21}$$

From Eqs. (20) and (21), Eq. (11a) gives

$$\nu = 2cf'(c)f^{-1}(c) + 1. \tag{22}$$

Hence, for the concrete problem where the value of ν is known in advance, the undetermined constants c can thus be determined by Eq. (22). Moreover, from Eqs. (9) and (19), we may finally obtain

$$w(r) = \left(\frac{a^4 q}{Eh}\right)^{1/3}\left(\frac{c}{2}\right)^{1/3}\left[g(c) - g\left(c\frac{r^2}{a^2}\right)\frac{r^2}{a^2}\right], \tag{23}$$

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and hence the maximum deflection of the circular membrane, at the central point ($r = 0$), is

$$w_m = w(0) = \left(\frac{a^4 q}{Eh}\right)^{1/3} \left(\frac{c}{2}\right)^{1/3} g(c). \tag{24}$$

The final results, Eqs. (23) and (24), indicate that they are the same as what have been obtained by using the stress-based solution method [20].

4 DISCUSSIONS

From Section 3 it may be seen that, our solving process is some more concise than that of classical solution [13, 20], since we here eliminate S_r from Eqs. (10a, b) and directly expand W into the power series of the x, i.e., $W(x) = Q^{1/3}\sum_{n=0}^{\infty} b_n x^n$ (while in classical solution, dw/dx was eliminated, and the power series method was not used until the transformation $xS_r(x) = 2^{-5/3}Q^{2/3}Z(x)$ was carried out [13, 20]).

Usually, it is considered that the number of undetermined constants in the general solution of a differential equation should be consistent with the order number of the differential equation. But from Eq. (12) and its general solution Eq. (14), it may be seen that, Eq. (12) is a third-order differential equation while its general solution Eq. (14) contains only two undetermined constants, b_0 and b_1. The reason for such an inconsistent phenomenon is that, in Eq. (12) the three-order derivative item is multiplied by x, consequently, after substituting Eq. (13) and its derivatives into Eq. (12), we may obtain the so-called recurrence relation

$$\begin{cases} 4b_1b_2 + 2b_1^5 = 0 \\ 18b_1b_3 + 20b_1^4b_2 = 0 \\ 30b_1^4b_3 + 80b_1^3b_2^2 + 48b_1b_4 = 0 \\ 40b_1^4b_4 + 100b_1b_5 - 18b_3^2 + 16b_2b_4 + 240b_1^3b_2b_3 + 160b_1^2b_2^3 = 0 \\ -72b_3b_4 + 60b_2b_5 + 320b_1^3b_2b_4 + 180b_1b_6 + 160b_1b_2^4 + 50b_1^4b_5 \\ +720b_1^2b_2^2b_3 + 180b_1^3b_3^2 = 0 \\ 294b_1b_7 + 64b_2^5 + 960b_1^2b_2^2b_4 + 480b_1^3b_3b_4 + 960b_1b_2^3b_3 + \\ 400b_1^3b_2b_5 + 144b_2b_6 + 60b_1^4b_6 + 1080b_1^2b_2b_3^2 - 96b_4^2 \\ -90b_3b_5 = 0 \\ \cdots\cdots \end{cases} \tag{25}$$

Then, from Eq. (25) it may be seen that, the first expression in Eq. (25) contains only b_1 and b_2, consequently b_2 can be expressed by b_1, similarly, the second expression contains only b_1, b_2 and b_3, consequently b_3 can finally be expressed by b_1, and by the same token, finally b_4, b_5, … can all be expressed by b_1. So, due to such a recurrence relation shown in Eq. (25), only b_0 and b_1 are left as arbitrary constants, i.e., the undetermined constants in the general solution Eq. (14). So it follows that, for the power series method, the number of undetermined constants in the general solution of a differential equation depends on the specified form of the differential equation.

5 CONCLUSIONS AND RECOMMENDATION

In this note, Föppl-Hencky membrane equation in the case of axisymmetric deformation was derived, and its power series solution was also presented by using the displacement-based solution method. In fact, the problem of axisymmetric deformation of thin circular plates [5, 21] can also be solved by using the similar method. Of course, the Föppl-Hencky membrane equation may also be solved via the perturbation method [22, 23]. Moreover, the solution presented here might have important significance to many applied areas, such as biological thin-film structures, architectural membrane structures, and the thin-film/substrate delamination studies.

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