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ARTICLE

Spanning trees in random regular uniform hypergraphs†

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Abstract

Let $\mathcal{G}_{n,r,s}$ denote a uniformly random r-regular s-uniform hypergraph on the vertex set $\{1, 2, \ldots, n\}$. We establish a threshold result for the existence of a spanning tree in $\mathcal{G}_{n,r,s}$, restricting to n satisfying the necessary divisibility conditions. Specifically, we show that when $s \geqslant 5$, there is a positive constant $\rho(s)$ such that for any $r \geqslant 2$, the probability that $\mathcal{G}_{n,r,s}$ contains a spanning tree tends to 1 if $r > \rho(s)$, and otherwise this probability tends to zero. The threshold value $\rho(s)$ grows exponentially with s. As $\mathcal{G}_{n,r,s}$ is connected with probability that tends to 1, this implies that when $r \leqslant \rho(s)$, most r-regular s-uniform hypergraphs are connected but have no spanning tree. When $s = 3, 4$ we prove that $\mathcal{G}_{n,r,s}$ contains a spanning tree with probability that tends to 1, for any $r \geqslant 2$. Our proof also provides the asymptotic distribution of the number of spanning trees in $\mathcal{G}_{n,r,s}$ for all fixed integers $r, s \geqslant 2$. Previously, this asymptotic distribution was only known in the trivial case of 2-regular graphs, or for cubic graphs.

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1. Introduction

A *hypergraph* $H = (V, E)$ consists of a set of vertices V and a multiset E of non-empty multisubsets of V, which we call *edges*. A hypergraph is *simple* if it has no repeated edges and no edge contains a repeated vertex. We focus on *uniform hypergraphs*, where every edge has the same size, and say that a hypergraph is s-uniform if every edge has size s. A graph is a simple 2-uniform hypergraph. For more background on hypergraphs, see [14].

In graph theory, a tree is a simple connected graph with no cycles, or equivalently a graph with the smallest number of edges amongst all connected graphs on a given vertex set. There are several different ways to generalise this notion to hypergraphs, involving different definitions of acyclicity in hypergraphs [11]. We use Berge acyclicity [7] to define trees in hypergraphs (hypertrees), see Section 2. In particular, the definition implies that any two distinct edges in the tree intersect in at most one vertex, and hence an s-uniform tree with n vertices has exactly $\frac{n-1}{s-1}$ edges. In fact, for n satisfying this divisibility condition, a tree is a connected hypergraph on n vertices with the smallest number of edges, exactly as in the graph case. We note also that our definition of trees in

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hypergraphs matches the definition given by Boonyasombat in [10], while Siu refers to the trees we consider as 'traditional hypertrees' [30, Section 1.2.1].

A spanning tree in a hypergraph H is a spanning subhypergraph of H which is a tree. Just as trees in graphs are well studied and extremely useful objects, trees in hypergraphs have various applications in a wide variety of areas, including game theory [29], relational databases [5], molecular optimisation [21] and network reliability [16]. For example, Warme [32] showed that the Steiner tree problem reduces to finding the minimum spanning tree in a hypergraph.

It is well known that a graph contains a spanning tree if and only if it is connected. However, this relation does not extend to hypergraphs: that is, there exist connected hypergraphs without spanning trees. In fact, our results imply that asymptotically almost all s-uniform regular hypergraphs are like this, provided the degree is not too large (depending on s); see Theorem 1.1 and Lemma 1.4. The property of containing a spanning tree can be thought of as a kind of 'optimal connectedness' of the hypergraph. This property is stronger than the usual notion of connectedness, which can be achieved using substantially overlapping edges and thus causing undesirable redundancy in various applications.

Given $r, s \geqslant 2$, let n be a positive integer such that $s \mid rn$, and let $\Gamma_{n,r,s}$ be the set of r-regular s-uniform simple hypergraphs on $[n]$. Denote by $\mathcal{G}_{n,r,s}$ a hypergraph chosen uniformly at random from $\Gamma_{n,r,s}$. Unless otherwise specified, all asymptotics in this paper are as $n \to \infty$, restricted to values of n, which satisfy the necessary divisibility conditions: that is, restricted to the set

$$\mathcal{N}_{(r,s)} = \{n \in \mathbb{Z}^+ : s \mid rn \text{ and } s-1 \mid n-1\}.$$

Our main result establishes a degree threshold for the existence of a spanning tree in $\mathcal{G}_{n,r,s}$, when $s \geqslant 5$, and proves that $\mathcal{G}_{n,r,s}$ contains a spanning tree with probability which tends to 1 when $s \in \{2, 3, 4\}$, except for the case $(r, s) = (2, 2)$. A 2-regular graph has a spanning tree if and only if it is connected (i.e. forms a Hamilton cycle). Thus it follows from [35, Equation (11)] that as $n \to \infty$,

$$\mathbb{P}(\mathcal{G}_{n,2,2} \text{ contains a spanning tree}) \sim \frac{1}{2} e^{3/4} \sqrt{\frac{\pi}{n}} \to 0. \tag{1}$$

Theorem 1.1. *Let $s \geqslant 2$ be a fixed integer. If $s \geqslant 5$ then there exists a positive constant $\rho(s)$ such that for any fixed integer $r \geqslant 2$, as $n \to \infty$ along $\mathcal{N}_{(r,s)}$,*

$$\mathbb{P}\left(\mathcal{G}_{n,r,s} \text{ contains a spanning tree}\right) \longrightarrow \begin{cases} 1 & \text{if } r > \rho(s), \\ 0 & \text{if } r \leqslant \rho(s). \end{cases}$$

Specifically, $\rho = \rho(s)$ is the unique real number in $(2, \infty)$ such that

$$(s-1)^{\rho}(\rho-1)^{s(\rho-1)} = \rho^{\rho s-\rho-s}(\rho s-\rho-s)^{(\rho s-\rho-s)/(s-1)}. \tag{2}$$

For $s \geqslant 5$ we have $\rho^-(s) < \rho(s) < \rho^+(s)$ where

$$\rho^-(s) = \frac{e^{s-2}}{s-1} - \frac{s-1}{2}, \qquad \rho^+(s) = \frac{e^{s-2}}{s-1} - \frac{s-3}{2}.$$

Furthermore, as $s \to \infty$,

$$\rho(s) = \frac{e^{s-2}}{s-1} - \frac{s^2-3s+1}{2(s-1)} + O_s(s^4\, e^{-s}).$$

Finally, if $s \in \{2, 3, 4\}$ then for any fixed integer $r \geqslant 2$,

$$\mathbb{P}\left(\mathcal{G}_{n,r,s} \text{ contains a spanning tree}\right) \longrightarrow \begin{cases} 1 & \text{if } (r, s) \neq (2, 2), \\ 0 & \text{if } (r, s) = (2, 2). \end{cases}$$

The value of $\rho(s)$ for $s = 5, \ldots, 12$ is shown in Table 1, together with the bounds $\rho^-(s)$ and $\rho^+(s)$. All values are rounded to three decimal places.

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Table 1. Values of $\rho(s)$ for $s = 5, \ldots, 12$, together with our bounds

s	5	6	7	8	9	10	11	12
$\rho^-(s)$	3.021	8.420	21.736	54.133	133.079	326.718	805.308	1996.906
$\rho(s)$	3.029	8.706	22.142	54.606	133.588	327.245	805.844	1997.444
$\rho^+(s)$	4.021	9.420	22.736	55.133	134.079	327.718	806.308	1997.906

Let $Y_{\mathcal{G}}$ be the number of spanning trees in $\mathcal{G}_{n,r,s}$. This random variable is our main object of study. Using asymptotic enumeration methods, Aldosari and Greenhill [2, Corollary 1.2.] established the following asymptotic expression for $\mathbb{E}Y_{\mathcal{G}}$ when $s \geqslant 3$:

$$\mathbb{E}Y_{\mathcal{G}} \sim \exp\left(\frac{rs-r-1}{2(r-1)}\right) \frac{(s-1)\sqrt{r-1}}{n\,(rs-r-s)^{\frac{s+1}{2(s-1)}}} \left(\frac{(s-1)^r\,(r-1)^{(r-1)s}}{r^{rs-r-s}\,(rs-r-s)^{\frac{rs-r-s}{s-1}}}\right)^{n/s}. \tag{3}$$

(In fact a more general result is proved in [2], which covers irregular degree sequences and allows s and the maximum degree to grow slowly with n.) In the graph case, the asymptotic formula for $\mathbb{E}Y_{\mathcal{G}}$ was known up to a constant factor by the results of McKay [24] (who also considered irregular, slowly growing degrees), and then this constant factor was calculated precisely in [19, Theorem 1.1].

The argument used to prove Theorem 1.1 also provides the asymptotic distribution of $Y_{\mathcal{G}}$, for any parameters $r, s \geqslant 2$.

Theorem 1.2. *Let $r, s, \geqslant 2$ be fixed integers. For all positive integers j, define*

$$\lambda_j = \frac{(r-1)^j(s-1)^j}{2j} \quad \text{and} \quad \zeta_j = \frac{\left(\frac{r}{r-1} - s + 1\right)^j - 2}{(r-1)^j(s-1)^j}.$$

Let $J(2) = 3$ and $J(s) = 2$ for $s \geqslant 3$. If $\mathbb{P}(Y_{\mathcal{G}} > 0) \to 1$ as $n \to \infty$ along $\mathcal{N}_{(r,s)}$, then the asymptotic distribution of $Y_{\mathcal{G}}$ satisfies

$$\frac{Y_{\mathcal{G}}}{\mathbb{E}Y_{\mathcal{G}}} \xrightarrow{d} \prod_{j=J(s)}^{\infty} (1+\zeta_j)^{Z_j}\, e^{-\lambda_j\zeta_j},$$

where $Z_j = \mathrm{Po}\,(\lambda_j)$ are independent Poisson random variables. Otherwise, the asymptotic distribution of $Y_{\mathcal{G}}$ is a point mass at zero.

Previously, the result of Theorem 1.2 was only known for two values of (r, s): when $(r, s) = (2, 2)$ the result follows trivially from (1), while for $(r, s) = (3, 2)$ (cubic graphs), the asymptotic distribution of $Y_{\mathcal{G}}$ was obtained by Greenhill, Kwan and Wind [19, Theorem 1.2]. The authors of [19] also conjectured an expression for the asymptotic distribution of $Y_{\mathcal{G}}$ when $s = 2$ and $r \geqslant 4$. Substituting $s = 2$ into Theorem 1.2 verifies that their conjecture is true.

Corollary 1.3. *The conjecture given in* [19, Conjecture 1.3] *holds. That is, the number $Y_{\mathcal{G}}$ of spanning trees in a random r-regular graph satisfies*

$$\frac{Y_{\mathcal{G}}}{\mathbb{E}Y_{\mathcal{G}}} \xrightarrow{d} \prod_{j=3}^{\infty} (1+\zeta_j)^{Z_j}\, e^{-\lambda_j\zeta_j}$$

with

$$\lambda_j = \frac{(r-1)^j}{2j}, \qquad \zeta_j = -\frac{2(r-1)^j - 1}{(r-1)^{2j}}$$

for $j \geqslant 3$.

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Theorems 1.1 and 1.2 are proved using the small subgraph conditioning method [26]. While many structural results about random regular graphs have been proved using this method (see [20, 35] for surveys), there are only two previously known results for hypergraphs with $s \geqslant 3$. Cooper, Frieze, Molloy and Reed [13] gave a threshold result for the existence of a perfect matching in $\mathcal{G}_{n,r,s}$, while Altman, Greenhill, Isaev and Ramadurai [3] proved a threshold result for the existence of loose Hamilton cycles in $\mathcal{G}_{n,r,s}$. Keeping only the most significant term, the threshold values for spanning trees, loose Hamilton cycles and perfect matchings are approximately

$$\frac{e^{s-2}}{s-1} \quad \text{(spanning trees)}, \qquad \frac{e^{s-1}}{s-1} \quad \text{(loose Hamilton cycles)}, \qquad e^{s-1} \quad \text{(perfect matchings)}$$

respectively. Hence (restricting to values of n satisfying the relevant divisibility conditions in each case), for a fixed s, as r increases, spanning trees appear first, followed by loose Hamilton cycles and then perfect matchings.

We close this section with some comments on connectedness in random regular uniform hypergraphs. It is well known that with probability tending to 1, random r-regular graphs are connected (indeed, r-connected) whenever $r \geqslant 3$, see [9, 34]. Dumitriu and Zhu [15] recently used spectral methods to investigate expansion properties of $\mathcal{G}_{n,r,s}$, and hence inferred [15, Lemma 6.2] that $\mathbb{P}(\mathcal{G}_{n,r,s} \text{ is connected}) \to 1$ when $r \geqslant s \geqslant 3$. For completeness we sketch a more elementary argument, which covers all $(r, s) \neq (2, 2)$.

Lemma 1.4. *Let $r, s \geqslant 2$ be fixed integers with $(r, s) \neq (2, 2)$. Then*

$$\mathbb{P}(\mathcal{G}_{n,r,s} \textit{ is connected}) \to 1$$

as n tends to infinity along $\mathcal{N}_{(r,s)}$.

Proof. Suppose that $A \subset [n]$ is a subset of vertices with $|A| = a$, where $1 \leqslant a \leqslant n/2$. The probability that $(A, [n] - A)$ is a cut in $\mathcal{G}_{n,r,s}$ is

$$\frac{|\Gamma_{a,r,s}|\,|\Gamma_{n-a,r,s}|}{|\Gamma_{n,r,s}|}. \tag{4}$$

It follows from (8), (9) and (10) that

$$|\Gamma_{n,r,s}| = \Theta(1)\ \frac{(rn)!}{(rn/s)!\,(s!)^{rn/s}\,(r!)^n}.$$

Substituting this into (4) and summing over all A with $1 \leqslant |A| \leqslant n/2$, we conclude that the probability that $\mathcal{G}_{n,r,s}$ is disconnected is $o(1)$ whenever $r, s \geqslant 2$ and $(r, s) \neq (2, 2)$. □

2. Preliminaries

Throughout, $\mathbb{N}$ denotes the non-negative integers and $(a)_b = a(a-1)\cdots(a-b+1)$ is the falling factorial.

A 1-*cycle* (or *loop*) is a hypergraph consisting of one edge, which contains a repeated vertex. A 2-*cycle* is a hypergraph consisting of two edges, which intersect in at least two vertices. For $j \geqslant 3$, a j-*cycle* is a hypergraph with j edges, which can be labelled $e_1, e_2, \ldots, e_j$ such that there exists distinct vertices $v_1, \ldots, v_j$ where $v_i \in e_i \cap e_{i+1}$ for $i = 1, \ldots, j$ (where $e_{j+1} \equiv e_1$).

A *(Berge) path* in H consists of a sequence $v_0, e_1, v_1, e_2, \ldots, e_j, v_j$ where $v_0, v_1, \ldots, v_j$ are distinct vertices, $e_1, \ldots, e_j$ are distinct edges and $v_{i-1}, v_i \in e_i$ for all $i = 1, \ldots, j$. A hypergraph is *connected* if there is a path between every pair of vertices. A *spanning tree* T in a hypergraph H is a connected spanning subhypergraph of H, which contains no j-cycles for all positive integers j. In particular, as T contains no 2-cycles it follows that edges of T overlap in at most one vertex.

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An s-uniform tree with t edges has $(s-1)t+1$ vertices, and the number of (labelled) s-uniform trees on $n=(s-1)t+1$ vertices is given by

$$\frac{n^{t-1}\,(n-1)!}{t!\,((s-1)!)^t}. \tag{5}$$

This formula was proved in [22, Corollary 1] and [31, Theorem 2]. See also [28]. Note that in the graph case $s=2$ we recover Cayley's formula.

Suppose that $\boldsymbol{\delta}=(\delta_1,\dots,\delta_n)$ is the degree sequence of an s-uniform tree on n vertices. Then $\boldsymbol{\delta}$ is a sequence of n positive integers such that

$$\delta_1+\cdots+\delta_n=\frac{s(n-1)}{s-1}=st,$$

and any such sequence $\boldsymbol{\delta}$ is called a *tree degree sequence*. Bacher [4, Theorem 1.1] proved that the number of labelled s-uniform trees on $n=(s-1)t+1$ vertices with degree sequence $\boldsymbol{\delta}$ is

$$\binom{t-1}{\delta_1-1,\dots,\delta_n-1}\frac{(n-1)!}{t!\,((s-1)!)^t}=\frac{(s-1)(n-2)!}{((s-1)!)^{\frac{n-1}{s-1}}}\prod_{i=1}^{n}\frac{1}{(\delta_i-1)!}. \tag{6}$$

This generalises the formula for the graph case proved by Moon [25].

Now we introduce some special families of cycles, which will be used in our analysis. An s-uniform 1-cycle is *loose* if the edge contains $s-1$ distinct vertices, and an s-uniform 2-cycle is *loose* if the intersection of the two edges has size 2. For $j\geqslant 3$, an s-uniform j-cycle is *loose* if

$$|e_k\cap e_\ell|=\begin{cases}1 \text{ if } k-\ell\equiv\pm1 \pmod j,\\ 0 \text{ otherwise}\end{cases}$$

for $k\neq\ell$. A loose j-cycle C contains $(s-1)j$ vertices.

Let v be a vertex in a loose j-cycle C. We say v is *C-external* (or *external*) if v has degree 2 in C. Otherwise, we say v has degree 1 and we say it is *C-internal* (or *internal*).

2.1 Configuration model

The configuration model for regular uniform hypergraphs is a generalisation of the configuration model for graphs, introduced by by Bollobás [8]. Take rn *points* in n cells, $B_1,\dots,B_n$, each containing r points. Then partition the points into rn/s subsets of size s, called *parts*. Each such partition P corresponds to a hypergraph $G(P)$, which may not be simple, obtained by replacing each cell by a vertex and replacing each part $\{a_1,\dots,a_s\}$ in the partition by an edge $\{v_{i_1},\dots,v_{i_s}\}$ such that $a_j\in B_{i_j}$ for $j=1,\dots,s$. If a part contains more than one point from the same cell then the corresponding edge in P is a loop, and if the partition has two parts that contain points from precisely the same cells, with the same multiplicity, then these parts produce a repeated edge in $G(P)$.

Each simple hypergraph corresponds to precisely $(r!)^n$ partitions, so an r-regular s-uniform simple hypergraph can be chosen uniformly at random by choosing a partition uniformly at random and rejecting the result if it has loops or multiple edges. A partition P is said to be simple if $G(P)$ is a simple hypergraph.

Denote the set of possible partitions by $\Omega_{n,r,s}$, and let $\mathcal{P}_{n,r,s}$ be a partition chosen uniformly at random from $\Omega_{n,r,s}$.

A *subpartition* P' is a subset of a partition $P\in\Omega_{n,r,s}$. A subpartition of P projects to a subhypergraph of $G(P)$.

The configuration model allows us to prove some properties of $\mathcal{G}_{n,r,s}$ by performing computations in $\mathcal{P}_{n,r,s}$. When $s\mid t$, the number of partitions of a set of t points into t/s parts of size s is

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$$p(t) = \frac{t!}{(t/s)!(s!)^{t/s}}.$$

Hence the number of partitions in $\Omega_{n,r,s}$ is

$$\left|\Omega_{n,r,s}\right| = p(rn) = \frac{(rn)!}{(rn/s)!\,(s!)^{rn/s}}. \tag{7}$$

Therefore, the number of r-regular s-uniform simple hypergraphs on n vertices is precisely

$$\left|\Gamma_{n,r,s}\right| = \frac{(rn)!\,\mathbb{P}(\text{Simple})}{(rn/s)!\,(s!)^{rn/s}(r!)^n}, \tag{8}$$

where 'Simple' is the event that the partition is simple. Thus, an asymptotic formula for $\left|\Gamma_{n,r,s}\right|$ can be found by estimating $\mathbb{P}(\text{Simple})$. When $s = 2$ the event "Simple" means no 1-cycles or 2-cycles, and Bender and Canfield [6] showed that

$$\mathbb{P}(\text{Simple}) \sim \exp\left(-(r^2-1)/4\right). \tag{9}$$

Cooper, Frieze, Molloy and Reed showed in [13] that for fixed integers $r \geqslant 2$ and $s \geqslant 3$,

$$\mathbb{P}(\text{Simple}) \sim \exp\left(-(r-1)(s-1)/2\right). \tag{10}$$

2.2 Small subgraph conditioning method for hypergraphs

Robinson and Wormald showed in [26, 27] that almost all r-regular graphs are Hamiltonian, for any fixed $r \geqslant 3$, using an analysis of variance technique now known as the *small subgraph conditioning method*. We restate the small subgraph conditioning method from [20] (with slightly different notation).

Theorem 2.1 (Janson [20, Theorem 1]). *Let $\lambda_j > 0$ and $\zeta_j \geqslant -1$, $j = 1, 2, \ldots$, be constants and suppose that for each n there are random variables $X_{j,n}$, $j = 1, 2, \ldots$, and Y_n (defined on the same probability space) such that $X_{j,n}$ is a non-negative integer valued and $\mathbb{E}Y_n \neq 0$ (at least for large n), and furthermore the following conditions are satisfied:*

(A1) $X_{j,n} \xrightarrow{d} Z_j$ *as* $n \to \infty$ *jointly for all j, where* $Z_j \sim \text{Po}(\lambda_j)$ *are independent Poisson random variables;*

(A2) *For any finite sequence* $x_1, \ldots, x_m$ *of non-negative integers,*

$$\frac{\mathbb{E}(Y_n | X_{1,n} = x_1, \ldots, X_{m,n} = x_m)}{\mathbb{E}Y_n} \to \prod_{j=1}^{m} (1+\zeta_j)^{x_j} e^{-\lambda_j \zeta_j} \quad \text{as } n \to \infty;$$

(A3) $\sum_{j \geqslant 1} \lambda_j \zeta_j^2 < \infty$;

(A4) $\frac{\mathbb{E}Y_n^2}{(\mathbb{E}Y_n)^2} \to \exp\left(\sum_{j \geqslant 1} \lambda_j \zeta_j^2\right)$ *as* $n \to \infty$.

Then

$$\frac{Y_n}{\mathbb{E}Y_n} \xrightarrow{d} W = \prod_{j=1}^{\infty} (1+\zeta_j)^{Z_j} e^{-\lambda_j \zeta_j} \quad \text{as } n \to \infty;$$

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moreover, this and the convergence in (A1) *hold jointly. The infinite product defining W converges asymptotically almost surely and in L^2, with*

$$\mathbb{E}W = 1 \qquad \textit{and} \qquad \mathbb{E}W^2 = \exp\left(\sum_{j\geqslant 1}\lambda_j\zeta_j^2\right) = \lim_{n\to\infty}\frac{\mathbb{E}Y_n^2}{(\mathbb{E}Y_n)^2}.$$

Furthermore, the event $W = 0$ equals, up to a set of probability zero, the event that $Z_j > 0$ for some j with $\zeta_j = -1$. In particular, $W > 0$ almost surely if and only if every $\zeta_j > -1$.

(In the above statement, we have corrected a typographical error from [20], which had $W > 0$ instead of $W = 0$ in the second-last sentence.)

Janson remarks in [20] that the index set $\mathbb{Z}^+$ may be replaced by any other countably infinite set, and that 0^0 is defined to be 1. We will apply Theorem 2.1 with the following random variables:

- Let Y be the number of s-uniform spanning trees in a random partition $P \in \mathcal{P}_{n,r,s}$.
- Let X_1 be the number of 1-cycles in a random partition $P \in \mathcal{P}_{n,r,s}$.
- For $j \geqslant 2$, let X_j be the number of loose j-cycles in a random partition $P \in \mathcal{P}_{n,r,s}$.

With $r, s \geqslant 2$ fixed, it is well known that $X_j \to Z_j$ as $n \to \infty$, where Z_j are asymptotically independent Poisson random variables with mean

$$\lambda_j = \frac{(r-1)^j(s-1)^j}{2j}. \tag{11}$$

This was proved for graphs ($s = 2$) by Bollobás [8], and by Cooper, Frieze, Molloy and Reed [13] when $s \geqslant 3$. To be more precise, Cooper, Frieze, Molloy and Reed worked with the random variable X'_j, the number of j-cycles (not necessarily loose), and showed that X'_j has the same asymptotic distribution as X_j, as the contribution to X'_j from non-loose j-cycles forms a negligible fraction of X'_j. This verifies that (A1) of Theorem 2.1 holds.

In order to verify condition (A2), the following lemma is helpful.

Lemma 2.2 (Janson [20, Lemma 1]). *Let $\lambda'_j \geqslant 0$, $j = 1, 2, \ldots$ be constants. Suppose that* (A1) *holds, that $Y_n \geqslant 0$ and that*

$$\text{(A2}')\qquad \frac{\mathbb{E}(Y_n(X_{1,n})_{x_1}\cdots(X_{m,n})_{x_m})}{\mathbb{E}Y_n} \to \prod_{j=1}^{m}(\lambda'_j)^{x_j} \quad \textit{as } n\to\infty,$$

for every finite sequence $x_1, \ldots, x_m$ of non-negative integers. Then condition (A2) *holds with $\lambda'_j = \lambda_j(1+\zeta_j)$.*

There are some challenges when applying the small subgraph conditioning method to regular uniform hypergraphs with $s \geqslant 3$. In the graph case, a partition is simple precisely when $X_1 = X_2 = 0$. For hypergraphs with $s \geqslant 3$, this is no longer true: a hypergraph is simple if and only if it has no 1-cycles and no repeated edges: some 2-cycles are allowed, as long as the two edges overlap in between 2 and $s-1$ vertices.

Fortunately, we can translate some asymptotic properties from the configuration model to random hypergraphs. For any event $\mathcal{E} \subseteq \Omega_{n,r,s}$, we have

$$\mathbb{P}(P \in \mathcal{E} \mid \text{Simple}) \leqslant \frac{\mathbb{P}(P \in \mathcal{E})}{\mathbb{P}(\text{Simple})}. \tag{12}$$

Altman, Greenhill, Isaev and Ramadurai proved the following lemma in [3].

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Lemma 2.3 ([3, Lemma 2.1]). *Fix integers $r, s \geqslant 2$. For any positive integer n such that $s \mid rn$, let $\widehat{P}$ be a uniformly random partition in $\Omega_{n,r,s}$ with no 1-cycles, let P_S be a uniformly random simple partition in $\Omega_{n,r,s}$ and let P be a uniformly random partition in $\Omega_{n,r,s}$. Let $Y : \Omega_{n,r,s} \to \mathbb{Z}$ be a random variable. Then as $n \to \infty$ along integers such that $s \mid rn$, the following two properties hold.*

(a) *If $\mathbb{P}(Y(P) \in A) = o(1)$, then $\mathbb{P}(Y(P_S) \in A) = o(1)$ for any $A \subseteq \mathbb{Z}$.*
(b) *$\mathbb{P}(Y(\widehat{P}) \in A) - \mathbb{P}(Y(P_S) \in A) = o(1)$ for any $A \subseteq \mathbb{Z}$.*

Property (a) follows from (12), while property (b) follows from (10) and the fact that the probability that two parts in a random partition $P \in \mathcal{P}_{n,r,s}$ give rise to a repeated edge is $o(1)$ (as remarked by Cooper, Frieze, Molloy and Reed in [13]).

Property (b) essentially tells us that the distribution of Y that arises from conditioning on $X_1 = 0$ is asymptotically equivalent to the distribution Y conditioned on 'Simple'. This allows us to apply the following corollary, very slightly adapted from [3, Corollary 2.6], which will be useful in the proof of Theorem 1.2.

Corollary 2.4. *Suppose that Y_n and $X_{j,n}$ satisfy conditions* (A1)–(A4) *of Theorem 2.1. Let $\widehat{Y}_n$ be the random variable obtained from Y_n by conditioning on the event $X_{1,n} = 0$. Then*

$$\frac{\widehat{Y}_n}{\mathbb{E}Y_n} \xrightarrow{d} e^{-\lambda_1\zeta_1} \prod_{j=2}^{\infty} (1+\zeta_j)^{Z_j} e^{-\lambda_j\zeta_j} \quad \text{as} \quad n \to \infty.$$

Moreover, if $\zeta_j > -1$ for all $j \geqslant 2$ then asymptotically almost surely $\widehat{Y}_n > 0$.

Proof. The statement of [3, Corollary 2.6] made the assumption that $\zeta_j > -1$ for all $j \geqslant 1$, and the proof used the final statement of Theorem 2.1. However, we can drop the assumption that $\zeta_1 > -1$ if we instead apply the second-last statement from Theorem 2.1. □

Before proceeding, we outline the structure of the rest of the paper. The first moment is calculated in Section 3, and the impact of short cycles on the number of spanning trees is investigated in Section 4. The second moment is calculated in Section 5. Finally, in Section 6 we analyse the value of the threshold when $s \geqslant 5$, and prove Theorems 1.1 and 1.2. Some technical proofs have been omitted here but can be found in [17, Appendix A].

3. First moment

We fix integers $r, s \geqslant 2$, where $(r, s) \neq (2, 2)$, and work in the configuration model $\mathcal{P}_{n,r,s}$, where $s \mid rn$ and $n = (s-1)t + 1$ for some $t \in \mathbb{N}$.

Lemma 3.1. *Let $r, s \geqslant 2$ be fixed integers with $(r, s) \neq (2, 2)$. Then as $n \to \infty$ along $\mathcal{N}_{(r,s)}$,*

$$\mathbb{E}Y \sim \frac{(s-1)\sqrt{r-1}}{n\,(rs-r-s)^{\frac{s+1}{2(s-1)}}} \left(\frac{(s-1)^r (r-1)^{(r-1)s}}{r^{rs-r-s}(rs-r-s)^{\frac{rs-r-s}{s-1}}} \right)^{n/s}.$$

Proof. Let $\mathcal{T}_n$ be the set of all s-uniform trees on n vertices. For a random partition $P \in \mathcal{P}_{n,r,s}$, we can write

$$\mathbb{E}Y = \sum_{P_T : G(P_T) \in \mathcal{T}_n} \mathbb{P}(P_T \subseteq P) = \sum_{P_T : G(P_T) \in \mathcal{T}_n} \frac{\left|\{P \in \Omega_{n,r,s} : P_T \subseteq P\}\right|}{\left|\Omega_{n,r,s}\right|}.$$

Selecting P_T uses up st points so, for a given P_T, the size of $\{P \in \Omega_{n,r,s} : P_T \subseteq P\}$ is $p(rn - st)$. This is the number of ways to partition the remaining $rn - st$ points after the points of the tree are

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selected. Recall that $\mathcal{D}_n$ is the set of possible tree degree sequences on n vertices. Given $\boldsymbol{\delta} \in \mathcal{D}_n$, define $\mathcal{T}_n(\boldsymbol{\delta})$ to be the set of trees with a degree sequence $\boldsymbol{\delta}$. We can write

$$\left|\Omega_{n,r,s}\right| \mathbb{E}Y = p(rn-st) \sum_{\boldsymbol{\delta}\in\mathcal{D}_n} \sum_{T\in\mathcal{T}_n(\boldsymbol{\delta})} \sum_{P_T\,:\,G(P_T)=T} 1. \tag{13}$$

Consider a subpartition that projects to a given tree $T \in \mathcal{T}_n(\boldsymbol{\delta})$, for a given $\boldsymbol{\delta} \in \mathcal{D}_n$. Exactly δ_i of the points in cell i must contribute to P_T, and there are $(r)_{\delta_j}$ ways to choose and order these points. So there are $\prod_{j=1}^{n} (r)_{\delta_j}$ possible subpartitions P_T, which project to the given spanning tree T. Therefore, using (6) for the second line, the number of subpartitions P_T, which project to some spanning tree is

$$\begin{aligned}
\sum_{\boldsymbol{\delta}\in\mathcal{D}_n} |\mathcal{T}_n(\boldsymbol{\delta})| \prod_{j=1}^{n} (r)_{\delta_j} &= \frac{(s-1)(n-2)!}{((s-1)!)^t} \left(\sum_{\boldsymbol{\delta}\in\mathcal{D}_n} \prod_{j=1}^{n} \frac{(r)_{\delta_j}}{(\delta_j-1)!} \right) \\
&= \frac{(s-1)(n-2)!}{((s-1)!)^t} [z^{st}] \left(\sum_{i=1}^{\infty} \frac{(r)_i}{(i-1)!} z^i \right)^n \\
&= \frac{(s-1)(n-2)!}{((s-1)!)^t} [z^{st}] \left(\sum_{i=1}^{\infty} rz \binom{r-1}{i-1} z^{i-1} \right)^n \\
&= \frac{r^n (s-1)(n-2)!}{((s-1)!)^t} \binom{(r-1)n}{t-1}.
\end{aligned} \tag{14}$$

Here square brackets denotes coefficient extraction. Substituting (14) into (13) and applying (7) gives, by definition of t,

$$\begin{aligned}
\mathbb{E}Y &= \frac{p(rn-st)}{p(rn)} \cdot \frac{r^n (s-1)(n-2)!}{((s-1)!)^t} \binom{(r-1)n}{t-1} \\
&= r^n s^{\frac{n-1}{s-1}} \frac{((r-1)n)!\,(n-1)!\,(rn/s)!}{(rn)!\,\left(\frac{n-1}{s-1}\right)!\,\left(\frac{(rs-r-s)n+s}{s(s-1)}\right)!}.
\end{aligned} \tag{15}$$

The result follows by applying Stirling's approximation. □

For future reference we note that by (7), Lemma 3.1 and Stirling's approximation,

$$\left|\Omega_{n,r,s}\right| \mathbb{E}Y \sim \frac{(s-1)\sqrt{(r-1)s}}{n(rs-r-s)^{\frac{s+1}{2(s-1)}}} \left(\frac{r^s\,(r-1)^{(r-1)s}\,n^{r(s-1)}}{(rs-r-s)^{\frac{rs-r-s}{s-1}}\,((s-2)!)^r\,e^{r(s-1)}} \right)^{n/s}. \tag{16}$$

In Section 6 we characterise pairs (r,s) for which $\mathbb{E}Y$ tends to infinity. This provides the threshold function $\rho(s)$ when $s \geqslant 5$ and, using (12), provides the negative half of the threshold result. In order to complete the proof, we must apply small subgraph conditioning.

4. Effect of short cycles

Fix a positive integer m and a sequence $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{N}^m$. Write $\ell = x_1 + \cdots + x_m$. Let $\mathcal{S}(\boldsymbol{x})$ be the set of sequences $(P_1, \ldots, P_\ell)$ of subpartitions such that $G(P_1), \ldots, G(P_{x_1})$ are distinct 1-cycles, and

$$G(P_{x_1+\cdots+x_{j-1}+1}), \ldots, G(P_{x_1+\cdots+x_{j-1}+x_j})$$

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are distinct loose j-cycles, for $j = 2, \dots, m$. Then let $\mathcal{S}^*(\boldsymbol{x})$ be the set of all $(P_1, \dots, P_\ell) \in \mathcal{S}(\boldsymbol{x})$ such that the cycles $G(P_1), \dots, G(P_\ell)$ are vertex disjoint. For a random partition $P \in \mathcal{P}_{n,r,s}$, we can write

$$\mathbb{E}[Y(X_1)_{x_1} \cdots (X_m)_{x_m}] = \sum_{(P_1,\dots,P_\ell)\in\mathcal{S}(\boldsymbol{x})} \sum_{P_T\,:\,G(P_T)\in\mathcal{T}_n} \mathbb{P}(P_1 \cup \cdots \cup P_\ell \cup P_T \subseteq P). \tag{17}$$

We will find that $\mathbb{E}[Y(X_1)_{x_1} \cdots (X_m)_{x_m}]$ is asymptotically dominated by the contribution from vertex-disjoint cycles. So we first evaluate

$$\begin{aligned}\Sigma^*(\boldsymbol{x}) &= \sum_{(P_1,\dots,P_\ell)\in\mathcal{S}^*(\boldsymbol{x})} \sum_{P_T\,:\,G(P_T)\in\mathcal{T}_n} \mathbb{P}(P_1 \cup \cdots \cup P_\ell \cup P_T \subseteq P) \\ &= \sum_{(P_1,\dots,P_\ell)\in\mathcal{S}^*(\boldsymbol{x})} \sum_{P_T\,:\,G(P_T)\in\mathcal{T}_n} \frac{\left|\{P \in \Omega_{n,r,s} : P_1 \cup \cdots \cup P_\ell \cup P_T \subseteq P\}\right|}{\left|\Omega_{n,r,s}\right|}. \end{aligned} \tag{18}$$

To perform this count, we condition on the intersections between P_T and each of the subpartitions $P_1, \dots, P_\ell$. We will use the lexicographical ordering on s-subsets of vertices to define a corresponding ordering on parts of P_i, by applying the lexicographical ordering to the set (or multiset) of s cells corresponding to the s points in the part. First suppose that $G(P_i)$ is a loose j-cycle with $j \geqslant 2$. The part of P_i, which is lexicographically least will be the starting part of P_i, and we fix a direction around P_i such that the second part visited is lexicographically smaller than the last part. Then we define a binary sequence $I_i \in \{0,1\}^j$ corresponding to P_i as follows: starting from the first part of P_i, in the fixed direction, if the k'th part of P_i belongs to $P_i \cap P_T$ then the k'th element of I_i is one; otherwise it is zero. All sequences in $\{0,1\}^j$ represent possible intersections, except for $(1, \dots, 1)$ because a tree contains no cycles. In the case that $G(P_i)$ is a 1-cycle then $I_i = (0)$. Denote the set of all possible intersection sequences for a cycle of length j by

$$\mathcal{I}_j = \{0,1\}^j \setminus \{(1, \dots, 1)\}$$

and define the Cartesian product

$$\mathcal{I}(\boldsymbol{x}) = \prod_{j=1}^{m} \mathcal{I}_j^{x_j}.$$

For $I \in \mathcal{I}_j$, let $U(I) \in \{1, \dots, j\}$ be the number of entries in I which equal zero. Given $\boldsymbol{I} = (I_1, \dots, I_\ell) \in \mathcal{I}(\boldsymbol{x})$, let $u_i = U(I_i)$for $i = 1, \dots, m$ and define $u = u(\boldsymbol{I}) = u_1 + \cdots + u_\ell$.

Given $(P_1, \dots, P_\ell, P_T)$, write $\iota(P_1, \dots, P_\ell, P_T) = \boldsymbol{I} \in \mathcal{I}(\boldsymbol{x})$ for the corresponding ℓ-tuple of intersection sequences. We can rewrite $\Sigma^*(\boldsymbol{x})$ as

$$\left|\Omega_{n,r,s}\right| \ \Sigma^*(\boldsymbol{x}) = \sum_{\boldsymbol{I}\in\mathcal{I}(\boldsymbol{x})} \sum_{\substack{(P_1,\dots,P_\ell,P_T):\\ (P_1,\dots,P_\ell)\in\mathcal{S}^*(\boldsymbol{x}),\\ G(P_T)\in\mathcal{T}_n,\\ \iota(P_1,\dots,P_\ell,P_T)=\boldsymbol{I}}} \left|\{P \in \Omega_{n,r,s} : P_1 \cup \cdots \cup P_\ell \cup P_T \subseteq P\}\right|.$$

For a given $\boldsymbol{I} \in \mathcal{I}(\boldsymbol{x})$, we evaluate the inner sum using the following process:

Step 1: Choose a sequence $(P_1, \dots, P_\ell) \in \mathcal{S}^*(\boldsymbol{x})$.
Step 2: Choose P_T with $G(P_T) \in \mathcal{T}_n$ such that $\iota(P_1, \dots, P_\ell, P_T) = \boldsymbol{I}$.
Step 3: Partition the remaining points arbitrarily.

Define the subpartition $Q = P_1 \cup \cdots \cup P_\ell$. Writing $\left|P'\right|$ for the number of parts in a subpartition P', we have

$$|Q| = \sum_{j=1}^{m} j\, x_j = \sum_{i=1}^{\ell} |P_i|\,.$$

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Lemma 4.1. *Let $r, s \geqslant 2$ be integers such that $(r,s) \neq (2,2)$ and fix $\boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{N}^m$. The number of ways to choose a sequence of subpartitions $(P_1, \dots, P_\ell)$ in $\mathcal{S}^*(\boldsymbol{x})$ is*

$$s_1(\boldsymbol{x}) \sim \left(\frac{(r-1)\, r^{s-1} n^{s-1}}{(s-2)!}\right)^{|Q|} \prod_{j=1}^{m} \frac{1}{(2j)^{x_j}}. \tag{19}$$

Proof. To begin, we claim that (19) is true when $\ell = 1$. First suppose that $j \geqslant 2$. We must show that the number of ways of selecting a subpartition P, which projects to a loose j-cycle C is asymptotically equal to

$$\frac{1}{2j}\left(\frac{(r-1)\, r^{s-1} n^{s-1}}{(s-2)!}\right)^{j}.$$

Recall that, in a loose j-cycle C, the C-external vertices have degree 2 and the C-internal vertices have degree 1. To specify a single loose j-cycle C, choose a sequence of $(s-1)j$ vertices in $(n)_{(s-1)j} \sim n^{(s-1)j}$ ways, then divide by $2j((s-2)!)^j$. Here, division by $2j$ adjusts for direction and starting point (where a starting point is a C-external vertex), and division by $((s-2)!)^j$ adjusts for the order of the $s-2$ C-internal vertices in each edge. To specify P_C, we choose two points for each C-external vertex and one point for each C-internal vertex in the configuration model, in $(r(r-1))^j r^{(s-2)j}$ ways. Hence (19) holds for a single loose j-cycle when $j \geqslant 2$.

When $j = 1$, the number of non-loose 1-cycles is $O(n^{s-2})$ while the number of loose 1-cycles is $\Theta(n^{s-1})$. There are $n\binom{n-1}{s-2} \sim \frac{n^{s-1}}{(s-2)!}$ ways to choose the vertices of a loose 1-cycle, where the first-chosen vertex is external, then there are $\binom{r}{2} r^{s-2}$ ways to choose points corresponding to these vertices. Multiplying these shows that (19) also holds when $j = 1$.

When $\ell > 1$, observe that this process can be iterated. The only change is that the next cycle must be disjoint from all previously selected cycles, ruling out $O(1)$ vertices. Hence the number of ways to select a sequence of $(s-1)j$ vertices for the next j-cycle is $\big(n - O(1)\big)^{(s-1)j} \sim n^{(s-1)j}$, and all remaining calculations are the same as above. This shows that (19) holds in general. □

Now suppose that a sequence of subpartitions $(P_1, \dots, P_\ell) \in \mathcal{S}^*(\boldsymbol{x})$ has been chosen. To perform Step 2, we construct an irregular configuration model $\mathcal{P}_{n',\boldsymbol{x}}$ from the points that are so far unused.

There are $n - (s-1)\,|Q|$ cells, which are not involved in any of $(P_1, \dots, P_\ell)$. Any cell which corresponds to an external vertex of some $G(P_i)$ has $r-2$ unused points, and any cell, which corresponds to an internal vertex of some $G(P_i)$ has $r-1$ unused points. Recall that $\boldsymbol{I} = (I_1, \dots, I_\ell)$ determines a collection of disjoint paths contained in the subpartitions $P_1, \dots, P_\ell$. This collection of paths will form the intersection of P_T and $P_1 \cup \cdots P_\ell$.

For each such path with at least one part, collect all the unused points and combine them together into an *irregular cell*. If such a path consists of k part then it contains of $k+1$ cells with $r-2$ points unused, and $(s-2)k$ cells with $r-1$ points unused. Thus, the resulting irregular cell has $(k+1)(r-2) + (s-2)k(r-1) = (rs-r-s)k + r - 2$ points. For each cell which corresponds to an external vertex of some $G(P_i)$, but which is not contained in the intersection $P_T \cap P_i$, we also form an irregular cell with $r-2$ points. Note, this matches the earlier formula with $k = 0$; we can think of these external vertices as a length-0 path in the intersection. Indeed, these cells are exactly those which are contained in two parts of P_i, which both correspond to a 0 in the intersection sequence I_i.

Recall that $u = u(\boldsymbol{I}) = u_1 + \cdots + u_\ell$ where $u_i = U(I_i)$ is the number of zero entries in I_i. Then u also equals the number of irregular cells identified so far, as the paths in the intersection $P_i \cap P_T$ (of length zero or more) are in one-to-one correspondence with the zero entries in I_i. The u irregular cells we have identified so far are called *external irregular cells*.

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Finally, for each cell which corresponds to an internal vertex of some $G(P_i)$, which is not involved in the intersection $P_T \cap P_i$, we form an *internal irregular cell* with $r-1$ points. There are $(s-2)u$ such cells.

To summarise the properties of our irregular configuration model:

- The total number of cells is $n' = n - (s-1)\,|Q| + (s-1)u$.
- There are $n - (s-1)\,|Q|$ regular cells with r points each.
- There are $(s-2)u$ internal irregular cells with $r-1$ points each.
- There are u external irregular cells. If an external irregular cell was collapsed from a path with k parts then it contains $(rs-r-s)k + r - 2$ points.

The number of ways to complete Step 2 equals the number of ways of choosing a subpartition P' in this irregular configuration model such that $G(P')$ is a spanning tree. The projection $T' = G(P')$ of this partition corresponds exactly to a tree $T \in \mathcal{T}_n$, with the subpaths determined by $\boldsymbol{I}$ contracted to single vertices.

For a sequence $I \in \mathcal{I}_j$ and $k \geqslant 0$, let $q_k(I)$ be the number of paths of length k in the intersection encoded by I. (Recall the length-0 paths correspond to cells, which belong to two parts in the j-cycle which are both encoded by 0 in I.) By a slight abuse of notation, write $q_k(\boldsymbol{I}) = \sum_{i=1}^{\ell} q_k(I_i)$.

The proof of the next result can be found in [17, Appendix A.1].

Lemma 4.2. *Fix $\boldsymbol{I} = (I_1, \dots, I_\ell) \in \mathcal{I}(\boldsymbol{x})$ and let $u = u(\boldsymbol{I})$. Fix a sequence of subpartitions $(P_1, \dots, P_\ell) \in \mathcal{S}^*(\boldsymbol{x})$ and let $Q = P_1 \cup \dots \cup P_\ell$. Then the number of ways to extend Q to a subpartition $Q \cup P_T$ consistent with $\boldsymbol{I}$, such that $G(P_T) \in \mathcal{T}_n$, is*

$$s_2(\boldsymbol{x}, \boldsymbol{I}) \sim \frac{\sqrt{r-1}\,(s-1)^2\,((s-1)!)^{\frac{1}{s-1}}}{(rs-r-s)^{\frac{3s-1}{2(s-1)}}\,n^2} \left(\frac{(rs-r-s)^{s+1}\,n^{s-1}}{(r-1)\,(s-1)^{s-1}\,(s-1)!}\right)^u \left(\frac{(s-2)!}{(r-1)\,r^{s-1}\,n^{s-1}}\right)^{|Q|} \times \left(\frac{r\,(r-1)^{r-1}\,(s-1)^{r-1}\,n}{e((s-1)!)^{\frac{1}{s-1}}\,(rs-r-s)^{\frac{rs-r-s}{s-1}}}\right)^n \prod_{k=0}^{m-1} \left(k + \frac{r-2}{rs-r-s}\right)^{q_k(\boldsymbol{I})}.$$

Finally, Step 3 completes the subpartition $P_T \cup Q$ to a partition $P \in \mathcal{P}_{n,r,s}$.

Lemma 4.3. *Given $\boldsymbol{I} \in \mathcal{I}(\boldsymbol{x})$, suppose that $(P_1, \dots, P_\ell) \in \mathcal{S}^*(\boldsymbol{x})$ is fixed, and P_T is a fixed subpartition with $\iota(P_1, \dots, P_\ell, P_T) = \boldsymbol{I}$. Let $u = u(\boldsymbol{I})$. The number of ways to complete Step 3 is*

$$s_3(\boldsymbol{x}, \boldsymbol{I}) \sim \frac{\sqrt{s}\,(rs-r-s)\,n}{(s-1)\,((s-1)!)^{\frac{1}{s-1}}} \left(\frac{(s-1)^{s-1}\,(s-1)!}{(rs-r-s)^{s-1}\,n^{s-1}}\right)^u \left(\frac{(rs-r-s)\,n}{e(s-1)\,((s-1)!)^{\frac{1}{s-1}}}\right)^{\frac{(rs-r-s)}{s}n}.$$

Proof. Out of the rn points in the original configuration model, $2\,|Q|$ points have been used for the external vertices in $Q = P_1 \cup \dots \cup P_\ell$ and $(s-2)\,|Q|$ points have been used for the internal vertices of Q. Finally, $\frac{s(n'-1)}{s-1}$ points have been used to complete the subpartition P_T. So there are

$$rn - 2\,|Q| - (s-2)\,|Q| - \frac{s(n'-1)}{s-1} = \frac{(rs-r-s)n}{s-1} - s\left(u - \frac{1}{s-1}\right)$$

points remaining. Hence, the number of ways to complete Step 3 is

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$$s_3(\boldsymbol{x}, \boldsymbol{I}) = p\left(\frac{(rs-r-s)n}{s-1} - s\Big(u - \frac{1}{s-1}\Big)\right) \tag{20}$$

and applying Stirling's approximation completes the proof. □

We use these expressions for s_1, s_2 and s_3 to prove the following result.

Lemma 4.4. *Let $r, s \geqslant 2$ be fixed integers with $(r, s) \neq (2, 2)$. For any fixed integer $m \geqslant 1$ and fixed sequence $(x_1, \ldots, x_m)$ of non-negative integers,*

$$\frac{\mathbb{E}[Y(X_1)_{x_1} \ldots (X_m)_{x_m}]}{\mathbb{E}Y} \longrightarrow \prod_{j=1}^{m} (\lambda_j(1+\zeta_j))^{x_j}$$

as $n \to \infty$ along $\mathcal{N}_{(r,s)}$, where for all $j \in \mathbb{Z}^+$,

$$\lambda_j = \frac{(r-1)^j(s-1)^j}{2j} \quad \textit{and} \quad \zeta_j = \frac{\left(\frac{r}{r-1} - s + 1\right)^j - 2}{(r-1)^j(s-1)^j}.$$

Proof. Recall the definition of Σ^* from (18). By definition of $s_1(\boldsymbol{x})$, $s_2(\boldsymbol{x}, \boldsymbol{I})$, $s_3(\boldsymbol{x}, \boldsymbol{I})$, we have

$$\Sigma^*(\boldsymbol{x}) = \sum_{\boldsymbol{I} \in \mathcal{I}(\boldsymbol{x})} \frac{s_1(\boldsymbol{x})\, s_2(\boldsymbol{x}, \boldsymbol{I})\, s_3(\boldsymbol{x}, \boldsymbol{I})}{\left|\Omega_{n,r,s}\right|\ \mathbb{E}Y}.$$

Combining Lemmas 4.1–4.3, then dividing by (16) and cancelling leads to

$$\begin{aligned}
\Sigma^*(\boldsymbol{x}) &\sim \prod_{j=1}^{m} \frac{1}{(2j)^{x_j}} \sum_{\boldsymbol{I} \in \mathcal{I}(\boldsymbol{x})} \left(\frac{(rs-r-s)^2}{r-1}\right)^{u} \prod_{k=0}^{m-1} \left(k + \frac{r-2}{rs-r-s}\right)^{q_k(\boldsymbol{I})} \\
&= \prod_{j=1}^{m} \frac{1}{(2j)^{x_j}} \sum_{\boldsymbol{I} \in \mathcal{I}(\boldsymbol{x})} \prod_{i=1}^{\ell} \left(\frac{(rs-r-s)^2}{r-1}\right)^{u_i} \prod_{k=0}^{m-1} \left(k + \frac{r-2}{rs-r-s}\right)^{q_k(I_i)} \\
&= \prod_{j=1}^{m} \xi_j^{x_j},
\end{aligned} \tag{21}$$

where

$$\xi_j = \frac{1}{2j} \sum_{I \in \mathcal{I}_j} \left(\frac{(rs-r-s)^2}{r-1}\right)^{U(I)} \prod_{k=0}^{j-1} \left(k + \frac{r-2}{rs-r-s}\right)^{q_k(I)}.$$

We will compute this sum with the help of a generating function. Because $(1, \ldots, 1) \notin \mathcal{I}_j$, we may identify a particular element in the sequence to be zero. By symmetry, we arbitrarily choose the last. Define the coefficients

$$c_{j,\ell} = \sum_{\substack{I \in \mathcal{I}_j: \\ U(I)=\ell \\ I_j = 0}} \mu^{\ell} \prod_{k=0}^{j-1} (k+\beta)^{q_k(I)},$$

where we let

$$\mu = \frac{(rs-r-s)^2}{r-1} \quad \text{and} \quad \beta = \frac{r-2}{rs-r-s}$$

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for convenience. Now, $c_{j,\ell}$ fixes the number of zeros in I to be ℓ, and assumes that $I_j = 0$ (i.e. the last entry of I is zero). Hence

$$\xi_j \sim \frac{1}{2j} \sum_{\ell=1}^{j} \frac{j\, c_{j,\ell}}{\ell} = \frac{1}{2} \sum_{\ell=1}^{j} \frac{c_{j,\ell}}{\ell}. \tag{22}$$

We now evaluate the coefficients $c_{j,\ell}$. Recall that $U(I)$ represents the number of zeros in the sequence I. We have $c_{j,1} = \mu\left(j - 1 + \beta\right)$, because the only sequence with 1 zero and the last element zero is $(1, \ldots, 1, 0)$. For $\ell \geqslant 2$, the sequence starts with k ones followed by a zero, for some $k \in \{0, \ldots, j-2\}$. Ranging over these possibilities gives

$$c_{j,\ell} = \mu \sum_{k=0}^{j-2} \left(k + \beta\right) c_{j-k-1,\ell-1}.$$

To solve this, define the generating function

$$F(x,y) = \sum_{j\geqslant 1} \sum_{\ell \geqslant 1} c_{j,\ell}\, x^j y^\ell.$$

By changing the order of summation and re-indexing, we have

$$\begin{aligned} F(x,y) - \sum_{j\geqslant 1} c_{j,1} x^j y &= \mu \sum_{j\geqslant 1} \sum_{\ell \geqslant 2} \sum_{k=0}^{j-2} \left(k+\beta\right) c_{j-k-1,\ell-1}\, x^j y^\ell \\ &= \mu \sum_{k\geqslant 0} \left(k+\beta\right) x^{k+1} y \sum_{j\geqslant k+2} \sum_{\ell' \geqslant 1} c_{j-k-1,\ell'}\, x^{j-k-1} y^{\ell'} \\ &= \mu \sum_{k\geqslant 0} \left(k+\beta\right) x^{k+1} y\, F(x,y). \end{aligned}$$

Thus, recalling that $c_{j,1} = \mu(j-1+\beta)$, we have

$$\begin{aligned} F(x,y) &= \mu \left[\sum_{j\geqslant 1} \left(j-1+\beta\right) x^j y + \sum_{k\geqslant 0} \left(k+\beta\right) x^{k+1} y\, F(x,y) \right] \\ &= \mu\, (F(x,y)+1) \sum_{k\geqslant 0} \left(k+\beta\right) x^{k+1} y. \end{aligned}$$

Recall that by differentiating both sides of $(1-x)^{-1} = \sum_{k\geqslant 0} x^k$, we have

$$\sum_{k\geqslant 0} k x^{k+1} = \frac{x^2}{(1-x)^2}.$$

Hence if we define

$$f(x) = \mu \sum_{k\geqslant 0} \left(k+\beta\right) x^{k+1} = \mu \left(\frac{x^2}{(1-x)^2} + \frac{\beta x}{1-x} \right),$$

we have $F(x,y) = y f(x)(F(x,y)+1)$ and thus

$$F(x,y) = \frac{f(x) y}{1 - f(x) y}.$$

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Now, going back to (22), we have

$$\xi_j \sim \tfrac{1}{2}\sum_{\ell=1}^{j}\frac{c_{j,\ell}}{\ell} = \tfrac{1}{2}[x^j]\sum_{\ell=1}^{j}\frac{1}{\ell}[y^{\ell-1}]\frac{f(x)}{1-f(x)y}.$$

Applying the Taylor expansion of $(1-z)^{-1}$ and $\log(1-z)$, we have

$$\xi_j \sim \tfrac{1}{2}[x^j]\sum_{\ell=1}^{j}\frac{1}{\ell}[y^{\ell-1}]\left(f(x)\sum_{k=0}^{\infty}(f(x)y)^k\right) = \tfrac{1}{2}[x^j]\sum_{\ell=1}^{j}\frac{f(x)^\ell}{\ell} = -\tfrac{1}{2}[x^j]\log(1-f(x)).$$

Now,

$$1-f(x) = \frac{\left(1-\left(\frac{r}{r-1}-s+1\right)x\right)(1-(r-1)(s-1)x)}{(1-x)^2},$$

so

$$\begin{aligned}
\xi_j &\sim \tfrac{1}{2}[x^j]\left(2\log(1-x) - \log\left(1-\left(\frac{r}{r-1}-s+1\right)x\right) - \log(1-(r-1)(s-1)x)\right)\\
&= \tfrac{1}{2}[x^j]\sum_{k=1}^{\infty}\frac{-2x^k + \left(\left(\frac{r}{r-1}-s+1\right)x\right)^k + ((r-1)(s-1)x)^k}{k}\\
&= \frac{\left(\frac{r}{r-1}-s+1\right)^j + (r-1)^j(s-1)^j - 2}{2j}\\
&= \lambda_j(1+\zeta_j)
\end{aligned}$$

for $j=1,\dots,m$. Substituting this into (21) implies that

$$\Sigma^*(\boldsymbol{x}) \sim \prod_{j=1}^{m}(\lambda_j(1+\zeta_j))^{x_j}.$$

To complete the proof, it remains to show that in (17), the sum over $\mathcal{S}(\boldsymbol{x})\setminus\mathcal{S}^*(\boldsymbol{x})$ is negligible. This is standard, but for completeness we sketch an argument. We adapt Steps 1–3 as above. There are $O(n^{(s-1)|Q|-1})$ ways to choose $(P_1,\dots,P_\ell)\in\mathcal{S}(\boldsymbol{x})\setminus\mathcal{S}^*(\boldsymbol{x})$, as then $Q=P_1\cup\cdots\cup P_\ell$ involves at most $(s-1)|Q|-1$ distinct cells. Now consider the number of ways to perform Steps 2 and 3, summed over all possibilities for the intersection $Q\cap P_T$. This is the number of ways to extend Q to $Q\cup P_T$, where P_T corresponds to a spanning tree $T\in\mathcal{T}_n$, and then extending $Q\cup P_T$ to a full partition. This is very similar to the calculations performed to evaluate $\mathbb{E}Y$, and the presence of Q only changes these calculations by a constant factor. Therefore, for a given Q, the total number of ways to perform Steps 2 and 3, summed over all possible intersections, and then divided by $\mathbb{E}Y$, is

$$O(1)\ \frac{p(rn-s|Q|)}{p(rn)} = O(n^{-(s-1)|Q|}).$$

Multiplying this with the $O(n^{(s-1)|Q|-1})$ ways to complete Step 1, we see that the sum over $(P_1,\dots,P_\ell)$ in (17), contributes $O(1/n)=o(1)$, as required. □

We now show that condition (A3) holds, under fairly weak conditions on (r,s).

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Lemma 4.5. *Fix integers $r, s \geqslant 2$ and r such that*

$$r \geqslant \begin{cases} 3 & \text{if } s = 2, \\ 2 & \text{if } s \in \{3, 4\}, \\ s-1 & \text{if } s \geqslant 5. \end{cases} \tag{23}$$

Then

$$\exp\left(\sum_{j=1}^{\infty} \lambda_j \zeta_j^2\right) = \frac{r^2\sqrt{s-1}}{\sqrt{\left(r^2 - rs + r + s - 1\right)(rs - r - s)(r-1)}} < \infty.$$

Proof. First, observe that

$$\frac{(\frac{r}{r-1} - s + 1)^2}{(r-1)(s-1)}, \qquad \frac{\frac{r}{r-1} - s + 1}{(r-1)(s-1)}, \qquad \frac{1}{(r-1)(s-1)}$$

are all less than 1 in absolute value. Next, we claim that

$$r^2 - rs + r + s - 1 > 0. \tag{24}$$

This condition is easily verified when $s \in \{2, 3, 4\}$ and r belongs to the stated range. When $s \geqslant 5$ we use the fact that

$$r \geqslant s - 1 > \frac{1}{2}\left(\sqrt{s^2 - 6s + 5} + s - 1\right),$$

which implies (24).

Therefore, using the Taylor expansion of $-\log(1-z)$ we obtain

$$\begin{aligned}
\sum_{j=1}^{\infty} \lambda_j \zeta_j^2 &= \tfrac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j} \left(\left(\frac{\left(\frac{r}{r-1} - s + 1\right)^2}{(r-1)(s-1)} \right)^j - 4 \left(\frac{\frac{r}{r-1} - s + 1}{(r-1)(s-1)} \right)^j + 4 \left(\frac{1}{(r-1)(s-1)} \right)^j \right) \\
&= -2 \log\left(1 - \frac{1}{(r-1)(s-1)}\right) + 2 \log\left(1 - \frac{\frac{r}{r-1} - s + 1}{(r-1)(s-1)}\right) \\
&\qquad - \tfrac{1}{2} \log\left(1 - \frac{\left(\frac{r}{r-1} - s + 1\right)^2}{(r-1)(s-1)}\right) \\
&= -2 \log\left(\frac{rs - r - s}{(r-1)(s-1)}\right) + 2 \log\left(\frac{r(rs - r - s)}{(r-1)^2(s-1)}\right) \\
&\qquad - \tfrac{1}{2} \log\left(\frac{(rs - r - s)(r^2 - rs + r + s - 1)}{(r-1)^3(s-1)}\right).
\end{aligned}$$

Taking the exponential of both sides establishes the result. □

Next, we investigate the parameters $\zeta_j(r, s)$.

Lemma 4.6. *Let $r, s \geqslant 2$ and recall that for all fixed integers $j \geqslant 1$,*

$$\zeta_j = \zeta_j(r, s) = \frac{\left(\frac{r}{r-1} - s + 1\right)^j - 2}{(r-1)^j (s-1)^j}.$$

Then

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i. $\zeta_j(2,2) = -1$ *for* $j \geqslant 1$;
ii. $\zeta_j(2,s) = -1$ *for* $s \geqslant 3$ *and* $j = 1$;

In all other cases, $\zeta_j(r,s) > -1$.

Proof. It is easy to check that (i) and (ii) hold. For (iii), note that $\zeta_j(r,s) > -1$ if and only if $f(r,s,j) > 0$, where

$$f(r,s,j) = \left(\frac{r}{r-1} - s + 1\right)^j - 2 + (r-1)^j(s-1)^j.$$

If $r = 2$ and $s \geqslant 3$ then $f(2,s,j) = (s-1)^j - (s-3)^j - 2$, which increases with j. So

$$f(2,s,j) \geqslant f(2,s,2) = 4s - 10 > 0$$

as $s \geqslant 3$. If $s = 2$, then $\left(\frac{r}{r-1} - s + 1\right)^j > 0$, so $f(r,2,j) > (r-1)^j - 2 \geqslant 0$ as $r \geqslant 3$ and $j \geqslant 1$.

It remains to show that $f(r,s,j) > 0$ when $r \geqslant 3$, $s \geqslant 3$ and $j \geqslant 1$. If j is even then $\left(\frac{r}{r-1} - s + 1\right)^j \geqslant 0$, so

$$f(r,s,j) \geqslant (r-1)^j(s-1)^j - 2 > 2^{2j} - 2 > 0.$$

Now suppose that j is odd. Since $r \geqslant 3$ and $s \geqslant 3$, we have $\frac{r}{r-1} - s + 1 < 0$. Thus

$$\begin{aligned} f(r,s,j) &= (r-1)^j(s-1)^j - \left(s - 1 - \frac{r}{r-1}\right)^j - 2 \\ &> \left((r-1)^j - 1\right)(s-1)^j - 2 \\ &\geqslant 2^j - 2 \geqslant 0. \end{aligned}$$

This completes the proof. □

5. Second moment

So far, conditions (A1)–(A3) of Theorem 2.1 have been verified. In this section we will assume that $r, s \geqslant 2$ are fixed integers such that $r > \rho(s)$ when $s \geqslant 5$, and that $(r,s) \neq (2,2)$ when $s \in \{2,3,4\}$. With this assumption, we will obtain an asymptotic expression for the second moment, verifying condition (A4). The following identity of Chu [12], which generalises Jensen's identity, will be useful.

Lemma 5.1 ([12]). *Let* m, b *be positive integers,* $x_1, \ldots, x_b$ *and* z *be complex numbers, and define* $\binom{-1}{0} = 1$. *Then*

$$\sum_{\substack{(k_1,\ldots,k_b)\in\mathbb{N}^b \\ k_1+\cdots+k_b=m}} \prod_{i=1}^{b} \binom{x_i + k_i z}{k_i} = \sum_{k=0}^{m} \binom{k+b-2}{k} \binom{x_1 + \cdots + x_b + mz - k}{m-k} z^k.$$

We write

$$\left|\Omega_{n,r,s}\right| \mathbb{E}Y^2 = \sum_{(P_{T_1}, P_{T_2})} \left|\{P \in \Omega_{n,r,s} : P_{T_1} \cup P_{T_2} \subseteq P\}\right|,$$

where the sum is over all pairs (P_{T_1}, P_{T_2}) such that $G(P_{T_1}) = T_2$ and $G(P_{T_2}) = T_2$, for some spanning trees T_1, T_2.

We perform this count by conditioning on the intersection between P_{T_1} and P_{T_2}, which will correspond to a union of disjoint trees. Let $b \in \{1 + (s-1)\ell : \ell = 0, 1, \ldots, \frac{n-1}{s-1}\}$ be the number of

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connected components in this intersection (we can show b must be of this form by adding up the number of vertices in each connected component). We break up the process into the following steps:

(1) Choose a partition $\nu = (\nu_1, \dots, \nu_b)$ of n, where $\nu_i > 0$, $s-1 \mid \nu_i - 1$ and $\sum_{i=1}^{b} \nu_i = n$. Here, ν_i represents the number of vertices in the ith connected component. (Later, we will divide by $b!$ to account for the assumption that the connected components are labelled).
(2) Choose a partition of the n vertices into b groups, where the size of the ith group is ν_i.
(3) In each group, choose a spanning tree on that group and a subpartition that projects to that tree.

We then collapse the unused points in each group to an irregular cell. The ith irregular cell will have $r\nu_i - \frac{s(\nu_i-1)}{s-1} = \frac{rs-r-s}{s-1}\nu_i + \frac{s}{s-1}$ points. In this irregular configuration model, we wish to partition two-part disjoint spanning trees T_1' and T_2', which will extend to T_1 and T_2.

(4) Choose $\boldsymbol{\delta}^{(1)}, \boldsymbol{\delta}^{(2)} \in \mathbb{N}^b$, the degree sequence of T_1' and T_2' respectively, such that, for all i,

$$\delta_i^{(1)}, \delta_i^{(2)} \geqslant 1, \quad \sum_{i=1}^{b} \delta_i^{(1)} = \sum_{i=1}^{b} \delta_i^{(2)} = \frac{s(b-1)}{s-1}, \quad \text{and} \quad \delta_i^{(1)} + \delta_i^{(2)} \leqslant \frac{rs-r-s}{s-1}\nu_i + \frac{s}{s-1}.$$

(5) Choose trees T_1', T_2' consistent with $\boldsymbol{\delta}^{(1)}$ and $\boldsymbol{\delta}^{(2)}$.
(6) Choose P_{T_1} and P_{T_2} such that there are no parts in common.
(7) Partition remaining points.

Then $\left|\Omega_{n,r,s}\right| \mathbb{E}Y^2$ is equal to the number of ways to complete the above process, summed over all b.

Let

$$\mathcal{S}_1(b) = \left\{ \nu \in \left\{ 1 + (s-1)\ell : \ell = 0, 1, \dots, \frac{n-1}{s-1} \right\}^b : \sum_{i=1}^{b} \nu_i = n \right\}$$

be the set of possible sequences ν from Step 1. The number of ways to complete Step 2 is

$$s_2 = \binom{n}{\nu_1, \dots, \nu_b} = n! \prod_{i=1}^{b} \frac{1}{\nu_i!}.$$

By (14), the number of ways to complete Step 3 is

$$s_3 = \frac{(s-1)^b\, r^n}{((s-1)!)^{\frac{n-b}{s-1}}} \prod_{i=1}^{b} (\nu_i - 2)! \binom{(r-1)\nu_i}{\frac{\nu_i - s}{s-1}} = \frac{(s-1)^b\, r^n}{((s-1)!)^{\frac{n-b}{s-1}}} \prod_{i=1}^{b} \frac{(\nu_i-2)!((r-1)\nu_i)!}{\left(\frac{\nu_i-s}{s-1}\right)! \left(\frac{rs-r-s}{s-1}\nu_i + \frac{s}{s-1}\right)!}.$$

Let

$$\mathcal{S}_4(\nu) = \left\{ (\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \boldsymbol{\eta}^{(3)}) \in (\mathbb{N}^b)^3 : \quad \eta_i^{(1)} + \eta_i^{(2)} + \eta_i^{(3)} = \frac{rs-r-s}{s-1}\nu_i - \frac{s-2}{s-1}, \right.$$
$$\left. \sum_{i=1}^{b} \eta_i^{(1)} = \sum_{i=1}^{b} \eta_i^{(2)} = \frac{b-s}{s-1} \right\}$$

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be the set of sequences arising from Step 4. By (6), the number of ways to complete Step 5 is

$$s_5 = \binom{\frac{b-1}{s-1}-1}{\delta_1^{(1)}-1,\ldots,\delta_b^{(1)}-1}\binom{\frac{b-1}{s-1}-1}{\delta_1^{(2)}-1,\ldots,\delta_b^{(2)}-1}\left(\frac{(b-1)!}{\left(\frac{b-1}{s-1}\right)!((s-1)!)^{\frac{b-1}{s-1}}}\right)^2.$$

As each cell in this irregular configuration model has $\frac{rs-r-s}{s-1}\nu_i + \frac{s}{s-1}$ points, the number of ways to complete Step 6 is

$$s_6 = \prod_{i=1}^{b}\left(\frac{rs-r-s}{s-1}\nu_i + \frac{s}{s-1}\right)_{\delta_i^{(1)}+\delta_i^{(2)}} = \prod_{i=1}^{b}\frac{\left(\frac{rs-r-s}{s-1}\nu_i + \frac{s}{s-1}\right)!}{\left(\frac{rs-r-s}{s-1}\nu_i + \frac{s}{s-1} - \delta_i^{(1)} - \delta_i^{(2)}\right)!}.$$

There are

$$\sum_{i=1}^{b} r\nu_i - \frac{s(\nu_i-1)}{s-1} - \delta_i^{(1)} - \delta_i^{(2)} = \frac{(rs-r-s)n}{s-1} - \frac{s(b-2)}{s-1}$$

points remaining, so the number of ways to complete Step 7 is

$$s_7 = p\left(\frac{(rs-r-s)n}{s-1} - \frac{s(b-2)}{s-1}\right) = \frac{\left(\frac{(rs-r-s)n}{s-1} - \frac{s(b-2)}{s-1}\right)!}{\left(\frac{rs-r-s}{s(s-1)}n - \frac{b-2}{s-1}\right)!(s!)^{\frac{rs-r-s}{s(s-1)}n-\frac{b-2}{s-1}}}.$$

It is convenient to work with non-negative variables, so we let

$$\eta_i^{(1)} = \delta_i^{(1)} - 1, \quad \eta_i^{(2)} = \delta_i^{(2)} - 1, \quad \eta_i^{(3)} = \frac{rs-r-s}{s-1}\nu_i - \frac{s-2}{s-1} - \eta_i^{(1)} - \eta_i^{(2)},$$

for $i = 1,\ldots,b$.

Combining everything, and dividing by $b!$ as promised earlier, we have

$$\begin{aligned}
\left|\Omega_{n,r,s}\right|\mathbb{E}Y^2 &= \sum_{\substack{b=1\\ s-1|b-1}}^{n}\frac{1}{b!}\sum_{\nu\in\mathcal{S}_1(b)}s_2s_3\sum_{(\boldsymbol{\eta}^{(1)},\boldsymbol{\eta}^{(2)},\boldsymbol{\eta}^{(3)})\in\mathcal{S}_4(\nu)}s_5s_6s_7\\
&= \sum_{\substack{b=1\\ s-1|b-1}}^{n}\frac{n!\,r^n(s-1)^b((b-1)!)^2}{b!((s-1)!)^{\frac{n+b-2}{s-1}}\left(\frac{rs-r-s}{s(s-1)}n-\frac{b-2}{s-1}\right)!\left(\left(\frac{b-1}{s-1}\right)!\right)^2(s!)^{\frac{rs-r-s}{s(s-1)}n-\frac{b-2}{s-1}}}\\
&\quad\times\sum_{\nu\in\mathcal{S}_1(b)}\left(\prod_{i=1}^{b}\frac{((r-1)\nu_i)!}{\nu_i(\nu_i-1)\left(\frac{\nu_i-s}{s-1}\right)!}\right)\\
&\quad\times\sum_{(\boldsymbol{\eta}^{(1)},\boldsymbol{\eta}^{(2)},\boldsymbol{\eta}^{(3)})\in\mathcal{S}_4(\nu)}\binom{\frac{b-1}{s-1}-1}{\eta_1^{(1)},\ldots,\eta_b^{(1)}}\binom{\frac{b-1}{s-1}-1}{\eta_1^{(2)},\ldots,\eta_b^{(2)}}\binom{\frac{(rs-r-s)n}{s-1}-\frac{s(b-2)}{s-1}}{\eta_1^{(3)},\ldots,\eta_b^{(3)}}.
\end{aligned}$$

Now we compute the sum over $\mathcal{S}_4(\nu)$ through the use of generating functions:

$$\begin{aligned}
&\sum_{(\boldsymbol{\eta}^{(1)},\boldsymbol{\eta}^{(2)},\boldsymbol{\eta}^{(3)})\in\mathcal{S}_4(\nu)}\binom{\frac{b-1}{s-1}-1}{\eta_1^{(1)},\ldots,\eta_b^{(1)}}\binom{\frac{b-1}{s-1}-1}{\eta_1^{(2)},\ldots,\eta_b^{(2)}}\binom{\frac{(rs-r-s)n}{s-1}-\frac{s(b-2)}{s-1}}{\eta_1^{(3)},\ldots,\eta_b^{(3)}}\\
&=\sum_{(\boldsymbol{\eta}^{(1)},\boldsymbol{\eta}^{(2)},\boldsymbol{\eta}^{(3)})\in\mathcal{S}_4(\nu)}\left([z_1^{\eta_1^{(1)}}\cdots z_b^{\eta_b^{(1)}}]\left(\sum_{i=1}^{b}z_i\right)^{\frac{b-s}{s-1}}\right)\left([z_1^{\eta_1^{(2)}}\cdots z_b^{\eta_b^{(2)}}]\left(\sum_{i=1}^{b}z_i\right)^{\frac{b-s}{s-1}}\right)
\end{aligned}$$

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$$\left([z_1^{\eta_1^{(3)}} \cdots z_b^{\eta_b^{(3)}}]\left(\sum_{i=1}^b z_i\right)^{\frac{(rs-r-s)n}{s-1}-\frac{s(b-2)}{s-1}}\right)$$

$$= [z_1^{\frac{rs-r-s}{s-1}v_1-\frac{s-2}{s-1}} \cdots z_b^{\frac{rs-r-s}{s-1}v_b-\frac{s-2}{s-1}}]\left(\sum_{i=1}^b z_i\right)^{\frac{(rs-r-s)n}{s-1}-\frac{b(s-2)}{s-1}}$$

$$= \binom{\frac{(rs-r-s)n}{s-1}-\frac{b(s-2)}{s-1}}{\frac{(rs-r-s)v_1}{s-1}-\frac{s-2}{s-1},\ldots,\frac{(rs-r-s)v_b}{s-1}-\frac{s-2}{s-1}}.$$

So

$$\left|\Omega_{n,r,s}\right| \mathbb{E}Y^2 = \sum_{\substack{b=1 \\ s-1|b-1}}^{n} \frac{n!\, r^n (r-1)^b ((b-1)!)^2 \left(\frac{(rs-r-s)n}{s-1}-\frac{b(s-2)}{s-1}\right)!}{b!((s-1)!)^{\frac{n+b-2}{s-1}} \left(\frac{rs-r-s}{s(s-1)}n-\frac{b-2}{s-1}\right)! \left(\left(\frac{b-1}{s-1}\right)!\right)^2 (s!)^{\frac{rs-r-s}{s(s-1)}n-\frac{b-2}{s-1}}}$$

$$\times \sum_{v\in\mathcal{S}_1(b)} \prod_{i=1}^b \binom{(r-1)v_i-1}{\frac{v_i-1}{s-1}}$$

Note that the summand is equal to $\left|\Omega_{n,r,s}\right| \mathbb{E}Y$ when $b=1$. For $b \geqslant 2$, we let $k_i = \frac{v_i-1}{s-1}$ and use Lemma 5.1, to see that

$$\sum_{v\in\mathcal{S}_1(b)} \prod_{i=1}^b \binom{(r-1)v_i-1}{\frac{v_i-1}{s-1}} = \sum_{\substack{k_1+\cdots+k_b=\frac{n-b}{s-1} \\ k_i\geqslant 0}} \prod_{i=1}^b \binom{(r-1)(s-1)k_i+r-2}{k_i}$$

$$= \sum_{k=0}^{\frac{n-b}{s-1}} \binom{k+b-2}{k}\binom{(r-1)n-b-k}{\frac{n-b}{s-1}-k}(r-1)^k.$$

Define

$$K = \{(\alpha,\beta) : \alpha,\beta \geqslant 0, \quad (s-1)\alpha+\beta \leqslant 1\}, \tag{25}$$

$$\mathcal{L} = \mathbb{Z}\times(s-1)\mathbb{Z} \tag{26}$$

and let K° denote the interior of K. Thus, dividing through by the expression in (7), we have

$$\mathbb{E}Y^2 = \mathbb{E}Y + \sum_{(k,b)\in(\mathcal{L}+(0,1))\cap nK} a_n(k,b), \tag{27}$$

where

$$a_n(k,b) = \begin{cases} 0 & \text{for } b \leqslant 1, \\ \dfrac{r^n(b-1)(r-1)^{k+b}\, s^{\frac{n+b-2}{s-1}}\, (k+b-2)!\, ((r-1)n-k-b)!\, (rn/s)!\, n!}{b\,k!\left(\left(\frac{b-1}{s-1}\right)!\right)^2\left(\frac{rs-r-s}{s(s-1)}n-\frac{b-2}{s-1}\right)!\left(\frac{n-(s-1)k-b}{s-1}\right)!\,(rn)!} & \text{otherwise.} \end{cases}$$

We now wish to apply Laplace's method to compute the asymptotic summation of this expression.

Greenhill, Janson and Ruciński [18] proved a version of Laplace's method for asymptotic summation, tailored for the small subgraph conditioning method. We refer to [18] for precise definitions.

Lemma 5.2 ([18, Lemma 6.3]). *Suppose the following:*

i. $\mathcal{L} \subset \mathbb{R}^m$ *is a lattice with full rank* m.

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ii. *$K \subset \mathbb{R}^m$ is a compact convex set with non-empty interior.*
iii. *$\varphi : K \to \mathbb{R}$ is a continuous function with a unique maximum at some interior point $x_0 \in K^\circ$.*
iv. *φ is a twice continuously differentiable in a neighbourhood of x_0 and the Hessian $H_0 := D^2\varphi(x_0)$ is strictly negative definite.*
v. *$\psi : K_1 \to \mathbb{R}$ is a continuous function on some neighbourhood $K_1 \subset K$ of x_0 with $\psi(x_0) > 0$.*
vi. *For each positive integer n there is a vector $\ell_n \in \mathbb{R}^m$.*
vii. *For each positive integer n there is a positive real number b_n and a function $a_n : (\mathcal{L} + \ell_n) \cap nK \to \mathbb{R}$ such that, as $n \to \infty$,*

$$a_n(\ell) = O(b_n e^{n\varphi(\ell/n)+o(n)}), \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK, \tag{28}$$

and

$$a_n(\ell) = b_n(\psi(\ell/n) + o(1))e^{n\varphi(\ell/n)}, \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK_1, \tag{29}$$

uniformly for ℓ in the indicated sets.

Then, as $n \to \infty$,

$$\sum_{(\mathcal{L}+\ell_n)\cap nK} a_n(\ell) \sim \frac{(2\pi)^{m/2}\psi(x_0)}{\det(\mathcal{L})\det(-H_0)^{1/2}}\, b_n n^{m/2} e^{n\varphi(x_0)}.$$

To apply this lemma, we define

$$b_n = \frac{(s-1)^2}{2\pi n^3}\left(\frac{(s-1)^{r/s}}{r^{\frac{rs-r-s}{s}}}\right)^n,$$

$$\psi(\alpha,\beta) = \frac{(r-1-\alpha-\beta)^{1/2}}{(\alpha+\beta)^{3/2}(rs-r-s(1+\beta))^{\frac{1}{2}+\frac{2}{s-1}}\beta^{1-\frac{2}{s-1}}\alpha^{1/2}(1-\beta-(s-1)\alpha)^{1/2}},$$

$$\varphi(\alpha,\beta) = (\alpha+\beta)\log(r-1) + g(\alpha+\beta) + g(r-1-\alpha-\beta) - \frac{2}{s-1}g(\beta) - g(\alpha) \tag{30}$$

$$- \frac{1}{s(s-1)}g(rs-r-s-s\beta) - \frac{1}{s-1}g(1-(s-1)\alpha-\beta), \tag{31}$$

where $g(x) = x\log x$ for $x > 0$ and $g(0) = 0$. The following result gives critical information about the function φ. The proof is technical and is omitted here, but can be found in [17, Appendix A.2].

Lemma 5.3. *Assume that $r, s \geqslant 2$ are fixed integers such that $r > \rho(s)$ when $s \geqslant 5$, and that $(r,s) \neq (2,2)$ when $s \in \{2,3,4\}$. The unique global maximum of φ over K occurs at the point (α_0, β_0) where*

$$\alpha_0 = \frac{1}{r(s-1)}, \quad \beta_0 = \frac{rs-r-s}{r(s-1)}.$$

The maximum value of φ over this domain equals

$$\varphi(\alpha_0,\beta_0) = 2(r-1)\log(r-1) - \frac{2(rs-r-s)}{s(s-1)}\log(rs-r-s) + \frac{r}{s}\log(s-1) - \frac{rs-r-s}{s}\log r.$$

Let H_0 be the Hessian of φ evaluated at the point (α_0, β_0). Then H_0 is strictly negative definite and

$$\det(-H_0) = \frac{r^3(s-1)^2\left(r^2-rs+r+s-1\right)}{(r-1)^2(rs-r-s)}.$$

Then Lemma 5.3 implies the following.

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Lemma 5.4. *Assume that $r, s \geqslant 2$ are fixed integers such that $r > \rho(s)$ when $s \geqslant 5$, and that $(r, s) \neq (2, 2)$ when $s \in \{2, 3, 4\}$. Then as $n \to \infty$ along $\mathcal{N}_{(r,s)}$,*

$$\frac{\mathbb{E}Y^2}{(\mathbb{E}Y)^2} \sim \frac{r^2\sqrt{s-1}}{\sqrt{\left(r^2 - rs + r + s - 1\right)(rs - r - s)(r-1)}}.$$

Proof. We apply Lemma 5.2 to compute the sum in (27). Now the conditions of Lemma 5.2 hold:

i. We defined $\mathcal{L} = \mathbb{Z} \times (s-1)\mathbb{Z}$, a lattice with rank $m = 2$ and $\det(\mathcal{L}) = s - 1$.
ii. The domain K, defined in (25), is compact and convex with a non-empty interior.
iii. The function $\varphi : K \to \mathbb{R}$ is a continuous function with a unique global maximum (α_0, β_0), by Lemma 5.3.
iv. The function $\varphi : K \to \mathbb{R}$ is twice differentiable in the interior of K, with a strictly negative definite Hessian, by Lemma 5.3.
v. Let K_1 be the open ball around (α_0, β_0) of sufficiently small radius, ensuring that $K_1 \subset K$. The function $\psi : K_1 \to \mathbb{R}$ is a continuous function with
$$\psi(\alpha_0, \beta_0) = \frac{r^{7/2}(s-1)^{5/2}}{\sqrt{r-1}(rs - r - s)^{\frac{2s}{s-1}}} > 0.$$
vi. Let ℓ_n be $(0, 1)$ for each n.
vii. This condition is verified by applying Stirling's approximation.

Thus, we can apply Lemma 5.2 to see that

$$\begin{aligned}\mathbb{E}Y^2 &\sim \frac{2\pi\ \psi(\alpha_0, \beta_0)}{\det(\mathcal{L})\det(-H_0)^{1/2}}\ b_n\, n\, e^{n\,\varphi(\alpha_0,\beta_0)} \\ &= \frac{r^2\sqrt{r-1}\,(s-1)^{5/2}\,(rs-r-s)^{\frac{1}{2}-\frac{2s}{s-1}}}{n^2\sqrt{r^2 - rs + r + s - 1}} \left(\frac{(s-1)^{r/s}\,(r-1)^{r-1}}{r^{\frac{rs-r-s}{s}}\,(rs-r-s)^{\frac{rs-r-s}{s(s-1)}}}\right)^{2n}.\end{aligned}$$

Dividing by $(\mathbb{E}Y)^2$, using the expression from Lemma 3.1, completes the proof. □

6. Threshold analysis

Define the logarithm of the base of the exponential factor as a function:

$$L(r, s) = L_s(r) = \frac{r}{s}\log(s-1) + (r-1)\log(r-1) - \frac{rs-r-s}{s}\log r - \frac{rs-r-s}{s(s-1)}\log(rs-r-s),$$

treating r as a continuous variable and $s \geqslant 2$ as a fixed positive integer. We restrict to $r \geqslant 2$, or $r \geqslant 3$ when $s = 2$. We want to determine when $L_s(r) > 0$ (which implies that $\mathbb{E}Y \to \infty$) and when $L_s(r) \leqslant 0$ (which implies that $\mathbb{E}Y = o(1)$).

The following can be checked using elementary calculus.

Lemma 6.1. *For $s \in \{2, 3, 4\}$, $L_s(r) > 0$ for $r \in (2, \infty)$. Furthermore, for $s \in \{3, 4\}$, $L_s(2) > 0$.*

The situation is quite different when $s \geqslant 5$, as Lemma 6.2 shows. The proof of Lemma 6.2 can be found in [17, Appendix A.3].

Lemma 6.2. *For a fixed integer $s \geqslant 5$, there exists a unique real number $\rho(s) > 2$ such that $L(\rho(s), s) = 0$,*

$$L(r, s) < 0 \quad \text{for } r \in [2, \rho(s)) \quad \text{and} \quad L(r, s) > 0 \text{ for } r \in (\rho(s), \infty).$$

Furthermore, if $s \geqslant 6$, then $\rho(s) > s$.

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The next lemma proves that $\rho(s)$ is exponential in s and lies strictly within an interval of unit width, and gives an asymptotic expression for $\rho(s)$ with exponentially small error, as $s \to \infty$. The proof is omitted here, but can be found in [17, Appendix A.4].

Lemma 6.3. *For a fixed integer $s \geqslant 5$, let $\rho(s)$ be the unique real number such that $L(\rho(s), s) = 0$. Then*

$$\frac{e^{s-2}}{s-1} - \frac{s-1}{2} < \rho(s) < \frac{e^{s-2}}{s-1} - \frac{s-3}{2}.$$

Furthermore, as $s \to \infty$,

$$\rho(s) = \frac{e^{s-2}}{s-1} - \frac{s^2 - 3s + 1}{2(s-1)} + O_s(s^4\, e^{-s}).$$

We can now prove our main result, Theorem 1.1, which gives us a threshold result for the existence of a spanning tree in $\mathcal{G}_{n,r,s}$ when $s \geqslant 5$.

Proof of Theorem 1.1. Lemma 6.2 proves the existence and uniqueness of $\rho(s)$ for $s \geqslant 5$, while Lemma 6.3 proves the upper and lower bounds on $\rho(s)$, and verifies the given asymptotic expression for $\rho(s)$.

For fixed $s \geqslant 2$ and $r \leqslant \rho(s)$ (but $(r, s) \neq (2, 2)$), the base of the exponential factor in Lemma 3.1 is less than 1, and thus we have $\mathbb{P}(Y \geqslant 1) \leqslant \mathbb{E}Y \to 0$. By Lemma 2.3, we conclude that $\mathbb{P}(Y_{\mathcal{G}} \geqslant 1) \to 0$.

Now suppose that $s \geqslant 6$ and $r > \rho(s)$, or $s \in \{2, 3, 4\}$ and $(r, s) \neq (2, 2)$. We verify the conditions of Theorem 2.1. Cooper, Frieze, Molloy and Reed [13] proved that condition (1) holds with λ_j defined in (11). Condition (A2) holds by Lemma 4.4, condition (A3) holds by Lemma 4.5, and condition (A4) holds by Lemma 5.4. By our assumptions on r, s, we have $\zeta_1 > -1$ for all $j \geqslant 2$, by Lemma 4.6. Hence $\widehat{Y} > 0$ a.a.s., where $\widehat{Y}$ is the random variable obtained from Y by conditioning on the event $X_1 = 0$. Then Lemma 2.3 shows that $Y_{\mathcal{G}} > 0$ a.a.s., as required. □

Finally, we provide the proof for Theorem 1.2, which gives the asymptotic distribution of the number of spanning trees in $\mathcal{G}_{n,r,s}$.

Proof of Theorem 1.2. If $\mathbb{P}(Y_{\mathcal{G}} > 0) \to 0$ then the result is immediate, as the asymptotic value of $Y_{\mathcal{G}}$ is zero with probability 1. For the remainder of the proof, suppose that $\mathbb{P}(Y_{\mathcal{G}} > 0) \to 1$.

We showed in the proof of Theorem 1.1 that conditions (A1)–(A4) of Theorem 2.1 hold for Y. This gives us

$$\frac{Y}{\mathbb{E}Y} \xrightarrow{d} \prod_{j=1}^{\infty} (1 + \zeta_j)^{Z_j} e^{-\lambda_j \zeta_j}.$$

For $s = 2$, the asymptotic distribution of $Y_{\mathcal{G}}$ is obtained from the asymptotic distribution of Y by conditioning on $X_1 = X_2 = 0$.

For $s \geqslant 3$, let $\widehat{Y}$ be the random variable obtained from Y by conditioning on the event $X_1 = 0$. We apply Corollary 2.4 to see that

$$\frac{\widehat{Y}}{\mathbb{E}Y} \xrightarrow{d} e^{-\lambda_1 \zeta_1} \prod_{j=2}^{\infty} (1 + \zeta_j)^{Z_j} e^{-\lambda_j \zeta_j}.$$

Applying Lemma 2.3(b) shows that $\widehat{Y}$ and $Y_{\mathcal{G}}$ have the same asymptotic distribution. Finally, combining (3) and Lemma 3.1 gives us $\mathbb{E}Y_{\mathcal{G}} \sim \exp(-\lambda_1 \zeta_1)\mathbb{E}Y$, and the result follows. □

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