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Transitive dendrite map with zero entropy

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Abstract. Hoehn and Mouron [Hierarchies of chaotic maps on continua. *Ergod. Th. & Dynam. Sys.* **34** (2014), 1897–1913] constructed a map on the universal dendrite that is topologically weakly mixing but not mixing. We modify the Hoehn–Mouron example to show that there exists a transitive (even weakly mixing) dendrite map with zero topological entropy. This answers the question of Baldwin [Entropy estimates for transitive maps on trees. *Topology* **40**(3) (2001), 551–569].

1. *Introduction*

Blokh [**4**] proved that a transitive map f of a connected topological graph is either non-invertible and has the relative specification property, or is an irrational rotation of the circle. In the former case, f has a dense set of periodic points and positive topological entropy. There are other connections between transitivity and topological entropy for one-dimensional maps (see [**2**]). Therefore it is natural to ask: for which compact metric spaces X does every transitive map $f : X \to X$ have positive topological entropy? A natural candidate for a class of continua with this property are dendrites. Dendrites generalize topological trees and appear in dynamics, for example as Julia sets for some complex polynomials [**5**, Theorem 4.2]. It is also known that maps on dendrites share some dynamical properties with graph maps (see, for example, [**1**, **9**, **10**, **12**, **14**, **15**]). In particular, it is known that if a dendrite D contains a free arc, then every transitive map $f : D \to D$ has positive topological entropy (see [**6**]). On the other hand, every dendrite D such that no subtree of D contains all free arcs admits a transitive map with arbitrarily small entropy (see [**16**]).

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In [**7**], Hoehn and Mouron constructed a family of self-maps on the Ważewski universal dendrite D_ω and showed that, among these maps, there are topologically weakly mixing but not mixing ones. We say that a map is a *Hoehn–Mouron map* if it is defined following the procedure described in [**7**]. Our main result is the following theorem (for the full statement, see §5).

MAIN THEOREM. (Short version) *There exists a weakly mixing, not mixing, proximal Hoehn–Mouron map $f_0 : D_\omega \to D_\omega$ with zero topological entropy.*

Our result gives an affirmative answer to the question of Baldwin [**3**, Problem 3]. *Do there exist any transitive maps f on a dendrite such that $h_{top}(f) = 0$?* The map f_0 is also a new example of a transitive map of a dendrite which does not have dense periodic points (because proximality excludes existence of more than one minimal point and every periodic point is minimal). This property of the Hoehn–Mouron example is also noted in [**1**]. Observe that dendrites have the fixed point property, and hence every continuous dendrite map has at least one fixed point. An example of a transitive dendrite map with only one fixed point and no other periodic points is given in [**15**], but the map constructed there has positive topological entropy.

In what follows we use the notation and facts introduced in [**7**], which we use as 'black boxes'. For more details we refer the reader to [**7**].

2. *Basic definitions*

We assume the reader is familiar with elementary theory of dynamical systems. Throughout this paper $\mathbb{N} = \{1, 2, \ldots\}$.

A *dendrite* is a compact, pathwise connected, locally connected, non-separating, and nowhere dense subset of the plane: equivalently, a locally connected continuum that contains no simple closed curves (see [**13**]). The *universal dendrite* (the *Ważewski dendrite*, see [**13**, §10.37, p. 181]) is a dendrite D_ω which is characterized (see [**18**]) by the following property. Each ramification point of D_ω has infinite order and, for each arc $A \subset D_\omega$, the set of ramification points of D_ω contained in A is dense in A.

Let $f : D_\omega \to D_\omega$ be a continuous map. Given non-empty open sets $U, V \subset D_\omega$, define $N(U, V) = \{n \in \mathbb{N} : f^{-n}(U) \cap V \neq \emptyset\}$. We recall that f is *transitive* if $N(U, V) \neq \emptyset$ for all non-empty open sets $U, V \subset D_\omega$. A map f is *weakly mixing* if the product map $f \times f$ is transitive. We say that f is mixing if the set $N(U, V) \neq \emptyset$ has finite complement in $\mathbb{N}$ for all non-empty open sets $U, V \subset D_\omega$. Equivalently, f is weakly mixing if $N(U, V)$ is thick for all non-empty open sets $U, V \subset X$. Recall that a set $A \subset \mathbb{N}$ is *thick*, if it contains arbitrarily long blocks of consecutive integers: that is, for every $n > 0$ there is $k \in \mathbb{N}$ such that $\{k, k + 1, \ldots, k + n - 1\} \subset A$. An f-invariant closed non-empty set $M \subset D_\omega$ is *minimal* if the orbit of every point $x \in M$ is dense in M. A point $x \in D_\omega$ is a *minimal point* if the closure of its orbit is a minimal set. We say that f is *proximal* if it has a unique minimal point. Note that if $x \in D_\omega$ is a unique minimal point for f, then x is fixed by f.

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3. *Description of D_ω*

Following Hoehn and Mouron we regard the Ważewski universal dendrite as the closure of an inductively constructed 'skeleton' in the plane. We parametrize the edges of the skeleton by elements of a set S defined below.

Let $\omega = \{0, 1, \ldots\}$ and let Q denote the set of dyadic rational numbers in the interval $(0, 1)$: that is, $x \in Q$ if and only if $x = p/2^q$ for some $p, q \in \mathbb{N}$ with $0 < p < 2^q$. Given $k \in \omega$, let $S_k = \omega \times (Q \times \omega)^k$ and $S = \{\emptyset\} \cup \bigcup_{k\in\omega} S_k$.

Let $c \in \mathbb{R}^2$ and let $\{B_j\}_{j\in\omega}$ be a family of straight line segments (edges) in $\mathbb{R}^2$ such that, for every $j \in \omega$, the point c is the endpoint of B_j, $\operatorname{diam} B_j < 1/2^j$ and $B_k \cap B_j = \emptyset$ for all $k < j$. We call the union $\bigcup_{j\in\omega} B_j$ a *fan* with *center* c and *beams* B_j.

We define the skeleton of D_ω as the union of an inductively constructed sequence of dendrites $D_0 \subset D_1 \subset D_2 \subset \cdots$. At step k, we attach to D_{k-1} a countable collection of straight line segments (edges). For each new edge e, we fix a linear homeomorphism $\psi_e : [0, 1] \to e$ such that $\{\psi_e(0)\} = e \cap D_k$. We refer to $\psi_e(t)$ as the *point at position* $t \in [0, 1]$ *along* e. These new edges are in a one-to-one correspondence with sequences in S_k.

The construction begins with a fan D_0 with center o and beams $\{B_j\}_{j\in\omega}$. Given $s \in S_0$ we define $A[s] = B_s$. In addition, we put $A[\emptyset] = \{o\}$. Assume now that we have defined D_k and $A[s]$ for some $k \in \omega$ and all $s \in S_k$. For each $s \in S_k$ and each dyadic rational $x = p/2^q$, we attach a fan $F_x = \bigcup_{j\in\omega} B_j$ to the point z at position $p/2^q$ along $A[s]$ in such a way that $F_x \cap D_k = \{z\}$ and $\operatorname{diam} F_x < 1/2^q \operatorname{diam} A[s]$. We also put $A[s \frown (p/2^q, j)] = B_j$, where $\frown$ denotes the concatenation operator. Furthermore, we demand that fans attached at step k be pairwise disjoint.

Finally, we define

$$D_\omega = \overline{\bigcup_{j\in\omega} D_j}.$$

For $j \in \omega$, we denote by E_j the subdendrite (homeomorphic with D_ω) defined as

$$E_j = \overline{\bigcup \{A[s] : s \in S,\ s \text{ begins with } j\}}.$$

Furthermore, we may write

$$D_\omega = \bigcup_{j=0}^{\infty} E_j \quad \text{where } E_i \cap E_j = \{o\} \text{ for } i \neq j.$$

4. *Hoehn–Mouron maps*

Hoehn and Mouron begin with an arbitrary set $Z \subset \omega$ and define, inductively, an auxiliary function $\alpha_Z : S\backslash\{\emptyset\} \to S$.

Given α_Z, they define a continuous transformation φ of the skeleton of D_ω described above. For any $s \in S\backslash\{\emptyset\}$, the transformation φ maps $A[s]$ onto $A[\alpha(s)]$ by an orientation preserving linear homeomorphism. Then the function $f_Z : D_\omega \to D_\omega$ is defined as a continuous extension of φ to all of D_ω.

The construction of α_Z guarantees that f_Z is continuous and, for each $s \in S\backslash\{\emptyset\}$, there is $n \in \omega$ such that $\alpha_Z^{n+1}(s) = \emptyset$. Then $n_Z(s)$ denotes the smallest such n, and hence $n_Z : S\backslash\{\emptyset\} \to \omega$.

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Unfortunately, α_Z is not well defined for every $Z \subset \omega$ and, due to the recursive nature of the construction, it is not easy to explicitly formulate conditions which guarantee the existence of α_Z. Nevertheless, Hoehn and Mouron proved that it is possible to find a set Z such that α_Z, n_Z and f_Z exist. The construction is based upon the following two claims† from [**7**].

Claim 17.1. If $s \in S$ is such that $\alpha_Z(s)$ is defined, then there exists $N = N_1(s) \in \mathbb{N}$ such that if $Z' \subset \omega$ is any other set with $Z' \cap [0, N] = Z \cap [0, N]$, then $\alpha_{Z'}(s) = \alpha_Z(s)$.

Claim 17.2. For each $s \in S$ and $N \in \omega$, there exists $M = M_2(s, N) \in \omega$ such that if $[N, N + M] \cap \omega \subset Z$, then $\alpha_Z(s)$ is defined.

To show that the resulting map is weakly mixing but not mixing one must ensure that Z satisfies two additional conditions dictated by the following claim that was shown in [**7**].

Claim 17.4. Suppose that for any $M \in \mathbb{N}$ there exists $N \in \omega$ such that $Z \cap [N, N + M] = \emptyset$. Then, for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\operatorname{diam} f_Z^n(E_0) < \varepsilon$.

Let $S^- = \bigcup_{j \in \omega} (\omega \times Q)^j$. Given $s \in S^-$ and $K, m \in \omega$, let $T_{s,K,m}$ be the set

$$T_{s,K,m} = \Big\{ s \frown \Big(m_0, \frac{p_1}{2^{q_1}}, m_1, \ldots, \frac{p_k}{2^{q_k}}, m_k \Big) \in S : 0 \le k \le K, \\ m \le m_0 \le m + K, \text{ and for each } 0 < i \le k \text{ we have} \\ 1 \le q_i \le K,\ 1 \le p_i < 2^{q_i},\ 0 \le m_i \le K \Big\}. \tag{1}$$

Note that $A[T_{s,K,m}] = \bigcup \{A[s] : s \in T_{s,K,m}\}$ is a subtree of D_ω with a root at the branching point of D_ω, determined by s. The following Claim 17.5 says that, given s, N, and K, if Z contains a long enough interval of consecutive integers, then there exists $m \in \omega$ such that some iteration of f_Z maps $A[T_{s,K,m}]$ onto $A[T_{\emptyset,K,0}]$.

Claim 17.5. For each $s \in S^-$ and $N, K \in \omega$, there exist $M, m, n \in \omega$ such that if $[N, N + M] \cap \omega \subset Z$, then $\alpha^n(T_{s,K,m}) = T_{\emptyset,K,0}$.

Our main observation concerning the Hoehn and Mouron construction is that the gaps in the set Z allow one to control return times to non-empty open sets $U \subset D_\omega \backslash \{o\}$. In particular, we prove that it is possible to find a set Z for which the map f does not admit any ergodic invariant measures apart from the Dirac measure concentrated on $\{o\}$.

5. *The main theorem*

MAIN THEOREM. *Let D_ω denote the Ważewski universal dendrite (see Figure 1). There exists a set $Z \subset \mathbb{N}$ such that the Hoehn–Mouron map $f_Z : D_\omega \to D_\omega$ has the following properties:*

(1) *f_Z is weakly mixing, but not mixing;*
(2) *all transitive points of f_Z are end points of D_ω;*
(3) *f_Z has a unique fixed point o, which is, in fact, a unique minimal point of f_Z (it follows that f_Z is proximal);*

† We have kept the numeration from [**7**].

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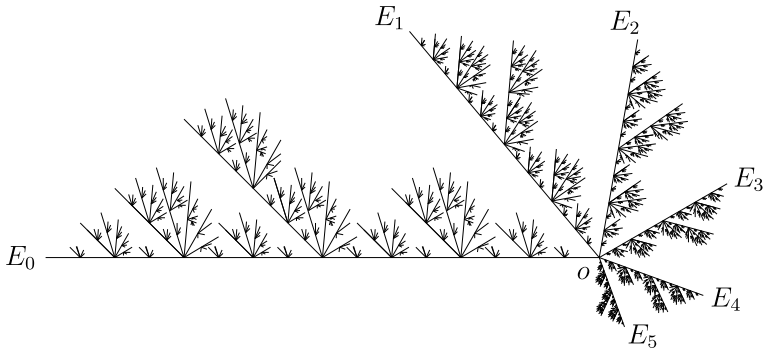


FIGURE 1. An approximation of the Ważewski universal dendrite and its subdendrites $E_0, \ldots, E_5$.

(4) *f_Z is uniquely ergodic, with the only f_Z-invariant Borel probability measure being the Dirac measure concentrated on o;*
(5) *f_Z is Li–Yorke chaotic, but not DC_2-chaotic; and*
(6) *f_Z has zero topological entropy.*

Proof. It remains to find a set Z such that the map f_Z exists and has all the properties listed in the main theorem. We define the set Z inductively. The main idea of the construction is as follows. Assume that at some point we have found a set $Z \subset \omega$ such that, for some $N \in \omega$, the restriction $Z \cap [0, N]$ suffices to define $\alpha_Z(s)$ for all $s \in S_0 \subset S$ (see Claim 17.1). Now we want to guarantee that $\alpha_Z(s)$ is also defined for each $s \in S'$, where $S' \subset S$ is finite. Given $N' \in \omega$ we can use Claim 17.2 to find $M' > N'$ large enough so that $[N', M'] \cap \omega \subset Z$ would allow us to define $\alpha_Z(s)$ for all $s \in S'$. By Claim 17.1, if we pick any $N' \geq N$, then setting $Z' = Z \cup [N', M']$ gives us a set Z' such that $Z' \cap [0, M']$ provides enough data to determine $\alpha_{Z'}(s)$ for all $s \in S_0 \cup S'$. Similarly, we can use Claim 17.5.

Construction of Z. Enumerate the elements of the countable set S^- as $s_0 = \emptyset$, $s_1, s_2, \ldots$. Let $N_0 = 0$ and $M_0 = 1$. Assume that for some $K \in \omega$ we have defined N_j and $M_j > N_j$ for $0 \leq j \leq K$. We set $N_{K+1} = \max\{L, (2^{M_K} + 1) \cdot M_K\}$. For each $0 \leq i \leq K$, we use Claim 17.5 to find m_i, n_i, M_i' such that $\alpha^{n_i}(T_{s_i, K, m_i}) = T_{\emptyset, K, 0}$ if $[N_{K+1}, N_{K+1} + M_i'] \cap \omega \subset Z$.

By Claim 17.2, we can take $M_{K+1} \geq N_{K+1} + \max\{M_i' : i = 0, \ldots, K\}$ large enough so that $[N_{K+1}, M_{K+1}] \subset Z$ would guarantee that $\alpha_Z(s)$ is well defined for every $s \in T_{s_i, K, m_i}$, where $0 \leq i \leq K$. Note that our choice of M_{K+1} guarantees that the conclusion of Claim 17.5 still holds. This concludes the induction step.

Take

$$Z = \bigcup_{K=1}^{\infty} [N_K, M_K] \cap \mathbb{N}.$$

The map f_Z, as defined in [**7**], is, for this choice of Z, weakly mixing but not mixing. The proof presented in the last two paragraphs of [**7**, p. 1907] applies verbatim as it requires only that f_Z be well defined and that the assumptions of Claims 17.4 and 17.5 hold.

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We now prove that the topological entropy of f_Z equals zero. Note that the definition of f_Z and [**7**, Claim 17.3] imply that

$$f_Z(E_j) = E_{j-1} \quad \text{for } j \geq 1, \tag{2}$$

$$f_Z^n(E_0) \subset \bigcup_{n+j-1 \in Z} E_j \quad \text{for } n = 1, 2, \ldots. \tag{3}$$

Let $U_j = E_j \backslash \{o\}$. Then U_j is an open subset of D_ω for $j \in \omega$. By (3), for each $n \in \mathbb{N}$,

$$f_Z^n(E_0) \subset \bigcup_{n+j-1 \in Z} E_j \quad \text{for } n = 1, 2, \ldots.$$

It follows that

$$f_Z^n(U_0) \subset \bigcup_{n+j-1 \in Z} U_j \cup \{o\} \quad \text{for } n = 1, 2, \ldots.$$

But, for our choice of Z, we know that if $N = M_K + 1$ for some $K \in \mathbb{N}$, then

$$f_Z^N(U_0) \subset \bigcup_{j=2^{M_K} \cdot M_K}^{\infty} U_j \cup \{o\},$$

and hence

$$f_Z^{N+l}(U_0) \subset \bigcup_{j=2^{M_K} \cdot M_K - l}^{\infty} U_j \cup \{o\} \quad \text{for } l = 1, 2, \ldots, 2^{M_K} \cdot M_K - 1.$$

Therefore, for each $K \in \mathbb{N}$,

$$f_Z^j(U_0) \cap U_0 = \emptyset \quad \text{for } j = M_K + 1, \ldots, (2^{M_K} + 1) \cdot M_K. \tag{4}$$

Let $x \in U_0$ and $N(x, U_0) = \{n \in \mathbb{N} : f_Z^n(x) \in U_0\}$. It follows from (4) that

$$\liminf_{n \to \infty} \frac{|N(x, U_0) \cap \{0, 1, \ldots, n-1\}|}{n} = 0. \tag{5}$$

If $\bar{\mu}(U_0) > 0$ for some f_Z-invariant Borel probability measure $\bar{\mu}$, then without loss of generality, we can assume that $\bar{\mu}$ is ergodic and then (5) would contradict the pointwise ergodic theorem. It follows that $\mu(U_0) = 0$ for every f_Z-invariant Borel probability measure μ. Moreover, we also have $\mu(U_j) = 0$ for every $j \in \mathbb{N}$, since

$$U_j \subset f_Z^{-j}(U_0) \quad \text{for } j = 1, 2, \ldots$$

(this follows easily from (2)). Hence the Dirac measure concentrated on o is the only f_Z invariant probability measure. By the variational principle [**17**], f_Z has topological entropy zero.

The statement concerning Li–Yorke chaos follows from f_Z being weakly mixing (see [**8**]). The statement about distributional chaos is a consequence of the fact that, for each $x, y \in D_\omega$, the pair (x, y) is proximal with upper Banach density one (because the Dirac measure concentrated on o is the only f_Z-invariant probability measure; see [**11**]). We refer the reader to [**8**, **11**] for the details and definitions of Li–Yorke and DC_2-chaos. □

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