Ergod. Th. & Dynam. Sys. (2021), **41**, 1751–1794 doi:10.1017/etds.2020.16
© *The Author(s) 2020. Published by Cambridge University Press*

Limit group invariants for non-free Cantor actions

STEVEN HURDER† and OLGA LUKINA‡

† *Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045, USA*
(e-mail: hurder@uic.edu)

‡ *Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria*
(e-mail: olga.lukina@univie.ac.at)

(*Received* 28 *April* 2019 *and accepted in revised form* 21 *January* 2020)

Abstract. A Cantor action is a minimal equicontinuous action of a countably generated group G on a Cantor space X. Such actions are also called generalized odometers in the literature. In this work, we introduce two new conjugacy invariants for Cantor actions, the stabilizer limit group and the centralizer limit group. An action is wild if the stabilizer limit group is an increasing sequence of stabilizer groups without bound and otherwise is said to be stable if this group chain is bounded. For Cantor actions by a finitely generated group G, we prove that stable actions satisfy a rigidity principle and furthermore show that the wild property is an invariant of the continuous orbit equivalence class of the action. A Cantor action is said to be dynamically wild if it is wild and the centralizer limit group is a proper subgroup of the stabilizer limit group. This property is also a conjugacy invariant and we show that a Cantor action with a non-Hausdorff element must be dynamically wild. We then give examples of wild Cantor actions with non-Hausdorff elements, using recursive methods from geometric group theory to define actions on the boundaries of trees.

Key words: minimal Cantor actions, continuous orbit equivalence, rigidity, non-Hausdorff groupoids
2010 Mathematics Subject Classification: 37B45, 37C15, 37C85 (Primary); 57S10 (Secondary)

1. *Introduction*

In this paper, we investigate the structure of non-free Cantor group actions and the relationship between the dynamics of the action and the algebraic properties of the group. One of our main results is the definition of direct limit groups, which are conjugacy

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

invariants of the action, and we investigate the relation between these new invariants and the dynamics of the action. Our results are illustrated by examples of Cantor actions in the literature and also those constructed in §8.

We assume as standing hypotheses that G is a countably infinite group, $\mathfrak{X}$ is a Cantor space, and $\Phi\colon G \to \mathrm{Homeo}(\mathfrak{X})$ is an action of G. We sometimes assume in addition that G is finitely generated and, when required, this hypothesis will be indicated. We denote the action by $(\mathfrak{X}, G, \Phi)$ and write $g \cdot x$ for $\Phi(g)(x)$ when the action is clear.

The action is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x) = \{g \cdot x \mid g \in G\}$ is dense in $\mathfrak{X}$. Let $d_{\mathfrak{X}}$ be a metric on $\mathfrak{X}$ compatible with the topology. The action $(\mathfrak{X}, G, \Phi)$ is *equicontinuous* with respect to $d_{\mathfrak{X}}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathfrak{X}$ and $g \in G$, $d_X(x, y) < \delta$ implies that $d_X(\Phi(g) \cdot x, \Phi(g) \cdot y) < \varepsilon$. This property is independent of the choice of the metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$.

In this paper, a *Cantor action* $(\mathfrak{X}, G, \Phi)$ is assumed to be minimal and equicontinuous. We also occasionally discuss actions which are not assumed to be minimal or equicontinuous and these will be called *general Cantor actions*.

Cantor actions can be divided into two types, *stable* and *wild*. For example, a free Cantor action of an abelian group G is stable. The stable property is a weaker form of the more well-known 'topologically free' property assumed in many works on the study of Cantor actions. Topologically free general Cantor actions have been extensively analyzed in the literature [**7**, **12**, **38**, **48**] and the authors' work in [**19**, **35**] extends some of these results to stable Cantor actions. The distinction between stable and topologically free actions is discussed in §6. On the other hand, the structure and properties of wild Cantor actions are less well known and in this work we study this class of actions in depth, obtaining new invariants and new classification results.

The first example of a wild Cantor action was constructed by Schori in [**50**], as the monodromy action for a weak solenoid which is not homogeneous. The 'Schori solenoid' gave a counter-example to a question raised by McCord in [**41**] and became the focus of further study. The analysis of this example in the authors' work with Dyer [**19**] introduced the notion of the wild property for a minimal equicontinuous action of a group G on a Cantor space $\mathfrak{X}$. It was observed by the authors in [**19**] that the Molino theory for foliated spaces developed in [**2**] applies for weak solenoids whose monodromy actions are stable, but fails for weak solenoids whose monodromy actions are wild.

The work [**36**] also gave a general method for constructing wild Cantor actions and used the wild property to show that these examples yield uncountably many classes of non-homeomorphic, non-homogeneous weak solenoids. This work suggested that a classification of weak solenoids up to homeomorphism requires a better understanding of the class of wild Cantor actions.

In the subsequent work [**35**], the authors studied the classification problem for stable Cantor actions and gave a generalization of the rigidity theorems for Cantor actions of Cortez and Medynets [**12**] and Li [**38**]. We showed that for stable Cantor actions, continuous orbit equivalence implies return equivalence of the actions. Our work also showed that if G is a finitely generated nilpotent group, then every Cantor action by G is stable. Thus, for Cantor actions by finitely generated nilpotent groups, one has a direct approach to their classification via the rigidity principle.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

The study of stable Cantor actions also highlighted the role of non-Hausdorff elements for the action, as discussed in §7.2 and Definition 7.3. An action with a non-Hausdorff element is at the opposite extreme of a stable action. Another theme of our work is to investigate wild actions with non-Hausdorff elements.

We now discuss the main results of this work. Sections 2 and 3 give the basic background material on Cantor actions that we require. This material overlaps with discussions and results in the authors' previous papers [**17**, **35**, **36**] and somewhat also with the works [**12**, **38**], but is discussed here in sufficient detail as necessary for the remainder of the paper.

Section 2 describes a model for a Cantor action $(\mathfrak{X}, G, \Phi)$ as a group action on a Cantor homogeneous space, $\mathfrak{X} \cong \mathfrak{G}(\Phi)/\mathfrak{G}(\Phi)_x$, where $\mathfrak{G}(\Phi) \subset \mathrm{Homeo}(\mathfrak{X})$ is a profinite group acting transitively on $\mathfrak{X}$ and $\mathfrak{G}(\Phi)_x \subset \mathfrak{G}(\Phi)$ is the isotropy subgroup of a point $x \in \mathfrak{X}$. The philosophy of our approach in this paper is that the study of the 'adjoint action' of the isotropy group $\mathfrak{G}(\Phi)_x$ on $\mathfrak{G}(\Phi)$ yields conjugacy invariants for the action.

Section 3 discusses the odometer model for a Cantor action and introduces a number of concepts about these actions which are key to subsequent sections. In particular, the stabilizer group chain of an odometer is defined in (22) and in §3.4 the centralizer group chain is defined.

Section 4 recalls the formal construction of direct limit groups, their equivalence, and properties of these groups. Then Definition 4.14 and Theorem 4.15 combined yield the following.

THEOREM 1.1. *Let* $(\mathfrak{X}, G, \Phi)$ *be a Cantor action. There are well-defined direct limit groups:*

(1) *the* stabilizer limit group $\Upsilon_s(\Phi)$*;*
(2) *the* centralizer limit group $\Upsilon_c(\Phi)$*,*

defined as the equivalence classes of the stabilizer and centralizer group chains associated to an odometer model for the action. Both $\Upsilon_s(\Phi)$ *and* $\Upsilon_c(\Phi)$ *are conjugacy class invariants of the action and there is an inclusion of direct limit groups* $\Upsilon_c(\Phi) \subset \Upsilon_s(\Phi)$*.*

For a Cantor action of an abelian group $G = \mathbb{Z}^d$, both of the limit group invariants in Theorem 1.1 are trivial and the action is stable. However, for Cantor actions of more general groups G, the stabilizer and centralizer limit group invariants can be highly non-trivial. One theme of this work is to use these limit group invariants to obtain a finer classification of non-free Cantor actions into the following subtypes of actions.

Definition 1.2. (Definition 4.18) A Cantor action $(\mathfrak{X}, G, \Phi)$ is said to be:

(1) *stable* if the stabilizer group $\Upsilon_s(\Phi)$ is bounded and *wild* otherwise;
(2) *algebraically stable* if the centralizer group $\Upsilon_c(\Phi)$ is bounded and *algebraically wild* otherwise;
(3) *wild of finite type* if the stabilizer group $\Upsilon_s(\Phi)$ is unbounded and represented by a chain of finite groups;
(4) *wild of flat type* if the stabilizer group $\Upsilon_s(\Phi)$ is unbounded and $\Upsilon_c(\Phi) = \Upsilon_s(\Phi)$;
(5) *dynamically wild* if the stabilizer group $\Upsilon_s(\Phi)$ is unbounded and is not of flat type.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

All of the possibilities in this definition can be realized by examples of Cantor actions. Section 5 discusses the notion of a *locally quasi-analytic* Cantor action $(\mathfrak{X}, G, \Phi)$ and the relation between this notion and the more usual notion of a topologically free action. The main result gives an interpretation of the stable property in terms of the 'analytic properties' of the adjoint action of the isotropy group $\mathcal{G}(\Phi)_x \subset \mathcal{G}(\Phi)$ for a homogeneous model for $\mathfrak{X} \cong \mathcal{G}(\Phi)/\mathcal{G}(\Phi)_x$.

THEOREM 1.3. (Theorem 5.3) *Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action; then Φ is stable if and only if the action of $\mathcal{G}(\Phi)$ on $\mathfrak{X}$ is locally quasi-analytic.*

Section 6 recalls the notion of a continuous orbit equivalence (COE) between Cantor actions and then discusses three notions of rigidity for Cantor actions. The work of Li in [**38**] gives cohomological criteria for when two COE Cantor actions, both of which are topologically free actions, are necessarily θ-conjugate, as defined in Definition 6.3. The work of Cortez and Medynets in [**12**] shows that two COE Cantor actions, both of which are free, are structurally conjugate (or virtually rigid as defined in Definition 6.7). The notion of return equivalence for Cantor actions was introduced in [**10**] and the authors showed in [**35**] that two COE Cantor actions, both of which are stable, are necessarily return equivalent as in Definition 6.8.

It is thus natural to ask how the invariants $\Upsilon_c(\Phi)$ and $\Upsilon_s(\Phi)$ behave under orbit equivalence. Our first result, and technically most involved, shows the following.

THEOREM 1.4. (Theorem 6.10) *Let $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', G', \Psi)$ be continuously orbit equivalent Cantor actions. If G is finitely generated and $(\mathfrak{X}', G', \Psi)$ is stable, then $(\mathfrak{X}, G, \Phi)$ is stable.*

We deduce from this a strong form of invariance for the wild property.

COROLLARY 1.5. *The wild property is an invariant of continuous orbit equivalence for the class of Cantor actions by finitely generated groups.*

Combining Corollary 1.5 with the results in [**35**] yields the following.

COROLLARY 1.6. (Theorem 6.18) *Let G and G' be finitely generated groups and suppose that the Cantor action $(\mathfrak{X}', G', \Psi)$ is stable. Let $(\mathfrak{X}, G, \Phi)$ be a general Cantor action which is continuously orbit equivalent to $(\mathfrak{X}', G', \Psi)$; then the actions are return equivalent.*

The above results give effective approaches to the classification of stable Cantor actions, up to conjugacy or continuous orbit equivalence. On the other hand, for wild Cantor actions, much less is known. The remainder of this work then investigates the properties of wild Cantor actions.

Section 7 introduces the notion of a non-Hausdorff element in $\mathcal{G}(\Phi)$ for a Cantor action $(\mathfrak{X}, G, \Phi)$. It was remarked by Renault in [**48**] that if the germinal groupoid associated to a group action is non-Hausdorff, then the action cannot be topologically free. In the authors' work [**35**], this result was extended to the observation that if $\mathcal{G}(\Phi)$ contains a non-Hausdorff element, then the action is wild. In this work, we give a more precise consequence. The stabilizer limit group $\Upsilon_s(\Phi)$ is said to be *of finite type* if each group

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

in a representative group chain is a finite group. Then we have the following result, which relates the direct limit invariants and other ideas introduced in this paper with the dynamics of a Cantor action.

THEOREM 1.7. *Let* $(\mathfrak{X}, G, \Phi)$ *be a Cantor action. Suppose that* $\mathfrak{G}(\Phi)$ *contains a non-Hausdorff element; then the action is dynamically wild and not of finite type. That is, there is a* proper *inclusion of direct limit groups* $\Upsilon_c(\Phi) \subset \Upsilon_s(\Phi)$*, and* $\Upsilon_s(\Phi)$ *is represented by an increasing chain of Cantor groups, which in particular are uncountable.*

The claims of Theorem 1.7 follow from Corollary 7.4 and Theorem 7.5.

COROLLARY 1.8. *Let* $(\mathfrak{X}, G, \Phi)$ *be a Cantor action for which the stabilizer direct limit group* $\Upsilon_s(\Phi)$ *has finite type; then the germinal groupoid* $\mathcal{G}(\mathfrak{X}, G, \Phi)$ *associated to the action is Hausdorff.*

The examples of wild Cantor actions given in [**36**, §8] have finite type, so Corollary 1.8 implies that there exist wild Cantor actions with Hausdorff germinal groupoids.

It seems to be a difficult problem to give criteria for a dynamically wild Cantor action which suffice to imply the existence of a non-Hausdorff element. Section 8 constructs examples of wild Cantor actions using the 'automata' approach, which is a well-known method in geometric group theory. This method defines a homeomorphism of the boundary of a tree, using a recursive definition along the branches of the tree. The examples constructed are inspired by the works of Nekrashevych in [**43**, **44**] and Pink in [**45**]. It would be very interesting to know if other methods of construction are possible and perhaps that the presence of a non-Hausdorff element for a Cantor action implies some form of underlying recursiveness for the action of its generators.

We conclude with the two general problems most relevant to the results of this work.

Problem 1.9. Give sufficient conditions for a pair $(\mathfrak{G}, \mathcal{D})$ to admit a non-Hausdorff element for the action of $\mathfrak{G}$ on $\mathfrak{X} \equiv \mathfrak{G}/\mathcal{D}$, where $\mathfrak{G}$ is a profinite group which is finitely generated in the topological sense and $\mathcal{D} \subset \mathfrak{G}$ is a 'totally not normal' closed subgroup.

The notion of a *totally not normal* subgroup is given in Definition 2.2.

Note that this problem is not just about the algebraic properties of the groups, as the wild property only emerges when considering the transitive action of $\mathfrak{G}$ on the quotient space $\mathfrak{X}$.

For the examples of wild Cantor actions of flat type given in [**36**, §8], the discriminant groups of these actions are an infinite product of finite groups. The second question asks whether there is a general result, that the discriminant groups of flat actions have a restricted algebraic structure.

Problem 1.10. Let $(\mathfrak{X}, G, \Phi)$ be a wild Cantor action of flat type, so the inclusion of direct limit groups $\Upsilon_c(\Phi) \subset \Upsilon_s(\Phi)$ is an equality. What restrictions are imposed on its discriminant group?

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

2. *The profinite model*

Given a Cantor action $(\mathfrak{X}, G, \Phi)$, let $\Phi(G) \subset \mathrm{Homeo}(X)$ denote the image subgroup. Introduce the closure $\mathfrak{G}(\Phi) \equiv \overline{\Phi(G)} \subset \mathrm{Homeo}(X)$ for the *uniform topology of maps*. (This corresponds to the *Ellis group* for the action, as defined in [**3**, **21**, **22**]; see also [**17**, §2].) That is, each element $\widehat{g} \in \mathfrak{G}(\Phi)$ is the uniform limit of a sequence of maps $\{\Phi(g_i) \mid i \geq 1\} \subset \Phi(G)$. By abuse of notation, we sometimes also denote the limiting element $\widehat{g}$ by (g_i). It was observed by Ellis in [**21**], by Auslander in [**3**, Ch. 3], and again by Glasner in [**31**, Proposition 2.5], that the assumption that the action is equicontinuous implies that $\mathfrak{G}(\Phi)$ is a separable profinite group.

For example, if G is an abelian group, then $\mathfrak{G}(\Phi)$ is a compact totally disconnected abelian group, which can be thought of as the group of asymptotic motions of the system. When G is non-abelian, the action closure $\mathfrak{G}(\Phi)$ can have more subtle algebraic properties. A main theme of this work is to examine the interplay between the algebraic structure of $\mathfrak{G}(\Phi)$ and the dynamics of the action Φ.

The philosophy behind our study of Cantor actions is to consider $\mathfrak{X}$ as a homogeneous space for $\mathfrak{G}(\Phi)$, in analogy to the case of homogeneous spaces of Lie type. Recall that for G a connected Lie group and $H \subset G$ a closed subgroup, the quotient space $X = G/H$ is a homogeneous G-space, and one studies the geometry of X using the structure of the Lie algebra $\mathfrak{g}$ of G and the adjoint action of H on $\mathfrak{g}$. See, for example, [**37**, Chs. X and XI]. For a Cantor action $(\mathfrak{X}, G, \Phi)$, there is no obvious analog of a Lie algebra associated to $\mathfrak{G}(\Phi)$. Nonetheless, one can investigate the properties of the adjoint action of the isotropy group of a point, localized to neighborhoods of x.

Let $\widehat{\Phi}\colon \mathfrak{G}(\Phi) \times \mathfrak{X} \to \mathfrak{X}$ denote the induced action of $\mathfrak{G}(\Phi)$ on $\mathfrak{X}$. For $\widehat{g} \in \mathfrak{G}(\Phi)$, we write its action on $\mathfrak{X}$ by $\widehat{g} \cdot x = \widehat{\Phi}(\widehat{g})(x)$. If the action $\Phi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is minimal, then the group $\mathfrak{G}(\Phi)$ acts transitively on $\mathfrak{X}$. The action $(\mathfrak{X}, G, \Phi)$ is said to be *faithful* if $\Phi(g) \cdot x = x$ for all $x \in \mathfrak{X}$ implies that g is the identity element. Equivalently, the action is faithful if the action map $\Phi\colon G \to \mathrm{Homeo}(\mathfrak{X})$ is injective. The action is *free* if for any $x \in \mathfrak{X}$, $\Phi(g) \cdot x = x$ implies that g is the identity element of G.

2.1. *The adjoint action.* Given $x \in \mathfrak{X}$, introduce the isotropy group at x,

$$\mathfrak{G}(\Phi)_x = \{\widehat{g} \in \overline{\Phi(G)} \mid \widehat{g} \cdot x = x\} \subset \mathrm{Homeo}(\mathfrak{X}), \tag{1}$$

which is a closed subgroup of $\mathfrak{G}(\Phi)$ and is thus either finite or is a Cantor group.

There is a natural identification $\mathfrak{X} \cong \mathfrak{G}(\Phi)/\mathfrak{G}(\Phi)_x$ of left G-spaces and thus the conjugacy class of $\mathfrak{G}(\Phi)_x$ in $\mathfrak{G}(\Phi)$ is independent of the choice of x. Moreover, if $\mathfrak{G}(\Phi)_x$ is the trivial group, then $\mathfrak{X}$ is identified with a profinite group and the action is free. However, there exist examples of free Cantor actions for which the group $\mathfrak{G}(\Phi)_x$ is non-trivial; the first such examples were constructed by Fokkink and Oversteegen in [**25**, §8] and further examples were constructed in [**19**, §10].

The action of $\mathfrak{G}(\Phi)_x$ on $\mathfrak{X}$ can be considered as induced by the adjoint action of $\mathfrak{G}(\Phi)_x$ on $\mathfrak{G}(\Phi)$. For $\widehat{h} \in \mathfrak{G}(\Phi)_x$ and $\widehat{g} \in \mathfrak{G}(\Phi)$, let $\mathbf{Ad}(\widehat{h})(\widehat{g}) = \widehat{h}\,\widehat{g}\,\widehat{h}^{-1}$. For $y \in \mathfrak{X}$, choose $\widehat{g} \in \mathfrak{G}(\Phi)$ such that $y = \widehat{g} \cdot \mathfrak{G}(\Phi)_x$. Then for $\widehat{h} \in \mathfrak{G}(\Phi)_x$ we have

$$\widehat{h} \cdot y = \widehat{h}\,\widehat{g} \cdot \mathfrak{G}(\Phi)_x = \widehat{h}\,\widehat{g}\,\widehat{h}^{-1} \cdot \mathfrak{G}(\Phi)_x = \mathbf{Ad}(\widehat{h})(\widehat{g}) \cdot \mathfrak{G}(\Phi)_x. \tag{2}$$

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

That is, the action of $\mathfrak{G}(\Phi)_x$ on $\mathfrak{X}$ can be considered as the factor of the adjoint action which is 'normal' to $\mathfrak{G}(\Phi)_x$. In the case of a homogeneous space of Lie type, this normal action is induced by the adjoint action on the Lie algebra of $\mathfrak{G}(\Phi)$ and is studied in terms of representation theory of the compact group $\mathfrak{G}(\Phi)_x$. For a Cantor action, we instead consider the restriction of the adjoint action to arbitrarily small neighborhoods of $x \in \mathfrak{X}$. If this localized action of $\mathfrak{G}(\Phi)_x$ stabilizes for arbitrarily small neighborhoods of x, then the action is said to be *stable* and otherwise it is *wild*. That is, for a stable Cantor action, there is a well-defined local geometric model for $\mathfrak{X}$ near x, while for a wild action there is no stable local model. We next make these statements precise.

2.2. *The finite model.* A profinite group, by definition, is the inverse limit of finite quotient groups. For an equicontinuous action $(\mathfrak{X}, G, \Phi)$, an analogous statement holds, that the action Φ is defined by an inverse limit of finite actions. The key concept to show this is that of clopen subsets of $\mathfrak{X}$ which are adapted to the action. We briefly recall some basic concepts.

Let $\mathrm{CO}(\mathfrak{X})$ denote the collection of all clopen (closed and open) subsets of the Cantor space $\mathfrak{X}$, which forms a basis for the topology of $\mathfrak{X}$. For $\phi \in \mathrm{Homeo}(\mathfrak{X})$ and $U \in \mathrm{CO}(\mathfrak{X})$, the image $\phi(U) \in \mathrm{CO}(\mathfrak{X})$. The following result is folklore and a proof is given in [**35**, Proposition 3.1].

PROPOSITION 2.1. *A minimal Cantor action* $(\mathfrak{X}, G, \Phi)$ *is equicontinuous if and only if, for the induced action* $\Phi_*\colon G \times \mathrm{CO}(\mathfrak{X}) \to \mathrm{CO}(\mathfrak{X})$*, the* G*-orbit of every* $U \in \mathrm{CO}(\mathfrak{X})$ *is finite.*

The proof that each $U \in \mathrm{CO}(\mathfrak{X})$ has finite orbit is essentially the same as what was called the 'coding method' for equicontinuous pseudogroup actions on Cantor transversals in [**9**], and discussed for group actions in [**17**, Appendix A], and for free actions in [**12**, §2].

We say that $U \subset \mathfrak{X}$ is *adapted* to the action $(\mathfrak{X}, G, \Phi)$ if U is a *non-empty clopen* subset and, for any $g \in G$, if $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$. The proof of Proposition 3.1 in [**35**] shows that given $x \in \mathfrak{X}$ and a clopen set $x \in W$, there is an adapted clopen set U with $x \in U \subset W$.

The key property of adapted sets is that for U adapted, the set of 'return times' to U,

$$G_U = \{g \in G \mid \varphi(g)(U) \cap U \neq \emptyset\}, \tag{3}$$

is a subgroup of G, called the *stabilizer* of U. Then for $g, g' \in G$ with $g \cdot U \cap g' \cdot U \neq \emptyset$, we have $g^{-1} g' \cdot U = U$ and hence $g^{-1} g' \in G_U$. Thus, the translates $\{g \cdot U \mid g \in G\}$ form a finite clopen partition of $\mathfrak{X}$ and are in one-to-one correspondence with the quotient space $X_U = G/G_U$. Then G acts by permutations of the finite set X_U and so the stabilizer group $G_U \subset G$ has finite index. The action of $g \in G$ on X_U is trivial precisely when g is a stabilizer of each coset $h \cdot G_U$, so $g \in C_U$, where

$$C_U = \bigcap_{h \in G} h G_U h^{-1} \subset G_U \tag{4}$$

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

is the largest normal subgroup of G contained in G_U. The action of the finite group $Q_U \equiv G/C_U$ on X_U by permutations is a finite approximation of the action of G on $\mathfrak{X}$, and the isotropy group of the identity coset $e \cdot G_U$ is $D_U \equiv G_U/C_U \subset Q_U$.

Thus, the finite model of a Cantor action $(\mathfrak{X}, G, \Phi)$ obtained from $U \in \mathrm{CO}(\mathfrak{X})$ yields the following data: the group Q_U, which acts on X_U, and the isotropy group $D_U \subset Q_U$ for the basepoint defined by the identity coset $x_U = e \cdot G_U \in X_U$. In addition to the group structures of Q_U and D_U, one can study how the group D_U is 'algebraically embedded' in Q_U.

Consider the adjoint action of D_U on Q_U, which for $h \in D_U$ is given by

$$\mathbf{Ad}_{Q_U}(h)\colon Q_U \to Q_U, \quad \mathbf{Ad}_{Q_U}(h)(k) = h\, k\, h^{-1}, \quad k \in Q_U. \tag{5}$$

The subgroup D_U is not a normal subgroup in Q_U and in fact a stronger condition holds. Let $h \in D_U$ and suppose that $k\, h\, k^{-1} \in D_U$ for all $k \in Q_U$. Then $k\, h\, k^{-1} = h'$ for some $h' \in D_U$, so $hk^{-1} = k^{-1}\, h' = k^{-1}\, h''\, h$ for $h'' = h'\, h^{-1}$. Hence, $h\, k^{-1}\, h^{-1} = k^{-1}\, h''$. As this holds for all $k \in Q_U$, the adjoint $\mathbf{Ad}_{Q_U}(h)$ acts trivially on X_U and hence h must be the identity in Q_U. In other words, for every non-trivial $h \in D_U$, there is a $k \in Q_U$ such that $k\, h\, k^{-1} \notin D_U$. This shows that the subgroup D_U satisfies the following condition, which was introduced in the works [**17**, **36**].

Definition 2.2. A subgroup $H \subset G$ is said to be *totally not normal* if for each non-trivial $h \in H$, there exists $g \in G$ such that $g\, h\, g^{-1} \notin H$.

2.3. *Induced actions.* We next consider the relations between the finite model constructed in §2.2 and the actions of G and $\mathfrak{G}(\Phi)$ on $\mathfrak{X}$. We first establish some technical results about restriction maps for adapted sets and then consider the behavior of the adjoint map for $\widehat{h} \in \mathfrak{G}(\Phi)_x$ under restriction to adapted sets for the action. Let U be adapted to the Cantor action $(\mathfrak{X}, G, \Phi)$.

The first observation is that each $\widehat{g} \in \mathfrak{G}(\Phi)$ is the uniform limit of a sequence of maps, $\widehat{g} = (g_i)$, and the translates of U form a finite partition of $\mathfrak{X}$ by closed sets; hence, the action of $\widehat{g}$ on the *partition* X_U equals the action by g_i for i sufficiently large. Thus, there is a well-defined epimorphism $\widehat{q}_U\colon \mathfrak{G}(\Phi) \to Q_U$, which maps $\mathfrak{G}(\Phi)_x$ onto D_U for $x \in U$.

Let (U, G_U, Φ_U) denote the action of G_U restricted to U, which is again a Cantor action. Define

$$H_U \equiv \Phi_U(G_U) \subset \mathrm{Homeo}(U), \quad K_U = \ker\{\Phi_U\colon G_U \to H_U\} \subset G_U \subset G. \tag{6}$$

LEMMA 2.3. *The action of H_U on U is minimal.*

Proof. As U is adapted, for each $x \in U$, the translates $H_U \cdot x = \{g \cdot x \in U \mid g \in G_U\}$ are dense. □

The group H_U is called the *holonomy* of the action Φ restricted to U, in analogy with the case of actions which arise from the holonomy of weak solenoids [**18**, **19**]. For each $g \in K_U$, the action $\Phi(g)$ fixes every point in U. Note that $K_{\mathfrak{X}}$ is the trivial group exactly when Φ is a faithful action, and K_U can be non-trivial even when the action Φ on $\mathfrak{X}$ is faithful.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Next, consider the closures of the restricted groups defined in (6). There is a subtle point to consider, that the closure can either be taken in the uniform topology on $\mathrm{Homeo}(\mathfrak{X})$ or, after restriction, in the uniform topology on $\mathrm{Homeo}(U)$. Let $\widehat{H}_U = \overline{\Phi_U(G_U)} \subset \mathrm{Homeo}(U)$ denote the closure of H_U in the uniform topology on $\mathrm{Homeo}(U)$. Then Lemma 2.3 implies the following.

COROLLARY 2.4. *Let $U \subset \mathfrak{X}$ be adapted; then $\widehat{H}_U$ acts transitively on U.*

The following is a key technical observation and is often used implicitly in the following.

LEMMA 2.5. *Let $U \subset \mathfrak{X}$ be adapted for the Cantor action $(\mathfrak{X}, G, \Phi)$. Then*

$$\widehat{H}_U = \{\widehat{g}|U \mid \widehat{g} \in \mathfrak{G}(\Phi), \widehat{g} \cdot U = U\}. \tag{7}$$

Proof. Let $\widehat{h} = (g_i) \in \widehat{H}_U$ for $g_i \in G_U$, where the sequence $\{\Phi_U(g_i)\}$ converges in the uniform topology in $\mathrm{Homeo}(U)$. Since $\mathfrak{G}(\Phi)$ is compact, there exists a subsequence $\{g_{i_\ell} \mid \ell \geq 0\}$ such that $\{\Phi(g_{i_\ell})\}$ converges uniformly in $\mathrm{Homeo}(\mathfrak{X})$ and hence yields $\widehat{g} = (g_{i_\ell}) \in \mathfrak{G}(\Phi)$. Moreover, as U is adapted, we can assume that each $\Phi(g_{i_\ell})(U) = U$, so $\widehat{g} \cdot U = U$. Then $\widehat{g}|U = \widehat{h}$, as uniform convergence on $\mathfrak{X}$ implies uniform convergence on U; hence, the actions agree on U. This shows that $\widehat{H}_U \subset \{\widehat{g}|U \mid \widehat{g} \cdot U = U\}$. The converse is immediate, since $\widehat{g} \in \mathfrak{G}(\Phi)$ with $\widehat{g} \cdot U = U$ implies, as noted above, that we can assume that $\widehat{g} = (g_i)$, where each $\Phi(g_i)(U) = U$, and hence $\widehat{g}|U \in \widehat{H}_U$. □

For adapted sets $V \subset U$, introduce the subgroups

$$\begin{aligned} \widehat{H}_{U,V} &\equiv \{\widehat{h} \in \widehat{H}_U \mid \widehat{h} \cdot V = V\} \subset \mathrm{Homeo}(U), \\ \widehat{K}_{U,V} &\equiv \{\widehat{h} \in \widehat{H}_{U,V} \mid \widehat{h}|V = \mathrm{id}\} \subset \mathrm{Homeo}(U). \end{aligned} \tag{8}$$

Note that $\widehat{H}_{\mathfrak{X},U} \subset \mathrm{Homeo}(\mathfrak{X})$ while $\widehat{H}_U \subset \mathrm{Homeo}(U)$, and Lemma 2.5 implies that there is a surjective restriction map $\rho_U \colon \widehat{H}_{\mathfrak{X},U} \to \widehat{H}_U$.

LEMMA 2.6. *Let $U \subset \mathfrak{X}$ be adapted for the Cantor action $(\mathfrak{X}, G, \Phi)$ with $x \in U$. Then*

$$\widehat{K}_{\mathfrak{X},U} \subset \mathfrak{G}(\Phi)_x \subset \widehat{H}_{\mathfrak{X},U} \subset \mathfrak{G}(\Phi). \tag{9}$$

Proof. For $\widehat{h} \in \widehat{K}_{\mathfrak{X},U}$, $\widehat{h} \cdot x = x$ and hence $\widehat{h} \in \mathfrak{G}(\Phi)_x$. For $\widehat{h} \in \mathfrak{G}(\Phi)_x$, we have $\widehat{h} \cdot U \cap U \neq \emptyset$; hence, $\widehat{h} \cdot U = U$ and so $\widehat{h} \in \widehat{H}_{\mathfrak{X},U}$. □

2.4. *Induced adjoint actions.* For $U \subset \mathfrak{X}$ adapted with $x \in U$, we consider the restriction of the adjoint action of $\mathfrak{G}(\Phi)_x$ to the subgroup $\widehat{H}_{\mathfrak{X},U} \subset \mathfrak{G}(\Phi)$. Let $\widehat{h} \in \mathfrak{G}(\Phi)_x$; then $\widehat{h} \in \widehat{H}_{\mathfrak{X},U}$ by Lemma 2.6, so the adjoint action of $\widehat{h}$ restricts to an action $\mathbf{Ad}_U(\widehat{h}) \colon \widehat{H}_{\mathfrak{X},U} \to \widehat{H}_{\mathfrak{X},U}$. Consider the induced action on the quotient space, $\mathbf{Ad}_U(\widehat{h}) \colon U \to U$.

LEMMA 2.7. *Let $U \in \mathrm{CO}(\mathfrak{X})$, $x \in U$. Then $\widehat{h} \in \mathfrak{G}(\Phi)_x$ acts as the identity on U if and only if*

$$\mathbf{Ad}_U(\widehat{h})(\widehat{g}) \cdot \mathfrak{G}(\Phi)_x = \widehat{g} \cdot \mathfrak{G}(\Phi)_x \quad \textit{for all } \widehat{g} \in \widehat{H}_{\mathfrak{X},U}. \tag{10}$$

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Proof. Suppose that $\widehat{h}$ acts as the identity on U, so $\widehat{h} \in \mathfrak{G}(\Phi)_x \cap \widehat{K}_{\mathfrak{X},U}$. For $y \in U$, by Corollary 2.4, we can choose $\widehat{g} \in \mathfrak{G}(\Phi)$ such that $y = \widehat{g} \cdot \mathfrak{G}(\Phi)_x$. Then $\widehat{g} \cdot U \cap U \neq \emptyset$; hence, $\widehat{g} \in \widehat{H}_{\mathfrak{X},U}$ and since $\widehat{h} \cdot y = y$, we have

$$\mathbf{Ad}_U(\widehat{h})(\widehat{g}) \cdot \mathfrak{G}(\Phi)_x = \widehat{h}\,\widehat{g}\,\widehat{h}^{-1} \cdot \mathfrak{G}(\Phi)_x = \widehat{h}\,\widehat{g} \cdot \mathfrak{G}(\Phi)_x = \widehat{h} \cdot y = y = \widehat{g} \cdot \mathfrak{G}(\Phi)_x, \quad (11)$$

which yields the identity (10). Conversely, if $\widehat{h} \in \mathfrak{G}(\Phi)_x$ satisfies (10), then (11) implies that $\widehat{h} \cdot y = y$ and that $\widehat{h} \in \mathfrak{G}(\Phi)_x \cap \widehat{K}_{\mathfrak{X},U}$. □

Let us also introduce the adjoint automorphisms on the isotropy subgroup $\mathfrak{G}(\Phi)_x$. For $\widehat{h} \in \mathfrak{G}(\Phi)_x$, let $\mathbf{Ad}_x(\widehat{h})\colon \mathfrak{G}(\Phi)_x \to \mathfrak{G}(\Phi)_x$ denote the adjoint action, given by $\mathbf{Ad}_x(\widehat{h})(\widehat{k}) = \widehat{h}\,\widehat{k}\,\widehat{h}^{-1}$ for $\widehat{k} \in \mathfrak{G}(\Phi)_x$.

Remark 2.8. Note that the condition (10) does not imply that $\mathbf{Ad}_x(\widehat{h})$ acts as the identity on the isotropy subgroup $\mathfrak{G}(\Phi)_x$, as the action of $\mathbf{Ad}_x(\widehat{h})$ is absorbed when passing to cosets of $\mathfrak{G}(\Phi)_x$. Examples show that the action of $\mathbf{Ad}_x(\widehat{h})$ can be non-trivial, while the induced quotient action on U is trivial. This observation lies behind the distinction between the definitions of the stabilizer and centralizer limit groups in §4.2, and the notions 'wild of flat type' and 'dynamically wild' actions in Definition 4.18.

3. *Odometer models*

The odometer model for a Cantor action $(\mathfrak{X}, G, \Phi)$ is obtained from a sequence of finite approximations of the action, as in §2. For example, a Vietoris solenoid is defined as the inverse limit of a sequence of finite coverings of $\mathbb{S}^1$, which is equivalent to considering a descending chain of finite-index subgroups of the fundamental group $\mathbb{Z}$ of $\mathbb{S}^1$. For a Cantor action, one considers a descending chain of finite-index subgroups of a fixed group G and then forms the inverse limit action associated to this chain. The odometer model for an action can be thought of as an 'algebraic model' for the action, which enables relating the algebraic properties of G with the dynamics of its action.

Downarowicz gives a survey of classical odometers in [**16**]. Some properties of odometer models for actions of non-abelian groups G are discussed by Cortez and Petite in the works [**13**, **14**] and by the authors in [**17**, **19**, **36**]. Section 2 of [**12**] discusses the literature for non-abelian odometers.

We first describe the construction of the odometer model as the inverse limit action obtained from finite approximations associated to adapted subsets of $\mathfrak{X}$. We then consider the equivalence of these odometer models and finally consider the adjoint action for these inverse limit actions.

3.1. *Group chains.*

For a choice of basepoint $x \in \mathfrak{X}$ and $\varepsilon > 0$, Proposition 2.1 implies that there exists an adapted clopen set $U \in \mathrm{CO}(\mathfrak{X})$ with $x \in U$ and $\mathrm{diam}(U) < \varepsilon$. Thus, given a basepoint x, by iterating this process one can always construct the following.

Definition 3.1. Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ is said to be an *adapted neighborhood basis* at $x \in \mathfrak{X}$ for the action Φ if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 1$ with $\bigcap_{\ell \geq 1} U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Given an adapted neighborhood basis $\mathcal{U}$ at x, for each $\ell \geq 0$ we can repeat the constructions of §2 for the adapted set U_ℓ. As a matter of notation, after fixing the collection $\mathcal{U}$, we use the subscript ℓ in place of the subscript U_ℓ when convenient. For example, $G_\ell \equiv G_{U_\ell}$ denotes the stabilizer group of U_ℓ. Then we obtain a descending chain of finite-index subgroups

$$\mathcal{G}_{\mathcal{U}}^{x} = \{G = G_0 \supset G_1 \supset G_2 \supset \cdots\}. \tag{12}$$

Note that each G_ℓ has finite index in G and is not assumed to be a normal subgroup. Also note that while the intersection of the chain $\mathcal{U}$ is a single point $\{x\}$, the intersection of the stabilizer groups in $\mathcal{G}_{\mathcal{U}}^{x}$ need not be the trivial group.

Next, set $X_\ell = G/G_\ell$ with basepoint $x_\ell = eG_\ell \in X_\ell$, where $e \in G$ is the identity element. Note that G acts transitively on the left on X_ℓ. The inclusion $G_{\ell+1} \subset G_\ell$ induces a natural G-invariant quotient map $p_{\ell+1}\colon X_{\ell+1} \to X_\ell$. Introduce the inverse limit

$$X_\infty \equiv \varprojlim \{p_{\ell+1}\colon X_{\ell+1} \to X_\ell \mid \ell > 0\}, \tag{13}$$

which is a Cantor space with the Tychonoff topology and basepoint $x_\infty = (x_\ell)$. The actions of G on the factors X_ℓ induce a minimal equicontinuous action on X_∞, denoted by $\Phi_x\colon G \times X_\infty \to X_\infty$.

For each $\ell \geq 0$, we have the 'partition coding map' $\Theta_\ell\colon \mathfrak{X} \to X_\ell$, which is G-equivariant. The maps $\{\Theta_\ell\}$ are compatible with the quotient maps in (13) and so induce a limit map $\Theta_x\colon \mathfrak{X} \to X_\infty$. The fact that the diameters of the clopen sets $\{U_\ell\}$ tend to zero implies that Θ_x is a homeomorphism. This is proved in detail in [**17**, Appendix A]. Moreover, $\Theta_x(x) = x_\infty \in X_\infty$, the basepoint of the inverse limit (13). Let X_∞ have a metric such that G acts on X_∞ by isometries and then let $d_{\mathfrak{X}}$ be the metric on $\mathfrak{X}$ induced by the homeomorphism $\widehat{\Theta}_x$. The minimal equicontinuous action (X_∞, G, Φ_x) is called the *odometer model* centered at x for the action $(\mathfrak{X}, G, \Phi)$.

3.2. *The Ellis group.* We next introduce the group chain model for the closure group $\mathfrak{G}(\Phi)$, as the inverse limit of the finite models for the action Φ introduced in §2. For each $\ell \geq 1$, let C_ℓ denote the largest normal subgroup (the *core*) of the stabilizer group G_ℓ, so

$$C_\ell = \bigcap_{g \in G} g\, G_\ell\, g^{-1} \subset G_\ell. \tag{14}$$

As G_ℓ has finite index in G, the same holds for C_ℓ. Observe that for all $\ell \geq 0$, we have $C_{\ell+1} \subset C_\ell$. Introduce the quotient group $Q_\ell = G/C_\ell$ with identity element $e_\ell = eC_\ell \in Q_\ell$. Then Q_ℓ acts transitively on the quotient set $X_\ell = G/G_\ell$. Let $\pi_{\ell+1}\colon Q_{\ell+1} \to Q_\ell$ be the quotient map induced by the inclusion $C_{\ell+1} \subset C_\ell$, which is equivariant for the left G-actions. Form the inverse limit group,

$$\widehat{G}_\infty \equiv \varprojlim \{\pi_{\ell+1}\colon Q_{\ell+1} \to Q_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} Q_\ell. \tag{15}$$

For $\ell \geq 0$, let $\widehat{q}_\ell\colon \widehat{G}_\infty \to Q_\ell$ be the projection map onto the ℓth factor. Let $\widehat{C}_\ell$ denote its kernel.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Give $\widehat{G}_\infty$ the relative topology induced by the product (Tychonoff) topology; that is, a basis for the topology of $\widehat{G}_\infty$ is given by the preimages of points for the maps $\widehat{q}_\ell$ for $\ell \geq 1$. For $\ell \geq 1$,

$$\widehat{C}_\ell = \widehat{q}_\ell^{-1}(e_\ell) = \{(g'_i) \in \widehat{G}_\infty \mid g'_i C_i = e_i \text{ for } 0 \leq i \leq \ell\} \tag{16}$$

is a clopen neighborhood of $\widehat{e} = (e_\ell) \in \widehat{G}_\infty$. Then, for any $\widehat{g} \in \widehat{G}_\infty$, the collection $\{\widehat{g}\ \widehat{C}_\ell \mid \ell \geq 0\}$ forms a neighborhood basis at $\widehat{g}$.

For each $\ell \geq 0$, let $\widehat{G}_\ell \subset \widehat{G}_\infty$ denote the clopen subgroup

$$\widehat{G}_\ell \equiv \bigcup_{g \in G_\ell / C_\ell} g\, \widehat{C}_\ell = \bigcup_{g \in G_\ell / C_\ell} \widehat{C}_\ell\, g. \tag{17}$$

For $\ell \geq 0$, let $q_\ell \colon G \to Q_\ell$ be the quotient map. The sequence of maps $\{q_\ell \mid \ell \geq 0\}$ induces a homomorphism $q_\infty \colon G \to \widehat{G}_\infty$ for which $q_\infty(e) = \widehat{e}$. Let $\widehat{\Theta} \colon \mathfrak{G}(\Phi) \to \widehat{G}_\infty$ denote the closure of q_∞ in the uniform topology on maps. We then have the following folklore result (see [**17**, Theorem 4.4] for a proof).

THEOREM 3.2. *Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action and suppose that $\{G_\ell \mid \ell \geq 0\}$ is the group chain associated to an adapted neighborhood basis $\mathcal{U}$ at $x \in \mathfrak{X}$. Then $\widehat{\Theta} \colon \mathfrak{G}(\Phi) \to \widehat{G}_\infty$ is an isomorphism of topological groups.*

Recall that for $\ell \geq 0$, $D_\ell \equiv G_\ell / C_\ell$ is the isotropy group at the basepoint $x_\ell = e_\ell G_\ell \in X_\ell$ of the action of Q_ℓ on X_ℓ. Moreover, by the definition of $\widehat{G}_\infty$, the quotient maps $\pi_{\ell+1} \colon Q_{\ell+1} \to Q_\ell$ in (15) are compatible with this identification.

LEMMA 3.3. *The image $\widehat{\Theta}(\mathfrak{G}(\Phi)_x) \subset \widehat{G}_\infty$ is the subgroup defined by*

$$\mathcal{D}_x \equiv \varprojlim \{\pi_{\ell+1} \colon G_{\ell+1}/C_{\ell+1} \to G_\ell / C_\ell \mid \ell \geq 0\} \subset \widehat{G}_\infty. \tag{18}$$

Proof. For $h \in \mathfrak{G}(\Phi)_x$, let $h_\ell = \widehat{q}_\ell(\widehat{\Theta}(h)) \in Q_\ell$; then $\widehat{h} = (h_\ell) \in \widehat{G}_\infty$. The action of h on X_ℓ fixes the coset $e_\ell G_\ell$ and hence $h_\ell \in G_\ell / C_\ell$ for each $\ell \geq 1$. Moreover, $\pi_{\ell+1}(h_{\ell+1}) = h_\ell$ as the sequence $(h_\ell) \in \widehat{G}_\infty$. Thus, $\widehat{h} \in \mathcal{D}_x$ is well defined in (18) and $\widehat{\Phi}_x(\widehat{h})(x_\infty) = x_\infty$. It follows that $\widehat{\Theta}(\mathfrak{G}(\Phi)_x) \subset \mathcal{D}_x$.

Conversely, given $\widehat{h} = (h_\ell) \in \mathcal{D}_x$, we have $\Phi(h_\ell) \in \mathrm{Homeo}(\mathfrak{X})$ for $\ell \geq 1$. Then, by Theorem 3.2, the sequence $(\Phi(h_\ell))$ of homeomorphisms of $\mathfrak{X}$ converges uniformly to a limit $\overline{h} \in \mathfrak{G}(\Phi)$. Note that multiplication by $\widehat{q}_\ell(\widehat{h}) \in Q_\ell$ fixes the coset eG_ℓ and hence the action of $\overline{h}$ on X_∞ leaves the clopen subset $U_\ell \subset X_\infty$ invariant for each $\ell \geq 0$. Hence, $\widehat{h} \cdot x = x$, so $\overline{h} \in \mathfrak{G}(\Phi)_x$.

Define $\widehat{\Theta}^{-1} \colon \mathcal{D}_x \to \mathfrak{G}(\Phi)_x \subset \mathrm{Homeo}(\mathfrak{X})$ by setting $\widehat{\Theta}^{-1}(\widehat{h}) = \overline{h}$. By the construction, $\widehat{\Theta}$ and $\widehat{\Theta}^{-1}$ are inverse maps and are each continuous; thus, $\widehat{\Theta} \colon \mathfrak{G}(\Phi)_x \to \mathcal{D}_x$ is a topological isomorphism. □

The group $\mathcal{D}_x$ is called the *discriminant* of the action, in analogy with the interpretation of $\widehat{G}_\infty$ as the profinite Galois group associated to the sequence of irregular coverings $Q_\ell \to X_\ell$.

COROLLARY 3.4. *The homeomorphism $\Theta_x \colon \mathfrak{X} \to X_\infty$ defined in §3.1 agrees with the induced map on quotients, $\widehat{\Theta} \colon \mathfrak{G}(\Phi)/\mathfrak{G}(\Phi)_x \to \widehat{G}_\infty / \mathcal{D}_x$.*

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

The abstract formula (18) is useful for computing the discriminant group, as in [**17**, §§6–8]. In particular, [**17**, Example 8.1] shows that the three-dimensional Heisenberg group admits group chains for which $\mathcal{D}_x$ is a Cantor group.

The intersection $K(\mathcal{G}_{\mathcal{U}}^x) = \bigcap_{\ell \geq 0} G_\ell$ is called the *kernel* of $\mathcal{G}_{\mathcal{U}}^x$ and need not be the trivial group. Let $NK(\mathcal{G}) \subset K(\mathcal{G})$ denote the largest normal subgroup of $K(\mathcal{G})$, which is also characterized as the kernel of the homomorphism $q_\infty \colon G \to \widehat{G}_\infty$.

3.3. *Equivalence of group chains.* The construction of the odometer model for a Cantor action $(\mathfrak{X}, G, \Phi)$ depends upon the choice of a basepoint $x \in \mathfrak{X}$ and an adapted neighborhood basis $\mathcal{U}$ at x, which then yields the group chain $\mathcal{G}_{\mathcal{U}}^x$ used for the construction. It is thus important to understand how the group chain $\mathcal{G}_{\mathcal{U}}^x$ depends upon these choices. This leads to introducing notions of equivalence between group chains.

Let G be a fixed group. A group chain in G is a sequence of nested subgroups

$$\mathcal{G} = \{G = G_0 \supset G_1 \supset G_2 \supset \cdots\}, \tag{19}$$

where each $G_{\ell+1}$ has finite index in G_ℓ for $\ell \geq 0$. The first notion of equivalence between group chains was used by Rogers and Tollefson in their study of equivalence of weak solenoids in [**49**] and corresponds to the standard concept of chain equivalence in terms of interlacing of the chains.

Definition 3.5. Two group chains $\{G_\ell \mid \ell \geq 0\}$ and $\{H_\ell \mid \ell \geq 0\}$ with $G_0 = H_0 = G$ are *equivalent* if and only if there is a group chain $\{K_\ell \mid \ell \geq 0\}$ in G and infinite subsequences $\{G_{\ell_k}\}_{k \geq 0}$ and $\{H_{j_k}\}_{k \geq 0}$ such that $K_{2k} = G_{\ell_k}$ and $K_{2k+1} = H_{j_k}$ for $k \geq 0$.

For example, suppose that $\mathcal{U}$ and $\mathcal{U}'$ are two choices of adapted neighborhood bases at x; then the corresponding group chains $\mathcal{G}_{\mathcal{U}}^x$ and $\mathcal{G}_{\mathcal{U}'}^x$ are equivalent in this sense [**17**, **25**].

The second notion of equivalence of group chains in G is a generalization of the above notion and was introduced by Fokkink and Oversteegen in [**25**] and further developed in the authors' work [**17**].

Definition 3.6. [**25**] Two group chains $\{G_\ell \mid \ell \geq 0\}$ and $\{H_\ell \mid \ell \geq 0\}$ with $G_0 = H_0 = G$ are *conjugate equivalent* if and only if there exists a sequence $(g_\ell) \subset G_0$ for which the compatibility condition $g_\ell G_\ell = g_{\ell+1} G_\ell$ for all $\ell \geq 0$ is satisfied and such that the group chains $\{g_\ell G_\ell g_\ell^{-1} \mid \ell \geq 0\}$ and $\{H_\ell \mid \ell \geq 0\}$ are equivalent.

The relation between the equivalences in Definitions 3.5 and 3.6 and their corresponding odometer models is given by the following theorem, which follows from results in [**25**]. The following result is stated and proved in [**17**].

THEOREM 3.7. *Let $\{G_\ell \mid \ell \geq 0\}$ and $\{H_\ell \mid \ell \geq 0\}$ be group chains, $G_0 = H_0 = G$, and let*

$$X_\infty = \varprojlim \{G/G_{\ell+1} \to G/G_\ell \mid \ell \geq 0\}, \quad Y_\infty = \varprojlim \{G/H_{\ell+1} \to G/H_\ell \mid \ell \geq 0\}.$$

Then the group chains $\{G_\ell \mid \ell \geq 0\}$ and $\{H_\ell \mid \ell \geq 0\}$ are equivalent *if and only if there exists a homeomorphism $\tau \colon X_\infty \to Y_\infty$ equivariant with respect to the G-actions on X_∞ and Y_∞ and such that $\tau(eG_\ell) = (eH_\ell)$. That is, the conjugating map is basepoint-preserving.*

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

The group chains $\{G_\ell \mid \ell \geq 0\}$ *and* $\{H_\ell \mid \ell \geq 0\}$ *are* conjugate equivalent *if and only if there exists a homeomorphism* $\tau : X_\infty \to Y_\infty$ *equivariant with respect to the* G*-actions on* X_∞ *and* Y_∞.

Remark 3.8. Given a Cantor action $(\mathfrak{X}, G, \Phi)$, let $\mathcal{G}(\Phi)$ denote the group chains in G which are *conjugate equivalent* to the group chain $\mathcal{G}_\mathcal{U}$ for some choice of basepoint $x \in \mathfrak{X}$ and adapted neighborhood basis $\mathcal{U}$ at x. Theorem 3.7 implies that an invariant defined for group chains in G, which is independent of the choice of a representative in $\mathcal{G}(\Phi)$, is an invariant for the conjugacy class of the action $(\mathfrak{X}, G, \Phi)$.

Remark 3.9. Let $\mathcal{G} = \{G = G_0 \supset G_1 \supset G_2 \supset \cdots\}$ be a group chain with associated odometer (X_∞, G, Φ_x). Let $NK(\mathcal{G}) \subset K(\mathcal{G})$ denote the largest normal subgroup of the kernel subgroup. It is immediate that $K(\mathcal{G})$ is an invariant of the equivalence class of $\mathcal{G}$, but examples show that it need not be invariant under conjugate equivalence. In contrast, it is immediate that $NK(\mathcal{G})$ is an invariant of conjugate equivalence and is identified with the kernel of the action map Φ_x on X_∞. Thus, $NK(\mathcal{G})$ is the trivial group whenever the action is faithful.

Applying these remarks to the group chains in $\mathcal{G}(\Phi)$ associated to a Cantor action $(\mathfrak{X}, G, \Phi)$, it follows that for a fixed basepoint $x \in \mathfrak{X}$, there is a well-defined kernel subgroup $K(\mathcal{G}^x_\mathcal{U}) \subset G$. On the other hand, given another choice of basepoint $y \in \mathfrak{X}$, its kernel $K(\mathcal{G}^y_\mathcal{U})$ need not equal $K(\mathcal{G}^x_\mathcal{U})$. In fact, it is possible for the kernel at x to be the trivial group and that at y to be a non-trivial subgroup. However, for free actions, they all agree and are trivial.

LEMMA 3.10. *A Cantor action* $(\mathfrak{X}, G, \Phi)$ *is free if and only if, for all* $x \in \mathfrak{X}$ *and any adapted neighborhood basis* $\mathcal{U}$ *at* x*, the kernel* $K(\mathcal{G}^x_\mathcal{U})$ *is the trivial group.*

3.4. *Adjoint group chains.* We next consider the adjoint actions for the discriminant group of an odometer model for a Cantor action $(\mathfrak{X}, G, \Phi)$. Assume that we have fixed an adapted neighborhood basis $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ at $x \in \mathfrak{X}$ and $\mathcal{G}^x_\mathcal{U} = \{G_\ell \mid \ell \geq 0\}$ is the associated group chain as in (12).

Recall that there are homeomorphisms $\mathfrak{X} \cong \mathfrak{G}(\Phi)/\mathfrak{G}(\Phi)_x$ and $X_\infty \cong \widehat{G}_\infty/\mathcal{D}_x$. By Theorem 3.2, there is an isomorphism $\widehat{\Theta} : \mathfrak{G}(\Phi) \to \widehat{G}_\infty$ of topological groups, which maps $\mathfrak{G}(\Phi)_x$ isomorphically to $\mathcal{D}_x$. For each $\ell \geq 0$, the clopen subgroup $\widehat{G}_\ell \subset \widehat{G}_\infty$ satisfies $\mathcal{D}_x \subset \widehat{G}_\ell$ and the quotient $\widehat{U}_\ell \equiv \widehat{G}_\ell/\mathcal{D}_x \subset \widehat{G}_\infty/\mathcal{D}_x \cong X_\infty$. Then Corollary 3.4 has a sharper statement, that for each $U_\ell \subset \mathfrak{X}$ in the adapted neighborhood basis $\mathcal{U}$ about x, we have that $\Theta_x : U_\ell \to \widehat{U}_\ell$ is a homeomorphism.

We introduce the localizations of the adjoint actions of $\mathcal{D}_x \subset \widehat{G}_\infty$ as defined in (18), in terms of the odometer model for the action. We first give a basic fact, where we recall that $\widehat{G}_\ell$ was defined in (17).

LEMMA 3.11. $\mathcal{D}_x = \bigcap_{\ell > 0} \widehat{G}_\ell$.

Proof. For $\widehat{h} \in \mathcal{D}_x$, by (18), we have $\widehat{h} = (h_i)$, where $h_i \in G_i$ for all $i \geq 1$ and $h_{i+1}^{-1} h_i \in G_i$. Since $G_{i+1} \subset G_i$ for all $i \geq 1$, given any $\ell \geq 1$, this implies that we can choose the

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

sequence (h_i) so that $h_i = h_\ell \in G_\ell$ for all $i \leq \ell$. Thus, if $\widehat{h} \in \mathcal{D}_x$, then $\widehat{h} \in \widehat{G}_\ell$ for all $\ell \geq 1$ and so $\mathcal{D}_x \subset \bigcap_{\ell>0} \widehat{G}_\ell$.

Conversely, suppose that $\widehat{h} \in \widehat{G}_\ell$ for all $\ell \geq 0$ and hence $\widehat{h} \cdot \widehat{G}_\ell = \widehat{G}_\ell$. Let $\widehat{U}_\ell \equiv \widehat{G}_\ell/\mathcal{D}_x \subset \widehat{G}_\infty/\mathcal{D}_x$. Then we have $\Theta_x(U_\ell) = \widehat{U}_\ell$, so that $\widehat{U}_\ell$ is identified with the clopen neighborhood $U_\ell \subset \mathfrak{X}$ of x. Since $\bigcap_{\ell>0} U_\ell = \{x\}$, it follows that $\widehat{h} \cdot x = x$ and so $\widehat{h} \in \mathcal{D}_x$. Thus, $\mathcal{D}_x = \bigcap_{\ell>0} \widehat{G}_\ell$. □

Now recall that in §2.3, for adapted sets $V \subset U$, we defined in (8) the groups $\widehat{H}_{U,V}$ and $\widehat{K}_{U,V}$. Their analogs for the odometer model of the action are the groups

$$\widehat{H}_{X_\infty,\widehat{U}_\ell} \equiv \{\widehat{g} \in \widehat{G}_\infty \mid \widehat{\Phi}_x(\widehat{g})(\widehat{U}_\ell) = \widehat{U}_\ell\} = \widehat{G}_\ell, \tag{20}$$

$$\widehat{K}_{X_\infty,\widehat{U}_\ell} \equiv \{\widehat{g} \in \widehat{G}_\ell \mid \widehat{\Phi}_x(\widehat{g}) \text{ acts as the identity on } \widehat{U}_\ell\} \subset \mathcal{D}_x. \tag{21}$$

For each $\ell \geq 0$, define

$$K_\ell \equiv \widehat{K}_{X_\infty,\widehat{U}_\ell}. \tag{22}$$

Then $\widehat{U}_{\ell+1} \subset \widehat{U}_\ell$ implies that $K_\ell \subset K_{\ell+1}$ for all $\ell \geq 0$. Thus, we obtain an increasing chain $\mathcal{K}(\mathcal{U}) = \{K_0 \subset K_1 \subset K_2 \subset \cdots\}$ of subgroups of $\mathcal{D}_x$. This is called the *stabilizer group chain*.

Next, Lemma 3.11 implies that for each $\ell \geq 0$, the adjoint action of $\widehat{h} \in \mathcal{D}_x$ restricts to a map $\mathbf{Ad}_\ell(\widehat{h})\colon \widehat{G}_\ell \to \widehat{G}_\ell$. Define

$$Z_\ell \equiv \ker\{\mathbf{Ad}_\ell\colon \mathcal{D}_x \to \mathrm{Homeo}(\widehat{G}_\ell)\} \subset \mathcal{D}_x. \tag{23}$$

Then $\widehat{G}_{\ell+1} \subset \widehat{G}_\ell$ implies that $Z_\ell \subset Z_{\ell+1}$ for all $\ell \geq 0$. Thus, we obtain another increasing chain $\mathcal{Z}(\mathcal{U}) = \{Z_0 \subset Z_1 \subset Z_2 \subset \cdots\}$ of subgroups of $\mathcal{D}_x$. This is called the *centralizer group chain*, due to the following observation. For a group H and a subgroup $K \subset H$, define the *centralizer* of K in H,

$$\mathrm{Cen}(H, K) = \{h \in H \mid h\,k = k\,h \text{ for all } k \in K\} \subset H. \tag{24}$$

Recall that $q_\infty\colon G \to \widehat{G}_\infty$ has dense image.

LEMMA 3.12. $Z_\ell = \mathcal{D}_x \cap \mathrm{Cen}(\widehat{G}_\infty, \widehat{G}_\ell) = \mathcal{D}_x \cap \mathrm{Cen}(\widehat{G}_\infty, q_\infty(G_\ell))$.

Proof. The equality $Z_\ell = \mathcal{D}_x \cap \mathrm{Cen}(\widehat{G}_\infty, \widehat{G}_\ell)$ is just a restatement of (23). The second equality follows from the observation that if $\widehat{h} \in \widehat{G}_\infty$ commutes with the image group $q_\infty(G_\ell)$, then it also commutes with every element in its closure $\widehat{G}_\ell$ and hence $\mathrm{Cen}(\widehat{G}_\infty, q_\infty(G_\ell)) \subset \mathrm{Cen}(\widehat{G}_\infty, \widehat{G}_\ell)$. □

The two group chains $\mathcal{Z}(\mathcal{U})$ and $\mathcal{K}(\mathcal{U})$ are closely related.

LEMMA 3.13. *There is an inclusion of group chains* $\iota_\infty\colon \mathcal{Z}(\mathcal{U}) \to \mathcal{K}(\mathcal{U})$.

Proof. By the definition (23), for each $\ell \geq 0$ and $\widehat{h} \in Z_\ell$, the action $\mathbf{Ad}_\ell(\widehat{h})$ on $\widehat{G}_\ell$ is trivial. Hence, by Lemma 2.7, the induced action $\widehat{\Phi}_x(\widehat{h})$ on $\widehat{U}_\ell \equiv \widehat{G}_\ell/\mathcal{D}_x$ is trivial. Thus, there is an inclusion map $\iota_\ell\colon Z_\ell \subset K_\ell$, which yields the inclusion map $\iota_\infty = (\iota_\ell)$ on group chains. □

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Consider $\widehat{h} \in Z_{\ell+1}$ such that $\widehat{h} \notin Z_\ell$. Then $\widehat{\Phi}_x(\widehat{h})$ acts trivially on $\widehat{U}_{\ell+1}$ and $\mathbf{Ad}_\ell(\widehat{h})$ acts non-trivially on $\widehat{G}_\ell$. However, it is possible that $\widehat{\Phi}_x(\widehat{h})$ also acts trivially on $\widehat{U}_\ell$, while the adjoint action $\mathbf{Ad}_\ell(\widehat{h})$ acts non-trivially on the subgroup $\mathcal{D}_x$ as discussed in Remark 2.8. The point is that comparing the subgroups $Z_\ell \subset K_\ell$ involves algebraic properties of the inclusion $\mathcal{D}_x \subset \widehat{G}_\infty$.

Finally, given $\widehat{h} \in \mathcal{D}_x$, which is represented by a sequence $\widehat{h} = (h_\ell)$ with $h_\ell = h \in G$ for all $\ell \geq 0$, then $h \in \widehat{G}_\ell$ for all $\ell \geq 0$, so $h \in K(\mathcal{G}^x_{\mathcal{U}})$. Conversely, recall that $NK(\mathcal{G}^x_{\mathcal{U}})$ is the largest normal subgroup of $K(\mathcal{G}^x_{\mathcal{U}})$ and so $NK(\mathcal{G}^x_{\mathcal{U}}) \subset C_\ell$ for all $\ell \geq 1$; hence, each $h \in NK(\mathcal{G}^x_{\mathcal{U}})$ acts trivially on $\widehat{G}_\infty$. Define the quotient group

$$\mathcal{D}^G_x \equiv K(\mathcal{G}^x_{\mathcal{U}})/NK(\mathcal{G}^x_{\mathcal{U}}) \subset \mathcal{D}_x; \tag{25}$$

then each $h \in \mathcal{D}^G_x$ is represented by a constant sequence (h) for $h \in G$. One can think of the elements of $\mathcal{D}^G_x$ as the 'G-rational points' of $\mathcal{D}_x$. Of course, if $K(\mathcal{G}^x_{\mathcal{U}})$ is trivial, then there are no such points. The examples in §8 illustrate actions with non-trivial G-rational points.

4. *Direct limit conjugacy invariants*

In this section, we use the direct limit construction for the group chains $\mathcal{Z}(\mathcal{U})$ and $\mathcal{K}(\mathcal{U})$ to obtain conjugacy invariants of a Cantor action $(\mathfrak{X}, G, \Phi)$. These are subtle invariants of an action, and each is independent of the other, as shown by Theorem 7.5 and the examples in §8. We first briefly recall the basic formulation of direct limits and then apply these ideas to obtain the *stabilizer limit group* $\Upsilon_s(\Phi)$ and the *centralizer limit group* $\Upsilon_c(\Phi)$ for a Cantor action $(\mathfrak{X}, G, \Phi)$.

4.1. *Direct limits.* We give the construction and properties of the direct limit in the category of groups. Basic references for this standard concept are Eilenberg and Steenrod [**20**, Ch. VIII, §2] and the text by Munkres [**42**, §73], which give proofs of the following results.

Definition 4.1. A directed system of groups $\mathcal{G}(G_\lambda, \phi^{\lambda'}_\lambda, \Lambda)$ over a directed set Λ is a collection of groups $\{G_\lambda \mid \lambda \in \Lambda\}$ and, for each $\lambda, \lambda' \in \Lambda$ with $\lambda < \lambda'$, a group homomorphism $\phi^{\lambda'}_\lambda : G_\lambda \to G_{\lambda'}$. We require that $\phi^\lambda_\lambda = \mathrm{id}_{G_\lambda}$ and, for $\lambda < \lambda' < \lambda''$, that $\phi^{\lambda''}_\lambda = \phi^{\lambda''}_{\lambda'} \circ \phi^{\lambda'}_\lambda$.

Definition 4.2. Let $\mathcal{G}(G_\lambda, \phi^{\lambda'}_\lambda, \Lambda)$ be a directed system of groups over a directed set Λ. Define an equivalence relation on the disjoint union $\bigcup_{\lambda\in\Lambda} G_\lambda$, where, for $g_\lambda \in G_\lambda$ and $g_{\lambda'} \in G_{\lambda'}$, we set $g_\lambda \sim g_{\lambda'}$ if for all $\lambda'' \in \Lambda$ with $\lambda < \lambda''$ and $\lambda' < \lambda''$; then $\phi^{\lambda''}_\lambda(g_\lambda) = \phi^{\lambda''}_{\lambda'}(g_{\lambda'})$.

Definition 4.3. The *direct limit* of a directed system of groups $\mathcal{G}(G_\lambda, \phi^{\lambda'}_\lambda, \Lambda)$ over a directed set Λ is defined to be the set of equivalence classes

$$\varinjlim \mathcal{G}(G_\lambda, \phi^{\lambda'}_\lambda, \Lambda) \equiv \bigcup_{\lambda\in\Lambda} G_\lambda \Big/ g_\lambda \sim g_{\lambda'}. \tag{26}$$

The group structure on $\varinjlim \mathcal{G}(G_\lambda, \phi^{\lambda'}_\lambda, \Lambda)$ is inherited from that on the summands. For $\lambda \in \Lambda$ and $g_\lambda \in G_\lambda$, let $[g_\lambda] \in \varinjlim \mathcal{G}(G_\lambda, \phi^{\lambda'}_\lambda, \Lambda)$ denote the equivalence class it determines.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Definition 4.4. A map Ξ between directed systems of groups $\mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ and $\mathcal{G}(H_\omega, \psi_\omega^{\omega'}, \Omega)$ is an order-preserving map $\xi\colon \Lambda \to \Omega$ and, for each $\lambda \in \Lambda$, a group homomorphism $\xi_\lambda\colon G_\lambda \to H_{\xi(\lambda)}$ such that for $\lambda < \lambda'$, we have

$$\xi_{\lambda'} \circ \phi_\lambda^{\lambda'} = \phi_{\xi(\lambda)}^{\xi(\lambda')} \circ \xi_\lambda\colon G_\lambda \to H_{\xi(\lambda')}.$$

PROPOSITION 4.5. *A map Ξ between directed systems of groups $\mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ and $\mathcal{G}(H_\omega, \psi_\omega^{\omega'}, \Omega)$ induces a homomorphism $\varinjlim \Xi\colon \varinjlim \mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda) \longrightarrow \varinjlim \mathcal{G}(H_\omega, \psi_\omega^{\omega'}, \Omega)$.*

We next recall some special cases of directed systems of groups and maps between directed systems of groups. We single out two special classes of maps.

PROPOSITION 4.6. *A map Ξ between directed systems of groups $\mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ and $\mathcal{G}(H_\omega, \psi_\omega^{\omega'}, \Omega)$ is a* monomorphism *if each of the maps $\xi_\lambda\colon G_\lambda \to H_{\xi(\lambda)}$ for $\lambda \in \Lambda$ is a monomorphism of groups. Then the induced map $\varinjlim \Xi\colon \varinjlim \mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda) \longrightarrow \varinjlim \mathcal{G}(H_\omega, \psi_\omega^{\omega'}, \Omega)$ is a group monomorphism.*

PROPOSITION 4.7. *A map Ξ between directed systems of groups $\mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ and $\mathcal{G}(H_\omega, \psi_\omega^{\omega'}, \Omega)$ is an* isomorphism *if each of the maps $\xi_\lambda\colon G_\lambda \to H_{\xi(\lambda)}$ for $\lambda \in \Lambda$ is an isomorphism of groups. Then the induced map $\varinjlim \Xi\colon \varinjlim \mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda) \longrightarrow \varinjlim \mathcal{G}(H_\omega, \psi_\omega^{\omega'}, \Omega)$ is a group isomorphism.*

A subset $\Lambda_0 \subset \Lambda$ of a directed set is said to be *cofinal* if for each $\lambda \in \Lambda$, there exists $\lambda' \in \Lambda_0$ with $\lambda < \lambda'$. Then we have the following.

PROPOSITION 4.8. *Let $\mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ be a directed systems of groups and $\Lambda_0 \subset \Lambda$ be a cofinal set. Then the inclusion induces a group isomorphism $\varinjlim \mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda_0) \cong \varinjlim \mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$.*

In addition, we introduce two special classes of direct limit systems.

Definition 4.9. A directed system of groups $\mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ is said to be *bounded* if there exists $\lambda_0 \in \Lambda$ such that for all $\lambda_0 < \lambda < \lambda'$, the map $\phi_\lambda^{\lambda'}$ is a group isomorphism. The directed system is said to be *unbounded* if no such λ_0 exists.

Definition 4.10. A directed system $\mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ has *finite type* if each group G_λ is finite.

For our purposes, the ordered sets Λ and Ω above will always be assumed to be a subset of the non-negative integers $\mathbb{N}$ with the natural order. All maps $\{\phi_\lambda^{\lambda'} \mid \lambda < \lambda' \in \Lambda\}$ in our applications will be group inclusions (possibly isomorphisms). For a morphism Ξ of directed systems of groups, the maps $\{\xi(\lambda) \mid \lambda \in \Lambda\}$ will be be group inclusions (possibly isomorphisms).

Now assume that $\Lambda \subset \mathbb{N}$; then $\varinjlim \mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ inherits a filtration from its definition.

Definition 4.11. The *height function* $\lambda\colon \varinjlim \mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda) \to \Lambda$ is defined for an equivalence class $[g]$ to be the least λ such that $[g] = [g_\lambda]$ for some $g_\lambda \in G_\lambda$.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Remark 4.12. Note that while the height function need not be preserved by a map between directed systems, it does yield a well-defined order, where $[g_1] \leq_\lambda [g_2]$ if $\lambda(g_1) < \lambda(g_2)$.

Remark 4.13. Suppose that there exists a group $\mathfrak{G}$ such that $G_\lambda \subset \mathfrak{G}$ for all $\lambda \in \Lambda$ and the maps $\phi_\lambda^{\lambda'}$ are inclusions of subgroups. Define

$$G_\infty \equiv \bigcup_{\lambda \in \Lambda} G_\lambda. \tag{27}$$

For $g_\lambda \in G_\lambda$ and $g'_{\lambda'} \in G_{\lambda'}$, there then exists $\lambda'' > \max\{\lambda, \lambda'\}$ so that we can consider g_λ, $g_{\lambda'} \in G_{\lambda''}$ and then $g_\lambda + g'_{\lambda'}$ is defined in $G_{\lambda''} \subset G_\infty$. That is, G_∞ is a subgroup of $\mathfrak{G}$.

Define a height function $\lambda_\infty \colon G_\infty \to \Lambda$ by $\lambda_\infty(g) \leq \lambda$ if $g \in G_\lambda$. Given the pair $\{G_\infty, \lambda_\infty\}$, define $\widetilde{G}_\lambda \equiv \{g \in G_\infty \mid \lambda_\infty(g) \leq \lambda\}$ and introduce maps given by inclusions. Let $\mathcal{G}(G_\infty, \lambda_\infty)$ denote the resulting direct limit of groups, which is a small abuse of notation. It is then immediate that there is an isomorphism of direct limits $\mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda) \cong \mathcal{G}(G_\infty, \lambda_\infty)$. Thus, one can view a direct limit of subgroups of $\mathfrak{G}$ as a subgroup $G_\infty \subset \mathfrak{G}$, equipped with a height function $\lambda_\infty \colon G_\infty \to \Lambda$.

In the case when the directed system of groups $\mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ is *bounded*, with λ_0 an upper bound, then the inclusion of the constant sequence $\mathcal{G}(G_{\lambda_0}, \mathrm{Id}_{G_{\lambda_0}}, \{\lambda_0\}) \subset \mathcal{G}(G_\lambda, \phi_\lambda^{\lambda'}, \Lambda)$ is an isomorphism and we have $G_\infty = G_{\lambda_0}$. In other words, the direct limit of a bounded direct limit collapses.

4.2. *The stabilizer and centralizer limit groups.* We now show that the chain of stabilizer subgroups defined by (22), and the chain of centralizer subgroups defined in Lemma 3.12, yield direct limit groups whose isomorphism classes are conjugacy invariants of Cantor actions.

Definition 4.14. Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action and $\mathcal{G}_\mathcal{U}^x = \{G_\ell \mid \ell \geq 0\}$ be the group chain associated to an adapted neighborhood basis $\mathcal{U}$ at $x \in \mathfrak{X}$. Let $\{K_\ell \mid \ell \geq 0\}$ be the chain of stabilizer groups associated to $\mathcal{G}_\mathcal{U}^x$ and let $\{Z_\ell \mid \ell \geq 0\}$ be the chain of centralizer groups associated to $\mathcal{G}_\mathcal{U}^x$. Then define:

(1) the *stabilizer (direct limit) group* is $\Upsilon_s^x(\Phi) = \varinjlim \mathcal{G}(K_\ell, \iota_\ell^{\ell'}, \mathbb{N})$;

(2) the *centralizer (direct limit) group* is $\Upsilon_c^x(\Phi) = \varinjlim \mathcal{G}(Z_\ell, \iota_\ell^{\ell'}, \mathbb{N})$;

where $\iota_\ell^{\ell'}$ denotes the inclusion map, for $0 \leq \ell \leq \ell'$.

We now come to the main result of this section.

THEOREM 4.15. *Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action. The direct limit isomorphism classes of $\Upsilon_s^x(\Phi)$ and $\Upsilon_c^x(\Phi)$ are invariants of the conjugacy class of the action Φ. These isomorphism classes are denoted by $\Upsilon_s(\Phi)$ and $\Upsilon_c(\Phi)$, respectively, where we suppress the basepoint x in the notation.*

Proof. Let $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', G, \Psi)$ be conjugate Cantor actions by a homeomorphism $h \colon \mathfrak{X}' \to \mathfrak{X}$. Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ be an adapted neighborhood basis at $x \in \mathfrak{X}$ for the action Φ and let $\mathcal{V} = \{V_\ell \subset \mathfrak{X}' \mid \ell \geq 0\}$ be an adapted neighborhood basis at $z \in \mathfrak{X}'$ for the action Ψ. Then $\mathcal{U}' = \{U'_\ell = h(V_\ell) \subset \mathfrak{X} \mid \ell \geq 0\}$ is an adapted neighborhood basis at

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

$y = h(z)$ for the action Φ, and the group chain in G associated to $\mathcal{V}$ and the action Ψ coincides with the group chain in G associated to $\mathcal{U}'$ and the action Φ. Thus, by Theorem 3.7, we need to show that the direct limits $\Upsilon_s^x(\Phi)$ and $\Upsilon_c^x(\Phi)$, associated to the group chain $\mathcal{G}_{\mathcal{U}}^x$ in G, are isomorphic as direct limits to the direct limits $\Upsilon_s^y(\Phi)$ and $\Upsilon_c^y(\Phi)$ associated to the group chain $\mathcal{G}_{\mathcal{U}'}^y$.

First consider the case where $x = y$, so we are given two adapted neighborhood bases at a common basepoint x, $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ and $\mathcal{U}' = \{U'_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$, with corresponding group chains $\mathcal{G}_{\mathcal{U}}^x = \{G_\ell \mid \ell \geq 0\}$ and $\mathcal{G}_{\mathcal{U}'}^x = \{G'_\ell \mid \ell \geq 0\}$.

As $\mathcal{U}$ and $\mathcal{U}'$ are both adapted neighborhood bases at x, there exist increasing sequences of indices $1 \leq i_1 < i_2 < i_3 < \cdots$ and $1 \leq j_1 < j_2 < j_3 < \cdots$ such that we have a descending sequence of adapted clopen sets at x, where $i_0 = j_0 = 0$,

$$\mathfrak{X} = U_0 = U'_0 \supset U_{i_1} \supset U'_{j_1} \supset U_{i_2} \supset U'_{j_2} \supset \cdots .$$

Then, by Theorem 3.7 and passing to a subsequence, we can assume without loss of generality that $i_\ell = \ell$ and $j_\ell = \ell$ for $\ell \geq 0$. Introduce a common refinement $\mathcal{U}'' = \{U''_\ell \mid \ell \geq 0\}$ of these chains of clopen sets, where $U''_{2\ell} = U_\ell$ and $U''_{2\ell-1} = U'_\ell$.

Let $\mathcal{G}_{\mathcal{U}''}^x = \{G''_\ell \mid \ell \geq 0\}$ be the group chain associated to $\mathcal{U}''$; then $G''_{2\ell} = G_\ell$ and $G''_{2\ell-1} = G'_\ell$. Let X_∞, X'_∞, and X''_∞ denote the inverse limit spaces defined as in (13) by the group chains $\mathcal{G}_{\mathcal{U}}^x$, $\mathcal{G}_{\mathcal{U}'}^x$, and $\mathcal{G}_{\mathcal{U}''}^x$, respectively. Let $\widehat{\mathcal{U}} = \{\widehat{U}_\ell \subset X_\infty \mid \ell \geq 0\}$ be the corresponding adapted basis at X_∞ and likewise $\widehat{\mathcal{U}}' = \{\widehat{U}'_\ell \subset X'_\infty \mid \ell \geq 0\}$ and $\widehat{\mathcal{U}}'' = \{\widehat{U}''_\ell \subset X''_\infty \mid \ell \geq 0\}$ for X'_∞ and X''_∞, respectively.

By the discussion in §3.1, there are homeomorphisms $\Theta_x \colon \mathfrak{X} \to X_\infty$, $\Theta'_x \colon \mathfrak{X} \to X'_\infty$, and $\Theta''_x \colon \mathfrak{X} \to X''_\infty$ which intertwine the G-actions on these spaces. Introduce the basepoint-preserving homeomorphisms $\tau = \Theta_x \circ (\Theta''_x)^{-1} \colon X''_\infty \to X_\infty$ and $\tau' = \Theta'_x \circ (\Theta''_x)^{-1} \colon X''_\infty \to X'_\infty$. The maps τ and τ' have simple descriptions in terms of sequences. Given $(g''_i) \in X''_\infty$, then $\tau(g''_i) = (g_i) \in X_\infty$, where $g_i = g''_{2i}$, and $\tau'(g''_i) = (g'_i) \in X'_\infty$, where $g'_i = g''_{2i-1}$.

Let $\widehat{G}_\infty$, $\widehat{G}'_\infty$, and $\widehat{G}''_\infty$ denote the inverse limit groups defined as in (15) by the group chains $\mathcal{G}_{\mathcal{U}}^x$, $\mathcal{G}_{\mathcal{U}'}^x$, and $\mathcal{G}_{\mathcal{U}''}^x$, respectively. By Theorem 3.2, there are topological isomorphisms

$$\widehat{\Theta} \colon \mathfrak{G}(\Phi) \to \widehat{G}_\infty, \quad \widehat{\Theta}' \colon \mathfrak{G}(\Phi) \to \widehat{G}'_\infty, \quad \widehat{\Theta}'' \colon \mathfrak{G}(\Phi) \to \widehat{G}''_\infty.$$

There are topological isomorphisms $\widehat{\tau} = \widehat{\Theta} \circ (\widehat{\Theta}'')^{-1} \colon \widehat{G}''_\infty \to \widehat{G}_\infty$ and $\widehat{\tau}' = \widehat{\Theta}' \circ (\widehat{\Theta}'')^{-1} \colon \widehat{G}''_\infty \to \widehat{G}'_\infty$. Given a sequence $\widehat{g} = (g''_i) \in \widehat{G}''_\infty$, then $\widehat{\tau}(g''_i) = (g_i = g''_{2i}) \in \widehat{G}_\infty$ and $\widehat{\tau}'(g''_i) = (g'_i = g''_{2i-1}) \in \widehat{G}'_\infty$.

We first consider the case of the stabilizer group chains. Let $\mathcal{D}''_x \subset \widehat{G}''_\infty$ denote the discriminant for the chain $\mathcal{G}_{\mathcal{U}''}^x$; then the restriction $\widehat{\tau} \colon \mathcal{D}''_x \to \mathcal{D}_x \subset \widehat{G}_\infty$ is an isomorphism by Lemma 3.3 and Corollary 3.4 and likewise for $\widehat{\tau}' \colon \mathcal{D}''_x \to \mathcal{D}'_x \subset \widehat{G}'_\infty$.

We show that the stabilizer groups associated to the chains $\mathcal{G}_{\mathcal{U}''}^x$ and $\mathcal{G}_{\mathcal{U}}^x$ are isomorphic, with the proof for $\mathcal{G}_{\mathcal{U}''}^x$ and $\mathcal{G}_{\mathcal{U}'}^x$ being the same. For each $\ell \geq 0$, by (21) and (22), we have

$$K_\ell \equiv \widehat{K}_{X_\infty, \widehat{U}_\ell} = \{\widehat{g} \in \widehat{G}_\ell \mid \widehat{\Phi}_x(\widehat{g}) \text{ acts as the identity on } \widehat{U}_\ell\} \subset \mathcal{D}_x,$$
$$K''_\ell \equiv \widehat{K}_{X''_\infty, \widehat{U}''_\ell} = \{\widehat{g} \in \widehat{G}''_\ell \mid \widehat{\Phi}''_x(\widehat{g}) \text{ acts as the identity on } \widehat{U}''_\ell\} \subset \mathcal{D}''_x.$$

The isomorphism $\widehat{\tau} \colon \widehat{G}''_{2\ell} \to \widehat{G}_\ell$ induces a homeomorphism $\tau_\ell \colon \widehat{U}''_{2\ell} = \widehat{G}''_{2\ell}/\mathcal{D}''_x \to \widehat{U}_\ell = \widehat{G}_\ell/\mathcal{D}_x$. Thus, $\widehat{\tau}$ induces an isomorphism $\widehat{\tau} \colon K''_{2\ell} = \widehat{K}_{X''_\infty, \widehat{U}''_{2\ell}} \to \widehat{K}_{X_\infty, \widehat{U}_\ell} = K_\ell$.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

For $0 \leq \ell \leq \ell'$, let $\phi_\ell^{\ell'} \colon K_\ell \subset K_{\ell'}$ and $\psi_\ell^{\ell'} \colon K''_\ell \subset K''_{\ell'}$ denote the inclusion maps. By Propositions 4.7 and 4.8, we obtain an isomorphism of direct limits

$$\widehat{\tau} \colon \varinjlim \mathcal{G}(K''_\ell, \psi_\ell^{\ell'}, \mathbb{N}) \to \varinjlim \mathcal{G}(K_\ell, \phi_\ell^{\ell'}, \mathbb{N}), \tag{28}$$

as was to be shown. Hence, the direct limits $\varinjlim \mathcal{G}(K_\ell, \phi_\ell^{\ell'}, \mathbb{N})$ and $\varinjlim \mathcal{G}(K'_\ell, \psi_\ell^{\ell'}, \mathbb{N})$ are isomorphic.

Next, consider the case of the centralizer group chains, given by

$$\begin{gathered} Z_\ell = \mathrm{Cen}(\widehat{G}_\infty, \iota_\infty(G_\ell)) \cap \mathcal{D}_x, \quad Z'_\ell = \mathrm{Cen}(\widehat{G}'_\infty, \iota'_\infty(G'_\ell)) \cap \mathcal{D}'_x, \\ Z''_\ell = \mathrm{Cen}(\widehat{G}''_\infty, \iota_\infty(G''_\ell)) \cap \mathcal{D}''_x, \end{gathered} \tag{29}$$

which define the group chains $\mathcal{Z}$, $\mathcal{Z}'$, and $\mathcal{Z}''$, respectively.

Recall that the topological isomorphisms $\widehat{\tau} \colon \widehat{G}''_\infty \to \widehat{G}_\infty$ and $\widehat{\tau}' \colon \widehat{G}''_\infty \to \widehat{G}'_\infty$ are given by $\widehat{\tau}(g_i) = (g_{2i})$ and $\widehat{\tau}'(g_i) = (g_{2i-1})$ for $\widehat{g} = (g_i) \in \widehat{G}''_\infty$. Then $\widehat{\tau}$ induces isomorphisms $\widehat{\tau}_\ell \colon Z''_{2\ell} \to Z_\ell$ and $\widehat{\tau}'_\ell \colon Z''_{2\ell-1} \to Z'_\ell$. Thus, as for the case of the stabilizer group chains, we obtain an isomorphism

$$\widehat{\tau} \colon \varinjlim \mathcal{G}(Z''_\ell, \psi_\ell^{\ell'}, \mathbb{N}) \to \varinjlim \mathcal{G}(Z_\ell, \phi_\ell^{\ell'}, \mathbb{N}), \tag{30}$$

as was to be shown. Hence, the direct limits $\varinjlim \mathcal{G}(Z_\ell, \phi_\ell^{\ell'}, \mathbb{N})$ and $\varinjlim \mathcal{G}(Z'_\ell, \phi_\ell^{\ell'}, \mathbb{N})$ are isomorphic.

Next, consider the case where x and y are distinct basepoints, and we are given adapted neighborhood bases $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ at x and $\mathcal{U}' = \{U'_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ at y, with corresponding group chains $\mathcal{G}_{\mathcal{U}}^x = \{G_\ell \mid \ell \geq 0\}$ and $\mathcal{G}_{\mathcal{U}'}^y = \{G'_\ell \mid \ell \geq 0\}$. Theorem 3.7 implies that the chains $\mathcal{G}_{\mathcal{U}}^x$ and $\mathcal{G}_{\mathcal{U}'}^y$ are conjugate equivalent. That is, choose $\widehat{g} = (g_i) \in \mathfrak{G}(\Phi)$ such that $\widehat{g} \cdot y = x$; then the collection $\mathcal{U}'' = \{g_i \cdot U'_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ is an adapted neighborhood basis at x and the associated group chain $\mathcal{G}_{\mathcal{U}''}^x = \{G''_\ell = g_\ell^{-1} \ \widehat{G}'_\ell \ g_\ell\}$ is equivalent to $\mathcal{G}_{\mathcal{U}}^x$.

A key point is that the isomorphism between $\widehat{G}'_\ell \subset \widehat{G}_\infty$ and $\widehat{G}''_\ell \subset \widehat{G}_\infty$ is induced by the conjugacy isomorphism $\mathbf{Ad}(g_\ell) \colon G_\ell \to g_\ell G_\ell g_\ell^{-1}$, which also conjugates the stabilizer groups K''_ℓ and K'_ℓ, and likewise for the centralizer groups Z''_ℓ and Z'_ℓ. Then, by Propositions 4.7 and 4.8, we obtain an isomorphism of direct limits. We are thus reduced to the case shown previously, where we are given direct limit groups obtained from adapted neighborhood systems centered at the same point x. □

Remark 4.16. The proof of Theorem 4.15 shows that the isomorphism between the direct limits is induced by a group isomorphism $\mathbf{Ad}(\widehat{g}) \colon \widehat{G}_\infty \to \widehat{G}_\infty$. By Remark 4.13, for $\mathfrak{G} = \widehat{G}_\infty$, the adjoint induces the isomorphism of subgroups $\mathbf{Ad}(\widehat{g}) \colon K_\infty \cong K'_\infty$ and $\mathbf{Ad}(\widehat{g}) \colon Z_\infty \cong Z'_\infty$. A key point is that this isomorphism preserves the height filtration on these groups as defined in Remark 4.13.

4.3. *Properties of the stabilizer and centralizer groups.* The direct limit groups $\Upsilon_s(\Phi)$ and $\Upsilon_c(\Phi)$ are invariants of Cantor actions whose dynamical implications will be considered in the following sections. We first give a restatement of Lemma 3.13 in the language of direct limits.

LEMMA 4.17. *For $x \in \mathfrak{X}$, there is an inclusion of direct limit groups $\Upsilon_c(\Phi) \subset \Upsilon_s(\Phi)$.*

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Examples show that the inclusion $\Upsilon_c(\Phi) \subset \Upsilon_s(\Phi)$ can be proper. One source of this distinction is noted in Remark 2.8, that the adjoint map for $\mathcal{D}_x$ restricted to $\mathcal{D}_x$ need not be trivial, while the action on U_ℓ is trivial. We next introduce five classes of Cantor actions.

Definition 4.18. A Cantor action $(\mathfrak{X}, G, \Phi)$ is said to be:

(1) *stable* if the stabilizer group $\Upsilon_s(\Phi)$ is bounded and is said to be *wild* otherwise;
(2) *algebraically stable* if the centralizer group $\Upsilon_c(\Phi)$ is bounded and is said to be *algebraically wild* otherwise;
(3) *wild of finite type* if the stabilizer group $\Upsilon_s(\Phi)$ is unbounded and represented by a chain of finite groups;
(4) *wild of flat type* if the stabilizer group $\Upsilon_s(\Phi)$ is unbounded and $\Upsilon_c(\Phi) = \Upsilon_s(\Phi)$;
(5) *dynamically wild* if the stabilizer group $\Upsilon_s(\Phi)$ is unbounded and is not of flat type.

Each of the properties in Definition 4.18 is a *conjugacy* invariant of the action by Theorem 4.15.

The notion of wild Cantor actions was introduced by the authors in [**36**, Definition 4.6], as part of the study of the homeomorphism types of weak solenoids in their works [**19**, **35**, **36**]. The definition of a wild action in [**36**] and in Definition 4.18 coincide. Moreover, the examples of wild actions constructed in [**36**, §8] are all of finite and flat type, while the examples of actions constructed in §8 below are dynamically wild and not of finite type. On the other hand, [**35**, Corollary 1.7] implies that if G is a finitely generated nilpotent group, then every Cantor action of G must be both stable and algebraically stable.

5. *Analytic regularity for stable Cantor actions*

In this section, we relate the bounded property of the stabilizer group $\Upsilon_s(\Phi)$ for a Cantor action $(\mathfrak{X}, G, \Phi)$ with 'analytic regularity' of the action. We first recall some background context.

5.1. *Locally quasi-analytic actions.*

Haefliger introduced in [**34**] the notion of a *quasi-analytic* topological action of a pseudogroup on a connected space X. Álvarez López and Candel in [**1**, Definition 9.4], and later Álvarez López and Moreira Galicia in [**2**, Definition 2.18], adapted the notion of a quasi-analytic topological action to the more general case where the action space X need not be connected. The authors formulated in [**35**, **36**] a notion of *locally quasi-analytic* Cantor actions and showed the relation between this condition and the stable property for the action.

The quasi-analytic condition for a Cantor action is a modification of the notion of a *topologically free* action for general Cantor actions, which first appeared in the work of Boyle and Tomiyama [**7**] in their study of flip-conjugacy. Renault showed in [**48**, §3] that an action is topologically free if and only if the associated action groupoid is *essentially principal*. We first recall the definition of a topologically free action and some properties of this definition. Topological freeness and related ideas are discussed in more detail in [**38**, §2].

The *isotropy group* at $x \in \mathfrak{X}$ of an action $(\mathfrak{X}, G, \Phi)$ is $G_x = \{g \in G \mid g \cdot x = x\}$. A point $x \in \mathfrak{X}$ is said to have trivial isotropy if G_x is the trivial group. All points in the orbit of x then also have trivial isotropy, so form a dense subset of $\mathfrak{X}$ for a minimal action.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Let $\mathrm{Fix}(g) = \{x \in \mathfrak{X} \mid \Phi(g) \cdot x = x\}$; then introduce the *isotropy set*

$$\mathrm{Iso}(\mathfrak{X}, G, \Phi) \equiv \{x \in \mathfrak{X} \mid \exists\, g \in G,\, g \neq \mathrm{id},\, g \cdot x = x\} = \bigcup_{e \neq g \in G} \mathrm{Fix}(g). \tag{31}$$

The action $(\mathfrak{X}, G, \Phi)$ is said to be *topologically free* if the set $\mathrm{Iso}(\mathfrak{X}, G, \Phi)$ is meager in $\mathfrak{X}$. Note that if $e \neq g \in G$ and $\Phi(g)$ acts trivially on $\mathfrak{X}$, then $\mathrm{Iso}(\mathfrak{X}, G, \Phi) = \mathfrak{X}$ and thus a topologically free action must be faithful. If the group G is abelian, it is an exercise to show that a faithful minimal Cantor action $(\mathfrak{X}, G, \Phi)$ must be topologically free; see, for example, [**35**, Corollary 2.3].

We next restrict attention to Cantor actions and use the special properties of equicontinuous actions to formulate local forms of the topologically free property. A Cantor action $(\mathfrak{X}, G, \Phi)$ is said to be *locally topologically free* if there exists $\varepsilon > 0$ such that for any adapted set U with $\mathrm{diam}(U) < \varepsilon$, the action of G_U on U is topologically free. There is another related notion, defined as follows.

Definition 5.1. [**1**, Definition 9.4] A Cantor action $(\mathfrak{X}, G, \Phi)$ is *locally quasi-analytic*, or simply *LQA*, if there exists $\varepsilon > 0$ such that for any adapted set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \varepsilon$, and for any adapted subset $V \subset U$ and elements $g_1, g_2 \in G$,

$$\text{if the restrictions } \Phi(g_1)|V = \Phi(g_2)|V, \text{ then } \Phi(g_1)|U = \Phi(g_2)|U. \tag{32}$$

That is, by [**35**, Proposition 2.2], the action of $H_U = \Phi(G_U)$ on U is topologically free.

If (32) holds for $U = \mathfrak{X}$, then the action of $\mathfrak{G}(\Phi)$ is topologically free.

Examples of equicontinuous Cantor actions $\mathfrak{X}$ which are locally quasi-analytic, but not quasi-analytic, are easily constructed, as given in [**19**, **36**], for example. There is also a generalization of the locally quasi-analytic property for the action of the profinite closure group $\mathfrak{G}(\Phi)$.

Definition 5.2. A Cantor action $(\mathfrak{X}, G, \Phi)$ is *locally completely quasi-analytic*, or simply *LCQA*, if there exists $\varepsilon > 0$ such that for any adapted set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \varepsilon$, and for any adapted subset $V \subset U$ and elements $\widehat{h}_1, \widehat{h}_2 \in \mathfrak{G}(\Phi)$,

$$\text{if the restrictions } \widehat{h}_1|V = \widehat{h}_2|V, \text{ then } \widehat{h}_1|U = \widehat{h}_2|U. \tag{33}$$

Equivalently, set $\widehat{h} = \widehat{h}_1\, \widehat{h}_2^{-1}$; then we have

$$\text{if the restriction } \widehat{h}|V = \mathrm{id}|V, \text{ then } \widehat{h}|U = \mathrm{id}|U. \tag{34}$$

Since $\Phi(G) \subset \mathfrak{G}(\Phi)$, the LCQA property implies the LQA property for a Cantor action. It is not known, however, if there exists a Cantor action $(\mathfrak{X}, G, \Phi)$ which is LQA but not LCQA. In particular, does there exist a free Cantor action which is not LCQA? All examples known to the authors which are not LCQA are also not LQA.

5.2. *Bounded stabilizer groups.* Recall from Definition 4.18 that a Cantor action $(\mathfrak{X}, G, \Phi)$ is stable if the stabilizer group $\Upsilon_s(\Phi)$ is bounded. Here is the main result of this section.

THEOREM 5.3. *Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action; then Φ is a locally completely quasi-analytic (LCQA) action if and only if its stabilizer limit group $\Upsilon_s(\Phi)$ is bounded.*

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Proof. Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action and $\mathcal{G}^x_{\mathcal{U}} = \{G_\ell \mid \ell \geq 0\}$ be the group chain associated to an adapted neighborhood basis $\mathcal{U}$ at $x \in \mathfrak{X}$. Let $\widehat{G}_\infty$ be defined as in (15); then, by Theorem 3.2, there is a topological isomorphism $\widehat{\Theta}\colon \mathfrak{G}(\Phi) \to \widehat{G}_\infty$ which induces an isomorphism $\widehat{\Theta}\colon \mathfrak{G}(\Phi)_x \to \mathcal{D}_x$.

Let $\mathcal{K}(\mathcal{U}) = \{K_\ell \subset \mathcal{D}_x \mid \ell \geq 0\}$ be the increasing chain of stabilizer subgroups defined in (22).

The map $\widehat{\Theta}$ induces a homeomorphism of G-spaces $\Theta_x\colon \mathfrak{G}(\Phi)/\mathfrak{G}(\Phi)_x \cong \mathfrak{X} \to \widehat{G}_\infty/\mathcal{D}_x \cong X_\infty$ by Corollary 3.4. Let X_∞ have a metric such that G acts on X_∞ by isometries and then let $d_{\mathfrak{X}}$ be the metric on $\mathfrak{X}$ induced by the homeomorphism Θ_x.

Recall that for $\ell \geq 0$, Θ_x identifies the clopen set $U_\ell \subset \mathfrak{X}$ with $\widehat{U}_\ell \equiv \widehat{G}_\ell/\mathcal{D}_x \subset \widehat{G}_\infty/\mathcal{D}_x$.

Assume that Φ is LCQA and let $\varepsilon > 0$ be as in Definition 5.2. Let $\ell_0 \geq 1$ be such that $\operatorname{diam}(U_\ell) < \varepsilon$ for all $\ell \geq \ell_0$. Thus, for $\ell > \ell_0$, the restricted action of $\widehat{G}_\ell$ on $\widehat{U}_\ell$ is quasi-analytic. Given $\widehat{h} \in \mathcal{D}_x$, suppose that $\widehat{h}$ acts trivially on $\widehat{U}_\ell$; then it must act trivially on U_{ℓ_0}. That is, $K_\ell \subset K_{\ell_0}$. As the converse $K_{\ell_0} \subset K_\ell$ always holds, this implies that $\Upsilon^x_s(\Phi)$ is a bounded direct limit group.

Conversely, assume that $\Upsilon^x_s(\Phi)$ is a bounded direct limit group. Then there exists $\ell_0 > 0$ such that $K_{\ell_0} = K_\ell$ for all $\ell \geq \ell_0$. Let $\varepsilon_0 > 0$ be such that the ball of radius ε_0 about x is contained in U_{ℓ_0}.

Set $\varepsilon = \varepsilon_0/2$. Let $U \subset \mathfrak{X}$ be an open set with $\operatorname{diam}(U) < \varepsilon$ and let V be an open subset with $V \subset U$. Then, by minimality and equicontinuity of the action Φ, there exists $g \in G$ such that $g \cdot x \in V$. By the choice of ε, we have that $U' = g^{-1} \cdot U \subset U_{\ell_0}$. In addition, as $\mathcal{U}$ is a neighborhood basis at x, there exists $\ell > 0$ such that $U_\ell \subset V' = g^{-1} \cdot V$.

Let $\widehat{h} \in \mathfrak{G}(\Phi)$ be such that $\widehat{h}$ acts as the identity on V; then $\widehat{h}' = g^{-1}\, \widehat{h}\, \cdot g$ acts as the identity on V' and thus also on U_ℓ, since $U_\ell \subset V'$. In particular, $\widehat{h}' \cdot x = x$, so $\widehat{h}' \in \mathfrak{G}(\Phi)_x$ and hence $\widehat{k} = \widehat{\Theta}(\widehat{h}') \in \mathcal{D}_x$. Moreover, $\widehat{k}$ acts as the identity on $\widehat{U}_\ell$ and hence $\widehat{k} \in K_\ell$. We are given that $K_\ell = K_{\ell_0}$, so $\widehat{h}' \in K_{\ell_0}$ and hence acts as the identity on $\widehat{U}_{\ell_0}$. Thus, $\widehat{h}'$ acts as the identity on $U' \subset U_{\ell_0}$ and so $\widehat{h}$ acts as the identity on U, as was to be shown. □

6. *Orbit equivalence and rigidity*

Let $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', G, \Psi)$ be general Cantor actions. A homeomorphism $h\colon \mathfrak{X} \to \mathfrak{X}'$ is a *conjugacy* between the two actions if $\Phi(g) = h^{-1} \circ \Psi(g) \circ h$ for all $g \in G$. A conjugacy map h thus preserves the structure of the orbits as G-spaces, which is used in constructing conjugacy invariants for an action. For example, this was used in Theorem 4.15 to show that the isomorphism classes of the direct limit groups $\Upsilon_s(\Phi)$ and $\Upsilon_c(\Phi)$ are invariants of the conjugacy class of a Cantor action $(\mathfrak{X}, G, \Phi)$.

An *orbit equivalence* between two actions is a bijective map $h\colon \mathfrak{X} \to \mathfrak{X}'$ which maps orbits of the action Φ to orbits of the action Ψ. In addition, one can impose additional assumptions on the map h, such as to assume that it is a measurable isomorphism with respect to given quasi-invariant measures on $\mathfrak{X}$ and $\mathfrak{X}'$. In the measurable category of actions, this yields the notion of *measurable orbit equivalence*, a topic which has been extensively studied. A celebrated result by Connes, Feldman and Weiss in [**11**] showed that for essentially free actions of amenable groups, any two such actions are measurably orbit equivalent. For example, if G is a nilpotent group, then any essentially free ergodic action of G is measurably orbit equivalent to an ergodic $\mathbb{Z}$-action. Thus, a measurable

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

orbit equivalence need not preserve the structure on orbits induced by the G-action. On the other hand, the results of Furman in [**26**] show that for measure-preserving actions of higher rank lattice groups, measure orbit equivalence implies virtual conjugacy of the actions and hence the orbit structures are preserved in this case. For further discussion of properties of measurable orbit equivalence, see, for example, the survey by Gaboriau [**27**].

For general Cantor actions, *continuous orbit equivalence* is the more natural notion to study. One assumes that the orbit equivalence h is a homeomorphism which, in addition, satisfies a 'locally constant' property, as stated precisely in Definition 6.1. This notion was introduced by Boyle in his thesis [**6**], see also [**32**], and has played a fundamental role in the classification of general Cantor actions in many subsequent works, as for example in [**5**, **28**, **29**]. The works of Renault [**48**] and Li [**38**, Theorem 1.2] use the notion of continuous orbit equivalence to classify the groupoid C^*-algebras associated to the action, in the case of topologically free actions. The related notion of the topological full group of an action has provided a rich source of examples of finitely generated groups with exceptional properties, as discussed for example in [**15**, **44**].

The *rigidity property* for Cantor actions states that topological orbit equivalence implies virtual conjugacy of the actions. In essence, this is saying that an orbit equivalence which locally preserves the orbit structures must preserve the orbit structures on a global scale. In this way, it can be seen as a property which is analogous to the measure rigidity property studied by Furman in [**26**]. The works of Cortez and Medynets in [**12**] and Li in [**38**] prove versions of topological rigidity.

In this section, we study the consequences of continuous orbit equivalence between Cantor actions, without any assumption that the actions are topologically free. Our main result is Theorem 6.10, which considers the relation between the stabilizer groups for continuously orbit equivalent actions. We deduce that the property of being stable, as well as algebraically stable, is an invariant of continuous orbit equivalence.

We first recall in §6.1 some notions from the theory of orbit equivalence for Cantor actions. In §6.2 we discuss various notions of rigidity for Cantor actions. Then in §6.3 the rigidity results from [**12**, **38**] are described. We then begin the proof of Theorem 6.10 in §6.3 and §2.3 contains some technical results needed to complete the proof. Theorem 6.18 gives an extension of Theorem 3.3 by Cortez and Medynets in [**12**] and [**35**, Theorem 1.5] by the authors.

6.1. *Continuous orbit equivalence.* Let $(\mathfrak{X}, G, \Phi)$ be a general Cantor action. Consider the equivalence relation on $\mathfrak{X}$ defined by the action,

$$\mathcal{R}(\mathfrak{X}, G, \Phi) \equiv \{(x, g \cdot x)) \mid x \in \mathfrak{X}, g \in G\} \subset \mathfrak{X} \times \mathfrak{X}. \tag{35}$$

Given actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', H, \Psi)$, we say that they are *orbit equivalent* if there exists a bijection $h\colon \mathfrak{X} \to \mathfrak{X}'$ which maps $\mathcal{R}(\mathfrak{X}, G, \Phi)$ to $\mathcal{R}(\mathfrak{X}', H, \Psi)$ and similarly for the inverse map h^{-1}.

Definition 6.1. Let $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', H, \Psi)$ be general Cantor actions. A *continuous orbit equivalence* between the actions is a homeomorphism $h\colon \mathfrak{X} \to \mathfrak{X}'$ which is an orbit equivalence and satisfies the locally constant conditions:

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

(1) for each $x \in \mathfrak{X}$ and $g \in G$, there exist $\alpha(g, x) \in H$ and an open set $x \in U_x \subset \mathfrak{X}$ such that $\Psi(\alpha(g, x)) \circ h|U_x = h \circ \Phi(g)|U_x$;

(2) for each $y \in \mathfrak{X}'$ and $k \in H$, there exist $\beta(k, y) \in G$ and an open set $y \in V_y \subset \mathfrak{X}'$ such that $\Phi(\beta(k, y)) \circ h|V_y = h \circ \Psi(k)|V_y$.

Note in particular that these conditions imply that the functions $\alpha\colon G \times \mathfrak{X} \to H$ and $\beta\colon H \times \mathfrak{X}' \to G$ are continuous, as the groups G and H have the discrete topology.

One special class of examples of continuous orbit equivalences are those for which the functions $\alpha(g, y)$ and $\beta(g, x)$ are constant in y. Then the identities (37) and (38) can be considered as defining 'time shifts' along the orbits, in analogy with the case when $G, H = \mathbb{Z}$.

Let $(\mathfrak{X}', H, \Psi)$ be a general Cantor action, let $h\colon \mathfrak{X} \to \mathfrak{X}'$ be a homeomorphism which implements a continuous orbit equivalence between $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', H, \Psi)$, and let

$$\Psi^h\colon H \times \mathfrak{X} \to \mathfrak{X}, \quad \text{where } \Psi^h(g)(x) = h^{-1} \circ \Psi(g) \circ h(x) \text{ for } g \in H. \tag{36}$$

Then the orbits for the actions Ψ^h and Φ are equal, so the identity map is an orbit equivalence between $\mathcal{R}(\mathfrak{X}, G, \Phi)$ and $\mathcal{R}(\mathfrak{X}, H, \Psi^h)$. Thus, for the study of orbit equivalence, it suffices to consider orbit equivalences for which h is the identity map. Then the data of the orbit equivalence is encapsulated in the continuous, hence locally constant, maps $\alpha\colon G \times \mathfrak{X} \to H$ and $\beta\colon H \times \mathfrak{X} \to G$. That is, for $y \in \mathfrak{X}$ and $g \in G$, there exists a clopen set $y \in U_{g,y} \subset \mathfrak{X}$ so that

$$\Psi(\alpha(g, y)) \cdot z = \Phi(g) \cdot z \quad \text{for } z \in U_{g,y} \tag{37}$$

and, for $k \in H$, there exists a clopen set $y \in V_{k,y} \subset \mathfrak{X}$ so that

$$\Phi(\beta(k, y)) \cdot z = \Psi(k) \cdot z \quad \text{for } z \in V_{k,y}. \tag{38}$$

Next, recall the notion of a cocycle over an action.

Definition 6.2. Let $\varphi\colon \Gamma \times X \to X$ be an action of the group Γ on a space X. A map $c\colon \Gamma \times X \to \Gamma'$ is a *cocycle over* the action if for all $\gamma, \gamma' \in \Gamma$ and $x \in X$, we have

$$c(\gamma' \cdot \gamma, x) = c(\gamma', \varphi(\gamma)(x)) \cdot c(\gamma, x). \tag{39}$$

For an action which is free, or more generally is topologically free, the identities (37) and (38) uniquely determine the values of $\alpha(g, y)$ and $\beta(k, y)$ for all $y \in \mathfrak{X}$ and $k \in G$ and thus the functions α and β are cocycles over the actions Φ and Ψ, respectively. However, if the actions are not topologically free, then the identities (37) and (38) no longer uniquely specify the values of $\alpha(g, y)$ and $\beta(k, y)$ and so they need not satisfy the cocycle identity.

For continuous orbit equivalent topologically free actions whose orbit functions α and β are constant, the cocycle identity (39) implies that they define group homomorphisms. This leads to the notion of *conjugacy* of Cantor actions, as was studied by Li in [**38**], and also the related notion of *structural conjugacy* by Cortez and Medynets in [**12**]. We use the following terminology.

Definition 6.3. The Cantor actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}, H, \Psi)$ are said to be θ*-conjugate* if there exists a group isomorphism $\theta\colon G \to H$ such that

$$\Phi(g)(x) = \Psi^\theta(x) \equiv \Psi(\theta(g))(x) \quad \text{for all } g \in G, x \in \mathfrak{X}. \tag{40}$$

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

The identity (40) implies that we can choose the function $\alpha(g, x) = \theta(g)$ in Definition 6.1.

For the case where $G = H = \mathbb{Z}$, the involution $\theta(n) = -n$ is the only non-trivial isomorphism and in this case θ-conjugacy is the same as flip-conjugacy as studied by Boyle and Tamiyama [**7**].

There is a related notion to Definition 6.3, where we assume that $\mathfrak{X} = \mathfrak{X}'$, $G = H$, and θ is the identity. Define the *center* of the action:

$$\mathrm{Aut}(\mathfrak{X}, G, \Phi) = \{h \in \mathrm{Homeo}(\mathfrak{X}) \mid h(\Phi(g) \cdot x) = \Phi(g) \cdot h(x) \text{ for all } x \in \mathfrak{X},\ g \in G\}. \tag{41}$$

Then we have the following relation between the direct limit group $\Upsilon_c^x(\Phi)$ and the center.

LEMMA 6.4. *Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action and $\{G_\ell \mid \ell \geq 0\}$ be the group chain associated to an adapted neighborhood basis $\mathcal{U}$ at $x \in \mathfrak{X}$. Then, for each $\ell \geq 1$, $Z_\ell \subset \mathrm{Aut}(\mathfrak{X}, G_\ell, \Phi)$.*

Proof. Let $\widehat{h} \in Z_\ell$ which acts on $\widehat{G}_\infty$ by left multiplication. Then, for any $\widehat{g} \in G_\ell$, we have $\widehat{g}\,\widehat{h} = \widehat{h}\,\widehat{g}$ by Lemma 3.12. Thus, for the isomorphism $\widehat{\Theta}\colon \mathfrak{G}(\Phi) \to \widehat{G}_\infty$, we have $h = \widehat{\Theta}^{-1}(\widehat{h}) \in \mathrm{Aut}(\mathfrak{X}, G_\ell, \Phi)$. □

Remark 6.5. Note that in the context of Lemma 6.4, while the action of G_ℓ on the clopen set U_ℓ is minimal, its action on $\mathfrak{X}$ cannot be minimal, as U_ℓ is adapted and hence invariant by the action of G_ℓ. Lemma 6.4 implies that when the action Φ is restricted to subactions by groups G_ℓ, the groups Z_ℓ arise as the centers of these restricted actions. That is, while Φ may have trivial center, if the direct limit group $\Upsilon_c^x(\Phi)$ is non-trivial, then there exists restricted actions with non-trivial 'symmetries'.

6.2. *Types of rigidity.* Rigidity for a dynamical system can be viewed as asserting that two dynamical systems which are equivalent in some weaker sense are also equivalent, possibly up to some finite indeterminacy, in some stronger sense. For Cantor systems, we formulate three notions of equivalence for their rigidity, each weaker than the previous one.

The first notion is based on θ-conjugacy as given in Definition 6.3.

Definition 6.6. A Cantor action $(\mathfrak{X}, G, \Phi)$ is *rigid* if, given a continuous orbit equivalence $h\colon \mathfrak{X} \to \mathfrak{X}'$ to a Cantor action $(\mathfrak{X}', G', \Psi)$, then Ψ^h as defined by (36) is θ-conjugate to Φ.

This is the form of the rigidity property used by Li [**38**].

It is elementary to construct examples of Cantor actions where G is a cross product of a normal subgroup by a finite quotient group, as in [**35**, Examples A.2 and A.3], such that two actions are continuous orbit equivalent but are not θ-conjugate actions. This is the situation considered by Cortez and Medynets in [**12**]. They accordingly introduced the weaker notion of *structural stability* and proved their rigidity results for free actions in these terms. Their notion of structural stability coincides with what we call virtual rigidity below.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Let $U \subset \mathfrak{X}$ be adapted for the Cantor action $(\mathfrak{X}, G, \Phi)$. Let $\Phi_U : G_U \times U \to U$ denote the restricted action of G_U on U. Similarly, for a Cantor action $(\mathfrak{X}', G', \Psi)$ with adapted set $V \subset \mathfrak{X}'$, let (V, G'_V, Ψ_V) denote the restricted action.

Definition 6.7. Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action. Then the action $(\mathfrak{X}, G, \Phi)$ is *virtually rigid* if, given a continuous orbit equivalence $h : \mathfrak{X} \to \mathfrak{X}'$ to a Cantor action $(\mathfrak{X}', G', \Psi)$, there exists an adapted set $U \subset \mathfrak{X}$ for the action Φ such that $V = h(U) \subset \mathfrak{X}'$ is adapted for the action Ψ and there is an isomorphism $\theta_U : G_U \to G'_V$ so that the action (U, G_U, Φ_U) is θ_U-conjugate to (V, G'_V, Ψ^h_V).

Note that both Definitions 6.6 and 6.7 are essentially only applicable for topologically free actions on the full space $\mathfrak{X}$. The third notion we consider is *return equivalence*, which was used in the authors' work [**36**] for the study of the homeomorphism types of weak solenoids. For the geometric applications in [**19**, **36**], the holonomy action on a transversal is the fundamental concept. Accordingly, return equivalence for Cantor actions is formulated in terms of the image group H_U for an adapted subset $U \subset \mathfrak{X}$ as defined in (6).

Let $U \subset \mathfrak{X}$ be adapted for the Cantor action $(\mathfrak{X}, G, \Phi)$. Let $\overline{\Phi}_U : H_U \times U \to U$ denote the induced action of $H_U = \Phi(G_U) \subset \mathrm{Homeo}(U)$ on U. Similarly, for a Cantor action $(\mathfrak{X}', G', \Psi)$ with adapted set $V \subset \mathfrak{X}'$, let $(V, H'_V, \overline{\Psi}_V)$ denote the induced action.

Definition 6.8. Two Cantor actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', G', \Psi)$ are *return equivalent* if there exist an adapted set $U \subset \mathfrak{X}$ for the action Φ, an adapted set $V \subset \mathfrak{X}'$ for the action Ψ, and a homeomorphism $h_U : U \to V$ which induces a θ_U-conjugacy between the action of H_U on U and the action of H'_V on V.

While the Definitions 6.7 and 6.8 are similar, the former requires an isomorphism between the subgroups $G_U \subset G$ and $G'_V \subset G'$, while the latter only requires an isomorphism between their respective image groups H_U and H'_V. Thus, Definition 6.8 is more relevant for the study of Cantor actions which are not topologically free.

For topologically free actions, an adaptation of the proof in [**12**, Theorem 3.3] shows that the fact that the isomorphism θ is induced by a continuous orbit equivalence h forces the groups G_U and G'_V to have equal indices in G and G', respectively. In Definition 6.8 of return equivalence, the homeomorphism h_U need not be induced by a homeomorphism of the space $\mathfrak{X}$, so there is no requirement that the groups H_U and H'_V have the same index with respect to any larger groups.

6.3. *Rigidity for Cantor actions.* The approach in the literature to proving rigidity for Cantor actions is to assume that both actions are free on an invariant dense subset $\mathfrak{Z} \subset \mathfrak{X}$ and then observe that the function α in Definition 6.1 satisfies a cocycle identity on $\mathfrak{Z}$ and hence on all of $\mathfrak{X}$ by continuity. The map $\alpha : G \times \mathfrak{X} \to H$ is called the 'orbit cocycle' for the orbit equivalence.

Then either a cohomological assumption as in Li [**38**], or a dynamical assumption as in Cortez and Medynets [**12**], is used to show that the cocycle α is cohomologous to a constant cocycle, which implies that the conditions of Definition 6.3 are satisfied.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

The results of Boyle and Tamiyama in [**7**] can be interpreted as saying that a minimal Cantor action by $\mathbb{Z}$ is rigid, as stated in [**38**, Theorem 3.2]. As G is abelian and the action is faithful in this case, there is a well-defined orbit cocycle and the authors showed that it is cohomologous to a constant.

Cortez and Medynets showed that *free* equicontinuous Cantor actions are rigid in [**12**, Theorem 3.3] and in fact their proof directly extends to show the following.

THEOREM 6.9. *Let G, G' be finitely generated groups, and $(\mathfrak{X}, G, \Phi)$, $(\mathfrak{X}', G', \Psi)$ be faithful Cantor actions. Suppose that the actions Φ and Ψ are topologically free. Then the actions are continuous orbit equivalent if and only they are virtually rigid in the sense of Definition 6.7.*

The proof of this result in [**12**] does not explicitly discuss the orbit cocycle. Rather, it uses the odometer model for an equicontinuous action to construct the isomorphism θ_U in Definition 6.7 directly from the action.

For G a finitely generated group, Theorems 1.3 and 1.4 in Li [**38**] give conditions on a topologically free Cantor action which imply that the orbit cocycle for the action is cohomologous to a constant cocycle. The discussion in [**38**, §4] explains the passage from a constant cocycle to a conjugacy.

The methods in [**12**, **38**] were used in the authors' work [**36**, §4] to show that stable actions which are continuous orbit equivalent are *return equivalent*. We also use these techniques in the proof of the following result, which will be used in the proof of our rigidity result (Theorem 6.18).

THEOREM 6.10. *Let $h\colon \mathfrak{X} \to \mathfrak{X}'$ be a continuous orbit equivalence between Cantor actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', G', \Psi)$. If G is finitely generated and $(\mathfrak{X}', G', \Psi)$ is stable, then $(\mathfrak{X}, G, \Phi)$ is stable.*

Proof. Following the discussion in §6.1, we can assume that $\mathfrak{X} = \mathfrak{X}'$ and h is the identity map. Fix a basepoint $x \in \mathfrak{X}$. Then, by Theorem 5.3, there exists $V_1 \subset \mathfrak{X}$ adapted to the action Ψ with $x \in V_1$ such that the action of $H_1' = \Psi(G'_{V_1}) \subset \mathrm{Homeo}(V_1)$ on V_1 is topologically free. Let $\mathcal{Z} \subset V_1$ be a dense subset which is H_1'-invariant and the restriction of the action of H_1' to $\mathcal{Z}$ is free.

Let $W \subset \mathfrak{X}$ be an adapted set for the action Φ with $x \in W \subset V_1$. Let $G_W \subset G$ be defined as in (3).

Let $\alpha\colon G \times \mathfrak{X} \to G'$ be the orbit function which satisfies the relation (37) and let $\alpha_W\colon G_W \times W \to G'$ denote its restriction to W.

For each $g \in G_W$ and $y \in W$, we have $\Phi(g)(y) \in W$. Let $h \in G'$ be such that $\Psi(h)(y) = \Phi(g)(y)$; then $W \subset V_1$ implies that $\Psi(h)(V_1) \cap V_1 \neq \emptyset$ and hence $\Psi(h) \in H_1' = \Psi_V(G'_{V_1}) \subset \mathrm{Homeo}(V_1)$. That is, the restriction of α to $G_W \times W$ induces a map $\widehat{\alpha}_W = \Psi_{V'} \circ \alpha\colon G_W \times W \to H_1'$, where $\Psi_{V'}\colon G_1' \to H_1'$.

The action of H_1' on V_1 is topologically free, so the proof of [**38**, Lemma 2.8] adapts to yield the following.

LEMMA 6.11. *$\widehat{\alpha}_W\colon G_W \times W \to H_1'$ satisfies the cocycle identity (39) for the restricted action $\Phi_W\colon G_W \times W \to W$.*

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

We next show that a properly chosen restriction of the cocycle $\widehat{\alpha}_W$ is a coboundary. The proof of this fact follows closely the proof of [**12**, Theorem 3.3] with the variation that we only assume that the action of H_1' is topologically free and do not assume that the action of G_W on W is topologically free.

Since G is finitely generated, the same holds for the subgroup G_W of finite index. Choose a symmetric generating set $\{g_i \mid 1 \le i \le m\}$ for G_W. That is, each $g \in G_W$ can be written as a product $g = g_{i_1}\, g_{i_2}\, \cdots\, g_{i_v}$, where each $1 \le i_\ell \le m$. The map $\alpha\colon G \times \mathfrak{X} \to G'$ is continuous and G' is discrete; hence, α_W is locally constant. Thus, there exists $\delta_1 > 0$ so that

$$\alpha_W(g_i, y) = \alpha_W(g_i, z) \text{ for } 1 \le i \le m \text{ and } y, z \in \mathfrak{X} \text{ with } d_{\mathfrak{X}}(y, z) < \delta_1. \tag{42}$$

Let $W' \subset W$ be an adapted set such that $x \in W'$ and $\operatorname{diam}(g \cdot W') < \delta_1$ for all $g \in G_W$.

The collection $\mathcal{W}' \equiv \{g \cdot W' \mid g \in G_W\}$ is a finite clopen partition of W, so there exists $0 < \delta_2 < \delta_1$ such that for any $y \in W$, there exists $g \in G_W$ such that $B_{d_{\mathfrak{X}}}(y, \delta_2) \subset g \cdot W'$, and $g \cdot W'$ is the unique element of the partition $\mathcal{W}'$ which contains it.

By the uniform continuity of the action Φ, there exists $0 < \delta_3 \le \delta_2$ such that

$$d_{\mathfrak{X}}(\Phi(g)(y), \Phi(g)(z)) < \delta_2 \text{ for all } y, z \in \mathfrak{X} \text{ with } d_{\mathfrak{X}}(y, z) < \delta_3 \quad \text{for all } g \in G. \tag{43}$$

Choose an adapted set U_1 such that $x \in U_1 \subset W'$ and $\operatorname{diam}(g' \cdot U_1) < \delta_3$ for all $g' \in G_{U_1}$. That is, once W' has been chosen, choose U_1 sufficiently small so that all orbits of points in U_1 under the action of $\Phi(G_W)$ have well-defined codings with respect to the partition $\mathcal{W}'$ of W formed by the translates of W'. (The coding method is discussed in detail in [**9**, §6].)

Set $G_1 = G_{U_1} \subset G_W$, let $H_1 = \Phi_{U_1}(G_1) \subset \operatorname{Homeo}(U_1)$, and let $\Phi_1\colon G_1 \to H_1$ denote the quotient map. We also denote the restricted action of G_1 by $\Phi_1\colon G_1 \times U_1 \to U_1$. Let $\widehat{\alpha}_1\colon G_1 \times U_1 \to H_1'$ denote the cocycle over Φ_1 obtained by restricting $\widehat{\alpha}_W$. The next result shows that $\widehat{\alpha}_1$ is a coboundary. The proof below uses a key idea from the proof of [**12**, Theorem 3.3].

PROPOSITION 6.12. *$\widehat{\alpha}_1\colon G_1 \times U_1 \to H_1'$ is induced by a homomorphism $\widehat{\theta}_1\colon G_1 \to H_1'$. That is, for $g \in G_1$ we have $\Psi(\widehat{\alpha}_1(g, x)) = \widehat{\theta}_1(g)$ and thus*

$$\Phi(g)(y) = \widehat{\theta}_1(g)(y) \quad \textit{for all } y \in U_1. \tag{44}$$

Proof. We first show that for $g \in G_1$, the function $\widehat{\alpha}_1(g, y)$ is constant in $y \in U_1$.

Recall that $\{g_i \mid 1 \le i \le m\}$ is a symmetric generating set for G_W. By the choice of δ_1, for $d_{\mathfrak{X}}(y, z) < \delta_1$ and $1 \le i \le m$, we have $\widehat{\alpha}_1(g_i, y) = \widehat{\alpha}_1(g_i, z)$. Thus, by the choice of W' and (42), for any $g' \in G_W$, the value of $\widehat{\alpha}_W(g_i, y)$ is constant for $y \in g' \cdot W'$.

Then by the choice of δ_3 so that (43) holds, and the choice of U_1 so that $\operatorname{diam}(g' \cdot U_1) < \delta_3$ for $g' \in G_W$, for $y, z \in U_1$ we have $g' \cdot y \in g' \cdot U_1$ and $g' \cdot z \in g' \cdot U_1$. Thus, $g' \cdot y$ and $g' \cdot z$ are contained in the same set of the partition $\mathcal{W}'$ and hence $\widehat{\alpha}_1(g_i, g' \cdot y) = \widehat{\alpha}_1(g_i, g' \cdot z)$ for any $1 \le i \le m$.

We now apply this recursively to the elements of G_1. Let $g \in G_1$; then $g = g_{i_1} \cdots g_{i_v} \in G_1$. Set $g' = g_{i_2} \cdots g_{i_v}$; then, for $y, z \in \mathcal{Z} \cap U_1$,

$$\begin{aligned} \widehat{\alpha}_1(g, y) &= \widehat{\alpha}_1(g_{i_1}\, g', y) = \widehat{\alpha}_W(g_{i_1}, g' \cdot y) \circ \widehat{\alpha}_W(g', y) \\ &= \widehat{\alpha}_W(g_{i_1}, g' \cdot z) \circ \widehat{\alpha}_W(g', z) = \widehat{\alpha}_W(g_{i_1}\, g', z) = \widehat{\alpha}_1(g, z). \end{aligned} \tag{45}$$

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

As the values of $\widehat{\alpha}_W(g,\, y) \in H_1'$ and $\widehat{\alpha}_W(g,\, z) \in H_1'$ are defined using the identity (37), the identity (45) holds for all $y,\, z \in U_1$ as the closure $\overline{\mathcal{Z} \cap U_1} = U_1$.

Thus, the calculation (45) shows that the value of $\widehat{\alpha}_1(g,\, y) \in H_1'$ does not depend on the choice of $y \in U_1$. We are given that $\Phi(g)(y) \in U_1$ for all $g \in G_1$, as U_1 is adapted to the action Φ. Thus, for $g,\, g' \in G_1$ and $y \in U_1$, we have $z = g' \cdot y \in U_1$ and so

$$\widehat{\alpha}_1(g\; g',\, y) = \widehat{\alpha}_1(g,\, z)\; \widehat{\alpha}_1(g',\, y) = \widehat{\alpha}_1(g,\, y)\; \widehat{\alpha}_1(g',\, y). \tag{46}$$

Fix $y \in U_1$ and define $\widehat{\theta}_1 \colon G_1 \to H_1'$ by setting $\widehat{\theta}_1(g) = \widehat{\alpha}_1(g,\, y)$; then (46) says that $\widehat{\theta}_1$ is a group homomorphism. Then (44) is just a restatement of the identity (37). □

Recall that $H_1 = \Phi_1(G_1) \subset \mathrm{Homeo}(U_1)$ and $H_1' = \Psi(G'_{V_1}) \subset \mathrm{Homeo}(V_1)$.

COROLLARY 6.13. *$\widehat{\theta}_1$ induces a homomorphism $\theta_1 \colon H_1 \to H_1'$.*

Proof. Suppose that $g,\, g' \in G_1$ satisfy $\Phi(g)|U_1 = \Phi(g')|U_1 \in H_1 \subset \mathrm{Homeo}(U_1)$. Then, by the defining identity (37), for $y \in \mathcal{Z} \cap U_1$, the action of H_1' is free on the orbit of y and hence $\widehat{\theta}_1(g) = \widehat{\alpha}_1(g,\, y) = \widehat{\alpha}_1(g',\, y) = \widehat{\theta}_1(g') \in H_1'$. Thus, $\widehat{\theta}_1$ induces a homomorphism $\theta_1 \colon H_1 \to H_1'$. □

6.4. *The induced map.* Recall that we are assuming the hypotheses of Theorem 6.10, so that $h \colon \mathfrak{X} \to \mathfrak{X}'$ is a continuous orbit equivalence between Cantor actions $(\mathfrak{X},\, G,\, \Phi)$ and $(\mathfrak{X}',\, G',\, \Psi)$, that G is finitely generated, and that $(\mathfrak{X}',\, G',\, \Psi)$ is stable.

The conclusion of Corollary 6.13 states that the identity (37), which defines the orbit equivalence between actions $(\mathfrak{X},\, G,\, \Phi)$ and $(\mathfrak{X},\, G',\, \Psi)$, induces a homomorphism $\theta_1 \colon H_1 \to H_1'$, but does not show that θ_1 is an isomorphism. We next show that the image of θ_1 is 'large enough' in the profinite topology defined by adapted bases for the actions, so that θ_1 induces a map between well-chosen odometer models for the actions Φ and Ψ. The idea is to recursively construct adapted neighborhood bases for these actions so that we can recursively apply the techniques of the proof of Proposition 6.12. First, set $U_0 = V_0 = \mathfrak{X}$ and let $U_1 \subset V_1$ be as in the proofs of Lemma 6.11 and Proposition 6.12. Then recursively choose:

(1) $V_{\ell+1}$ adapted to Ψ with $x \in V_{\ell+1} \subset U_\ell$ for $\ell \geq 1$;
(2) U_ℓ adapted to Φ with $x \in U_\ell \subset V_\ell$ for $\ell \geq 2$

such that $\bigcap_{\ell \geq 1}\; U_\ell = \bigcap_{\ell \geq 1}\; V_\ell = \{x\}$. We obtain adapted neighborhood bases $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ for $(\mathfrak{X},\, G,\, \Phi)$ at x, and $\mathcal{V} = \{V_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ for $(\mathfrak{X},\, G',\, \Psi)$ at x.

Let $G_\ell \equiv G_{U_\ell} \subset G$ for $\ell \geq 0$; then $\mathcal{G}_{\mathcal{U}} = \{G_\ell \mid \ell \geq 0\}$ is a group chain associated to the action Φ.

Let $G'_\ell \equiv G'_{V_\ell} \subset G'$ for $\ell \geq 0$; then $\mathcal{G}'_{\mathcal{V}} = \{G'_\ell \mid \ell \geq 0\}$ is a group chain associated to the action Ψ.

Set $H_\ell \equiv H_{U_\ell} = \Phi(G_\ell)|U_\ell \subset \mathrm{Homeo}(U_\ell)$ and $H'_\ell \equiv H'_{V_\ell} = \Psi(G'_\ell)|V_\ell \subset \mathrm{Homeo}(V_\ell)$.

Let $(U_1,\, G_1,\, \Phi_1)$ denote the restricted action $\Phi_1 \colon G_1 \times U_1 \to U_1$ of Φ. Let $\overline{\Phi}_1 \colon H_1 \times U_1 \to U_1$ be the induced action by $H_1 \equiv \Phi_1(G_1) \subset \mathrm{Homeo}(U_1)$.

Let $(V_1,\, G'_1,\, \Psi_1)$ denote the restricted action $\Psi_1 \colon G'_1 \times V_1 \to V_1$ of Ψ. Let $\overline{\Psi}_1 \colon H'_1 \times V_1 \to V_1$ be the induced action by $H'_1 \equiv \Psi_1(G'_1) \subset \mathrm{Homeo}(V_1)$.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

We next use the existence of the map β defined by the identity (38) to prove the following result, which is used to show that the map θ_1 induces a map between the odometer models for the induced Cantor actions $(U_1, H_1, \overline{\Phi}_1)$ and $(V_1, H'_1, \overline{\Psi}_1)$. Note that it is not assumed that the action of H_1 on U_1 is topologically free and as a consequence the choice of the map β in (38) need not be unique. Correspondingly, the map θ_1 need not be a monomorphism.

LEMMA 6.14. *For all $\ell \geq 1$, the subgroup $H'_\ell \subset H'_1$, defined as follows, satisfies*

$$H'_{\ell+1} \equiv \Psi_1(G'_{\ell+1}) \subset \widehat{\theta}_1(G_\ell) = \theta_1 \circ \Phi_1(G_\ell) \subset \Psi_1(G'_\ell) \equiv H'_\ell \subset H'_1. \tag{47}$$

Proof. Fix $\ell \geq 1$ and let $k \in G'_{\ell+1}$ and $y \in V_{\ell+1}$; then by the identity (38) there exist $g = \beta(k, y) \in G$ and an open set $V_{k,y} \subset \mathfrak{X}$ such that $\Phi(g)(z) = \Psi(k)(z)$ for all $z \in V_{k,y}$. Note that we are not assuming that the function β is a cocycle, just the fact that it exists. Then, by the definition of $\widehat{\alpha}_1 \colon G_1 \times U_1 \to H'_1$ as the restriction of $\widehat{\alpha}_W$, we have that $\widehat{\alpha}_1(g) = \Psi(k)$ when restricted to $V_{k,y}$.

Recall that the action of $H'_1 = \Psi_1(G'_{V_1})$ on V_1 is topologically free and H'_1 acts freely on the dense H'_1-invariant subset $\mathcal{Z} \subset V_1$. The subset $V_{k,y} \cap V_1$ is open in V_1 and contained in the closure of $\mathcal{Z} \cap V_{k,y} \cap V_1$; hence, the maps $\Phi(g)$ and $\Psi(k)$ agree on an open subset of V_1 and hence agree on V_1 since the homomorphism θ_1 does not depend on the choice of U_1.

It is given that $k \in G'_{\ell+1}$ and $y \in V_{\ell+1} \subset U_\ell$, so $\Psi(k)(y) \in V_{\ell+1} \subset U_\ell$ and thus $\Phi(g)(U_\ell) \cap U_\ell \neq \emptyset$; hence, $g \in G_\ell$. Thus, by (44), we have

$$\Psi(k)|V_1 = \Phi(g)|V_1 = \widehat{\theta}_1(g)|V_1 \in \widehat{\theta}_1(G_\ell), \tag{48}$$

which establishes the first inclusion in (47).

To show the second inclusion in (47), let $g \in G_\ell$; then note that $\Phi(g)(U_\ell) \cap U_\ell \neq \emptyset$. Then by (44) we have $\widehat{\theta}_1(g) \cdot U_\ell \cap U_\ell \neq \emptyset$ and hence $\widehat{\theta}_1(g) \cdot V_\ell \cap V_\ell \neq \emptyset$ and so $\widehat{\theta}_1(g) \in H'_\ell$. □

Next, we examine the relation between the discriminant groups for the Cantor actions $(\mathfrak{X}, \mathfrak{G}(\Phi), \widehat{\Phi})$ and $(U_k, H_k, \overline{\Phi}_k)$, for $k \geq 1$, where $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ is the adapted neighborhood basis at x chosen above for $(\mathfrak{X}, G, \Phi)$. Let $\widehat{G}_\infty$ be the profinite group associated to the group chain $\mathcal{G}_\mathcal{U}$ and let $\widehat{\Phi}_x \colon \widehat{G}_\infty \times X_\infty \to X_\infty$ be the associated action. We show that the wild property for this action is preserved by restriction.

By Theorem 3.2, there is a topological isomorphism $\widehat{\Theta} \colon \mathfrak{G}(\Phi) \to \widehat{G}_\infty$ which conjugates the action $\widehat{\Phi}$ to the action $\widehat{\Phi}_x$, so we may make all calculations in terms of the action $\widehat{\Phi}_x \colon \widehat{G}_\infty \times X_\infty \to X_\infty$.

Recall that for $\ell \geq 1$, the clopen set $\widehat{U}_\ell \equiv \widehat{G}_\ell / \mathcal{D}_x \subset \widehat{G}_\infty / \mathcal{D}_x \cong X_\infty$ is adapted to the action $\widehat{\Phi}_x$ with stabilizer $\widehat{G}_\ell \subset \widehat{G}_\infty$ defined by (17). Recall that $\Phi_\ell \colon G_\ell \times \widehat{U}_\ell \to \widehat{U}_\ell$ denotes the restriction of the action Φ_x to $\widehat{U}_\ell$. Now, for $\ell' \geq \ell \geq 0$, define

$$H_{\ell,\ell'} \equiv \Phi_\ell(G_{\ell'}) \subset \Phi_\ell(G_\ell) \subset \mathrm{Homeo}(\widehat{U}_\ell) \tag{49}$$

and let $\widehat{H}_{\ell,\ell'}$ denote the closure of $H_{\ell,\ell'}$ in the uniform topology on $\mathrm{Homeo}(\widehat{U}_\ell)$. Note that $H_{\ell,\ell} = H_\ell$ and $\widehat{H}_{\ell,\ell} = \widehat{H}_\ell$ as defined previously. Then the restricted action Φ_ℓ induces a

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

homomorphism $\widehat{\Phi}_\ell \colon \widehat{G}_\ell \to \widehat{H}_\ell$, which is a surjection, as seen using the same method of proof as for Lemma 2.5.

Recall that for all $\ell \geq 0$, $\mathcal{D}_x \subset \widehat{G}_\ell$, so that there is a well-defined isotropy action $\rho_\ell \colon \mathcal{D}_x \to \mathrm{Homeo}(\widehat{U}_\ell)$. Then by (22) we have $K_\ell = \ker\{\rho_\ell \colon \mathcal{D}_x \to \mathrm{Homeo}(\widehat{U}_\ell)\}$. Thus, K_ℓ is a normal subgroup of $\mathcal{D}_x$ and, in particular, K_ℓ is normal in $K_{\ell'}$ for $\ell' > \ell$.

For $k \geq 0$, by Lemma 3.11, we have $\mathcal{D}_x = \bigcap_{\ell \geq 0} \widehat{G}_\ell = \bigcap_{\ell \geq k} \widehat{G}_\ell$.

For $k \geq 0$, recall that $H_k = \Phi_k(G_k) \subset \mathrm{Homeo}(\widehat{U}_k)$ and let $\overline{\Phi}_k \colon H_k \times \widehat{U}_k \to \widehat{U}_k$ denote the quotient action induced from the restricted action $(\widehat{U}_k, G_k, \Phi_k)$. Then we have the following.

PROPOSITION 6.15. *For $k \geq 0$, the stabilizer direct limit group $\Upsilon_s(\overline{\Phi}_k)$ for the induced action $(\widehat{U}_k, H_k, \overline{\Phi}_k)$ is represented by the quotient group chain $\{K_\ell/K_k \mid \ell \geq k\}$.*

Proof. First, observe that $\mathcal{H}_k \equiv \{H_{k,\ell} \mid \ell \geq k\}$ is an adapted group chain for the action $(\widehat{U}_k, H_k, \overline{\Phi}_k)$ at $x \in \widehat{U}_k$. Recall that $\widehat{H}_k \subset \mathrm{Homeo}(\widehat{U}_k)$ is the closure of the group H_k.

Let $A_k = \ker\{\widehat{\Phi}_k \colon \widehat{G}_k \to \widehat{H}_k\}$, which is a closed normal subgroup of $\widehat{G}_k$. Then for $\ell > k$ we have $\widehat{H}_{k,\ell} \cong \widehat{G}_\ell/(\widehat{G}_\ell \cap A_k)$ and the discriminant $\overline{\mathcal{D}}_k$ of the action $(\widehat{U}_k, H_k, \overline{\Phi}_k)$ is given by $\overline{\mathcal{D}}_k = \bigcap_{\ell \geq k} \widehat{H}_{k,\ell}$.

Suppose that $\widehat{h} \in \mathcal{D}_x$; then the image $\overline{\Phi}_k(\widehat{h}) \in \widehat{H}_{k,\ell}$ for all $\ell \geq k$ and hence $\widehat{\Phi}_k(\widehat{h}) \in \overline{\mathcal{D}}_k$. Moreover, as $\widehat{G}_k$ is sequentially compact, the restricted map $\widehat{\Phi}_k \colon \mathcal{D}_x \to \overline{\mathcal{D}}_k$ is onto with kernel the closed normal subgroup $K_k \equiv \mathcal{D}_x \cap A_k$. Thus, there is an isomorphism $\widehat{\Phi}_k \colon \mathcal{D}_x/K_k \cong \overline{\mathcal{D}}_k$.

For each $\ell > k$, let $\overline{\rho}_{k,\ell} \colon \overline{\mathcal{D}}_k \to \widehat{H}_\ell$ be the restriction of the map $\overline{\Phi}_k \colon \widehat{H}_{k,\ell} \to \widehat{H}_\ell \subset \mathrm{Homeo}(\widehat{U}_\ell)$ and let $\overline{K}_{k,\ell} = \ker\{\overline{\Phi}_k\}$. Then the stabilizer group $\Upsilon_s(\overline{\Phi}_k)$ is represented by the chain $\{\overline{K}_{k,\ell} \mid \ell \geq k\}$.

Let $\widehat{k} \in \overline{K}_{k,\ell}$ and so $\widehat{k}$ acts as the identity on $\widehat{U}_\ell$. Choose $\widehat{h} \in \mathcal{D}_x$ with $\widehat{\Phi}_k(\widehat{h}) = \widehat{k}$; then $\widehat{h}$ also acts as the identity on $\widehat{U}_\ell$ and hence $\widehat{h} \in K_\ell$. Moreover, if $\widehat{h} \in K_\ell$ and $\widehat{\Phi}_k(\widehat{h}) = \mathrm{id}$, then $\widehat{h} \in K_\ell \cap A_k = K_k$. It follows that $\overline{K}_{k,\ell} \cong K_\ell/K_k$ for $\ell \geq k$, which yields the claim. □

COROLLARY 6.16. *For $k \geq 1$, $(\mathfrak{X}, G, \Phi)$ is wild if and only if $(\widehat{U}_k, H_k, \overline{\Phi}_k)$ is wild.*

Proof. Let $\ell \geq 0$. Then $(\mathfrak{X}, G, \Phi)$ is a wild action if and only if the chain of subgroups $\{K_\ell \mid \ell \geq 0\}$ as above is unbounded, if and only if the chain of quotient groups $\{K_\ell/K_k \mid \ell \geq k\}$ is unbounded. Then, by the proof of Proposition 6.15 above, we have $\overline{K}_{k,\ell} \cong K_\ell/K_k$, so the action is wild if and only if the chain $\{\overline{K}_{k,\ell} \mid \ell \geq k\}$ is unbounded, if and only if the stabilizer group $\Upsilon_s(\overline{\Phi}_k)$ is unbounded. □

We can now complete the proof of Theorem 6.10. We are given the Cantor actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}, G', \Psi)$, where G and G' are finitely generated groups, Φ and Ψ have the same orbits, and satisfy the conditions (37) and (38). We assume that the action $(\mathfrak{X}, G, \Phi)$ is wild and derive a contradiction to the assumption that the action $(\mathfrak{X}', G', \Psi)$ is stable.

By Proposition 6.12, there are a clopen set $U_1 \subset \mathfrak{X}$ adapted to the action Φ, a clopen set $V_1 \subset \mathfrak{X}$ adapted to the action Ψ, and a homomorphism $\theta_1 \colon H_1 \to H_1'$ which satisfies the condition (44).

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ be an adapted neighborhood basis for $(\mathfrak{X}, G, \Phi)$ at x and let $\mathcal{V} = \{V_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ be an adapted neighborhood basis for $(\mathfrak{X}, G', \Psi)$ at x, where U_1 and V_1 are chosen so that Proposition 6.12 holds, and the chains $\mathcal{U}$ and $\mathcal{V}$ chosen as above so that the conclusions of Lemma 6.14 hold. We use the notation of the proofs of these results in the following as well.

Let $\mathcal{G}_\mathcal{U} = \{G_\ell \mid \ell \geq 0\}$ be the group chain associated to $\mathcal{U}$ and let $\mathcal{H}_\mathcal{U} = \{H_{1,\ell} \mid \ell \geq 1\}$ be the group chain for the induced action $(U_1, H_1, \overline{\Phi}_1)$.

Let $\widehat{G}_\infty$ be the profinite group associated to the group chain $\mathcal{G}_\mathcal{U}$ with discriminant group $\mathcal{D}_x \subset \widehat{G}_\infty$.

Let $\widehat{H}_\infty$ be the profinite group associated to the group chain $\mathcal{H}_\mathcal{U}$ with discriminant group $\overline{\mathcal{D}}_1 \subset \widehat{H}_\infty$.

Let $\mathcal{K} = \{K_\ell \mid \ell \geq 0\}$ be the group chain representing $\Upsilon_s^x(\Phi)$ and let $\overline{\mathcal{K}} \cong \{\overline{K}_{1,\ell} \cong K_\ell/K_1 \mid \ell \geq 1\}$ be the group chain representing $\Upsilon_s^x(\overline{\Phi}_1)$. Then, by Corollary 6.16, the chain $\overline{\mathcal{K}}$ is unbounded.

Next, let $\mathcal{G}'_\mathcal{V} = \{G'_\ell \mid \ell \geq 0\}$ be the group chain associated to $\mathcal{V}$ and let $\mathcal{H}'_\mathcal{V} = \{H'_{1,\ell} \mid \ell \geq 1\}$ be the group chain for the induced action $(V_1, H'_1, \overline{\Psi}_1)$. Note that $H'_{1,1} = H'_1$.

Let $\widehat{G}'_\infty$ be the profinite group associated to the group chain $\mathcal{G}'_\mathcal{V}$ with discriminant group $\mathcal{D}'_x \subset \widehat{G}'_\infty$.

Let $\widehat{H}'_\infty$ be the profinite group associated to the group chain $\mathcal{H}'_\mathcal{V}$ with discriminant group $\overline{\mathcal{D}}'_1 \subset \widehat{H}'_\infty$.

Let $\mathcal{K}' = \{K'_\ell \mid \ell \geq 0\}$ be the group chain representing $\Upsilon_s^x(\Psi)$ and let $\overline{\mathcal{K}}' = \{\overline{K}'_{1,\ell} \cong K'_\ell/K'_1 \mid \ell \geq 1\}$ be the group chain representing $\Upsilon_s^x(\overline{\Psi}_1)$.

The homomorphism $\theta_1 \colon H_1 \to H'_1$ satisfies the condition (47), which implies that

$$H'_{1,\ell+1} \subset \theta_1(H_{1,\ell}) \subset H'_{1,\ell}, \tag{50}$$

so it induces a map $\widehat{\tau}_1 \colon \widehat{H}_\mathcal{U} \to \widehat{H}'_\mathcal{V}$. Then (50) and Lemma 3.11 imply that $\widehat{\tau}_1$ restricts to a surjection $\widehat{\tau}_1 \colon \overline{\mathcal{D}}_1 \to \overline{\mathcal{D}}'_1$ which satisfies $\widehat{\tau}_1(\overline{K}_\ell) = \overline{K}'_\ell$. As a result, by Theorem 3.7, we obtain maps

$$\mathfrak{X} \supset U_1 \cong \widehat{H}_\infty/\overline{\mathcal{D}}_1 \xrightarrow{\widehat{\tau}_*} \widehat{H}'_\infty/\overline{\mathcal{D}}'_1 \cong V_1 \subset \mathfrak{X}. \tag{51}$$

That is, $\widehat{\tau}_* \colon U_1 \to V_1$ is the map induced from the homomorphism $\widehat{\tau}_1$ which is H_1-equivariant.

We claim that if $\widehat{k} \in \overline{K}_{\ell+1}$ and $\widehat{k} \notin \overline{K}_\ell$, then $\widehat{\tau}_1(\widehat{k}) \notin \overline{K}'_{\ell-1}$ and hence $\overline{K}'_{\ell-1} \neq \overline{K}'_{\ell+1}$. Let $\widehat{h} \in \mathcal{D}_x$ be chosen so that it has image $\widehat{k} \in \overline{\mathcal{D}}_1$. Then $\widehat{k} \notin \overline{K}_\ell$ implies that the restricted action of $\Phi_\ell(\widehat{h}) \colon \widehat{U}_\ell \to \widehat{U}_\ell$ is not the identity, while the restricted action $\Phi_{\ell+1}(\widehat{h}) \colon \widehat{U}_{\ell+1} \to \widehat{U}_{\ell+1}$ is the identity.

By the definition of the conjugating map θ_1 in Proposition 6.12, and more precisely the relation (44), the map $\theta_1(\Phi_1(\widehat{h})) \in \mathrm{Homeo}(\widehat{V}_1)$ restricts to $\Phi_\ell(\widehat{h})$ on $\widehat{U}_\ell$ and so is not the identity on $\widehat{U}_\ell$. Since $\widehat{U}_\ell \subset \widehat{V}_{\ell-1}$ by construction, this implies that $\theta_1(\Phi_2(\widehat{h}))|\widehat{V}_{\ell-1} \in \mathrm{Homeo}(\widehat{V}_{\ell-1})$ is also not the identity and hence $\widehat{\tau}_1(\widehat{k}) \notin \overline{K}'_{\ell-1}$.

Thus, if the chain $\mathcal{K} = \{K_\ell \mid \ell \geq 0\}$ is unbounded, then the chain $\overline{\mathcal{K}}' = \{\overline{K}'_\ell \mid \ell \geq 1\}$ is unbounded and hence the action is $(V_1, H'_1, \overline{\Psi}_1)$ is wild. This contradicts the choice of V_1 and completes the proof of Theorem 6.10. □

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Remark 6.17. Here is a simple example that illustrates one of the main ideas in the above proof of Theorem 6.10. Let $\Phi\colon \mathbb{Z}^n \times \mathfrak{X} \to \mathfrak{X}$ be a faithful action for $n \geq 2$. So, $(\mathfrak{X}, \mathbb{Z}^n, \Phi)$ is an odometer action of the abelian group $\mathbb{Z}^n$ and is a free action. Let $\theta \in \mathbf{SL}(\mathbf{n}, \mathbb{Z})$ be a non-identity matrix, which acts via conjugation on $\mathbb{Z}^n$. Let $\Psi = \Phi^\theta$ be the action of $\mathbb{Z}^n$ on $\mathfrak{X}$ as defined by (6.3), which is again free. Then the identity map is a continuous orbit equivalence between $(\mathfrak{X}, \mathbb{Z}^n, \Phi)$ and $(\mathfrak{X}, \mathbb{Z}^n, \Psi)$. The map θ_1 obtained in Proposition 6.12 is then just the restriction of θ to a subgroup of finite index in $\mathbb{Z}^n$. For a choice of group chains adapted to the actions, the map θ_1 need not preserve the subgroups in these chains, but the map $\widehat{\tau}_*$ in (51) is the map on the inverse limit space $\mathfrak{X}$ induced by the action of Θ. The proof above is a generalization of this example to the more general (and more complicated) non-abelian setting.

Finally, we give an application of Theorem 6.10 and the techniques used in its proof to obtain an extension of Theorem 3.3 by Cortez and Medynets in [**12**] and [**35**, Theorem 1.5].

THEOREM 6.18. *Let G and G' be finitely generated groups and suppose that the Cantor action $(\mathfrak{X}', G', \Psi)$ is stable. Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action which is continuously orbit equivalent to $(\mathfrak{X}', G', \Psi)$; then the actions are return equivalent.*

Proof. By Theorem 6.10, the action $(\mathfrak{X}, G, \Phi)$ is stable. Thus, in the proof of Theorem 6.10, we can choose the clopen set $U_1 \subset \mathfrak{X}$ so that the restricted action $\overline{\Phi}_1\colon H_1 \times U_1 \to U_1$ is topologically free.

Then, in the proof of Lemma 6.14, the choice of $g = \beta(k, y) \in G$ such that $\Phi(g)(z) = \Psi(k)(z)$ is unique on U_1, so the map θ_1 is injective. That is, the homomorphism $\theta_1\colon H_1 \to H'_{V_1}$ is an isomorphism onto its image. Set $H''_1 = \theta_1(H_1) \subset H'_{V_1}$.

Then (50) implies that the group chain $\mathcal{H}'_{\mathcal{V}} = \{H'_{1,\ell} \mid \ell \geq 1\}$ for the induced action $(V_1, H'_1, \overline{\Psi}_1)$ is equivalent, in the sense of Definition 3.5, to the group chain $\mathcal{H}'' = \{H''_\ell \mid \ell \geq 1\}$, where $H''_1 = H'_{V_1}$ and $H''_\ell = \theta_1(H_{1,\ell})$ for $\ell \geq 1$. Let $W'_1 = \widehat{\tau}_*(U_1) \subset V'_1$ be the clopen subset corresponding to the truncated chain $\{H'_{1,\ell} \mid \ell \geq 1\}$, where $\widehat{\tau}_*$ is the map in (51). Let $h_1\colon U_1 \to W'_1$ denote the map $\widehat{\tau}_*$ onto its image, which then conjugates the induced action $(U_1, H_1, \overline{\Phi}_1)$ with the Cantor action $(W'_1, H''_\ell, \overline{\Psi}_1)$ which is induced from $(\mathfrak{X}', G', \Psi)$. Thus, the actions Φ and Ψ are return equivalent. □

7. *Wild actions and non-Hausdorff elements*

In this section, we consider Cantor actions $(\mathfrak{X}, G, \Phi)$ for which the stabilizer direct limit group $\Upsilon_s(\Phi)$ is unbounded and derive some of their dynamical properties. Of special interest will be the existence of *non-Hausdorff elements* for the action, which are defined in §7.1.

The notion of a non-Hausdorff element is local in $\mathfrak{X}$ and our first result is a generalization of Proposition 3.1 by Renault in [**48**]. Proposition 7.2 below implies that a Cantor action with a non-Hausdorff element has unbounded stabilizer group $\Upsilon_s(\Phi)$. Then, in Theorem 7.5, we show that a Cantor action with a non-Hausdorff element must be dynamically wild. The converse implication, which is to give criteria for the existence of non-Hausdorff elements, is a much more subtle problem.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

7.1. *The germinal groupoid.* The reduced C^*-algebra $C_r^*(\mathfrak{X}, G, \Phi)$ associated to a Cantor action $(\mathfrak{X}, G, \Phi)$ is an invariant of the continuous orbit equivalence class of the action. The study of the K-theory of $C_r^*(\mathfrak{X}, G, \Phi)$ offers another approach to the classification of Cantor actions, as used for example in the work [**30**]. This C^*-algebra can also be constructed using the germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \Phi)$ associated to the action, as discussed for example by Renault in [**46**]. Then one can ask how the properties of the C^*-algebra are related to the dynamical properties of the action, as discussed by Renault in [**47**].

In the work [**48**], Renault assumed that the Cantor action is topologically free, and thus the germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is a Hausdorff topological space, in order to avoid technical difficulties that arise otherwise. In fact, as remarked in Corollary 1.8, there are also wild actions for which $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is still Hausdorff. Thus, the case where $\mathcal{G}(\mathfrak{X}, G, \Phi)$ has non-Hausdorff topology may be considered to be exceptional and the fact that the topology is non-Hausdorff has implications for the algebraic structure of $C_r^*(\mathfrak{X}, G, \Phi)$, as discussed in [**8**, **24**].

Recall first the definition of the germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \Phi)$. For $g_1, g_2 \in G$, we say that $\Phi(g_1)$ and $\Phi(g_2)$ are *germinally equivalent* at $x \in \mathfrak{X}$ if $\Phi(g_1)(x) = \Phi(g_2)(x)$ and there exists an open neighborhood $x \in U \subset \mathfrak{X}$ such that the restrictions agree, $\Phi(g_1)|U = \Phi(g_2)|U$. We then write $\Phi(g_1) \sim_x \Phi(g_2)$. For $g \in G$, denote the equivalence class of $\Phi(g)$ at x by $[g]_x$. The collection of germs $\mathcal{G}(\mathfrak{X}, G, \Phi) = \{[g]_x \mid g \in G, x \in \mathfrak{X}\}$ is given the sheaf topology and forms an *étale groupoid* modeled on $\mathfrak{X}$. We recall the following formulation of the Hausdorff property that was given by Winkelnkemper.

PROPOSITION 7.1. [**51**, Proposition 2.1] *The germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is Hausdorff at $[g]_x$ if and only if, for all $[g']_x \in \mathcal{G}(\mathfrak{X}, G, \Phi)$ with $g \cdot x = g' \cdot x$, if there exists a sequence $\{x_i\} \subset \mathfrak{X}$ which converges to x such that $[g]_{x_i} = [g']_{x_i}$ for all i, then $[g]_x = [g']_x$.*

Winkelnkemper showed in [**51**, Proposition 2.3] that for a smooth foliation $\mathcal{F}$ of a connected manifold M for which the associated holonomy pseudogroup $\mathcal{G}_\mathcal{F}$ is generated by real analytic maps, then $\mathcal{G}_\mathcal{F}$ is a Hausdorff space. For Cantor actions, an analogous result holds for the LQA property.

PROPOSITION 7.2. *If an action $(\mathfrak{X}, G, \Phi)$ is locally quasi-analytic, then $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is Hausdorff.*

Proof. Assume that $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is not Hausdorff. Then there exist $g \in G$ and $x \in \mathfrak{X}$ such that $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is non-Hausdorff at $[g]_x$. By Proposition 7.1, there exist $[g']_x \in \mathcal{G}(\mathfrak{X}, G, \Phi)$ with $g \cdot x = g' \cdot x$ and a sequence $\{x_i\} \subset \mathfrak{X}$ which converges to x such that $[g]_{x_i} = [g']_{x_i}$ for all i, but $[g]_x \neq [g']_x$. Let $g'' = g^{-1}g' \in G$; then $g'' \cdot x = x$ and $[g'']_{x_i} = [\mathrm{id}]_{x_i}$ for all i, but $[g'']_x \neq [\mathrm{id}]_x$.

The fact that the germ $[g'']_x \neq [\mathrm{id}]_x$ means that the action of $\Phi(g'')$ is not the identity in any open neighborhood of x. On the other hand, the germinal equalities $[g]_{x_i} = [g']_{x_i}$ for $i \geq 1$ imply that there exists a sequence of open sets $x_i \in W_i \subset \mathfrak{X}$ for which the restriction of $\Phi(g'')$ to W_i is the identity. Hence, there does not exist $\varepsilon > 0$ such that $\Phi(g)|U$ is quasi-analytic for any open neighborhood $x \in U$ with $\mathrm{diam}\,(U) < \varepsilon$. Thus, the action $(\mathfrak{X}, G, \Phi)$ is not locally quasi-analytic. □

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

7.2. *Non-Hausdorff elements in the closure.* Recall that by Theorem 5.3, a Cantor action $(\mathfrak{X}, G, \Phi)$ is a locally completely quasi-analytic (LCQA) action if and only if the stabilizer group $\Upsilon_s(\Phi)$ is bounded. Based on Proposition 7.1, we introduce the following notion.

Definition 7.3. Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action and let $\mathfrak{G}(\Phi) \subset \mathrm{Homeo}(\mathfrak{X})$ denote the closure of the action. Then $\widehat{g} \in \mathfrak{G}(\Phi)$ is a *non-Hausdorff element* at $x \in \mathfrak{X}$ if:

(1) $\widehat{g} \cdot x = x$; and there exists

(2) a sequence $\{x_i \mid i \geq 1\} \subset \mathfrak{X}$ converging to x; and

(3) clopen subsets $x_i \in W_i \subset \mathfrak{X}$

such that for any clopen subset $x \in U \subset \mathfrak{X}$, the restriction of $\widehat{g}$ to U is not the identity, while for all $i \geq 1$, we have $\widehat{g} \cdot W_i = W_i$ and the restriction of $\widehat{g}$ to W_i is the identity.

We then have the following consequence of Proposition 7.2, which gives a connection between the unbounded property for $\Upsilon_s(\Phi)$ and the dynamics of the action.

COROLLARY 7.4. *Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action. Suppose that $\mathfrak{G}(\Phi)$ contains a non-Hausdorff element; then the stabilizer limit group $\Upsilon_s(\Phi)$ is unbounded. That is, the action is wild.*

Proof. Assume that $\mathfrak{G}(\Phi)$ contains a non-Hausdorff element; then, by Proposition 7.2 applied to the action of $\mathfrak{G}(\Phi)$, the action $(\mathfrak{X}, G, \Phi)$ is not locally completely quasi-analytic. Then, by Theorem 5.3, $\Upsilon_s(\Phi)$ is unbounded. □

7.3. *Non-Hausdorff dynamics.* We next give in Theorem 7.5 a sharper version of the conclusion of Corollary 7.4. As an application, Corollary 7.6 implies that the examples of wild actions constructed in [**36**, §9] cannot have non-Hausdorff elements in $\mathfrak{G}(\Phi)$. The arboreal constructions in §8 yield examples of Cantor actions with non-Hausdorff elements in $\Phi(G)$.

The difficulty in constructing non-Hausdorff elements can be seen from the following considerations. The hypothesis that $\Upsilon_s(\Phi)$ is unbounded implies that for $x \in \mathfrak{X}$, there exist a sequence of adapted clopen neighborhoods of x and elements in $\mathfrak{G}(\Phi)_x$ which do not act as the identity on this open neighborhood, but do act as the identity on some smaller clopen neighborhoods. On the other hand, the non-Hausdorff property asserts that there is some fixed element $\widehat{h} \in \mathfrak{G}(\Phi)_x$ so that for any adapted neighborhood with $x \in U$, the restricted action of $\Phi(\widehat{h})$ is not the identity on U, but the action is the identity on some clopen subset $V \subset U$. The distinction is that the non-Hausdorff condition in Proposition 7.1 is a statement about the local action of a fixed element $\widehat{h}$, while the wild hypothesis is a statement about the behavior of a sequence of elements in $\mathfrak{G}(\Phi)$. We use the interplay of these two notions in the proof of Theorem 7.5.

Recall from Definition 4.10 that the limit group $\Upsilon_s(\Phi)$ for a Cantor action $(\mathfrak{X}, G, \Phi)$ has finite type if it is represented by an increasing chain $\mathcal{K} = \{K_\ell \mid \ell \geq 0\}$, where each K_ℓ is a finite group. Recall from Definition 4.18 that a Cantor action $(\mathfrak{X}, G, \Phi)$ is dynamically wild if the stabilizer group $\Upsilon_s(\Phi)$ is unbounded and there is a proper inclusion $\Upsilon_c(\Phi) \subset \Upsilon_s(\Phi)$.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

THEOREM 7.5. *Let* $(\mathfrak{X}, G, \Phi)$ *be a Cantor action with a non-Hausdorff element* $\widehat{h} \in \mathfrak{G}(\Phi)_x$*; then* $\Upsilon_s(\Phi)$ *does not have finite type and the action is dynamically wild.*

Proof. Let $x \in \mathfrak{X}$ with $\widehat{g} \in \mathfrak{G}(\Phi)_x$ a non-Hausdorff element at x. Then, by Definition 7.3, for any adapted clopen set U with $x \in U$, the restriction of $\widehat{g}$ to U is not the identity. In addition, there exist a sequence $\{x_i \mid i \geq 1\} \subset \mathfrak{X}$ of distinct points converging to x, with $\widehat{g} \cdot x_i = x_i$ for all $i \geq 1$, and clopen subsets $x_i \in W_i \subset \mathfrak{X}$ such that $\widehat{g} \cdot W_i = W_i$ and the restriction of $\widehat{g}$ to W_i is the identity. It follows that $x \notin W_i$ for all $i \geq 1$.

Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ be an adapted neighborhood basis for the action Φ at x.

Let $\mathcal{G}^x_{\mathcal{U}} = \{G_\ell \mid i \geq 0\}$ be the group chain associated to $\mathcal{U}$. Let $\{K_\ell \mid \ell \geq 0\}$ be its chain of stabilizer groups and $\{Z_\ell \mid \ell \geq 0\}$ be its chain of centralizer groups as in Definition 4.14.

Recall from Corollary 3.4 that the map $\widehat{\Theta}$ in Theorem 3.2 induces a homeomorphism of G-spaces $\Theta_x \colon \mathfrak{G}(\Phi)/\mathfrak{G}(\Phi)_x \cong \mathfrak{X} \to \widehat{G}_\infty/\mathcal{D}_x \cong X_\infty$.

For each $i \geq 1$, choose $\widehat{g}_i \in \widehat{G}_\infty$ such that $x_i = \widehat{g}_i \cdot x$.

As $x_i \in W_i$ and $x \notin W_i$, there exists $\ell_i > i$ such that $U_{\ell_i} \cap W_i = \emptyset$ and $\widehat{g}_i \cdot U_{\ell_i} \subset W_i$; hence, $\widehat{g}_i \cdot U_{\ell_i} \cap U_{\ell_i} = \emptyset$. Moreover, as x_i limits to x there exists $j > i$ such that $x_j \in U_{\ell_i}$ and $x_j \notin W_i$.

As $\{U_\ell \mid \ell \geq 0\}$ is a neighborhood system about x, by passing to subsequences chosen recursively, we can assume that for $\ell \geq 1$, we have

$$U_\ell \cap W_\ell = \emptyset, \quad x_\ell = \widehat{g}_\ell \cdot x, \quad \widehat{g}_\ell \cdot U_\ell \subset W_\ell, \quad x_{\ell+1} \notin W_\ell. \tag{52}$$

Set $\widehat{h}_\ell = \widehat{g}_\ell^{-1}\ \widehat{g}\ \widehat{g}_\ell$. Then $\widehat{g} \cdot x_\ell = x_\ell$ implies that $(\widehat{g}_\ell^{-1}\ \widehat{g}\ \widehat{g}_\ell) \cdot x = x$ and hence $\widehat{h}_\ell \cdot x = x$, so $\widehat{h}_\ell \in \mathcal{D}_x$.

Since $\widehat{g}$ acts as the identity on W_ℓ and $\widehat{g}_\ell \cdot U_\ell \subset W_\ell$, we obtain that $\widehat{h}_\ell \in K_\ell$.

Set $y_\ell = \widehat{g}_\ell^{-1} \cdot x$ and observe that $\widehat{h}_\ell \cdot y_\ell = y_\ell$ and that $y_\ell \notin U_{\ell+1}$ as $\widehat{g}_\ell \cdot U_{\ell+1} \cap U_{\ell+1} = \emptyset$.

We claim that this implies that each K_ℓ is not a finite group and hence $\Upsilon_s(\Phi)$ does not have finite type.

For $\ell \geq 0$, recall from (17) that $\widehat{G}_\ell = \widehat{C}_\ell\ \mathcal{D}_x$ is a clopen subset of $\widehat{G}_\infty$. Then Θ_x identifies the clopen set $U_\ell \subset \mathfrak{X}$ with $\widehat{G}_\ell/\mathcal{D}_x \subset \widehat{G}_\infty$. Thus, $\widehat{C}_\ell$ acts transitively on the quotient space

$$\widehat{U}_\ell = \widehat{G}_\ell/\mathcal{D}_x = (\widehat{C}_\ell\ \mathcal{D}_x)/\mathcal{D}_x = \widehat{C}_\ell/(\widehat{C}_\ell \cap \mathcal{D}_x).$$

It is given that $\widehat{C}_\ell \subset \widehat{G}_\ell$ is a normal subgroup of $\widehat{G}_\infty$ and hence for any $g \in G$ we have $g\ \widehat{C}_\ell\ g^{-1} = \widehat{C}_\ell$; thus, $\widehat{C}_\ell$ also acts transitively on the clopen set $\widehat{U}^g_\ell \equiv g \cdot \widehat{U}_\ell$.

For each $\widehat{k} \in \widehat{C}_\ell$, define the conjugate element $\widehat{h}^{\widehat{k}}_\ell = \widehat{k}\ \widehat{h}_\ell\ \widehat{k}^{-1}$. Since $\widehat{h}_\ell$ acts as the identity on U_ℓ, and $\widehat{k} \cdot U_\ell = U_\ell$, we have $\widehat{h}^{\widehat{k}}_\ell \in K_\ell$.

Finally, recall that $\widehat{g}$ is not the identity on any clopen neighborhood of x, and $\widehat{h}_\ell = \widehat{g}_\ell^{-1}\ \widehat{g}\ \widehat{g}_\ell$ satisfies $\widehat{h}_\ell \cdot y_\ell = y_\ell$ and is not the identity map in any open neighborhood of y_ℓ. That is, the map $\widehat{h}_\ell$ has a non-trivial germ at y_ℓ. Thus, each conjugate map $\widehat{h}^{\widehat{k}}_\ell$ has a fixed point at $\widehat{y} = \widehat{k} \cdot y_\ell \in U_\ell^{\widehat{g}_\ell^{-1}}$ and acts non-trivially on any neighborhood of $\widehat{y}$. Let $\mathcal{B} = \{\widehat{h}^{\widehat{k}}_\ell \mid \widehat{k} \in \widehat{C}_\ell\} \subset K_\ell$ be the collection of all such maps.

For $\phi \in \mathrm{Homeo}(\widehat{U}_\ell^{\widehat{g}_\ell^{-1}})$, let $\mathrm{Fix}(\phi) = \{\widehat{y} \in \widehat{U}_\ell^{\widehat{g}_\ell^{-1}} \mid \phi(\widehat{y}) = \widehat{y}\}$ and let $\mathrm{Germ}(\phi, \widehat{U}_\ell^{\widehat{g}_\ell^{-1}})$ denote the subspace of $\mathrm{Fix}(\phi)$ such that ϕ has non-trivial germ at $\widehat{y}$. If $\widehat{y} \in W$ is a clopen

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

subset of Germ(ϕ, $\widehat{U}_\ell^{\widehat{g}_\ell^{-1}}$), then $W \subset \mathrm{Fix}(\phi)$ and then the germ of ϕ is trivial at $\widehat{y}$. This is a contradiction, so the set Germ(ϕ, $\widehat{U}_\ell^{\widehat{g}_\ell^{-1}}$) has no interior.

Now suppose that the collection $\mathcal{B}$ is countable. Then the set of points $\widehat{y} \in \widehat{U}_\ell^{\widehat{g}_\ell^{-1}}$ for which there exists some $\phi \in \mathcal{B}$ for which $\widehat{y}$ is a fixed point with non-trivial holonomy is a countable union of subsets of $\widehat{U}_\ell^{\widehat{g}_\ell^{-1}}$ without interior. Since $\widehat{U}_\ell^{\widehat{g}_\ell^{-1}}$ is a Cantor space and hence is Baire, by the Baire category theorem, this union has no interior. This contradicts the previous observation that every point of $\widehat{U}_\ell^{\widehat{g}_\ell^{-1}}$ is a fixed point for some $\phi \in \mathcal{B}$ with non-trivial germ. Thus, $\mathcal{B}$ must be an uncountable collection. In particular, K_ℓ is an uncountable group for all $\ell \geq 0$.

Note that the above Baire argument is based on the same ideas as in the proofs of Theorem 1 by Epstein, Millet and Tischler in [**23**] and Theorem 3.6 by Renault in [**48**].

Next, we show that the action $(\mathfrak{X}, G, \Phi)$ is dynamically wild. Suppose not; then we have $\Upsilon_c(\Phi) = \Upsilon_s(\Phi)$, which implies that there exists an increasing subsequence $\{\ell_i \mid i \geq 1\}$ such that the inclusion $Z_{\ell_i} \subset K_{\ell_i}$ is an isomorphism for all $i \geq 1$. Then for each $i \geq 1$ we have the collection of maps $\{\widehat{h}_{\ell_i}^{\widehat{k}} \mid \widehat{k} \in \widehat{C}_{\ell_i}\} \subset Z_{\ell_i}$. For simplicity, set $\ell_i = \ell$.

Recall that by construction for $\ell \geq 1$, the maps h_ℓ have non-trivial germinal holonomy at y_ℓ. Moreover, since $\widehat{g}_\ell^{-1}$ conjugates y_ℓ to x, $\widehat{h}_\ell$ is non-Hausdorff at y_ℓ. Therefore, there is a clopen subset $W' \subset \widehat{U}_\ell^{\widehat{g}_\ell^{-1}}$ such that $\widehat{h}_\ell$ is the identity on W'.

Since $\widehat{C}_\ell$ preserves and acts transitively on $\widehat{U}_\ell^{\widehat{g}_\ell^{-1}}$, there is $\widehat{k} \in \widehat{C}_\ell$ such that $\widehat{k} \cdot y_\ell \in W'$. Then the element $\widehat{h}_\ell^{\widehat{k}}$ fixes an open neighborhood of y_ℓ. But this contradicts the fact that $\widehat{h}_\ell^{\widehat{k}} = \widehat{k}\widehat{h}_\ell\widehat{k}^{-1} = \widehat{h}_\ell$ is not the identity map on any clopen neighborhood of the fixed point y_ℓ. Thus, $Z_\ell = K_\ell$ is impossible. □

COROLLARY 7.6. *Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action with discriminant group $\mathcal{D}_x$ at $x \in \mathfrak{X}$. Suppose that for each $\widehat{g} \in \mathcal{D}_x$, the intersection $\mathcal{D}_x \cap \{\widehat{h}^{-1}\, \widehat{g}\, \widehat{h} \mid \widehat{h} \in \widehat{G}_\infty\}$ is finite; then the germinal groupoid of the action is Hausdorff.*

Proof. Suppose that $\widehat{g} \in \mathcal{D}_x$ is a non-Hausdorff element. Then the proof of Theorem 7.5 shows that the set of conjugacy classes of $\widehat{g}$ in $\mathcal{D}_x$ is an infinite set. □

It is a basic question to find a converse to the conclusion of Theorem 7.5. That is, to find sufficient conditions so that a Cantor action which is dynamically wild must have a non-Hausdorff element.

In the next §8, examples of wild Cantor actions are constructed using the 'automata' method, which is a well-known technique in geometric group theory, and defines a homeomorphism using a recursive definition along the branches of a tree. It would be very interesting to understand if the use of automata is the only approach to constructing actions with non-Hausdorff elements or whether there are possibly alternative general methods for their construction.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

8. *Examples of tree automorphisms*

In this section, we present examples constructed as actions on the boundary of an infinite binary tree to illustrate some of the properties of wild Cantor actions. The boundary of a tree T is identified with the set of all infinite paths in the tree and can also be viewed as a collection of infinite sequences of 0's and 1's. It is a Cantor set by a standard argument. To construct our examples, we use an approach well known in geometric group theory, of defining a homeomorphism of the Cantor set at the boundary recursively, by specifying how it acts at each level of the tree. For example, the Grigorchuk group and the Basilica group are usually defined in this way; see [**33**, **43**] and other works.

8.1. *Actions on trees.* We start by explaining the notation and the method for the recursive construction of a homeomorphism of the boundary of a tree.

Let T be a binary tree, that is, T consists of the vertex set $V = \bigsqcup_{n\geq 0} V_n$ and of the set of edges E with the following properties: for $n \geq 0$:

(1) $\mathrm{card}(V_n) = 2^n$;
(2) for each $v \in V_n$, there are exactly two vertices in V_{n+1} joined to v by edges;
(3) for each $v \in V_n$ with $n > 0$, there is exactly one vertex in V_{n-1} joined to v by an edge.

We denote by $\mathcal{P}(T)$ the set of all infinite connected paths in T.

It is sometimes convenient to label the vertices in V by 0's and 1's as follows. The single vertex in V_0 is not labelled; the two vertices in V_1 are labelled one by 0 and the other by 1. Now suppose that the vertices in V_n are labelled by words of length n consisting of 0's and 1's. Since there are 2^n such distinct words, we can assign to each vertex in V_n a unique word, and to every word of length n we can assign a vertex. Proceed to label the vertices in V_{n+1} as follows. Let $v \in V_n$ be labelled by a word $s_1 s_2 \cdots s_n$. Since T is binary, there are two vertices, w_1 and w_2, in V_{n+1} which are joined to v by edges. Label w_1 by $s_1 s_2 \cdots s_n 0$ and w_2 by $s_1 s_2 \cdots s_n 1$.

It follows that every infinite sequence $s_1 s_2 \cdots$, where $s_i \in \{0, 1\}$ for $i \geq 1$, corresponds to a unique path s in the tree T. Namely, s passes through the vertex labelled by s_1 in V_1, by $s_1 s_2$ in V_2, and, inductively, s passes through the vertex labelled by $s_1 s_2 \cdots s_n$ in V_n for $n \geq 1$.

Next, let $w = w_1 w_2 \cdots w_n$ be a word of length n in 0's and 1's. We denote by wT the subtree of T which contains all infinite paths which start with w, that is, the path space of wT is given by

$$\mathcal{P}(wT) = \{s_1 s_2 \cdots s_m \cdots \in \mathcal{P}(T) \mid s_i = w_i \text{ for } 1 \leq i \leq n\}.$$

Clearly every path in $\mathcal{P}(wT)$ contains the vertex in V_n labelled by w. The set $\mathcal{P}(wT)$ is a clopen subset of $\mathcal{P}(T)$. There is an obvious homeomorphism

$$\psi_w : \mathcal{P}(wT) \to \mathcal{P}(T) : w_1 w_2 \cdots w_n s_{n+1} s_{n+2} \cdots \mapsto s_{n+1} s_{n+2} \cdots ,$$

which induces a homeomorphism between wT and T which preserves the paths. Also, we can write $T = v_0 \cup 0T \cup 1T$, where v_0 is the unique vertex in V_0.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

We now explain the recursive definition of homeomorphisms of $\mathcal{P}(T)$. Let σ be the non-trivial permutation of a set of two elements, that is, if this set is $\{0, 1\}$, then σ interchanges 0 and 1. Denote by $\mathrm{Aut}(T)$ the automorphism group of the tree T, that is, an element $h \in \mathrm{Aut}(T)$ is a homeomorphism of $\mathcal{P}(T)$ which for all $n \geq 1$ restricts to a permutation of V_n. Let $h, g \in \mathrm{Aut}(T)$; then define an automorphism (h, g) of T by declaring that it restricts to the trivial permutation of V_1 and, for every $s = s_1 s_2 \cdots \subset 0T$, one has

$$(h, g)(s) = \psi_0^{-1} \circ h \circ \psi_0(s) \tag{53}$$

and, for every $s \subset 1T$, one has

$$(h, g)(s) = \psi_1^{-1} \circ g \circ \psi_1(s). \tag{54}$$

That is, the automorphism h is applied to the branch of the tree through the first vertex $\{0\}$ of V_1, and g is applied to the branch of the tree through the second vertex $\{1\}$ of V_1. We compose the maps on the left.

We now give an example of a recursive definition of a homeomorphism of T. Define a such that

$$a = (a, 1)\sigma, \tag{55}$$

where 1 denotes the identity in $\mathrm{Aut}(T)$. Then, for every $s = s_1 s_2 \cdots$, the map σ maps s_1 to $(s_1 + 1) \mod 2$ and fixes the rest of the sequence, so we can write, with a slight abuse of notation,

$$\sigma(s) = \sigma(s_1) s_2 s_3 \cdots .$$

Next, we apply $(a, 1)$ to $\sigma(s)$. If $\sigma(s_1) = 1$, then, by the rule (54), the element a acts as the identity map on $\sigma(s_1)T = 1T$. If $\sigma(s_1) = 0$, then, by (53),

$$a(s) = \sigma(s_1) a(s_2 s_3 \cdots) = 0 a(s_2 s_3 \cdots),$$

so we need to compute $a(s_2 s_3 \cdots)$. We compute that $\sigma(s_2) = (s_2 + 1) \mod 2$ and then we have to apply a to $\sigma(s_1)\sigma(s_2) s_3 \cdots = 0\sigma(s_2) s_3 \cdots$, that is, the image of s is computed recursively. For example,

$$a(0001110^\infty) = 1001110^\infty, \quad a(11001^\infty) = 00101^\infty, \quad a(1^\infty) = 0^\infty,$$

where 1^∞ denotes an infinite sequence of 1's and 0^∞ denotes an infinite sequence of 0's.

The element a, defined by (55), generates a group isomorphic to the integers $\mathbb{Z}$, which acts freely and transitively on every vertex set V_n, $n \geq 1$. Such an arboreal action is often called an *odometer*, a *standard odometer*, or an *adding machine* in the literature [**43**, **45**]. In this paper, we use the term 'odometer' in a more general sense; that is, an odometer is an equicontinuous minimal action of *any* group G on a Cantor set $\mathfrak{X}$. To distinguish between the two notions, we use the term *cyclic odometer* for the action generated by a single element a as described above.

8.2. *Rigidity of Cantor actions.* In this section, we illustrate the discussion in §6.2 of various definitions of rigidity. We present an example of orbit equivalent actions, where one of the actions is free, and another one is stable but not topologically free. This example motivates the introduction of the notion of *return equivalence* in §6.2.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

Let T be a binary tree with the Cantor set boundary $\mathcal{P}(T)$ and set $\mathfrak{X} = \mathcal{P}(T)$. Let $a_1 = (a_1, 1)\sigma$ be the cyclic odometer defined in (55). Let $a_2 = (a_1, 1)$, that is, a_2 acts as a cyclic odometer on the clopen set $\mathcal{P}(0T)$ and as the identity map on $\mathcal{P}(1T)$. We also note that $a_2|\mathcal{P}(0T) = a_1^2|\mathcal{P}(0T)$.

Define $H = \langle a_1 \rangle$ and $G = \langle a_1, a_2 \rangle$, so we have the actions $(\mathfrak{X}, G)$ and $(\mathfrak{X}, H)$, where we omit Φ and Ψ from the notation since G and H are already defined as subgroups of Homeo$(\mathfrak{X})$. Both actions are minimal, since the orbit of any path $(v_n)_{n\geq 1} \in \mathcal{P}(T)$ under the powers of a_1 is dense in $\mathfrak{X}$. It is easy to see that the group G is non-commutative.

The action $(\mathfrak{X}, G)$ is not topologically free, as a_2 acts as the identity on the clopen set $\mathcal{P}(1T)$. On the other hand, $(\mathfrak{X}, H)$ is a free action of a cyclic odometer.

We now show that the actions $(\mathfrak{X}, G)$ and $(\mathfrak{X}, H)$ are continuously orbit equivalent. Let $h : \mathfrak{X} \to \mathfrak{X}$ be the identity map; then we define the maps $\alpha : \mathfrak{X} \times G \to H$ and $\beta : \mathfrak{X} \times H \to G$ (as in §6.1) as follows.

Since a_2 acts on $0T$ as a_1^2 and on $1T$ as the identity map, for any $x \in \mathfrak{X}$ the orbits of the action of G and H coincide as sets. Since the action of H is free, given $x \in \mathfrak{X}$ and $g \in G$, there exists a unique element $h \in H$ such that $g \cdot x = h \cdot x$. We define $\alpha(g, x) = h$. We show that α is constant when restricted to $\mathcal{P}(0T)$ and to $\mathcal{P}(1T)$.

So, let $\mathcal{U} = \{\mathcal{P}(0T), \mathcal{P}(1T)\}$ be a partition of $\mathcal{P}(T)$. Any element $g \in G$ can be written as a finite word $g = a_1^{k_n} a_2^{k_{n-1}} \cdots a_1^{k_2} a_2^{k_1}$ with $k_j \in \mathbb{Z}$ for $1 \leq j \leq n$. The actions of any power of a_2 and of even powers of a_1 preserve $\mathcal{P}(wT)$, and the actions of odd powers of a_1 interchange $\mathcal{P}(0T)$ and $\mathcal{P}(1T)$. It follows that x and y are in the same set of the partition $\mathcal{U}$ if and only if $a_2^{k_1} \cdot x$ and $a_2^{k_1}(y)$ are in the same set of $\mathcal{U}$. Implementing induction on the number of elements in the word decomposition of g, we obtain that x and y are in the same set of $\mathcal{U}$ if and only if $g \cdot x$ and $g \cdot y$ are in the same set of $\mathcal{U}$.

Define the map $\beta : H \times \mathfrak{X} \to G$ by $\beta(a_1^k, x) = a_1^k$. The map β is constant on $\mathfrak{X}$, so is continuous. Thus, for $h = \mathrm{id}$, α and β yield a continuous orbit equivalence between the actions $(\mathfrak{X}, G)$ and $(\mathfrak{X}, H)$. By Theorem 1.4, the action $(\mathfrak{X}, G)$ is stable.

We also note that the map $\alpha_x : G \to H$ obtained by specializing $\alpha_x = \alpha(-, x)$ for $x \in \mathcal{P}(T)$ is not injective. For example, for $x \in \mathcal{P}(T)$, we have

$$\alpha_x^{-1}(a_1^2) \supset \{a_1^2, a_2\}.$$

Note that this example can be considered as an explicit form of the construction in [**35**, Example A.4].

8.3. *Dynamically wild actions.* Let T be a binary tree with the Cantor set boundary $\mathcal{P}(T)$; we set $\mathfrak{X} = \mathcal{P}(T)$. Let $r \geq 3$, $2 \leq s \leq r$, and consider a group G generated by the homeomorphisms

$$a_1 = \sigma,\ a_{s+1} = (a_s, a_r),\ a_i = (a_{i-1}, 1) \text{ for } 3 \leq i \leq s,\ s+1 \leq i \leq r. \tag{56}$$

Groups generated by the recursive rules (56) arise as iterated monodromy groups of quadratic post-critically finite polynomials [**4**, **43**]. They have been extensively studied, for example, in [**4**] and also in other works. For example, in [**45**], it was shown that the closure of the action of an iterated monodromy group, associated to a post-critically finite quadratic polynomial with strictly pre-periodic post-critical orbit of length r, where the

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

periodic part has length $r-1$, is conjugate to the closure of the action of the group G generated by (56).

Let G be a group generated by (56) and let $\mathfrak{G}(\Phi)$ be the closure of the action. It was shown in [**39**, Theorem 1.3] (see also [**40**]) that G contains non-Hausdorff elements, namely the generators $a_{s+1}, \ldots, a_r$ are non-Hausdorff. Then, by Theorem 7.5, the action of the group G is dynamically wild; that is, there is a proper inclusion of the direct limit groups $\Upsilon_c(\Phi) \subset \Upsilon_s(\Phi)$.

8.4. *Infinite stabilizer group.* Theorem 7.5 shows that if a group G, or more generally its closure $\mathfrak{G}(\Phi)$, has a non-Hausdorff element, then the groups $K_0 \subset K_1 \subset K_2 \subset \cdots \mathcal{D}_x$ in the stabilizer chain $\mathcal{K}(\mathcal{U})$ must be infinite. In specific examples it may be difficult to compute the groups K_ℓ, $\ell \geq 1$, explicitly. We now give an example of an action for which this can be done.

Let T be a binary tree with the Cantor set boundary $\mathcal{P}(T)$; we set $\mathfrak{X} = \mathcal{P}(T)$ and let G be generated by (56) with $r = 2s$ and $s \geq 2$.

By definition, $a_2, \ldots, a_s, a_{s+2}, \ldots, a_{2s}$ are the identity on the clopen set $\mathcal{P}(1T)$ of all sequences starting with digit 1. Since $a_{s+1} = (a_s, a_{2s})$, and a_{2s} is the identity on $\mathcal{P}(1T)$, a_{s+1} is the identity on the set $\mathcal{P}(11T)$ of all sequences starting with a finite word 11.

Choose a path $x \in \mathcal{P}(11T)$, that is, $x = 11x_3x_4 \ldots$ and, for $\ell \geq 1$, let U_ℓ be a clopen neighborhood

$$U_\ell = \{s_1s_2s_3 \ldots \mid x_i = s_i \text{ for } 1 \leq i \leq \ell\}.$$

Then every $U_\ell \subset \mathcal{P}(11T)$ and so the generators $a_2, \ldots, a_{2s}$ act trivially on U_ℓ; in other words,

$$\langle a_2, \ldots, a_s, a_{s+1}, \ldots, a_{2s} \rangle \subset K_\ell.$$

Consider the compositions $a_i a_j \in K_\ell$ with $i < j$. By [**45**, Proposition 3.1.9] if $j = i + s$, then $a_i a_j$ has infinite order. In particular, $a_s a_{2s}$ has infinite order. It follows that $G \cap K_\ell$ is infinite and so K_ℓ is a Cantor group.

Acknowledgement. O. Lukina is supported by FWF Project P31950-N35.

References

[**1**] J. Álvarez López and A. Candel. Equicontinuous foliated spaces. *Math. Z.* **263** (2009), 725–774.
[**2**] J. Álvarez López and M. Moreira Galicia. Topological Molino's theory. *Pacific J. Math.* **280** (2016), 257–314.
[**3**] J. Auslander. *Minimal Flows and their Extensions (North-Holland Mathematics Studies, 153)*. North-Holland, Amsterdam, 1988.
[**4**] L. Bartholdi and V. Nekrashevych. Iterated monodromy groups of quadratic polynomials, I. *Groups Geom. Dyn.* **2** (2008), 309–336.
[**5**] S. Bezuglyi and K. Medynets. Full groups, flip conjugacy, and orbit equivalence of Cantor minimal systems. *Colloq. Math.* **110** (2008), 409–429.
[**6**] M. Boyle. Topological orbit equivalence and factor maps in symbolic dynamics. *PhD Thesis*, University of Washington, 1983.
[**7**] M. Boyle and J. Tomiyama. Bounded topological orbit equivalence and C^*-algebras. *J. Math. Soc. Japan* **50** (1998), 317–329.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

[**8**] J. Brown, L. Clark, L. Orloff and C. Farthing. Simplicity of algebras associated to étale groupoids. *Semigroup Forum* **88** (2014), 433–452.
[**9**] A. Clark and S. Hurder. Homogeneous matchbox manifolds. *Trans. Amer. Math. Soc.* **365** (2013), 3151–3191.
[**10**] A. Clark, S. Hurder and O. Lukina. Classifying matchbox manifolds. *Geom. Topol.* **23** (2019), 1–38.
[**11**] A. Connes, J. Feldman and B. Weiss. An amenable equivalence relation is generated by a single transformation. *Ergod. Th. & Dynam. Sys.* **1** (1981), 431–450.
[**12**] M. I. Cortez and K. Medynets. Orbit equivalence rigidity of equicontinuous systems. *J. Lond. Math. Soc. (2)* **94** (2016), 545–556.
[**13**] M.-I. Cortez and S. Petite. G-odometers and their almost one-to-one extensions. *J. Lond. Math. Soc. (2)* **78** (2008), 1–20.
[**14**] M. I. Cortez and S. Petite. On the centralizers of minimal aperiodic actions on the Cantor set. *Preprint*, 2018, arXiv:1807.04654.
[**15**] Y. de Cornulier. Groupes pleins-topologiques (d'après Matui, Juschenko, Monod, . . .). *Astérisque* **361** (2014), 183–223.
[**16**] T. Downarowicz. Survey of odometers and Toeplitz flows. *Algebraic and Topological Dynamics (Contemporary Mathematics, 385)*. American Mathematical Society, Providence, RI, 2005, pp. 7–37.
[**17**] J. Dyer, S. Hurder and O. Lukina. The discriminant invariant of Cantor group actions. *Topology Appl.* **208** (2016), 64–92.
[**18**] J. Dyer, S. Hurder and O. Lukina. Growth and homogeneity of matchbox manifolds. *Indag. Math. (N.S.)* **28** (2017), 145–169.
[**19**] J. Dyer, S. Hurder and O. Lukina. Molino theory for matchbox manifolds. *Pacific J. Math.* **289** (2017), 91–151.
[**20**] S. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology*. Princeton University Press, Princeton, NJ, 1952.
[**21**] R. Ellis. A semigroup associated with a transformation group. *Trans. Amer. Math. Soc.* **94** (1969), 272–281.
[**22**] R. Ellis and W. H. Gottschalk. Homomorphisms of transformation groups. *Trans. Amer. Math. Soc.* **94** (1969), 258–271.
[**23**] D. B. A. Epstein, K. C. Millet and D. Tischler. Leaves without holonomy. *J. Lond. Math. Soc. (2)* **16** (1977), 548–552.
[**24**] R. Exel. Non-Hausdorff étale groupoids. *Proc. Amer. Math. Soc.* **139** (2011), 897–907.
[**25**] R. Fokkink and L. Oversteegen. Homogeneous weak solenoids. *Trans. Amer. Math. Soc.* **354**(9) (2002), 3743–3755.
[**26**] A. Furman. Orbit equivalence rigidity. *Ann. of Math. (2)* **150** (1999), 1083–1108.
[**27**] D. Gaboriau. Orbit equivalence and measured group theory. *Proc. Int. Congress Mathematicians, III*. Hindustan Book Agency, New Delhi, 2010, pp. 1501–1527.
[**28**] T. Giordano, H. Matui, I. Putnam and C. Skau. Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems. *Invent. Math.* **179** (2010), 119–158.
[**29**] T. Giordano, I. Putman and C. Skau. Full groups of Cantor minimal systems. *Israel J. Math.* **111** (1999), 285–320.
[**30**] T. Giordano, I. Putman and C. Skau. $\mathbb{Z}^d$-odometers and cohomology. *Groups Geom. Dyn.* **13** (2019), 909–938.
[**31**] E. Glasner. Enveloping semigroups in topological dynamics. *Topology Appl.* **154** (2007), 2344–2363.
[**32**] E. Glasner and B. Weiss. Weak orbit equivalence of Cantor minimal systems. *Internat. J. Math.* **6** (1995), 559–579.
[**33**] R. I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), 939–985.
[**34**] A. Haefliger. Pseudogroups of local isometries. *Differential Geometry (Santiago de Compostela, 1984) (Research Notes in Mathematics, 131)*. Ed. L. A. Cordero. Pitman, Boston, MA, 1985, pp. 174–197.
[**35**] S. Hurder and O. Lukina. Orbit equivalence and classification of weak solenoids. *Indiana Univ. Math. J.* to appear. *Preprint*, 2018, arXiv:1803.02098.
[**36**] S. Hurder and O. Lukina. Wild solenoids. *Trans. Amer. Math. Soc.* **371** (2019), 4493–4533.
[**37**] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry. Vol. II (Interscience Tracts in Pure and Applied Mathematics, 15)*. Interscience–John Wiley, New York, 1969.
[**38**] X. Li. Continuous orbit equivalence rigidity. *Ergod. Th. & Dynam. Sys.* **38** (2018), 1543–1563.
[**39**] O. Lukina. Galois groups and Cantor actions. Submitted. *Preprint*, 2018, arXiv:1809.08475.
[**40**] O. Lukina. Arboreal Cantor actions. *J. Lond. Math. Soc. (2)* **99** (2019), 678–706.
[**41**] C. McCord. Inverse limit sequences with covering maps. *Trans. Amer. Math. Soc.* **114** (1965), 197–209.
[**42**] J. Munkres. *Elements of Algebraic Topology*. Addison-Wesley, Menlo Park, CA, 1984.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press

[**43**] V. Nekrashevych. *Self-similar Groups (Mathematical Surveys and Monographs, 117)*. American Mathematical Society, Providence, RI, 2005.
[**44**] V. Nekrashevych. Palindromic subshifts and simple periodic groups of intermediate growth. *Ann. of Math. (2)* **187** (2018), 667–719.
[**45**] R. Pink. Profinite iterated monodromy groups arising from quadratic polynomials. *Preprint*, 2013, arXiv:1307.5678.
[**46**] J. Renault. *A Groupoid Approach to C*-Algebras (Lecture Notes in Mathematics, 793)*. Springer, Berlin, 1980.
[**47**] J. Renault. Transverse properties of dynamical systems. *Representation Theory, Dynamical Systems, and Asymptotic Combinatorics (American Mathematical Society Translations Series 2, 217)*. American Mathematical Society, Providence, RI, 2006, pp. 185–199.
[**48**] J. Renault. Cartan subalgebras in C^*-algebras. *Irish Math. Soc. Bull.* **61** (2008), 29–63.
[**49**] J. T. Rogers Jr. and J. L. Tollefson. Homogeneous inverse limit spaces with non-regular covering maps as bonding maps. *Proc. Amer. Math. Soc.* **29** (1971), 417–420.
[**50**] R. Schori. Inverse limits and homogeneity. *Trans. Amer. Math. Soc.* **124** (1966), 533–539.
[**51**] E. Winkelnkemper. The graph of a foliation. *Ann. Global Anal. Geom.* **1** (1983), 51–75.

https://doi.org/10.1017/etds.2020.16 Published online by Cambridge University Press