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On lambda operations on mixed motives

by

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Abstract

We study the natural λ-ring structure on the Grothendieck ring of the triangulated category of mixed motives. Basic properties of a natural notion of characteristic-like series are developed in the context of equivariant objects.

Key Words: lambda ring, Grothendieck ring, motivic zeta function.

Mathematics Subject Classification 2010: Primary 14C15 - Secondary 19L20.

1. Introduction

In this note we consider certain natural lambda operations on mixed motives and then formulate some applications of this structure. The structure is natural because it is induced from the natural tensor structure on the category $D = DM_{\mathrm{gm}}(\mathrm{Spec}(F),\mathbb{Q})$ of geometric motives of Voevodsky [35] for a field F of characteristic zero. Other motivic categories can also be considered.

Recall (§2) that for a (pure or mixed) motive X and an integer $n > 0$, we have a natural action of the symmetric group Σ_n on the n-th tensor power $X^{\otimes n}$. This gives the objects $\mathrm{Alt}^n(X)$, $\mathrm{Sym}^n(X)$, and more generally the Schur objects $S_\pi(X)$ by following the general categorical definition of Deligne [13]. It is of certain interest to consider the power series

$$\zeta(X,t) = \sum_{n\geq 0} \mathrm{cl}(\mathrm{Sym}^n(X))t^n$$

as an element of the power series ring on the Grothendieck ring $K_0(D)$ where the class of a motive X in K_0 is denoted by $\mathrm{cl}(X)$. This Grothendieck ring is defined as a quotient of the Grothendieck ring of the underlying additive category of D by the subgroup generated by all expressions $\mathrm{cl}(Z) - \mathrm{cl}(X) - \mathrm{cl}(Y)$ for distinguished triangles $X \to Z \to Y \to X[1]$. The above series is a $K_0(D)$-variant of the so-called (motivic) zeta function of a motive X, which was considered by Kapranov [22, 1.3]. Let $\lambda(X)$ denote $\zeta(X,-t)^{-1}$. It follows from (4.5) that

$$\lambda(X) = \sum_{n\geq 0} \mathrm{cl}\big(\mathrm{Alt}^n(X)\big)t^n.$$

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We shall consider (4.1) the assignment $\mathrm{cl}(X) \mapsto \lambda(X)$ which, by a result of Guletskii [18], is well-defined on $K_0(D)$ and defines a λ-ring structure on $K_0(D)$.

The problem of rationality of the zeta functions has been posed (André [3, 13.3]) for all (pure) motives X. In fact if X is a mixed Tate motive, a motive of an abelian variety, or a smooth curve, or a reductive group, or a product of such motives, etc., then $\zeta(X,t)$ is, by (4.11), a rational function.

Questions similar to the above rationality one and concerning the elements of the Grothendieck ring $K_0(\mathfrak{A})$ of a tensor category may be examined after applying ring homomorphisms $K_0(\mathfrak{A}) \to A$ to different commutative rings A. We shall consider a trace homomorphism $\mathrm{tr}\colon K_0(D) \to \mathbb{Z}$ (defined after 3.1) for this purpose.

Any λ-ring is equipped with certain operations called Adams operations ψ_n. As in the classical case of vector spaces, the latter, when considered in the case of motives, can be more efficiently applied in the categorical context of motives with actions of finite groups. We shall explore these from an abstract point of view in a separate section. For a motive X equipped with an action of a group G, we may consider for any $g \in G$ the characteristic series

$$\overline{\chi}_X(g) := \sum_{n\geq 0} \mathrm{tr}(g; \mathrm{Alt}^n(X))t^n$$

where the trace tr is defined immediately after 3.1. As shown in (5.1), the latter is an element of $\Lambda(\mathbb{Z}) := 1 + t\mathbb{Z}[[t]]$ which is in fact a λ-ring (cf. discussion after 2.2). We shall show in (3.12) that in the general case of motives equipped with actions of finite groups, we have the formula

$$\psi_n\big(\overline{\chi}_X(g)\big) = \overline{\chi}_X(g^n)$$

in the λ-ring $\Lambda(\mathbb{Z})$. It follows that for any motive X and any finite group G of exponent n, the series $\overline{\chi}_{\psi_n(X)}(g)$ is by construction (5.5) always rational. It follows also from the methods developed in (5.8, 5.9) that power series expressions such as

$$\sum_{n\geq 0} \mathrm{tr}(f; \mathrm{Sym}^n(X))t^n$$

considered as an element of $\Lambda(\overline{\mathbb{Q}})$ and where $f\colon X \to X$ is an endomorphism of a motive X, is rational for a large class of pairs (X, f). For example this is the case for arbitrary X and automorphisms f of finite order and also for pairs (X, f) where X is any motive for which $\mathrm{End}(X)$ is finite dimensional and f is any endomorphism.

The reader may also want to look at [30] where similar results regarding λ-operations and K_0 of categories of motives are obtained. In a forthcoming note we

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shall use results of the current paper to define and study γ-filtrations in the settings of motivic categories.

Throughout the paper k (and F) denote fields of characteristic zero.

2. lambda operations on the Grothendieck ring

We let $\mathfrak{A}$ be a $k-$linear tensor category in the sense of [31, I.0.1.2, I.2.4] where k is a field of characteristic zero. This means in particular that $\mathfrak{A}$ is equipped with compatible associativity, commutativity, and unital constraints. The unit is denoted by $\mathbb{1}$ and k-linearity means that $\mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ is merely a k-algebra. We shall always assume that $\mathfrak{A}$ is essentially small and also Karoubian, i.e. every projector $p = p^2 \in \mathrm{End}(X)$ has a kernel.

The tensor structure of $\mathfrak{A}$ gives an action of $\Sigma_2 = \{1, \tau\}$ on each $X^{\otimes 2}$ by $\tau \mapsto \psi_{X,X}$ where ψ is the commutativity constraint of $\mathfrak{A}$. This is naturally extended to a k-algebra homomorphism

$$\xi_X : k\Sigma_n \to \mathrm{End}_{\mathfrak{A}}(X^{\otimes n}), \quad \sigma \mapsto \sigma_X.$$

Now we recall some constructions from [13, 1.3]. For a vector space V of finite dimension over k and an object X of $\mathfrak{A}$, define $V \otimes_k X$ to be the object of $\mathfrak{A}$ representing the covariant functor

$$Y \mapsto \mathrm{Hom}_k(V, \mathrm{Hom}_{\mathfrak{A}}(X, Y)).$$

Next define $\underline{\mathrm{Hom}}(V, X) := V^\vee \otimes_k X$. If a finite group G of order m acts on X, define X^G to be the image of the idempotent $m^{-1}\sum g_X$ where the sum is taken over $g \in G$ and $g_X \in \mathrm{End}_{\mathfrak{A}}(X)$ denotes the endomorphism given by the action. Note that by functoriality of the construction above, the object $Y = \underline{\mathrm{Hom}}(V, X)$ has an action of G. Denote Y^G by $\underline{\mathrm{Hom}}_G(V, X)$. In particular, following [13, 1.4], associated to any finite dimensional k-representation V of the symmetric group Σ_n we can define

$$S_V(X) := \underline{\mathrm{Hom}}_{\Sigma_n}(V, X^{\otimes n}). \tag{1}$$

This is functorial in V and X and commutes with tensor functors. If V is the alternating representation Alt^n or the symmetric representation Sym^n, we denote $S_V(X)$ by $\mathrm{Alt}^n(X)$ or $\mathrm{Sym}^n(X)$ respectively. In general if $V = V_\pi$ corresponds to a partition π of an integer $n =: |\pi|$, then $S_V(X)$ is usually denoted by $S_\pi(X)$ and $X \mapsto S_\pi(X)$ is called a Schur functor.

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Let $K_0(\mathfrak{A})$ be the Grothendieck group of the underlying additive category $\mathfrak{A}$; that is the quotient of the free abelian group on the set of isomorphism classes of objects of $\mathfrak{A}$ by the subgroup generated by all $[X \oplus Y] - [X] - [Y]$ for X, Y objects of $\mathfrak{A}$ with classes $[X]$ and $[Y]$ in the free abelian group (see also [SGA 6, Exp. I, 6.3]). For an element X of the set of isomorphism classes of objects of $\mathfrak{A}$, define the element

$$\lambda(X) = \lambda_t(X) := \sum_{n \geq 0} \mathrm{cl}\big(\mathrm{Alt}^n(X)\big) t^n \in 1 + tK_0(\mathfrak{A})[[t]] =: \Lambda(K_0(\mathfrak{A})). \tag{2}$$

Lemma 2.1 $\mathrm{cl}(X) \mapsto \lambda(X)$ *is a well-defined monomorphism* $K_0(\mathfrak{A}) \to 1 + tK_0(\mathfrak{A})[[t]] \subseteq K_0(\mathfrak{A})[[t]]^\times$ *of abelian groups. In other words,* $K_0(\mathfrak{A})$ *is equipped with a natural pre-*λ*-ring structure.*

Proof: We must show that $\lambda(X \oplus Y) = \lambda(X)\lambda(Y)$. We compare the coefficients of t^n in both sides. By [13, Prop. 1.8] we have the formula

$$\mathrm{Alt}^n(X \oplus Y) \simeq \coprod_{p+q=n} \mathrm{Alt}^p(X) \otimes \mathrm{Alt}^q(Y) \tag{3}$$

for each $n \geq 0$. It follows that λ is a homomorphism. The injectivity follows from the fact that by definition $\mathrm{Alt}^1(X) = X$. □

Remark 2.2 The fact that $K_0(\mathfrak{A})$ is a pre-λ-ring has some ring-theoretic consequences. For example every torsion element of $K_0(\mathfrak{A})$ must be nilpotent. This is proved for any pre-λ-ring in [16, §0] where the result is called G. Segal's lemma. In particular $K_0(\mathfrak{A})$ can not be a non-zero torsion abelian group. See also 3.2 below.

For any commutative ring A consider $\Lambda(A) = 1 + tA[[t]]$. This is a subgroup of the abelian group $A[[t]]^\times$. This structure is denoted by $(a,b) \mapsto ab$. There is also a multiplication $(a,b) \mapsto a \star b$ on this abelian group making $\Lambda(A)$ into a ring. Moreover $\Lambda(A)$ has a pre-λ-ring structure. We will make only formal use of this structure. A λ-ring can be defined to be a pre-λ-ring A such that the structure $\lambda = \lambda_t : A \to \Lambda(A)$ is a ring homomorphism compatible with pre-λ-ring structures (i.e. a λ-homomorphism of pre-λ-rings). For all of these see [SGA 6, Exp. V, §2] and for equivalent definitions see [5, I] where pre-λ-rings (resp. λ-rings) are called λ-rings (resp. special λ-rings).

Proposition 2.3 *The pre-*λ*-ring* $K_0(\mathfrak{A})$ *is a* λ*-ring.*

Proof: See [19, 4.2]. □

Remark 2.4 The existence of a λ-ring structure on Grothendieck rings of various types of tensor categories has already been studied. In [5], the case of representation rings is treated. It is shown in [34] that the Grothendieck ring of certain tensor

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categories (with extra structures) admits a natural λ-ring structure. In the case of arbitrary (abelian) tensor categories, the result is stated in [11, Ch. 3, 5.1] without proof. In [19], the case of arbitrary additive tensor categories is considered. In [7] the case of the tensor triangulated category of mixed Tate motives is treated.

Remark 2.5 If $\mathfrak{A}$ is in addition abelian with a bi-exact tensor structure, then the assignment (2) is well-defined on the Grothendieck ring $K_0(\mathfrak{A})$ defined in terms of exact sequences. This follows from the proof of [13, 1.19]; for any exact sequence $0 \to X \to Z \to Y \to 0$ and any partition π, the object $S_\pi(Z)$ has a finite filtration whose associated graded object is $S_\pi(Y \oplus X)$. It then follows that $K_0(\mathfrak{A})$ is a λ-ring.

In the case of the category ($k\Sigma_n$-fgmod) of finitely generated $k\Sigma_n$-modules, taking the Grothendieck ring gives the representation ring

$$R_k(\Sigma_n) := K_0(k\Sigma_n\text{-fgmod}).$$

Note that the category ($k\Sigma_n$-fgmod) is abelian semi-simple and hence $R_k(\Sigma_n)$ is, as an abelian group, free on the finite set $\mathrm{Irr}_k(\Sigma_n)$ of irreducible k-representations of Σ_n. Set $R_k(\Sigma_0) = \mathbb{Z}$. The abelian graded group

$$R_k := \coprod_{n \geq 0} R_k(\Sigma_n)$$

has the structure of an associative commutative algebra as described below. The product of $u := \mathrm{cl}(U) \in R_k(\Sigma_p)$ and $v := \mathrm{cl}(V) \in R_k(\Sigma_q)$ is given by

$$uv := \mathrm{cl}\big(\mathrm{Ind}_{\Sigma_p \times \Sigma_q}^{\Sigma_{p+q}}(U \otimes_k V)\big) \in R_k(\Sigma_{p+q}) \tag{4}$$

where $\Sigma_p \times \Sigma_q \to \Sigma_{p+q}$ is, for example, the natural embedding onto the subgroup of elements of Σ_{p+q} consisting of those fixing the subset $\{1,2,\cdots,p\}$ and its complement. If $V = V_\mu$ and $U = V_\eta$, then the coefficient of $\mathrm{cl}(V_\pi)$ in the product uv is denoted by $c_{\lambda,\mu}^{\pi}$. Let us mention that R_k is in fact a Hopf algebra: the co-product of $w = \mathrm{cl}(W) \in R_k(\Sigma_n)$ is given by

$$\Delta(w) := \sum_{p+q=n} \mathrm{cl}\big(\mathrm{Res}_{\Sigma_p \times \Sigma_q}^{\Sigma_n}(W)\big) \in R_k \otimes R_k \tag{5}$$

where we have used the natural identification $R_k(\Sigma_p \times \Sigma_q) \simeq R_k(\Sigma_p) \otimes R_k(\Sigma_q)$. A verification of the above Hopf algebra structure is given in [26].

Now let Λ_n be the ring of symmetric polynomials in n variables, i.e. the subring of $\mathbb{Z}[x_1,\cdots,x_n]$ consisting of elements f with $\sigma f = f$ for all $\sigma \in \Sigma_n$.

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The substitution $x_{n+1} = 0$ defines a morphism $\Lambda_{n+1} \to \Lambda_n$ of graded rings. The graded ring of symmetric functions is defined to be the projective limit

$$\Lambda := \varprojlim^{\mathrm{gr}} \Lambda_n$$

taken in the category of graded rings (see [27, I, §2]). It is known ([4], [24], etc.) that R_k is isomorphic to Λ. We use the isomorphism θ from [24, III, §1] given by $\mathrm{cl}(V_\pi) \mapsto s^\pi$ where s^π is the Schur function associated to π. See [26, Th. A] for all possible (Hopf-)algebra isomorphisms (and related necessary definitions). Also note that Λ is a polynomial algebra $\mathbb{Z}[\lambda^1, \lambda^2, \cdots]$ on infinitely many variables λ^i (also denoted by e_i) with $\theta^{-1}(e_i) = \mathrm{cl}(\mathrm{Alt}^i)$. More precisely, e_i is a symmetric function given as the limit of the polynomials $\sum x_{j_1} \cdot \ldots \cdot x_{j_i} \in \Lambda_n$ where the sum is taken over all $1 \leq j_1 < \ldots < j_i \leq n$. The corresponding partition to it (i.e. to Alt^i) is $(i)^t := (1, \ldots, 1)$. We will often use the identification $R_k \simeq \Lambda$ without mentioning θ.

The object $R_k \simeq \Lambda$ has one more structure; it is a free monogeneric λ-ring. For a proof and explanation see [5, §1] and [24, III, 1.]. To use this, let $\mathfrak{A}$ and k be as in the beginning of this section. Let the collection of $\lambda^i \colon K_0(\mathfrak{A}) \to K_0(\mathfrak{A})$ for $i = 0, 1, 2, \ldots$, be the λ-structure given by 2.1, i.e. we require $\sum_{n \geq 0} \lambda^i(x) t^i = \lambda(x) \in K_0(\mathfrak{A})[[t]]$ for all $x \in K_0(\mathfrak{A})$. Let $x \in K_0(\mathfrak{A})$ be an element. The fact that Λ is a polynomial ring shows that there is a homomorphism

$$s(x) \colon R_k \to K_0(\mathfrak{A}), \quad \lambda^i \mapsto \lambda^i(x) \tag{6}$$

of rings. We denote the value of $s(x)$ at a virtual representation v of Σ_n, i.e. an element of $R_k(\Sigma_n)$, by $s^v(x)$. If $v = \mathrm{cl}(V_\pi)$ corresponds to the partition π, then the element $s^v(x) \in K_0(\mathfrak{A})$ may be denoted by $s^\pi(x)$.

Lemma 2.6 *Let $x = \mathrm{cl}(X)$ and $v = \mathrm{cl}(V)$. Then $s^v(x) = \mathrm{cl}(S_V(X))$.*

Proof: The ring R_k is, as an abelian group, freely generated by the classes $w = \mathrm{cl}(W)$ where W is an irreducible k-representation of Σ_n for some $n \geq 0$. The assignment $\mathrm{cl}(W) \mapsto \mathrm{cl}(S_W(X))$ then extends to a group homomorphism, denoted by $w \mapsto s_1^w(x)$. We must show that this is the same as $s(x)$. By definition of $s(x)$ and (2) these two agree on $v = \mathrm{cl}(\mathrm{Alt}^i)$. These generate R_k as a ring. Therefore it is enough to show that $s_1(x)$ is a ring homomorphism. Note that by definition (4) we have, for $v = \mathrm{cl}(V_\mu)$ and $u = \mathrm{cl}(V_\eta)$ corresponding to respective partitions μ and η, the equality $vu = \sum c^\pi_{\mu,\eta} \mathrm{cl}(V_\pi)$ where the sum is over the set A of all partitions π with $|\pi| = |\mu| + |\eta|$. Now we need the result [13, Prop. 1.6] that

$$S_\mu(X) \otimes S_\eta(X) \simeq \coprod_{\pi \in A} S_\pi(X)^{\oplus c^\pi_{\mu,\eta}}.$$

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Therefore $s_1^v(x)s_1^u(x) = s_1^{vu}(x)$ in $K_0(\mathfrak{A})$. That is, $s_1(x)$ is a ring homomorphism. The result follows. □

Remark 2.7 For any (pre-)λ-ring A let $\sigma\colon A \to A[[t]]$ be the corresponding opposite structure, i.e. $\sigma(a) = \sum \sigma^n(a)t^n := \lambda_{-t}(a)^{-1}$. Consider this in the ring Λ and denote $\sigma^i(\lambda^1)$ by σ^i. Under the isomorphism $\Lambda \simeq R_k$ fixed above each σ^i corresponds to the class of Sym^i (cf. [24, III, §2]). It then follows from 2.6 that for each object X of $\mathfrak{A}$ we have

$$\lambda(X)^{-1} = \lambda(-\mathrm{cl}(X)) = \sum_{n\geq 0}(-1)^n \mathrm{cl}(\mathrm{Sym}^n(X))t^n. \tag{7}$$

By definition of σ^i we have: $\lambda^i(-x) = (-1)^i\sigma^i(x)$ for all $x \in K_0(\mathfrak{A})$ and all $i \geq 0$.

Lemma 2.8 *For any $x, y \in K_0(\mathfrak{A})$ and each partition π with transpose π^t we have*

$$s^\pi(-x) = (-1)^{|\pi|}s^{\pi^t}(x) \text{ and } s^\pi(x+y) = \sum_{\mu,\eta} c^\pi_{\mu,\eta} s^\mu(x)s^\eta(y). \tag{8}$$

Proof: (See also [27, Ch. 1, pp. 25, 43 & 74]). By using the second equation we reduce the first assertion to the case $x = \mathrm{cl}(X)$ where the assertion follows for $\pi = (i)^t$ by 2.7 and then as in the proof of 2.6 the general case also follows. For the second equation consider the composite

$$R_k \xrightarrow{\Delta} R_k \otimes_{\mathbb{Z}} R_k \xrightarrow{s(x)\otimes s(y)} K_0(\mathfrak{A}) \otimes K_0(\mathfrak{A}) \xrightarrow{m} K_0(\mathfrak{A})$$

where Δ is the comultiplication on R_k (cf. (5)) and m is the multiplication on $K_0(\mathfrak{A})$. The map is an algebra homomorphism and its value at $v = \mathrm{cl}(V_\pi)$ is exactly the right hand side of the second equation above: use Frobenius reciprocity (see e.g. [13, 1.5]). The map is easily seen to agree with $s(x+y)$ on elements $v = \mathrm{Alt}^i$ (or Sym^i). Therefore the two maps agree on all of R_k. This proves the second assertion. □

Remark 2.9 In general situations we may also consider the assignment

$$\lambda_\Sigma\colon K_0(\mathfrak{A}) \to 1 + tK_0(\mathfrak{A}) \otimes_{\mathbb{Z}} R_k[[t]]$$
$$\mathrm{cl}(X) \mapsto \sum_\pi \mathrm{cl}(S_\pi(X)) \otimes \mathrm{cl}(V_\pi)t^{|\pi|}$$

where the sum is taken over all partitions π (of respective integers $|\pi| \geq 0$). Similarly to 2.1 this gives a well-defined homomorphism of groups. It easily follows from 2.6 and 2.8 that

$$\lambda_\Sigma(x) = \sum_\pi s^\pi(x) \otimes \mathrm{cl}(V_\pi)t^{|\pi|}$$

for all $x \in K_0(\mathfrak{A})$.

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Now we consider the case of certain tensor triangulated categories. Consider the category $\mathfrak{A}' := K^b(\mathfrak{A})$ of bounded complexes over $\mathfrak{A}$ with morphisms being up to homotopy. We note that $\mathfrak{A}'$ is a triangulated category and has a tensor structure which is exact in both arguments. Define $K_0(\mathfrak{A}')$ to be the Grothendieck group of $\mathfrak{A}'$ with respect to the distinguished triangles (cf. [SGA 5, Exp. VIII]). Note that $\mathrm{cl}(Z[m]) = (-1)^m \mathrm{cl}(Z)$ for any object Z of $\mathfrak{A}'$ and any $m \in \mathbb{Z}$. Let $\lambda(X) \in 1 + tK_0(\mathfrak{A}')[[t]]$ be given by (2).

Proposition 2.10 $\mathrm{cl}(X) \mapsto \lambda(X)$ *defines a* λ*-ring structure on* $K_0(\mathfrak{A}')$.

Proof: We must show that if $\Delta : Y \to X \to Z \to Y[1]$ is a distinguished triangle in $K^b(\mathfrak{A})$, then

$$\lambda(X) = \lambda(Y)\lambda(Z).$$

It is clear that if $X \simeq X'$ in $K^b(\mathfrak{A})$, then $\lambda(X) = \lambda(X')$. By definition, we may assume that Δ is a standard distinguished triangle, i.e. $Z^n = Y^{n+1} \oplus X^n$ with standard differentials. We can view each object of $\mathfrak{A}$ as a complex concentrated in degree zero. It is then enough to show that for each bounded complex X we have

$$\lambda(X) = \prod_{n \in \mathbb{Z}} \lambda(X^n)^{(-1)^n}. \tag{9}$$

To prove this we first use the result [7, Proposition 3.2] that for any integer $i \geq 0$ and any object T of $\mathfrak{A}'$ we have $\mathrm{Alt}^i(T[1]) \simeq \mathrm{Sym}^i(T)[i]$. Use the identity (7) from 2.7 written for the underlying additive category $\mathfrak{A}''$ of $\mathfrak{A}'$ and then apply to it the canonical ring homomorphism $K_0(\mathfrak{A}'') \to K_0(\mathfrak{A}')$ and conclude that $\lambda(T[1]) = \lambda(T)^{-1}$ in $K_0(\mathfrak{A}')[[t]]$. Therefore the right hand side of (9) is the product of all $\lambda(X^n[n])$. Finally let

$$\mathrm{gr}^\sigma(X) = \coprod_{n \in \mathbb{Z}} X^n[n]$$

be the associated graded object of the dummy filtration. Note that $\mathrm{cl}(X) = \mathrm{cl}(\mathrm{gr}^\sigma(X))$ and more generally for each partition π we have $\mathrm{cl}(\mathrm{gr}^\sigma(S_\pi(X))) \simeq \mathrm{cl}(S_\pi(\mathrm{gr}^\sigma(X)))$. In particular $\lambda(X) = \lambda(\mathrm{gr}^\sigma(X))$. The result follows. □

Remark 2.11 In many interesting cases the λ-ring $K_0(\mathfrak{A})$ is augmented. See below (5.3) for an applicable argument.

For the additive category $\mathfrak{A}$ as above consider the functor $\mathfrak{A} \to \mathfrak{A}'$ given by viewing each object of $\mathfrak{A}$ as a complex concentrated in degree zero: i.e. $X \mapsto X[0] = X$ where $X[0]$ is a complex whose degree n component is X for $n = 0$ and zero otherwise. Since the triangle $X \to X \oplus Y \to Y \to X[1]$ with zero differential is distinguished in $K^b(\mathfrak{A})$ we obtain a homomorphism

$$i_* : K_0(\mathfrak{A}) \to K_0(\mathfrak{A}'), \quad \mathrm{cl}(X) \mapsto \mathrm{cl}(X[0]). \tag{10}$$

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We note that since $\mathfrak{A}$ is Karoubian, the map i_* is by [17, 3.2.1] an isomorphism with an inverse given by sending $\mathrm{cl}(K)$ to the sum over $i \in \mathbb{Z}$ of $(-1)^i \mathrm{cl}(K^i)$. The following is clear from the discussion above.

Proposition 2.12 *With the above notation, the homomorphism i_* is an isomorphism of λ-rings, i.e. the diagram*

$$\begin{array}{ccc} K_0(\mathfrak{A}) & \xrightarrow{i_*} & K_0\big(K^b(\mathfrak{A})\big) \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} \\ \Lambda\big(K_0(\mathfrak{A})\big) & \xrightarrow{\Lambda(i_*)} & \Lambda\big(K_0\big(K^b(\mathfrak{A})\big)\big) \end{array}$$

is commutative.

We do not know whether for an arbitrary Karoubian linear tensor triangulated category D the natural definition $\mathrm{cl}(X) \mapsto \mathrm{cl}(\mathrm{Alt}^i(X))$ is well-defined on $K_0(D)$. However finer examples of tensor triangulated categories than those from 2.10 can also be considered. For example if $D = D^b(\mathfrak{A})$ is the bounded derived category of a tensor abelian category, then it can be shown that the ring $K_0(D)$ is equipped with its natural λ-structure and in view of 2.5 this structure is in a sense similar to 2.12 compatible with the isomorphism $K_0(D) \simeq K_0(\mathfrak{A})$ given by the Euler characteristic. To see this repeat the proof 2.10 with necessary changes: for example note that if $X \simeq X'$ in D, then by functoriality $\mathrm{Alt}^n(X) \simeq \mathrm{Alt}^n(X')$ and hence $\lambda(X) = \lambda(X')$; also replace the identity (9) by products of $\lambda(H^n(X)[n])$ and gr^σ by $\mathrm{gr}^\tau : D \to D$ given by

$$X \mapsto \coprod_{n \in \mathbb{Z}} H^n(X)[n]$$

which being by [8] a tensor functor (i.e. by Künneth formula) commutes with Schur functors.

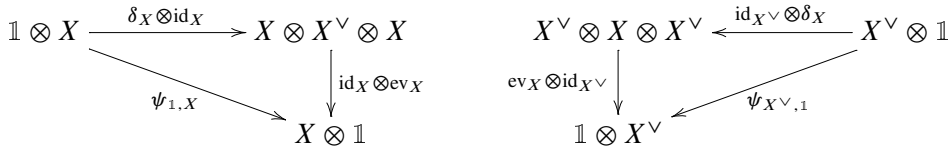
3. Group action and character

In this section we deal with rigid categories. Let $\mathfrak{A}$ be as in §2; i.e. a Karoubian tensor category with $\mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ containing a field k of characteristic zero. We will assume that $\mathfrak{A}$ is rigid ([31, I, §5.1]), i.e. every object of $\mathfrak{A}$ is dualizable in the sense of definition below. The following definition, its variants and related materials can be found in [31], [12], [15], and other places.

Definition 3.1 An object X of $\mathfrak{A}$ is said to be dualizable if there is an object $X^\vee$ together with morphisms $\delta_X : \mathbb{1} \to X \otimes X^\vee$ and $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$ such that both

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of the diagrams



commute.

If X is dualizable, then the dual $(X^\vee, \delta, \mathrm{ev})$ is unique. Next we recall the notion of the trace. For an endomorphism f of a dualizable object X define $\mathrm{tr}(f;X)$ to be the composite

$$\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{id}} X \otimes X^\vee \xrightarrow{\psi_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1}$$

This defines the $\mathrm{End}_{\mathfrak{A}}(\mathbb{1})$-linear homomorphism $\mathrm{tr}\colon \mathrm{End}_{\mathfrak{A}}(X) \to \mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ called the trace and for which we use the notation $f \mapsto \mathrm{tr}(f;X)$. Basic properties of the trace, that is multiplicativity, $(\oplus-)$additivity, and commutativity are given in [31, I, §5].

Remark 3.2 Consider the trace homomorphism

$$\mathrm{tr}\colon K_0(\mathfrak{A}) \to \mathrm{End}_{\mathfrak{A}}(\mathbb{1}), \quad x = \mathrm{cl}(X) \mapsto \mathrm{tr}(\mathrm{id};X).$$

This is, by basic properties of the trace, a well-defined ring homomorphism. We note that for any partition π there is a universal polynomial $d_\pi(x)$ with rational coefficients such that for each object X we have

$$\mathrm{tr}(\mathrm{id}_X; S_\pi(X)) = d_\pi\big(\mathrm{tr}(\mathrm{id}_X;X)\big).$$

For $\pi = (i)^t$ this is the result [12, 7.2] with $d_\pi(x) = (i!)^{-1}x(x-1)\cdots(x-i+1)$ which easily extends to the case of arbitrary π. In particular if we endow $\mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ with its natural λ-ring structure (cf. [SGA 6, Exp. V, 2.6]); $\lambda^i(a) = (i!)^{-1}a(a-1)\cdots(a-i+1)$ for all $a \in \mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ and all $i \geq 0$, then it can be stated that tr is a λ-morphism. Therefore if $\mathrm{tr}(\mathrm{id};X) \in \mathbb{Z}$ for all X, then $K_0(\mathfrak{A})$ is an augmented λ-ring in the sense of [5, §1].

We let G be a group (or monoid). Define the category $\mathrm{Rep}_{\mathfrak{A}}(G)$ to be a category with objects being G-objects in $\mathfrak{A}$, i.e. (X,ρ) where X is an object in $\mathfrak{A}$ and $\rho\colon kG \to \mathrm{End}_{\mathfrak{A}}(X)$ is a $k-$algebra homomorphism, and morphisms being G-morphisms, i.e. morphisms in $\mathfrak{A}$ compatible with the action of G. We use the notation X for (X,ρ) and g_X or g for $\rho(g)$.

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Example 3.3 For $G = 1$ the category $\mathrm{Rep}_{\mathfrak{A}}(G)$ is nothing but $\mathfrak{A}$. Also if $G = \mathbb{N}$, then $\mathrm{Rep}_{\mathfrak{A}}(G)$ is the category $\underline{\mathrm{End}}(\mathfrak{A})$ of endomorphisms of $\mathfrak{A}$; objects are pairs (X, f) with X in $\mathfrak{A}$ and $f \in \mathrm{End}_{\mathfrak{A}}(X)$ and morphisms are appropriate commutative diagrams. Similarly $\mathrm{Rep}_{\mathfrak{A}}(\mathbb{Z}) = \underline{\mathrm{Aut}}(\mathfrak{A})$.

It is also clear that $\mathrm{Rep}_{\mathfrak{A}}(G)$ is naturally equipped with a tensor structure such that the forgetful functor $\mathrm{Rep}_{\mathfrak{A}}(G) \to \mathfrak{A}$ is a strict tensor functor and where the action of G on the tensor product of two $G-$objects is given by the diagonal action. Note that for the unit of the category $\mathrm{Rep}_{\mathfrak{A}}(G)$, i.e. $\mathbb{1}$ with the trivial G-action, we have $\mathrm{End}(\mathbb{1}) = \mathrm{End}_{\mathfrak{A}}(\mathbb{1})$. We will write

$$R_{\mathfrak{A}}(G) := K_0\big(\mathrm{Rep}_{\mathfrak{A}}(G)\big).$$

From the above, this is a commutative associative unital ring and could be called the ring of $k-$representations of G in $\mathfrak{A}$. Moreover by 2.1 we have the following (imprecisely stated) result.

Proposition 3.4 *$R_{\mathfrak{A}}(G)$ is naturally a λ-ring.*

This is in view of 3.3 a generalization of 2.1. We note that if $\mathfrak{A}$ is abelian, then so is naturally $\mathrm{Rep}_{\mathfrak{A}}(G)$ in which an exact sequence may be called G-equivariant. In view of 2.5, in the latter case we may consider the above K_0 with respect to G-equivariant exact sequences. According to 3.2 the λ-structure on $R_{\mathfrak{A}}(G)$ is augmented if $\mathrm{tr}(\mathrm{id}; X) \in \mathbb{Z}$ for all objects X of $\mathfrak{A}$. We immediately note the following example.

Example 3.5 Let $\mathfrak{A}$ be an abelian tensor category as in 2.5 and G a finite group. If $\mathfrak{A}$ is, as an abelian category, semisimple, then $\mathrm{Rep}_{\mathfrak{A}}(G)$ is a semisimple abelian tensor category.

For any ring A, let $FC(G, A)$ be the ring of A-valued central functions on G, i.e. functions $t : G \to A$ with $t(gh) = t(hg)$ and where the ring structure on $FC(G, A)$ is defined value-wise. Note that if A is a (pre-)λ-ring, then $FC(G, A)$ is also a (pre-)λ-ring with value-wise definitions $\lambda^i(f)(g) = \lambda^i(f(g))$. For a rigid category $\mathfrak{A}$, consider the assignment

$$\chi : R_{\mathfrak{A}}(G) \to FC(G, \mathrm{End}_{\mathfrak{A}}(\mathbb{1})), \quad \mathrm{cl}(X) \mapsto \chi_X := \big(g \mapsto \mathrm{tr}(g_X; X)\big).$$

Lemma 3.6 *χ is a well-defined ring homomorphism for any rigid additive (resp. additionally abelian) category $\mathfrak{A}$ as above (resp. 2.5).*

Proof: This is clear for additive categories. In the case of abelian categories the assertion follows from the fact that for any exact G-equivariant sequence $0 \to X \to Z \to Y \to 0$ we have

$$\mathrm{tr}(g_Z; Z) = \mathrm{tr}(g_X; X) + \mathrm{tr}(g_Y; Y)$$

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which follows from [14, 3.5]. □

We shall find a formula which even in the classical case of $\mathfrak{A} = k$-fgmod looks interesting for its own. Let X be in $\mathrm{Rep}_{\mathfrak{A}}(G)$ and consider the ring homomorphism

$$s(x)\colon R := R_{\mathbb{Q}} \to R_{\mathfrak{A}}(G), \quad v \mapsto s^v(x)$$

corresponding to $x = \mathrm{cl}(X)$ constructed in (6). Let ϑ be any character of Σ_n, i.e. an element of $R_{\mathbb{Q}}$. As usual (cf. [33, 14.1]) we view ϑ as a central function on Σ_n; this is given by $\sigma \mapsto \vartheta(\sigma) \in \mathbb{Z}$. By 3.6 we may consider the ring homomorphism $\mathrm{tr}(g;-)\colon R_{\mathfrak{A}}(G) \to \mathrm{End}(\mathbb{1})$ given by $\mathrm{cl}(X) \mapsto \mathrm{tr}(g_X;X)$.

Lemma 3.7 *For any $g \in G$ we have*

$$\mathrm{tr}(g;s^{\vartheta}(x)) = \frac{1}{\#\Sigma_n} \sum_{\sigma\in\Sigma_n} \vartheta(\sigma^{-1})\mathrm{tr}(g^{\otimes n}\sigma;X^{\otimes n}) \in \mathrm{End}(\mathbb{1}).$$

Proof: Both sides of the equation are additive in ϑ and hence it is enough to show the equality for an irreducible character ϑ. This corresponds to a K_0-class $\mathrm{cl}(V)$ where V is an irreducible $\mathbb{Q}$-representation of Σ_n. In this case the element

$$q_{\vartheta} := \frac{\vartheta(1)}{\#\Sigma_n} \sum_{\sigma\in\Sigma_n} \vartheta(\sigma^{-1})\sigma \in \mathrm{End}_{\mathfrak{A}}(X^{\otimes n})$$

is a projector and has the image $V \otimes_{\mathbb{Q}} S_V(X)$. Since $\vartheta(1) = \dim_{\mathbb{Q}}(V) \neq 0$ we have by usual properties of the trace and 2.6 the equalities

$$\begin{aligned} \mathrm{tr}(g;S_V(X)) &= \frac{1}{\vartheta(1)}\mathrm{tr}(\mathrm{id}\otimes g;V\otimes_{\mathbb{Q}} S_V(X)) \\ &= \frac{1}{\vartheta(1)}\mathrm{tr}(g^{\otimes n}q_{\vartheta};X^{\otimes n}) \\ &= \frac{1}{\#\Sigma_n}\sum_{\sigma\in\Sigma_n}\vartheta(\sigma^{-1})\mathrm{tr}(g^{\otimes n}\sigma;X^{\otimes n}). \end{aligned} \tag{11}$$

□

Remark 3.8 Let $X_1, X_2, \ldots X_n$ be objects of $\mathfrak{A}$ and $u_i\colon X_i \to X_{i+1}$ morphisms where $X_{n+1} = X_1$. The morphism $u = \otimes u_i$ can be considered as an endomorphism of $Z = \otimes X_i$. The result [12, 7.2] states that

$$\mathrm{tr}(u;Z) = \mathrm{tr}(u_n \circ u_{n-1} \circ \cdots \circ u_1;X_1).$$

This can be used to show that if $\sigma \in \Sigma_n$ is a permutation with disjoint cyclic decomposition $\sigma = \prod\sigma_j$ with σ_j being a cycle of order m_j and $g \in \mathrm{End}_{\mathfrak{A}}(X)$,

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then $\mathrm{tr}(g^{\otimes n}\sigma; X^{\otimes n}) = \prod \mathrm{tr}(g^{m_j}; X)$. In fact it is enough to consider the case where σ is an $n-$cycle in which case the assertion follows from the result cited above by setting $X_i = X$ and $u_i = g$ for all $i \leq n$.

Example 3.9 In view of 2.6 the result 3.7 implies expected formulas such as $2\chi_{\mathrm{Sym}^2(X)}(g) = \chi_X^2(g) + \chi_X(g^2)$ found in the classical setting (cf. [33, Ch. 2, Prop. 3]). In fact the character ϑ of Sym^2 is given by $\sigma \mapsto 1$ for $\Sigma_2 = \{\mathrm{id}, \sigma\}$ and hence

$$\mathrm{tr}(g; \mathrm{cl}(\mathrm{Sym}^2(X))) = \frac{1}{2}\big(\mathrm{tr}(g^{\otimes 2}; X^{\otimes 2}) + \mathrm{tr}(g^{\otimes 2}\sigma; X^{\otimes 2})\big).$$

By 3.8 we have $\mathrm{tr}(g^{\otimes 2}\sigma; X^{\otimes 2}) = \mathrm{tr}(g^2; X)$. The assertion follows.

As in the classical case define $\overline{\chi}\colon R_{\mathfrak{A}}(G) \to FC(G, \Lambda(\mathrm{End}(\mathbb{1})))$ by

$$\mathrm{cl}(X) \mapsto \overline{\chi}_X := \big(g \mapsto \sum_{n\geq 0} \mathrm{tr}(g; \mathrm{Alt}^n(X))t^n\big). \tag{12}$$

Proposition 3.10 *The assignment* $\mathrm{cl}(X) \mapsto \overline{\chi}_X$ *is a* λ*-homomorphism of* λ*-rings.*

Proof: Set $A := \mathrm{End}_{\mathfrak{A}}(\mathbb{1})$. First note that for the additive category $\mathfrak{A}' := \mathrm{Rep}_{\mathfrak{A}}(G)$ we have by 3.6 the ring homomorphism $\chi\colon K_0(\mathfrak{A}') \to FC(G, A)$. By 2.3 the homomorphism

$$\lambda\colon K_0(\mathfrak{A}') \to 1 + tK_0(\mathfrak{A}')[[t]] = \Lambda(K_0(\mathfrak{A}'))$$

is a $\lambda-$homomorphism. It follows from the functoriality of $B \mapsto \Lambda(B)$ that $\Lambda(\chi)\circ\lambda$ is a $\lambda-$homomorphism. But this composition is nothing but $\overline{\chi}$. □

Remark 3.11 If $\mathfrak{A}$ is abelian as in 2.5, then the above assertions remains true.

Every λ-ring A is equipped with certain operations $\psi_n\colon A \to A$ for $n \geq 1$ which are called the Adams operations and defined by

$$-t\frac{d\lambda(x)}{dt}/\lambda(x) = \sum_{n\geq 1} \psi_n(x)(-t)^n. \tag{13}$$

Each ψ_n is a homomorphism of $\lambda-$rings. For this and more see [5, §5].

Theorem 3.12 *For any object* X *of* $\mathrm{Rep}_{\mathfrak{A}}(G)$ *and any* $g \in G$*, we have*

$$\overline{\chi}_X(g^n) = \psi_n\big(\overline{\chi}_X(g)\big).$$

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Proof: We recall that in this statement $\mathfrak{A}$ is a rigid Karoubian tensor category with $\mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ containing a field of characteristic zero and G is a monoid. For any integer $m \geq 1$, let $L_{(m)}$ be a virtual representation of $\mathbb{Q}\Sigma_m$ which under the isomorphism $\Lambda \simeq R$ mentioned and fixed in §2 corresponds to the power sum function $p_m = x_1^m + x_2^m + \cdots$. It is known ([24, III, §2]) that (the virtual character of) $L_{(m)}$ is given by $\sigma \mapsto m$ if σ is a $m-$cycle and zero otherwise. We use the formula in 3.7 (for $\vartheta = L_{(m)} \in R$ and $x = \mathrm{cl}(X)$) and the result [12, 7.2] (as formulated in 3.8) to obtain

$$\begin{aligned} \mathrm{tr}(g; s^{\vartheta}(x)) &= \frac{1}{(m-1)!} \sum_{m-\text{cycles}} \mathrm{tr}(g^m; X) \\ &= \mathrm{tr}(g^m; X). \end{aligned} \tag{14}$$

Now it is enough to apply $\chi \circ s(x) \colon R \to R_{\mathfrak{A}}(G) \to FC(G, \mathrm{End}(\mathbb{1}))$ to the identity

$$\sum_{k=1}^{m} (-1)^k p_k e_{m-k} = -m e_m$$

in $\Lambda \simeq R$ where e_i's correspond to alternating representations and p_j's are the above power sums. The result is an identity between coefficients in (13) so that

$$-t \frac{d\overline{\chi}_X(g)}{dt} / \overline{\chi}_X(g) = \sum_{m \geq 1} \mathrm{tr}(g^m; X)(-t)^m$$

in the $\lambda-$ring $\Lambda(\mathrm{End}(\mathbb{1})) = 1 + t\mathrm{End}(\mathbb{1})[[t]]$. Using the formula $\overline{\chi} = \Lambda(\chi) \circ \lambda$ from the proof of 3.10, the definition (12), and the fact that ψ_n is a λ-morphism we conclude that $\overline{\chi}_{\psi_n(X)}(g) = \overline{\chi}_X(g^n)$ where $\psi_n(X) = \psi_n(x)$ for $x = \mathrm{cl}(X) \in R_{\mathfrak{A}}(G)$. But $\overline{\chi}$ being by 3.10 a λ-morphism is easily seen to commute with all Adams operations and hence $\overline{\chi}_{\psi_n(X)} = \psi_n(\overline{\chi}_X)$. The result follows. □

Remark 3.13 Note that as the proof shows the formula in 3.12 states nothing but the equality

$$\overline{\chi}_{\psi_n(X)}(g) = \overline{\chi}_X(g^n)$$

as elements of $\mathrm{End}(\mathbb{1})[[t]]$. We also remark that the above result should be related to [21, 15.6].

4. Mixed motives

In this section we consider the category $D = DM_{\mathrm{gm}}(\mathrm{Spec}(F), \mathbb{Q})$ of geometric motives of Voevodsky [35] over the field F and with rational coefficients. This

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is by the result of *ibid.*, 4.3, a (Karoubian) rigid tensor triangulated category. Note that because of the existence and well behavior of motives with coefficients in any commutative ring, i.e. the existence of an exact tensor functor $D \to DM_{\mathrm{gm}}(\mathrm{Spec}(F),k)$ for any ring extension $\mathbb{Q} \subseteq k$, most of the results below are valid for more general coefficient rings.

We want to consider a λ-ring structure on $K_0(D)$ where K_0 is defined with respect to distinguished triangles (cf. [SGA 5, Exp. VIII]). We consider the assignment $X \mapsto \lambda(X)$ given by

$$\lambda(X) = \sum_{n\geq 0} \mathrm{cl}\big(\mathrm{Alt}^n(X)\big)t^n \in K_0(D)[[t]]^{\times} \tag{15}$$

on the set of isomorphism classes of mixed motives.

Theorem 4.1 $\mathrm{cl}(X) \mapsto \lambda(X)$ *gives the structure of a* λ*-ring to* $K_0(D)$.

Proof: This is a special case of [18, Th. 1]. □

Remark 4.2 Let $M := M_{Ch}(\mathrm{Spec}(F),\mathbb{Q})$ be the (opposite) category of pure Chow motives constructed in [28, §5]. This is a rigid $\mathbb{Q}$-linear tensor category. We let $\lambda\colon K_0(M) \to \Lambda(K_0(M))$ be the natural λ-ring structure from 2.1 (and 2.3). By [35, 2.1.4] there is a natural embedding $i\colon M \to D_{\mathrm{add}}$ of tensor categories where D_{add} is the underlying additive category of D. Therefore by considering the tensor functor $D_{\mathrm{add}} \to D$ we have the homomorphism $\rho\colon K_0(M) \to K_0(D)$. According to [9, 6.4] this is an isomorphism. This being induced by tensor functors, is a λ-isomorphism.

Remark 4.3 Let $\mathfrak{A}$ be the Karoubian envelope of the additive category $SmCor_k$ of finite correspondences as defined in [35, 2.1]. Consider the category $\mathfrak{A}' = K^b(\mathfrak{A})$. There is an exact functor $m\colon \mathfrak{A}' \to D$ of tensor triangulated categories. In view of 2.10, $K_0(m)$ is a λ-morphism. This together with 4.2 and the results of [9, 6.5] can be used to obtain explicit computations of $\lambda^i(x)$ for $x = \mathrm{cl}(M^c(X))$ where $M^c(X)$ is (from [35, 2.2]) the motive with compact support of a F-scheme X of finite type. We leave it to the interested reader to work out the details.

Remark 4.4 In view of the general arguments in λ-rings (e.g. [5]), the result 4.1 implies that the natural map $\mathbb{Z} \to K_0(D)$ is injective. In fact $m = 0$ in $K_0(D)$ implies that $1 = \lambda(m) = (1+t)^m$ and the claim follows by comparing the degrees. More importantly we will see in 5.3 that $K_0(D)$ is an augmented λ-ring.

Following the general definitions in [22, 1.3], for any mixed motive X define

$$\zeta(X,t) = \sum_{n\geq 0} \mathrm{cl}(\mathrm{Sym}^n(X))t^n \in 1 + tK_0(D)[[t]] = \Lambda(K_0(D)). \tag{16}$$

Lemma 4.5 $\zeta(X,t) = \lambda_{-t}(X[1]) = \lambda_{-t}(X)^{-1}$.

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Proof: Note that $\mathrm{cl}(X) + \mathrm{cl}(X[1]) = 0$ in $K_0(D)$ and use 2.7. □

For $X = M(S)$ where $S \to \mathrm{Spec}(F)$ is a scheme of finite type, by abuse of notations we denote the function $\zeta(M(S),t)$ by $\zeta(S,t)$. Let us call this expression rational if it can be written as a fraction fg^{-1} with $f,g \in K_0(D)[t]$ and $g(0) = 1$.

Proposition 4.6 *$\zeta(X,t)$ depends only on the class of the motive X in the Grothendieck ring $K_0(D)$. Moreover the map*

$$\zeta\colon K_0(D) \to \Lambda(K_0(D)), \quad \mathrm{cl}(X) \mapsto \zeta(X,t)$$

is well-defined.

Proof: This follows directly from 4.1. □

Corollary 4.7 *If $\zeta(X,t) \in K_0(M_{Ch}(Spec(F),\mathbb{Q}))[[t]]$ is rational for every pure motive X, then so is $\zeta(Y,t)$ for every mixed motive Y.*

Proof: Since the classes $\mathrm{cl}(M)$ for pure Chow motives M generate $K_0(D)$, the result follows from 4.6. □

Example 4.8 Let us say that a morphism $f : X \to S$ in $Sm/\mathrm{Spec}(F)$ is a Y-bundle for a smooth scheme $Y \to \mathrm{Spec}(F)$ of finite type, if S can be covered by a finite set A of open subschemes U with $f^{-1}(U) \to U$ being U-isomorphic to $\mathrm{pr}_U : U \times_F Y \to U$. An argument identical to [17, Prop. 5 (ii)] shows that

$$\mathrm{cl}(M(X)) = \mathrm{cl}(M(S)) \cdot \mathrm{cl}(M(Y)) \in K_0(D).$$

In particular if $\zeta(M(S),t)$ and $\zeta(M(Y),t)$ are rational power series, then $\zeta(M(X),t)$ is also a rational function.

Corollary 4.9 *Let X be a motive. Then $\zeta(X,t)$ is rational if and only if $\zeta(X^\vee,t)$ is so.*

Proof: Indeed $\mathrm{Alt}^n(X^\vee)$ is isomorphic to $\mathrm{Alt}^n(X)^\vee$. Therefore $J : K_0(D) \to K_0(D)$ defined by $\mathrm{cl}(X) \mapsto \mathrm{cl}(X^\vee)$ is a homomorphism of λ-rings. The result follows. □

Let us mention that in the case $X = M(S)$ where $S \to \mathrm{Spec}(F)$ is a scheme of finite type, the above statement can be reformulated in terms of $\zeta(X,t)$ and $\zeta(M^c(S),t)$. Now let $R = R_F \subseteq D$ be the full subcategory of D formed by motives X for which $\zeta(X,t)$ is rational.

Corollary 4.10 *R is a tensor triangulated subcategory of D.*

Proof: This follows from 4.1. □

Remark 4.11 The subcategory R contains many of the better understood motives. For example R contains the following motives.

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1. Every mixed Tate motive.
2. The motive of every abelian variety.
3. The motive of every projective curve.
4. The motive of any split reductive group

and of course by 4.10 any tensor product of the motives above. More precisely we note that (1) and hence (4) follows from [7, 5.8], for (2) and (3) first note that $\zeta(X,t)$ is already a polynomial (resp. the inverse of a polynomial) for any X with $\mathrm{Sym}^n(X) = 0$ (resp. $\mathrm{Alt}^n(X) = 0$) for large enough n. Now in view of 4.6 and 2.8 the results follow from [23] and [32] (see also [3, 13.3]).

Remark 4.12 We were informed (in Oberwolfach in May 2010) by Prof. C. Weibel that in his joint work with C. Mazza (now appeared as [30]) results similar to the above have been found.

We now consider the Adams operations recalled in (13) on mixed motives. As above, D denotes the Karoubian rigid $\mathbb{Q}$-linear tensor triangulated category $DM_{\mathrm{gm}}\big(\mathrm{Spec}(F),\mathbb{Q}\big)$ of geometric motives.

Example 4.13 Let $A = K_0(D)$ with its natural λ-ring structure 4.1. Let $x = \mathrm{cl}(M(\mathbb{P}^1_F))$. We note that for $\alpha = x - 1$ we have $\lambda^i(\alpha) = 0$ for $i \geq 2$. In fact we have $\alpha = \mathrm{cl}(\mathbb{Q}(1)[2]) = \mathrm{cl}(\mathbb{Q}(1))$ and the action of Σ_2 on $\mathbb{Q}(1)$ is by [35, 2.1.5] trivial and hence (by e.g. [13, 1.7]) we have $\mathrm{Alt}^i(\mathbb{Q}(1)) = 0$ for all $i \geq 2$. It follows that for each $n \geq 1$ we have

$$\psi_n(x) = 1 + \alpha^n.$$

This simple case can be generalized as will follow.

Let $D_{\mathrm{tate}} \subseteq D$ be the full triangulated subcategory of (mixed) Tate motives, i.e. the triangulated subcategory generated by $\mathbb{Q}(n)$, $n \in \mathbb{Z}$ (see [25, §3] and [36, §5]). Any motive contained in the essential image of D_{tate} is called a (mixed) Tate motive. For example, the motive of every homogeneous variety is Tate whereas the motive of a general abelian variety is not.

Example 4.14 Let X be a mixed motive and $x = \mathrm{cl}(X) \in K_0(D)$. Assume that X is mixed Tate. Then there is a polynomial $f(a,b) \in \mathbb{Z}[a,b]$ in two variables such that for all integers $n > 0$ we have

$$\psi_n(x) = f(\mathrm{cl}(L^{\otimes n}),\mathrm{cl}(L^{\otimes -n})) \in K_0(D)$$

where L is the Lefschetz (or Tate) motive; $L \oplus \mathbb{Q} = M(\mathbb{P}^1_F)$ (that is $\mathrm{cl}(L) = \alpha$ in 4.13). In fact the tensor exact embedding $D_{\mathrm{tate}} \to D$ of tensor triangulated

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categories gives a ring homomorphism $j : K_0(D_{\text{tate}}) \to K_0(D)$. By [7, §5] and 4.1 these two rings are equipped with their natural λ-ring structure coming from the tensor structure of the respective categories and hence j is a λ-morphism. Now by [7, 5.2] there is a an isomorphism

$$\mathbb{Z}[T,T^{-1}] \xrightarrow{\simeq} K_0(D_{\text{tate}}), \quad T \mapsto \text{cl}(M(\mathbb{P}^1_F)) - 1 = \text{cl}(L),$$

where T is an indeterminate. The claim follows from the fact that ψ_n is a ring homomorphism.

Example 4.15 The proof of 4.14 shows that, with notations there, if X is mixed Tate and effective, then f can be chosen to be the unique polynomial in one variable so that $\psi_n(x) = f(\text{cl}(L^{\otimes n})) \in K_0(D)$ for all $n \geq 1$.

Example 4.16 Consider the smooth group scheme $\mathbf{GL}_2 \to \text{Spec}(F)$. Its geometric motive is in fact mixed Tate. Let $x = \text{cl}(M(\mathbf{GL}_2))$. In view of [6, 5.6] we have $x = (1-\alpha)(1-\alpha^2)$ and hence by 4.15 we have $\psi_n(x) = (1-\alpha^n)(1-\alpha^{2n})$.

Proposition 4.17 *Let X be a motive. If $\zeta(X,t)$ is rational, then so is $\zeta(\psi_n(X),t)$ for every $n \geq 1$.*

Proof: The Adams operations are λ-morphisms. □

5. Characteristic series

We consider the category of motives equipped with an action of a group. The most important structure of the category of motives $D = DM_{\text{gm}}\big(\text{Spec}(F),\mathbb{Q}\big)$ needed for our discussion is the rigidity (see §3). First we note the following essential property.

Lemma 5.1 *If $f : X \to X$ is an automorphism of finite order of a motive X, then* $\text{tr}(f;X) \in \mathbb{Z}$.

This may be checked after applying an appropriate tensor functor R with values in the category of graded vector spaces (over any field of characteristic zero); $\text{tr}(f;X) = \text{tr}(Rf;RX)$. Such functors exist and are called the realization functors (see [20]).

We now let G be a group and consider the category of G-objects in D_{add} (cf. 4.2). As in the general case, this category, denoted by $\text{Rep}_D(G)$, is a Karoubian $\mathbb{Q}$-linear tensor category. It has one more structure. Namely let us call a distinguished triangle $\Delta : X \to Z \to Y \to X[1]$ to be G-equivariant if all morphisms are G-morphisms. We define the group

$$R_D(G) := K_0^D\big(\text{Rep}_D(G)\big)$$

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to be the quotient of the free group on the set of isomorphism classes of objects of $\mathrm{Rep}_D(G)$ by the subgroup generated by all $\mathrm{cl}(Z)-\mathrm{cl}(X)-\mathrm{cl}(Y)$ for G-equivariant triangles Δ. Note that for $G=1$, the group $K_0^D\big(\mathrm{Rep}_D(G)\big)$ is the same as $K_0(D)$.

Let G be a torsion group and consider D_{add}. There is a natural surjective homomorphism $K_0\big(\mathrm{Rep}_{D_{\mathrm{add}}}(G)\big) \twoheadrightarrow R_D(G)$. From definition and notations of (12) and 5.1 we have the ring homomorphism

$$\overline{\chi}\colon K_0\big(\mathrm{Rep}_{D_{\mathrm{add}}}(G)\big) \to FC(G,\Lambda(\mathbb{Z})) \quad \text{(cf. 3.10)}.$$

Theorem 5.2 *The homomorphism $\overline{\chi}$ descends to a $\lambda-$homomorphism of $\lambda-$rings*

$$\overline{\chi}\colon R_D(G) \to FC(G,\Lambda(\mathbb{Z})), \quad \mathrm{cl}(X) \mapsto \overline{\chi}_X.$$

Proof: The proof is identical to that of 3.10 provided we prove a triangulated version of the result 3.6 used in that proof, i.e. show that the homomorphism

$$\chi\colon R_D(G) \to FC(G,\mathbb{Z}), \quad \mathrm{cl}(X) \mapsto \chi_X := \big(g \mapsto \mathrm{tr}(g;X)\big)$$

is well-defined. This amounts to proving that for a G-equivariant distinguished triangle $\Delta : X \to Z \to Y \to X[1]$ in D we have $\chi_Z = \chi_X + \chi_Y$ as functions from $G \to \mathbb{Z}$. Let $g \in G$ and H the subgroup it generates. Let $L/\mathbb{Q}$ be the finite extension generated by a primitive $\#H$-th root of unity. Let $D' := DM_{\mathrm{gm}}(\mathrm{Spec}(F),L)$. There is a tensor exact functor $D \to D'$ denoted by $X \mapsto X'$. The image of G-objects under this functor are again G-objects. It is enough to prove that $\chi_{Z'} = \chi_{X'} + \chi_{Y'}$. The action of H on Δ' extends uniquely to an action of LH. The group algebra LH is split semisimple and the Wedderburn-Artin theorem gives the decomposition

$$Z' \simeq \coprod_{V \in Irr_L H} V \otimes_L S_V(Z') \tag{17}$$

where $Irr_L H$ is the finite set of isomorphism classes of irreducible $L-$ representations of H and $S_V(Z') := \underline{\mathrm{Hom}}_H(V,Z')$ is an object of D' representing the contravariant functor

$$Y \mapsto \mathrm{Hom}_{LH}(V,\mathrm{Hom}_{D'}(Y,Z'))$$

on D' (cf. §2 and [13, 1.3]). The isomorphism (17) is H-equivariant (where H acts trivially on each $S_V(Z')$) and hence

$$\mathrm{tr}(g;Z') = \sum_V \chi_V(g)\mathrm{tr}(\mathrm{id};S_V(Z')) \tag{18}$$

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where $\chi_V : G \to L$ is the usual character of V. The above formula holds for X' and Y' as well. Moreover the triangle

$$S_V(X') \to S_V(Z') \to S_V(Y') \to S_V(X'[1])$$

being a direct summand of Δ' is distinguished in D'. Therefore the statement is reduced to the case $G = 1$, that is $\mathrm{tr}(\mathrm{id}; S_V(Z')) = \mathrm{tr}(\mathrm{id}; S_V(X')) + \mathrm{tr}(\mathrm{id}; S_V(Y'))$ which holds by the additivity of traces [29, Thm. 0.1]. □

Lemma 5.3 *$K_0(D)$ is an augmented λ-ring.*

Proof: Indeed the trace homomorphism $\mathrm{tr}: K_0(D) \to \mathbb{Z}$ is by 5.1 and the additivity of $\mathrm{tr}(\mathrm{id}; -)$ as used in 5.2 well-defined and by 3.2 an augmentation, i.e. a λ-homomorphism between λ-rings. □

Remark 5.4 In view of 2.2, it is an important question to determine whether $K_0(D)$ (or equivalently $K_0(M)$ in 4.2) can have any non-trivial nilpotent element.

Example 5.5 Consider the case $G = 1$. We note that the corresponding homomorphism

$$\overline{\chi}: K_0(D) \to \Lambda(\mathbb{Z}), \quad \mathrm{cl}(X) \mapsto \overline{\chi}_X(1)$$

from 5.2 has an image in the subgroup of rational series. To see this first consider $x = \mathrm{cl}(X)$ with $\mathrm{tr}(\mathrm{id}; X) \geq 0$. In this case $\overline{\chi}(x)$ is a polynomial because $m := \mathrm{tr}(\mathrm{id}; X)$ is a non-negative integer and and hence by 3.2 we have $\mathrm{tr}(\mathrm{id}; \mathrm{Alt}^n(X)) = 0$ for all $n > m$. Next note that every element of $K_0(D)$ can be written as a linear combination of elements $x = \mathrm{cl}(X)$ with $\mathrm{tr}(\mathrm{id}; X) \geq 0$ and that $\overline{\chi}(-x)\overline{\chi}(x) = 1$. The claim follows. This example can be generalized for automorphisms of finite order (see below).

Assume that G is a finite group of exponent m. Let $L/\mathbb{Q}$ be any field containing a primitive m-th root of unity. Let $D' = DM_{\mathrm{gm}}(\mathrm{Spec}(F), L)$. Denote by $R_L(G)$ the representation ring of G in the category of finite dimensional L-vector spaces. Note that there is a natural homomorphism $i: R_L(G) \to R_{D'}(G)$ defined by $\mathrm{cl}(V) \mapsto \mathrm{cl}(V \otimes_L \mathbb{1})$. We also consider the homomorphism $j: K_0(D') \to R_{D'}(G)$ given by viewing each object of D' as a G-object with the trivial action. We may consider the diagrams

$$\begin{array}{ccccc}
R_L(G) \otimes K_0(D') & \xrightarrow{i \otimes j} & R_{D'}(G) \otimes R_{D'}(G) & \xrightarrow{m} & R_{D'}(G) \\
\downarrow{\scriptstyle \overline{\chi} \otimes \overline{\chi}} & & \downarrow{\scriptstyle \overline{\chi} \otimes \overline{\chi}} & & \downarrow{\scriptstyle \overline{\chi}} \\
FC(G, \Lambda(L)) \otimes \Lambda(\mathbb{Z}) & \longrightarrow & FC(G, \Lambda(L)) \otimes FC(G, \Lambda(L)) & \xrightarrow{m} & FC(G, \Lambda(L))
\end{array}$$

in which we use different $\overline{\chi}$'s from 5.2 and 5.5, the homomorphisms denoted by m are respective multiplications, and where $\Lambda(\mathbb{Z})$ is viewed as a subring of constant

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central functions. These diagrams are commutative; the one on the left is by definition commutative and the one on the right is commutative because $\overline{\chi}$ is a ring homomorphism.

Also note that all the rings in the above diagram are λ-rings; if R and S are λ-rings, then the ring $R \otimes_{\mathbb{Z}} S$ has a λ-ring structure λ which is defined by requiring the natural maps $r \mapsto r \otimes 1$ from $R \to R \otimes_{\mathbb{Z}} S$ and the similar one for S to be λ-morphisms (this means that using certain universal polynomials each element of the form $\lambda^n(r \otimes s) = \lambda^n(r \otimes 1 \cdot 1 \otimes s)$ is determined). This construction is also mentioned in [24, I, p. 21]. In the diagram above, we denote $m \circ (i \otimes j)$ by ω_L; that is

$$\begin{aligned} \omega_L \colon R_L(G) \otimes K_0(D') &\to R_{D'}(G) \\ \mathrm{cl}(V) \otimes \mathrm{cl}(X) &\mapsto \mathrm{cl}(V \otimes_L X). \end{aligned}$$

Proposition 5.6 *ω_L is an isomorphism of λ-rings.*

Proof: To prove the bijectivity of ω_L, we construct an inverse. Define

$$\alpha_L(\mathrm{cl}(X)) = \sum_{V \in Irr_L G} \mathrm{cl}(V) \otimes \mathrm{cl}(S_V(X))$$

from $R_{D'}(G) \to R_L(G) \otimes K_0(D')$. By the formula (17) in the proof of 5.2 this is indeed an inverse to ω_L. By definition, $\omega_L = m \circ (i \otimes j)$ is a ring homomorphism. It is also evident that i and j are λ-morphisms. Finally it is easily seen that the multiplication homomorphism of any λ-ring is a λ-morphism. □

Corollary 5.7 *For any G-object X in $DM_{\mathrm{gm}}(Spec(F), \mathbb{Q})$ we have*

$$\overline{\chi}_X(g) = \prod_{V \in Irr_L G} \overline{\chi}_V(g) \star \overline{\chi}_{S_V(X)}(1)$$

where $\star$ denotes the multiplication in the λ-ring $\Lambda(\overline{\mathbb{Q}})$.

Proof: First note that $\overline{\chi}_X(g) \in \Lambda(\mathbb{Z})$ by 5.1. With notations as in (the proof of) 5.6 we have $\mathrm{cl}(X) = \omega_L \circ \alpha_L(\mathrm{cl}(X))$. Now we can use the commutative diagram appearing before 5.6. The result follows immediately. □

Corollary 5.8 *For any automorphism $f : X \to X$ of finite order of a motive X in $DM_{\mathrm{gm}}(Spec(F), \mathbb{Q})$, the power series*

$$\overline{\chi}_X(f) = \sum_{n \geq 0} \mathrm{tr}(f; \mathrm{Alt}^n(X)) t^n \in \mathbb{Z}[[t]]$$

is rational as an element of $\overline{\mathbb{Q}}[[t]]$.

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Proof: Set $G = \mathbb{Z}/m\mathbb{Z}$ where m is the order of f. Let G act on X via $n \mapsto f^{\circ n}$. Use the result 5.7 and note that $\overline{\chi}_V(g)$ is a polynomial with coefficients being algebraic integers (in L introduced above) and $\overline{\chi}_{S_V(X)}(1)$ (similar to 5.5) is a rational power series and belongs to $\Lambda(\mathbb{Z})$. Now it is enough to note that rational series form a subring of $\Lambda(\overline{\mathbb{Q}})$. □

Remark 5.9 It is clear from the above and the general results on representation theory of symmetric groups that if $G = \Sigma_n$, then 5.6 holds for $L = \mathbb{Q}$. Also note that the above rationality result is also valid for any endomorphism f contained in a finite dimensional subalgebra of $\mathrm{End}(X)$. Moreover by definition (16) and 5.2, in a result such as 5.8 we can replace $\mathrm{Alt}^n(X)$'s by $\mathrm{Sym}^n(X)$'s.

Remark 5.10 We note that the series $\overline{\chi}_X(f)$ in 5.8 are even rational in $\mathbb{Q}[[t]]$. This follows from the fact that for a series with coefficients in a field being rational is equivalent to the vanishing of certain (Henkel) determinants defined in terms of the coefficients of the series; cf. [10, IV, §4, exerc. 1].

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REFERENCES

1. *Cohomologie l−adique et Fonctions L*. Séminaire de Géométrie Algébrique du Bois-Marie 1965-1966 (SGA 5), dirigé par A. Grothendieck avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou et J.-P. Serre. Lecture Notes in Mathematics **589**, Springer-Verlag, Berlin, 1977

2. *Théorie des intersections et théorème de Riemann-Roch*. Séminaire de Géométrie Algébrique du Bois-Marie 1966-1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre. Lecture Notes in Mathematics **225**, Springer-Verlag, Berlin, 1971

3. André, Yves, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)* Panoramas et Synthèses [Panoramas and Syntheses] **17**, Société Mathématique de France, Paris, 2004.

4. Atiyah, M. F., *Power operations in K-theory*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 165–193.

5. Atiyah, M. F., Tall, D. O., *Group representations, λ-rings and the J-homomorphism*, Topology **8** (1969), 253–297.

https://doi.org/10.1017/is013005008jkt231 Published online by Cambridge University Press

6. Biglari, S., *Motives of Reductive Groups*, Amer. J. Math. **134**(1) (2012), 235–257.
7. Biglari, S., *On finite dimensionality of mixed Tate motives*, J. $K-$Theory **4**(1) (2009), 145–161.
8. Biglari, S., *A Künneth formula in tensor triangulated categories*, J. Pure Appl. Algebra **210**(3) (2007), 645–650.
9. Bondarko, M. V., *Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky versus Hanamura*, *J. Inst. Math. Jussieu* **8**(1) (2009), 39–97.
10. Bourbaki, N., Éléments de mathématique. Algèbre. Chapitres 4 à 7, Masson, Paris, 1981.
11. Davydov, A. A., *Monoidal categories*, J. Math. Sci. (New York) **88**(4) (1998), 457–519.
12. Deligne, P., *Catégories tannakiennes*, The Grothendieck Festschrift II, Progr. Math. **87**, Birkhäuser Boston, Boston, MA, 1990, pp. 111–195.
13. Deligne, P., *Catégories tensorielles*, Mosc. Math. J. **2**(2) (2002), 227–248.
14. Deligne, P., *La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel*, Tata Inst. Fund. Res., Mumbai, 2007, pp. 209–273.
15. Dold, A., Puppe, D., *Duality, trace, and transfer*, Proceedings of the International Conference on Geometric Topology (Warsaw, 1978), 1980, pp. 81–102.
16. Dress, A. W. M., *Induction and structure theorems for orthogonal representations of finite groups*, Ann. of Math. (2) **102**(2) (1975), 291–325.
17. Gillet, H., Soulé, C., *Descent, motives and K-theory*, J. Reine Angew. Math. **478** (1996), 127–176.
18. Guletskii, V. I., *Zeta functions in triangulated categories*, Math. Notes **87**(3-4) (2010), 345–354.
19. Heinloth, F., *A note on functional equations for zeta functions with values in Chow motives*, Ann. Inst. Fourier (Grenoble) **57**(6) (2007), 1927–1945.
20. Huber, A., *Realization of Voevodsky's motives*, J. Algebraic Geom. **9**(4) (2000), 755–799, Corrigendum: *ibid.* **13**(1) (2004), 195–207.
21. Kahn, B., *Zeta functions and motives*, Pure Appl. Math. Q. **5**(1) (2009), 507–570.
22. M. Kapranov, *The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups*, 2000, arXiv:math/0001005.
23. Kimura, S.-I., *Chow groups are finite dimensional, in some sense*, Math. Ann. **331**(1) 2005, 173–201.
24. Knutson, D., *$\lambda-$rings and the representation theory of the symmetric group*, Lecture Notes in Mathematics **308**, Springer-Verlag, Berlin, 1973.

https://doi.org/10.1017/is013005008jkt231 Published online by Cambridge University Press

25. Levine, M., *Tate motives and the vanishing conjectures for algebraic K-theory*, In Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **407**, Kluwer Acad. Publ., Dordrecht, 1993, pp. 167–188.

26. Liulevicius, A., *Arrows, symmetries and representation rings*, J. Pure Appl. Algebra **19** (1980), 259–273.

27. Macdonald, I. G, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, Oxford University Press, New York, 1995.

28. Manin, Ju. I, *Correspondences, motifs and monoidal transformations*, Math. USSR-Sb. **6** (1968), 439–470,

29. May, J. P., *The additivity of traces in triangulated categories*, Adv. Math. **163**(1) (2001), 34–73.

30. Mazza, C., Weibel, C., *Schur-finiteness in λ-rings*, Journal of Algebra **374** (2013), 66–78.

31. Saavedra Rivano, N., *Catégories Tannakiennes*, Lecture Notes in Mathematics **265**, Springer-Verlag, Berlin, 1972.

32. Šermenev, A. M., *The Motif of an Abelian variety*, Functional Anal. Appl. **8** (1974), 47–53.

33. Serre, J.-P., *Représentations linéaires des groupes finis*, 2ième éd., Hermann, Paris, 1971.

34. Swan, R. G., *A splitting principle in algebraic K-theory*, Representation theory of finite groups and related topics (Proc. Sympos. Pure Math. XXI, Univ. Wisconsin, Madison, Wis., 1970), Amer. Math. Soc., Providence, R.I., 1971, pp. 155–159.

35. Voevodsky, V., *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud. **143**, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238,

36. Voevodsky, V., *Motives over simplicial schemes*, J. K-Theory **5**(1) (2010), 1–38.

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