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# A note on the best constant for uncentered maximal functions with measure on $\mathbb{R}$

Zhipeng Wu and Qingdong Guo

*Abstract.* Let $M_\mu$ be the uncentered Hardy–Littlewood maximal operator with a Borel measure $\mu$ on $\mathbb{R}$. In this note, we verify that the norm of $M_\mu$ on $L^p(\mathbb{R},\mu)$ with $p\in(1,\infty)$ is just the upper bound $\theta_p$ obtained by Grafakos and Kinnunen and reobtain the norm of $M_\mu$ from $L^1(\mathbb{R},\mu)$ to $L^{1,\infty}(\mathbb{R},\mu)$. Moreover, the norm of the "strong" maximal operator $N_{\vec{\mu}}^n$ on $L^p(\mathbb{R}^n,\vec{\mu})$ is also given.

## 1 Introduction

Let $\mu$ be a Borel measure on $\mathbb{R}^n$; that is, every Borel set is $\mu$-measurable and $\mu(K)<\infty$ for any compact set $K\subset\mathbb{R}^n$. For a $\mu$-locally integrable function $f:\mathbb{R}^n\to[0,\infty]$, the uncentered maximal function of $f$ with respect to $\mu$ is defined by

$$M_\mu f(x):=\sup_{B\ni x}\frac{1}{\mu(B)}\int_B f\,d\mu,\quad x\in\mathbb{R}^n, \tag{1.1}$$

where the supremum is taken over all open balls $B$ containing $x$. The integral average in (1.1) is equal to $f(x)$ if $\mu(B)=0$.

As we know, when $n=1$, the operator $M_\mu$ always maps $L^1(\mathbb{R},\mu)$ into the Lorenz space $L^{1,\infty}(\mathbb{R},\mu)$ and maps $L^p(\mathbb{R},\mu)$ into itself for $1<p\le\infty$ (see [1, 10]). It is therefore natural to investigate the best constant $C_1^\mu$ such that for any $f\in L^1(\mathbb{R},\mu)$ and $\lambda>0$,

$$\mu(\{x\in\mathbb{R}:M_\mu f(x)>\lambda\})\le\frac{C_1^\mu}{\lambda}\|f\|_{L^1(\mathbb{R},\mu)}, \tag{1.2}$$

as well as the best constant $C_p^\mu$ such that for any $f\in L^p(\mathbb{R},\mu)$,

$$\|M_\mu f\|_{L^p(\mathbb{R},\mu)}\le C_p^\mu\|f\|_{L^p(\mathbb{R},\mu)}. \tag{1.3}$$

In their remarkable work [7], Grafakos and Montgomery-Smith showed that when $\mu$ is the $n$-dimensional Lebesgue measure, the norm of the "strong" uncentered maximal operator on $L^p(\mathbb{R}^n)$ for $p\in(1,\infty)$ tends to $\infty$ as $n\to\infty$. Moreover, when $n=1$, they also showed that the best constant $C_p^\mu$ for $p\in(1,\infty)$ is the unique positive solution to the equation

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$$(p-1)x^p - px^{p-1} - 1 = 0. \tag{1.4}$$

When $\mu$ is a non-zero Borel measure satisfying the following assumption.

**Assumption (A):** For any point $a \in \mathbb{R}$, $\mu(\{a\}) = 0$,

Bernal [1] proved that the best constant $C_1^\mu$ in (1.2) is 2. On the other hand, Grafakos and Kinnunen in [6] showed that for a general Borel measure $\mu$ on $\mathbb{R}$, the best constants in (1.2) and (1.3), respectively, have upper bounds 2 and $\theta_p$ with $1 < p < \infty$, where $\theta_p$ is the unique positive solution of equation (1.4) (see Lemma 2.1 below). Recently, there are still scholars actively engaged in this field. Wu et al. [11] obtained the best constant of truncated Hardy–Littlewood maximal function on $L^1(\mathbb{R})$. Qin et al. [9] established the limiting weak type behaviors of the strong maximal operators. There are numerous papers on the best constants of maximal functions (see [2, 4, 8]).

The purpose of this note is twofold. For $p = 1$, by an alternative method, we reobtain the result of Bernal in [1]. That is, we have the conclusion as follows.

***Theorem 1.1*** *Let $\mu$ be a non-zero Borel measure on $\mathbb{R}$ that satisfies Assumption (A). Then, $C_1^\mu = 2$.*

The second purpose of this note is to study the best constant $C_p^\mu$ of $M_\mu$ with a Borel measure $\mu$ for $p \in (1, \infty)$. Under some additional assumptions, we verify that the upper bound $\theta_p$ obtained in [6] is just the norm of $M_\mu$. More precisely, we have the following result.

***Theorem 1.2*** *Let $\mu$ be a Borel measure on $\mathbb{R}$ that satisfies Assumption (A) and the following condition that is*

***Assumption (B):*** *There exists a non-negative locally integrable function $w$ on $\mathbb{R}$, such that for any subset $A \subset \mathbb{R}$, $\mu(A) = \int_A w(t)dt$, and*

$$\int_{-\infty}^{0} w(t)dt = \int_{0}^{\infty} w(t)dt = \infty.$$

*Then, $C_p^\mu = \theta_p$, where $\theta_p$ is the unique positive solution of equation (1.4) with $1 < p < \infty$.*

***Remark 1.3*** (i) It is not clear whether Assumption (B) is necessary.

(ii) Let $w \in A_p$, the class of Muckenhoupt weights (see, for example, [5]). It is not difficult to see that the Borel measure $\mu = wdx$ satisfies Assumptions (A) and (B). This implies that for a given $w \in A_p$, the norm of $M_\mu$ on $L^p(\mathbb{R}, \mu)$ or from $L^1(\mathbb{R}, \mu)$ to $L^{1,\infty}(\mathbb{R}, \mu)$ is independent of $w$.

We mention that when $p = 1$, by the result in [6], the best constant $C_1^\mu \le 2$ (see Lemma 2.1 below). Thus, to show Theorem 1.1, it suffices to prove the reverse inequality. A novelty of this note is that we take a new method to achieve this aim by finding some suitable "test functions." This method is also used to prove Theorem 1.2.

At the end of this article, we also briefly discuss the "strong" maximal operator $N_{\vec{\mu}}^n$ with the measure $\vec{\mu}$ on $\mathbb{R}^n$. By an argument similar to Theorem 1.2, we show that the operator norm of $N_{\vec{\mu}}^n$ on $L^p(\mathbb{R}^n, \vec{\mu})$ is $\theta_p^n$, $1 < p < \infty$. Moreover, we point out that $\theta_p^n$ grows exponentially with $n$, as $n \to \infty$.

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## 2 Proofs of Theorems 1.1 and 1.2

In this section, we firstly provide the proofs of Theorems 1.1 and 1.2, and then we consider the $n$-dimensional maximal operator $N^n_{\vec{\mu}}$ in the end. Let $M_\mu f$, $C_1^\mu$, and $C_p^\mu$ be as in (1.1)–(1.3), respectively, and $\theta_p$ be the unique positive solution of (1.4). For convenience, we write $Mf := M_\mu f$, $C_1 := C_1^\mu$, and $C_p := C_p^\mu$. We begin with recalling the following result by Grafakos and Kinnunen in [6].

***Lemma 2.1*** *Let $\mu$ be a Borel measure on $\mathbb{R}$ and $p \in (1,\infty)$. Then, $C_1 \le 2$ and $C_p \le \theta_p$.*

Theorem 1.1 is a consequence of Lemma 2.1 and the following proposition.

***Proposition 2.2*** *Let $\mu$ be a non-zero Borel measure on $\mathbb{R}$ satisfying Assumption (A). Then, there exists a collection $\{(a_i,b_i)\}_{i=1}^\infty$ of open intervals such that*

$$a_i < a_{i+1} < b_{i+1} < b_i, \quad \int_{a_i}^{b_i} d\mu > 0, \quad \lim_{i\to\infty} \int_{a_i}^{b_i} d\mu = 0, \tag{2.1}$$

*and*

$$\lim_{i\to\infty} \frac{\sup\limits_{t>0} t\mu(\{x\in\mathbb{R} : M\chi_{(a_i,b_i)}(x) > t\})}{\mu((a_i,b_i))} = 2.$$

**Proof:** We claim that there exists $x_0 \in \mathbb{R}$, such that $\mu(I) > 0$ for any open interval $I \ni x_0$. Otherwise, for any $x \in \mathbb{R}$, there exists an open interval $I_x \ni x$ such that $\mu(I_x) = 0$. By the Lindelöf covering theorem, we deduce that there exists a countable collection $\{I_j\}_{j=1}^\infty$ of open intervals, such that $\mu(I_j) = 0$ for any $j \in \mathbb{N}$, and

$$\mathbb{R} = \bigcup_{j\in\mathbb{N}} I_j.$$

Therefore,

$$0 < \mu(\mathbb{R}) = \mu\Big(\bigcup_{j\in\mathbb{N}} I_j\Big) \le \sum_{j=1}^\infty \mu(I_j) = 0,$$

which yields a contradiction. From this claim, we see that there exists a collection $\{(a_i,b_i)\}_{i=1}^\infty$ of open intervals satisfying (2.1).

Let $f_i(x) := \chi_{(a_i,b_i)}(x)$, where $i \in \mathbb{N}$ and $x \in \mathbb{R}$. We then see that for $x < a_i$,

$$Mf_i(x) = \sup_{a_i\le y\le b_i} \frac{\int_{a_i}^y d\mu}{\int_x^y d\mu} = \frac{\int_{a_i}^{b_i} d\mu}{\int_x^{b_i} d\mu}.$$

We can treat similarly when $x > b_i$. Therefore, we conclude that for any $i$,

$$Mf_i(x) = \begin{cases} \dfrac{\int_{a_i}^{b_i} d\mu}{\int_x^{b_i} d\mu}, & x < a_i; \\ 1, & a_i \le x \le b_i; \\ \dfrac{\int_{a_i}^{b_i} d\mu}{\int_{a_i}^{x} d\mu}, & x > b_i. \end{cases}$$

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By (2.1), we see that $\mu((a_i, b_i)) \in (0, \infty)$. Without loss of generality, we may further assume that

$$\gamma_i := \max\Big\{\frac{\int_{a_i}^{b_i} d\mu}{\int_{-\infty}^{b_i} d\mu}, \frac{\int_{a_i}^{b_i} d\mu}{\int_{a_i}^{\infty} d\mu}\Big\} < 1.$$

Moreover, since $Mf_i$ is continuous, increasing on $(-\infty, a_i)$, and decreasing on $(b_i, \infty)$, we have that

$$E_i := \{x \in \mathbb{R} : Mf_i(x) = \lambda\} \neq \varnothing \text{ for } \gamma_i < \lambda < 1.$$

Let

$$\alpha_i := \inf\{y : y \in E_i\},$$

$$\beta_i := \sup\{y : y \in E_i\}.$$

Then, by the continuity of $Mf_i$, we see that

$$\alpha_i < a_i, \quad \beta_i > b_i,$$

$$\alpha_i, \beta_i \in E_i,$$

$$\int_{a_i}^{b_i} d\mu = \lambda \int_{\alpha_i}^{b_i} d\mu, \quad \int_{a_i}^{b_i} d\mu = \lambda \int_{a_i}^{\beta_i} d\mu,$$

and

$$\{x \in \mathbb{R} : Mf_i(x) > \lambda\} = (\alpha_i, \beta_i).$$

These facts imply that

$$\begin{aligned}\lambda\mu(\{x \in \mathbb{R} : Mf_i(x) > \lambda\}) &= \lambda \int_{\alpha_i}^{\beta_i} d\mu \\ &= \lambda \int_{\alpha_i}^{b_i} d\mu + \lambda \int_{a_i}^{\beta_i} d\mu - \lambda \int_{a_i}^{b_i} d\mu \\ &= (2 - \lambda) \int_{a_i}^{b_i} d\mu.\end{aligned}$$

Hence,

$$\begin{aligned}\sup_{\lambda>0} \lambda\mu(\{x \in \mathbb{R} : Mf_i(x) > \lambda\}) &\geq \sup_{\gamma_i<\lambda<1} \lambda\mu(\{x \in \mathbb{R} : Mf_i(x) > \lambda\}) \\ &= \sup_{\gamma_i<\lambda<1} (2 - \lambda) \int_{a_i}^{b_i} d\mu \\ &= (2 - \gamma_i)\mu((a_i, b_i)).\end{aligned}$$

By letting $i \to \infty$, we deduce that

$$\varliminf_{i\to\infty} \frac{\sup_{\lambda>0} \lambda\mu(\{x \in \mathbb{R} : M\chi_{(a_i,b_i)}(x) > \lambda\})}{\mu((a_i, b_i))} \geq \varliminf_{i\to\infty} (2 - \gamma_i) = 2. \tag{2.2}$$

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On the other hand, from Lemma 2.1, we also have

$$\overline{\lim_{i\to\infty}} \frac{\sup_{\lambda>0} \lambda\mu(\{x\in\mathbb{R}: M\chi_{(a_i,b_i)}(x)>\lambda\})}{\mu((a_i,b_i))} \le 2.$$

This, together with (2.2), yields that Proposition 2.2 holds and the proof is finished. ∎

To present the proof of Theorem 1.2, we also need the following proposition.

***Proposition 2.3*** *Let $T(x) := \frac{x^{1-\frac{1}{p}}+1}{x+1}$, $x>0$ and $1<p<\infty$. Then, the maximum value of the function $T$ is $\frac{p-1}{p}\theta_p$.*

**Proof:** We firstly see that

$$T'(x) = \frac{-\frac{1}{p}x^{1-\frac{1}{p}} + (1-\frac{1}{p})x^{-\frac{1}{p}} - 1}{(x+1)^2}, \quad x>0.$$

It is not difficult to prove that there exists a unique point $x_0>0$ such that $T'(x_0)=0$ and $\max_{x>0} T(x) = T(x_0)$. Therefore,

$$\max_{x>0} T(x) = \frac{x_0^{1-\frac{1}{p}}+1}{x_0+1} = \frac{x_0^{1-\frac{1}{p}} + \frac{\frac{1}{p}x_0^{1-\frac{1}{p}}}{(1-\frac{1}{p})x_0^{-\frac{1}{p}}-1}}{x_0 + \frac{\frac{1}{p}x_0^{1-\frac{1}{p}}}{(1-\frac{1}{p})x_0^{-\frac{1}{p}}-1}} = \frac{p-1}{p}x_0^{-\frac{1}{p}}.$$

Since $x_0^{-\frac{1}{p}}>0$ satisfies (1.4), we set $\theta_p := x_0^{-\frac{1}{p}}$, which implies

$$\max_{x>0} T(x) = \frac{p-1}{p}\theta_p.$$

∎

Next, we give the proof of Theorem 1.2.

**Proof of Theorem 1.2:** Let $1<p<\infty$ and

$$f(x) := \int_0^x w(t)dt, \quad F(x) := |f(x)|^{-\frac{1}{p}}, \quad x\in\mathbb{R}.$$

From assumptions on $\mu$, we see that $f$ is well-defined, continuous, and increasing on $\mathbb{R}$ and absolutely continuous in any closed interval on $\mathbb{R}$. Also let

$$\mathscr{A} := \left\{H : H \text{ is a maximal closed interval such that } \int_H w(t)dt = 0\right\},$$

$E := \cup_{H\in\mathscr{A}} H$, and $E^c := \mathbb{R}\backslash E$. Obviously, $\mathscr{A}$ is a countable set, and $\mu(E)=0$.

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By Proposition 2.3 and Assumption (B), we see that for any $x \in E^c \cap (0,\infty)$, $f(x) > 0$ and

$$\begin{aligned} MF(x) &= \sup_{a<x<b} \frac{\int_a^b F(t)d\mu(t)}{\int_a^b w(t)dt} \\ &\geq \sup_{a<0} \frac{\int_a^x F(t)d\mu(t)}{\int_a^x w(t)dt} \\ &= \sup_{a<0} \frac{p}{p-1} \frac{|f(x)|^{1-\frac{1}{p}} + |f(a)|^{1-\frac{1}{p}}}{|f(x)| + |f(a)|} \\ &= \frac{p}{p-1} F(x) \sup_{a<0} \frac{|\frac{f(a)}{f(x)}|^{1-\frac{1}{p}} + 1}{|\frac{f(a)}{f(x)}| + 1} \\ &= \frac{p}{p-1} F(x) \frac{p-1}{p} \theta_p \\ &= \theta_p F(x). \end{aligned}$$

The above inequality also holds for $x \in E^c \cap (-\infty, 0)$. Hence,

$$MF(x) \geq \theta_p F(x), \quad \mu - a.e. \quad x \in \mathbb{R}.$$

Observe that $F \notin L^p(\mathbb{R},\mu)$. So instead, we consider the following function:

$$F_\varepsilon(x) := F(x) \min\{|f(x)|^\varepsilon, |f(x)|^{-\varepsilon}\} \in L^p(\mathbb{R},\mu), \quad \varepsilon > 0. \tag{2.3}$$

We claim that

$$MF_\varepsilon(x) \geq \frac{1}{\varepsilon + \frac{1}{p'}} F_\varepsilon(x) \frac{1 + \theta_p^{1-p-\varepsilon p}}{1 + \theta_p^{-p}}, \quad \mu - a.e. \quad x \in \mathbb{R}. \tag{2.4}$$

In fact, assume that $x \in E^c \cap (0,\infty)$, and set

$$\varphi(u) := \sup\{t : f(t) \leq u f(x)\}, \quad u \in \mathbb{R}. \tag{2.5}$$

We denote by $m$ the Lebesgue measure. From the property of $f$, $\varphi(u)$ is well-defined, strictly increasing, right continuous, and $m - a.e.$ differentiable on $\mathbb{R}$. We can further verify the following fact:

$$\varphi(1) \geq x,$$

$$f(\varphi(u)) = u f(x), \quad u \in \mathbb{R},$$

$$w(t) = 0, \quad m - a.e. \quad t \in [x, \varphi(1)],$$

and

$$w(\varphi(u))\varphi'(u) = f(x), \quad m - a.e. \quad u \in \mathbb{R}.$$

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From these facts, we deduce that

$$\begin{aligned}
MF_\varepsilon(x) &\geq \sup_{y\leq 0} \frac{\int_y^x F_\varepsilon(t)d\mu(t)}{\int_y^x w(t)dt} \\
&\geq \sup_{y\leq 0} \frac{\int_y^x |f(t)|^{-\frac{1}{p}} \min\{|f(t)|^\varepsilon, |f(t)|^{-\varepsilon}\} w(t)dt}{|f(x)|+|f(y)|} \\
&= \sup_{y\leq 0} \frac{\int_y^{\varphi(1)} |f(t)|^{-\frac{1}{p}} \min\{|f(t)|^\varepsilon, |f(t)|^{-\varepsilon}\} w(t)dt}{|f(x)|+|f(y)|} \\
&\geq \frac{\int_{\varphi(-\theta_p^{-p}f(x))}^{\varphi(1)} |f(t)|^{-\frac{1}{p}} \min\{|f(t)|^\varepsilon, |f(t)|^{-\varepsilon}\} w(t)dt}{|f(x)|+|\theta_p^{-p}f(x)|} \\
&= \frac{\int_{-\theta_p^{-p}}^1 |u|^{-\frac{1}{p}} |f(x)|^{-\frac{1}{p}} \min\{|uf(x)|^\varepsilon, |uf(x)|^{-\varepsilon}\} f(x)du}{|f(x)|+|\theta_p^{-p}f(x)|} \\
&= |f(x)|^{-\frac{1}{p}} \frac{\int_{-\theta_p^{-p}}^1 |u|^{-\frac{1}{p}} \min\{|uf(x)|^\varepsilon, |uf(x)|^{-\varepsilon}\} du}{1+\theta_p^{-p}} \\
&= |f(x)|^{-\frac{1}{p}} \min\{|f(x)|^\varepsilon, |f(x)|^{-\varepsilon}\} \frac{\int_{-\theta_p^{-p}}^1 |u|^{-\frac{1}{p}} \frac{\min\{|uf(x)|^\varepsilon, |uf(x)|^{-\varepsilon}\}}{\min\{|f(x)|^\varepsilon, |f(x)|^{-\varepsilon}\}} du}{1+\theta_p^{-p}} \\
&\geq F_\varepsilon(x) \frac{\int_{-\theta_p^{-p}}^1 |u|^{\varepsilon-\frac{1}{p}} du}{1+\theta_p^{-p}} \\
&= \frac{1}{\varepsilon+\frac{1}{p'}} F_\varepsilon(x) \frac{1+\theta_p^{1-p-\varepsilon p}}{1+\theta_p^{-p}},
\end{aligned}$$

where in the fifth equation, we use integration by substitution with $t = \varphi(u)$ (see [3]). When $x \in E^c \cap (-\infty, 0)$, we can treat similarly. Therefore, the claim holds.

Then, from (2.4), it follows that

$$\lim_{\varepsilon\to 0} \frac{\|MF_\varepsilon\|_{L^p(\mathbb{R},\mu)}}{\|F_\varepsilon\|_{L^p(\mathbb{R},\mu)}} \geq p' \frac{1+\theta_p^{1-p}}{1+\theta_p^{-p}} = \theta_p,$$

which means $C_p \geq \theta_p$. Hence, we complete the proof of Theorem 1.2. ∎

Finally, we briefly discuss the $n$-dimensional case. Following the method of Grafakos and Montgomery-Smith [7], we present a complete adaptation of their argument here for completeness. Denote by $\mathbf{x} := (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. For a non-negative locally integrable function $f$ on $\mathbb{R}^n$ and $w_j(t)$ on $\mathbb{R}$, $j = 1, \ldots, n$, satisfying the conditions in Theorem 1.2, we define the "strong" maximal function on $\mathbb{R}^n$, by

$$N_{\vec{\mu}}^n(f)(\mathbf{x}) := \sup_{a_1<x_1<b_1} \cdots \sup_{a_n<x_n<b_n} \frac{\int_{a_n}^{b_n} \ldots \int_{a_1}^{b_1} f(\mathbf{y}) d\vec{\mu}(\mathbf{y})}{\int_{a_n}^{b_n} \ldots \int_{a_1}^{b_1} d\vec{\mu}(\mathbf{y})},$$

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where $d\vec{\mu}(\mathbf{y}) := w_1(y_1)\dots w_n(y_n)dy_1\dots dy_n$. Clearly, $N_{\vec{\mu}}^1 = M_\mu$. Moreover, observe that

$$N_{\vec{\mu}}^n \le M_\mu^{(1)} \circ \cdots \circ M_\mu^{(n)},$$

where $M_\mu^{(j)}$ denotes the maximal operator $M_\mu$ on $\mathbb{R}$ applied to the $x_j$ coordinate. This implies that the operator norm of $N_{\vec{\mu}}^n$ on $L^p(\mathbb{R}^n, \vec{\mu})$ is less than or equal to $\theta_p^n$, $1 < p < \infty$. By considering the function

$$G_\varepsilon(\mathbf{x}) := \prod_{j=1}^n F_\varepsilon(x_j), \quad \mathbf{x} \in \mathbb{R}^n,$$

where $F_\varepsilon$ is as in (2.3), and using an argument similar to Theorem 1.2, we can obtain the reverse inequality. We state this result as follows.

***Corollary 2.4*** *For $1 < p < \infty$, the operator norm of $N_{\vec{\mu}}^n$ on $L^p(\mathbb{R}^n, \vec{\mu})$ is $\theta_p^n$.*

Furthermore, it is easy to verify $\frac{p}{p-1} < \theta_p < \frac{2p}{p-1}$, which implies that the operator norm of $N_{\vec{\mu}}^n$ on $L^p(\mathbb{R}^n, \vec{\mu})$ grows exponentially with $n$, as $n \to \infty$.

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*School of Mathematical Sciences, Xiamen University, Xiamen 361005, People's Republic of China*
*e-mail*: wuzhp@stu.xmu.edu.cn

*School of Mathematics and Statistics, Xiamen University of Technology, Xiamen 361024, People's Republic of China*
*e-mail*: qingdongmath@stu.xmu.edu.cn

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