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# Convergence Analysis for a Three-Level Finite Difference Scheme of a Second Order Nonlinear ODE Blow-Up Problem

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**Abstract.** We consider the second order nonlinear ordinary differential equation $u''(t) = u^{1+\alpha}$ ($\alpha > 0$) with positive initial data $u(0) = a_0$, $u'(0) = a_1$, whose solution becomes unbounded in a finite time $T$. The finite time $T$ is called the blow-up time. Since finite difference schemes with uniform meshes can not reproduce such a phenomenon well, adaptively-defined grids are applied. Convergence with mesh sizes of certain smallness has been considered before. However, more iterations are required to obtain an approximate blow-up time if smaller meshes are applied. As a consequence, we consider in this paper a finite difference scheme with a rather larger grid size and show the convergence of the numerical solution and the numerical blow-up time. Application to the nonlinear wave equation is also discussed.

**AMS subject classifications**: 65L12

**Key words**: Blow-up, numerical blow-up time, finite difference method, nonlinear ODE.

## 1. Introduction

In this paper, we consider a second order nonlinear ordinary differential equation

$$u''(t) = u^{1+\alpha}(t), \qquad u(0) = a_0 > 0, \qquad u'(0) = a_1 > 0, \tag{1.1}$$

and its finite difference analogue. Here, $\alpha > 0$ is a parameter and $'$ denotes the differentiation. It is easy to show that the solution of (1.1) blows up in finite time $T$. In fact, multiplying the first equation of (1.1) by $u'(t)$, we have

$$u'(t) = \left(\frac{2}{2+\alpha}u^{2+\alpha} + C\right)^{\frac{1}{2}}, \tag{1.2}$$

where $C = a_1^2 - 2a_0^{2+\alpha}/(2+\alpha)$. Therefore, for $a_0, a_1 > 0$, the solution becomes unbounded in finite time $T = \int_{a_0}^{\infty} ds/g(s)$, where

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$$g(s)=\left(\frac{2}{2+\alpha}s^{2+\alpha}+C\right)^{\frac{1}{2}}. \tag{1.3}$$

This phenomenon is known as blow-up and the finite time $T$ is called the blow-up time.

It is known that a scheme with uniform time meshes can not reproduce the finite-time blow-up phenomenon and thus adaptively-defined time meshes are considered to be necessary for such problems. See for instance [1, 4, 8, 11, 18]. Consequently, we consider the following finite difference analogue of (1.1)

$$\frac{1}{\tau_n}\left(V^{n+1}-V^n\right)=(U^n)^{1+\alpha} \quad \text{and} \quad V^n=\frac{U^n-U^{n-1}}{\Delta t_{n-1}}, \tag{1.4}$$

where the grid size $\Delta t_n$ is given adaptively by

$$\Delta t_n=\tau\cdot\min\left\{1,\frac{1}{(U^n)^\gamma}\right\} \quad \left(0<\gamma<\frac{\alpha}{2}\right). \tag{1.5}$$

Here, $\tau$ is a prescribed constant and $\tau_n=(\Delta t_{n-1}+\Delta t_n)/2$. $t_0=0$ and $t_n=t_{n-1}+\Delta t_{n-1}$ $(\forall n\geq 1)$ denote the grid points. The discrete initial data is given by

$$U^0=a_0>0 \quad \text{and} \quad \frac{U^1-U^0}{\Delta t_0}=a_1>0. \tag{1.6}$$

We now set the numerical blow-up time $T(\tau)$ by

$$T(\tau)=\lim_{n\to\infty}t_n=\sum_{n=0}^{\infty}\Delta t_n.$$

Then we are going to prove in this paper

**Theorem 1.1.** *Let $\{U^n\}$ be the solution of* (1.4)(1.6). *Let $T$ denote the blow-up time of the solution of* (1.1) *and let $T_0$ be an arbitrary number such that $0<T_0<T$. Then there exist positive constant $C$ and $\tau_0$, depending only on $T_0$ and the initial data, such that*

$$\left|U^n-u(t_n)\right|\leq C\tau$$

*holds so far as $t_n\leq T_0$ and $0<\tau\leq\tau_0$.*

**Theorem 1.2.** *Let $\{U^n\}$ be the solution of* (1.4)(1.6). *Let $T$ denote the blow-up time of* (1.1). *Then we have*

$$T(\tau)\to T \quad as \quad \tau\to 0.$$

The conclusions of Theorem 1.1 and 1.2 themselves are not important. As a matter of fact, we may reduce the second order nonlinear ODE (1.1) to the first order ODE (1.2) and then apply the method given in [8] for the computation of the numerical solution and the

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numerical blow-up time. The importance of Theorem 1.1 and 1.2 lies in its application to the nonlinear wave equation [10, 12–14]

$$u_{tt}(t,x) = u_{xx}(t,x) + u^{1+\alpha}(t,x) \quad (\alpha > 0), \tag{1.7}$$

whose solution captures common blow-up features arising in various physical models. For instance, (1.7) is exactly the nonlinear Klein-Gordon equation

$$u_{tt}(t,x) - u_{xx}(t,x) + m^2 u(t,x) = u^{1+\alpha}(t,x) \quad (\alpha > 0),$$

which describes self-focusing waves in nonlinear optics [2] if we put the constant $m = 0$. See also [15] and the references therein. The first author [5] considered the blow-up problem (1.7) and the finite difference analogue

$$\frac{1}{\tau_n}\left(\frac{U_j^{n+1} - U_j^n}{\Delta t_n} - \frac{U_j^n - U_j^{n-1}}{\Delta t_{n-1}}\right) = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} + (U_j^n)^{1+\alpha}, \tag{1.8}$$

where $h$ is the spatial grid size, $x_j = jh$ are the spatial grid points. $\Delta t_n$ is given by

$$\Delta t_n = \tau \cdot \min\left\{1, \frac{1}{\|U^n\|_p^\gamma}\right\}, \tag{1.9}$$

where $\tau$ is a prescribed number, $\tau_n = (\Delta t_n + \Delta t_{n-1})/2$ and $\|U^n\|_p$ denotes the discrete $L^p$-norm. $t_0 = 0$, $t_n = t_{n-1} + \Delta t_{n-1}$ $(\forall n \geq 1)$ are the temporal grid points. $U_j^n$ denotes the approximation of the solution $u$ of (1.7) at $(t_n, x_j)$. Let $\lambda = \tau/h \leq 1$ be fixed for stability. The numerical blow-up time is defined by

$$T(\tau, h) = \sum_{n=0}^{\infty} \Delta t_n.$$

Then the author showed that the phenomenon of finite-time blow-up is numerically reproduced by the scheme (1.8). That is, the author showed $T(\tau, h) < \infty$. Moreover, the author proved the convergence of the numerical blow-up time under the assumption that the numerical solutions converge to the real solutions. Although the numerical experiments suggested that the computation proceeds stably, we could not prove the convergence for the scheme (1.8) analytically.

To show the stability or convergence for a two-level difference scheme with nonuniform time mesh is fundamental because the stability estimate can be obtained directly from the estimate of the solution at each time step. However, the situation becomes much more complicated for a three-level scheme like (1.4) or (1.8) since not only the estimate of the solution of previous steps but the relationship between consecutive two temporal increments have to be taken into account in the stability analysis. In fact, to show the convergence of three-level schemes, we often need some non-increasing discrete energy estimates. See for example [16, 20]. Unfortunately, such estimates are not always available if $\Delta t_n$ is varied.

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Although the author [7] constructed a "modified" discrete energy for the three-level finite difference scheme

$$\frac{1}{\tau_n}\left(\frac{U_j^{n+1}-U_j^n}{\Delta t_n}-\frac{U_j^n-U_j^{n-1}}{\Delta t_{n-1}}\right)=\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{h^2} \tag{1.10}$$

of the linear wave equation $u_{tt}=u_{xx}$, the stability depends strongly on the monotonicity of the ratio $\Delta t_n/\Delta t_{n-1}$. See also [16, 17] in which similar stability criteria were derived for semi-implicit schemes. However, it is difficult to investigate the behavior of $\Delta t_n/\Delta t_{n-1}$ $(n \geq 0)$ for the scheme (1.8) since the temporal grid size defined by (1.9) decreases indefinitely.

As a consequence, we first neglect the diffusion term of (1.7) to simplify the problem and rigorously prove the convergence of the numerical solution and the numerical blow-up time as a first step for the numerical study of the nonlinear blow-up problem (1.7). This is the motivation for considering (1.1).

**Remark 1.1.** The convergence of the numerical blow-up time of (1.4) for the case $\alpha/2 \leq \gamma$ has been proved in [6], while the current case $0<\gamma<\alpha/2$ remains open due to certain technical problems. It should be noted that the larger $\gamma$ is, the smaller the grid size becomes. Consequently, more iterations are required to obtain an approximate blow-up time if a larger $\gamma$ is applied. Although, intuitively, smaller grids give more accurate estimate for the solution, the computation becomes unnecessarily slow. See for example Table 1 in [8] and Table 1, 2 below. This is why we consider in this paper the case of $0<\gamma<\alpha/2$.

We also remark here that the proof given in [6] for the case $\gamma \geq \alpha/2$ can not be applied to the current case $0<\gamma<\alpha/2$. Due to the largeness of the grid size, we have to investigate the behavior of the numerical solution in more detail. In fact, as will be shown in the following section, convergence of the current case depends strongly on the monotonicity of $\Delta t_n/\Delta t_{n-1}$.

We put from now on $\alpha=1$ for simplicity. The proofs for other choices of $\alpha$ can be carried out in exactly the same way.

We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3. The computational results will be presented in Section 4. We also report some numerical results for the nonlinear wave equation (1.7) in the same section. Finally, the paper is ended with a brief conclusion.

## 2. Proof of Theorem 1.1

Before we prove the main theorems, we first note that, for $0<a_0<1$, the first several time-step lengths are uniform. Therefore, (1.4) reads as

$$\frac{U^{n+1}-2U^n+U^{n-1}}{\tau^2}=(U^n)^2,$$

which, together with the discrete initial data (1.6), is a scheme of order $O(\tau)$. Moreover, the positivity of $a_0$ and $a_1$ implies that $U^n$ is increasing in $n$ and that $U^{n_0} \geq 1$ for a certain

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$n_0$. At the time $t_m$ when $U^m > 1$ for the first time, $|U^m - u(t_m)| = O(\tau)$. Thus, by regarding $t_m$ as the origin of $t$ axis, we may assume, without loss of generality, that $a_0 \geq 1$. In other words, the time grid size $\Delta t_n$ is given by

$$\Delta t_n = \frac{\tau}{(U^n)^\gamma} \quad (\forall n \geq 0). \tag{2.1}$$

We begin the proof for Theorem 1.1 with several lemmas.

**Lemma 2.1.** *Let $\{U^n\}$ be a solution of* (1.4)(1.6). *Then it holds that*

$$V^{n+1} > V^n \ (\forall n \geq 1) \quad \textit{and} \quad U^{n+1} > U^n \ (\forall n \geq 0).$$

*Moreover, we have $U^n \to \infty$ as $n \to \infty$.*

Lemma 2.1 can be easily proved by the positivity of the initial data. We thus omit it.

**Lemma 2.2.** *Assume that*

$$2a_0^3 - 3a_1^2 \geq 0. \tag{2.2}$$

*Then*

$$\frac{U^k}{U^{k+1}} \leq \frac{U^{k-1}}{U^k} \quad (\forall k \geq 1). \tag{2.3}$$

*In particular, we have*

$$\frac{\Delta t_{k+1}}{\tau_{k+1}} \leq \frac{\Delta t_k}{\tau_k} \quad (\forall k \geq 1). \tag{2.4}$$

*Proof.* By (1.4), one has

$$U^2 = a_0 + a_1 \Delta t_0 + a_1 \Delta t_1 + \Delta t_1 \tau_1 (U^1)^2.$$

Note also that, for $k = 1$, (2.3) can be written as

$$U^2 \geq a_0 + 2a_1 \Delta t_0 + \frac{a_1^2}{a_0} (\Delta t_0)^2.$$

Therefore, to show the validity of (2.3) for $k = 1$ is equivalent to show

$$a_1(\Delta t_1 - \Delta t_0) + \Delta t_1 \tau_1 (U^1)^2 - \frac{a_1^2}{a_0} (\Delta t_0)^2 \geq 0. \tag{2.5}$$

To this end, since, by the monotonicity of $\{U^n\}$, (2.1), and the fact that $0 < \gamma < 1/2$,

$$\begin{aligned}
a_1(\Delta t_1 - \Delta t_0) =& a_1 \Delta t_0 \left( \left( \frac{U^0}{U^1} \right)^\gamma - 1 \right) \\
\geq& a_1 \Delta t_0 \left( \frac{(U^0)^{\frac{1}{2}} - (U^1)^{\frac{1}{2}}}{(U^1)^{\frac{1}{2}}} \right) \\
\geq& - a_1 \Delta t_0 \cdot \frac{1}{2} \frac{U^1 - U^0}{(U^1)^{\frac{1}{2}} (U^0)^{\frac{1}{2}}} \\
\geq& - \frac{a_1 (\Delta t_0)^2}{2U^0} \frac{U^1 - U^0}{\Delta t_0} = - \frac{(a_1 \Delta t_0)^2}{2a^0},
\end{aligned}$$

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we have

$$a_1(\Delta t_1-\Delta t_0)+\Delta t_1\tau_1(U^1)^2-\frac{a_1^2}{a_0}(\Delta t_0)^2\geq-\frac{3(a_1\Delta t_0)^2}{2a_0}+(\Delta t_1U^1)^2.$$

Observe that

$$\frac{\Delta t_1U^1}{\Delta t_0U^0}=\left(\frac{U^1}{U^0}\right)^{1-\gamma}\geq 1,$$

which implies $\Delta t_1U^1\geq\Delta t_0U^0=\Delta t_0a_0$. Now it follows from the assumption (2.2) that

$$a_1(\Delta t_1-\Delta t_0)+\Delta t_1\tau_1(U^1)^2-\frac{a_1^2}{a_0}(\Delta t_0)^2\geq\frac{(\Delta t_0)^2}{2a_0}\left(-3a_1^2+2a_0^3\right)\geq 0.$$

Next, we assume that (2.3)(2.4) hold for $k=n-1$. Namely,

$$\frac{U^{n-1}}{U^n}\leq\frac{U^{n-2}}{U^{n-1}}\quad\text{and}\quad\frac{\Delta t_n}{\tau_n}\leq\frac{\Delta t_{n-1}}{\tau_{n-1}}.\tag{2.6}$$

For $k=n$, we first note that

$$\begin{aligned}
&U^{n+1}U^{n-1}-(U^n)^2\\
=&\left[U^n+\frac{\Delta t_n}{\Delta t_{n-1}}(U^n-U^{n-1})+\tau_n\Delta t_n(U^n)^2\right]U^{n-1}-(U^n)^2\\
=&U^nU^{n-1}\left[1+\frac{\Delta t_n}{\Delta t_{n-1}}\left(1-\frac{U^{n-1}}{U^n}\right)+\tau_n\Delta t_nU^n-\frac{U^n}{U^{n-1}}\right]\\
\geq&U^nU^{n-1}\left[1+\left(\frac{U^{n-1}}{U^n}\right)^{\gamma}\left(1-\frac{U^{n-1}}{U^n}\right)+\tau_{n-1}\Delta t_{n-1}U^{n-1}-\frac{U^n}{U^{n-1}}\right].
\end{aligned}$$

Here, use has been made of $\Delta t_n/\tau_n\leq\Delta t_{n-1}/\tau_{n-1}$, which implies

$$\frac{\tau_n\Delta t_nU^n}{\tau_{n-1}\Delta t_{n-1}U^{n-1}}\geq\left(\frac{\Delta t_n}{\Delta t_{n-1}}\right)^2\frac{U^n}{U^{n-1}}=\left(\frac{U^n}{U^{n-1}}\right)^{1-2\gamma}\geq 1.$$

Put $f(x)=x^{\gamma}(1-x)$. It is elementary to show that $f(x)$ is decreasing on $[\gamma/(1+\gamma),1]$ and increasing on $[0,\gamma/(1+\gamma))$. If $U^{n-1}/U^n\geq\gamma/(1+\gamma)$, then one has by (2.6)

$$f\left(\frac{U^{n-1}}{U^n}\right)\geq f\left(\frac{U^{n-2}}{U^{n-1}}\right),$$

which, together with (1.4), implies

$$\begin{aligned}
U^{n+1}U^{n-1}-(U^n)^2\geq&U^nU^{n-1}\left[1+\left(\frac{U^{n-1}}{U^n}\right)^{\gamma}\left(1-\frac{U^{n-1}}{U^n}\right)+\tau_{n-1}\Delta t_{n-1}U^{n-1}-\frac{U^n}{U^{n-1}}\right]\\
\geq&U^nU^{n-1}\left[1+\left(\frac{U^{n-2}}{U^{n-1}}\right)^{\gamma}\left(1-\frac{U^{n-2}}{U^{n-1}}\right)+\tau_{n-1}\Delta t_{n-1}U^{n-1}-\frac{U^n}{U^{n-1}}\right]=0.
\end{aligned}$$

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On the other hand, since $f$ attains its maximum at $\gamma/(1+\gamma)$, one has by (1.4) and (2.6)

$$\begin{aligned}1 =& \frac{U^{n-1}}{U^n}\left[1+\left(\frac{U^{n-2}}{U^{n-1}}\right)^{\gamma}\left(1-\frac{U^{n-2}}{U^{n-1}}\right)\right]+\frac{\tau^2}{2}(U^{n-1})^{1-2\gamma}\left[\frac{U^{n-1}}{U^n}+\left(\frac{U^{n-1}}{U^n}\right)^{1-\gamma}\left[\frac{(U^{n-1})^2}{U^nU^{n-2}}\right]^{\gamma}\right]\\ \le& \frac{U^{n-1}}{U^n}\left[1+\left(\frac{U^{n-2}}{U^{n-1}}\right)^{\gamma}\left(1-\frac{U^{n-2}}{U^{n-1}}\right)\right]+\frac{\tau^2}{2}(U^{n-1})^{1-2\gamma}\left[\frac{U^{n-1}}{U^n}+\left(\frac{U^{n-1}}{U^n}\right)^{1-\gamma}\right]\\ \le& \frac{\gamma}{1+\gamma}\left[1+f\left(\frac{\gamma}{1+\gamma}\right)\right]+\frac{\tau^2}{2}(U^{n-1})^{1-2\gamma}\left[\frac{\gamma}{1+\gamma}\left(1+\left(\frac{1+\gamma}{\gamma}\right)^{\gamma}\right)\right]\end{aligned}$$

if $U^{n-1}/U^n < \gamma/(1+\gamma)$. This implies

$$\frac{\tau^2}{2}(U^{n-1})^{1-2\gamma} \ge \frac{\frac{1}{\gamma}-f\left(\frac{\gamma}{1+\gamma}\right)}{1+\left(\frac{1+\gamma}{\gamma}\right)^{\gamma}} \equiv c.$$

From the definition of $f$, it is not difficult to see that $c$ is positive. Note also that we have by (1.4)

$$\begin{aligned}\frac{U^{n+1}}{U^n} =& 1+\left(\frac{U^{n-1}}{U^n}\right)^{\gamma}\left(1-\frac{U^{n-1}}{U^n}\right)+\frac{\tau^2}{2}(U^n)^{1-2\gamma}\left(1+\left(\frac{U^n}{U^{n-1}}\right)^{\gamma}\right)\\ \ge& 1+\left(\frac{U^{n-1}}{U^n}\right)^{\gamma}\left(1-\frac{U^{n-1}}{U^n}\right)+\frac{\tau^2}{2}(U^{n-1})^{1-2\gamma}\left(1+\left(\frac{U^n}{U^{n-1}}\right)^{\gamma}\right).\end{aligned}$$

Let $p(x) = x^{\gamma}(1-x)+(U^{n-1})^{1-2\gamma}(1+x^{-\gamma})\tau^2/2$. Then

$$p'(x) \le x^{\gamma-1}(\gamma-(1+\gamma)x-c\gamma x^{-2\gamma}) \le \begin{cases} x^{\gamma-1}(\gamma-(1+\gamma)x) \le 0 & , \text{ if } x \ge \frac{\gamma}{1+\gamma} \\ x^{\gamma-1}\left(\gamma-c\gamma\left(\frac{\gamma}{1+\gamma}\right)^{-2\gamma}\right) \le 0 & , \text{ if } 0<x<\frac{\gamma}{1+\gamma}\end{cases}.$$

Here, we have used the fact that

$$1-c\left(\frac{\gamma}{1+\gamma}\right)^{-2\gamma} \le 0, \tag{2.7}$$

for $0<\gamma<1/2$. Since it is elementary to show (2.7), we provide the graph of the function $R(\gamma) = c-(\gamma/(1+\gamma))^{2\gamma}$ instead of a proof. See Fig. 1.

It now follows that $p(x)$ is monotone decreasing for all $x>0$. Consequently,

$$\begin{aligned}\frac{U^{n+1}}{U^n} \ge& 1+\left(\frac{U^{n-1}}{U^n}\right)^{\gamma}\left(1-\frac{U^{n-1}}{U^n}\right)+\frac{\tau^2}{2}(U^{n-1})^{1-2\gamma}\left(1+\left(\frac{U^n}{U^{n-1}}\right)^{\gamma}\right)\\ \ge& 1+\left(\frac{U^{n-2}}{U^{n-1}}\right)^{\gamma}\left(1-\frac{U^{n-2}}{U^{n-1}}\right)+\frac{\tau^2}{2}(U^{n-1})^{1-2\gamma}\left(1+\left(\frac{U^{n-1}}{U^{n-2}}\right)^{\gamma}\right)=\frac{U^n}{U^{n-1}}.\end{aligned}$$

In either case, (2.3) holds for $k=n$. Now (2.3) follows by induction. □

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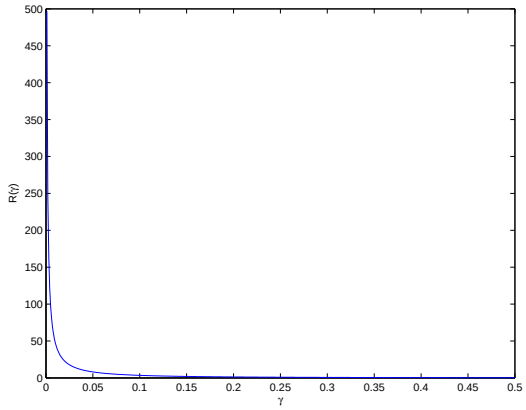



Figure 1: $R(\gamma)$ for $0 < \gamma < 1/2$.

We are now in a position to show the convergence of the numerical solution.

***Proof of Theorem 1.1***

By Taylor's formula, one has

$$\frac{1}{\tau_n}\left(\frac{u(t_{n+1})-u(t_n)}{\Delta t_n}-\frac{u(t_n)-u(t_{n-1})}{\Delta t_{n-1}}\right)=(u(t_n))^2+r_n, \tag{2.8}$$

where

$$r_n=u'''(t_n)\frac{\Delta t_n-\Delta t_{n-1}}{3}+\frac{1}{\tau_n}\left[u^{(4)}(t_n+\theta_1\Delta t_n)\frac{(\Delta t_n)^3}{4!}+u^{(4)}(t_n-\theta_2\Delta t_n)\frac{(\Delta t_{n-1})^3}{4!}\right],$$

for some $0<\theta_1,\theta_2<1$. Let $E^n=U^n-u(t_n)$. Then we have

$$\frac{1}{\tau_n}\left(\frac{E^{n+1}-E^n}{\Delta t_n}-\frac{E^n-E^{n-1}}{\Delta t_{n-1}}\right)=E^n(U^n+u(t_n))-r_n. \tag{2.9}$$

Put $F^{n+1}=(E^{n+1}-E^n)/\Delta t_n$ $(\forall n\geq 0)$. Multiply (2.9) by $F^{n+1}$, the left-hand side becomes

$$\begin{aligned}L.H.S.=&\frac{1}{\tau_n}(F^{n+1}-F^n)\left(\frac{F^{n+1}+F^n}{2}+\frac{F^{n+1}-F^n}{2}\right)\\=&\frac{1}{2\tau_n}\left[(F^{n+1})^2-(F^n)^2+(F^{n+1}-F^n)^2\right],\end{aligned}$$

while the right-hand side becomes

$$\begin{aligned}R.H.S.=&\left[\left(\frac{E^{n+1}+E^n}{2}-\frac{E^{n+1}-E^n}{2}\right)(U^n+u(t_n))-r_n\right]F^{n+1}\\=&\frac{1}{2\Delta t_n}\left[(E^{n+1})^2-(E^n)^2-(E^{n+1}-E^n)^2\right](U^n+u(t_n))-r_nF^{n+1}.\end{aligned}$$

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Now we have

$$\begin{aligned}&\frac{1}{2\tau_n}\big[(F^{n+1})^2-(F^n)^2+(F^{n+1}-F^n)^2\big]\\=&\frac{1}{2\Delta t_n}\big[(E^{n+1})^2-(E^n)^2-(E^{n+1}-E^n)^2\big]\big(U^n+u(t_n)\big)-r_nF^{n+1},\end{aligned}$$

which, together with the monotonicity of $\{U^n\}$ and $\{u(t_n)\}$, implies

$$\frac{\Delta t_n}{\tau_n}(F^{n+1})^2-(E^{n+1})^2\big(U^{n+1}+u(t_{n+1})\big)\le\frac{\Delta t_n}{\tau_n}(F^n)^2-(E^n)^2\big(U^n+u(t_n)\big)-2\Delta t_nr_nF^{n+1}.$$

Let $L=\frac{1}{3}\max_{t\in[0,T_0]}|u'''(t)|$, $R=\frac{1}{24}\max_{t\in[0,T_0]}|u^{(4)}(t)|$ and $K=\max_{t\in[0,T_0]}|u(t)|$. Since

$$|r_n|\le L(\Delta t_{n-1}-\Delta t_n)+\frac{R}{\tau_n}\big[(\Delta t_n)^3+(\Delta t_{n-1})^3\big],$$

we have by (2.4)

$$\begin{aligned}&\frac{\Delta t_n}{\tau_n}(F^{n+1})^2-(E^{n+1})^2\big(U^{n+1}+u(t_{n+1})\big)\\\le&\frac{\Delta t_n}{\tau_n}(F^n)^2-(E^n)^2\big(U^n+u(t_n)\big)+2\tau_n\frac{\Delta t_n}{\tau_n}|r_n||F^{n+1}|\\\le&\frac{\Delta t_n}{\tau_n}(F^n)^2-(E^n)^2\big(U^n+u(t_n)\big)\\&+2\tau_n\left(L(\Delta t_{n-1}-\Delta t_n)+\frac{R}{\tau_n}\big[(\Delta t_n)^3+(\Delta t_{n-1})^3\big]\right)\\\le&\frac{\Delta t_{n-1}}{\tau_{n-1}}(F^n)^2-(E^n)^2\big(U^n+u(t_n)\big)\\&+2\tau_n\left(L(\Delta t_{n-1}-\Delta t_n)+\frac{R}{\tau_n}\big[(\Delta t_n)^3+(\Delta t_{n-1})^3\big]\right)\end{aligned}\tag{2.10}$$

as long as $|F^{n+1}|\Delta t_n/\tau_n\le 1$. Now it follows

$$\begin{aligned}&\frac{\Delta t_n}{\tau_n}(F^{n+1})^2-(2K+1)(E^{n+1})^2\\\le&\frac{\Delta t_n}{\tau_n}(F^{n+1})^2-(E^{n+1})^2\big(U^{n+1}+u(t_{n+1})\big)\\\le&\frac{\Delta t_{n-1}}{\tau_{n-1}}(F^n)^2-(E^n)^2\big(U^n+u(t_n)\big)\\&+2\tau_n\left(L(\Delta t_{n-1}-\Delta t_n)+\frac{R}{\tau_n}\big[(\Delta t_n)^3+(\Delta t_{n-1})^3\big]\right)\\\le&\frac{\Delta t_1}{\tau_1}(F^2)^2-(E^2)^2(U^2+u(t_2))+L\big[(\Delta t_1)^2-(\Delta t_n)^2\big]\\&+2R\left(\sum_{k=2}^{n}\big((\Delta t_{k-1})^3+(\Delta t_k)^3\big)\right),\end{aligned}$$

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as long as $|E^{n+1}| < 1$. Taking (2.10) into account, we have

$$\begin{aligned}
&\frac{\Delta t_n}{\tau_n}(F^{n+1})^2 - (2K+1)(E^{n+1})^2 \\
\le &\frac{\Delta t_1}{\tau_1}(F^1)^2 - (E^1)^2\big(U^1 + u(t_1)\big) + L\big[(\Delta t_0)^2 - (\Delta t_n)^2\big] \\
&+ 2R\left(\sum_{k=1}^{n}\big((\Delta t_{k-1})^3 + (\Delta t_k)^3\big)\right) \\
\le &\frac{\Delta t_1}{\tau_1}(F^1)^2 + L(\Delta t_0)^2 + 2R(\Delta t_0)^2\left(\sum_{k=1}^{n}(\Delta t_{k-1} + \Delta t_n)\right) \\
\le &(F^1)^2 + (L + 4RT_0)(\Delta t_0)^2,
\end{aligned}$$

as long as $|F^{n+1}|\Delta t_n/\tau_n \le 1$ and $|E^{n+1}| \le 1$. Since $E^0 = 0$ and $|E^1| = O((\Delta t_0)^2)$, there exists a positive constant $Q$, depending on $a_0$ and $a_1$, such that

$$|F^1| = \frac{|E^1|}{\Delta t_0} \le Q\Delta t_0.$$

Put $M = \sqrt{Q^2 + L + 4RT_0}$. Then we have

$$\frac{\Delta t_n}{\tau_n}(F^{n+1})^2 \le (2K+1)(E^{n+1})^2 + (M\Delta t_0)^2 \le \big(\sqrt{2K+1}|E^{n+1}| + M\Delta t_0\big)^2,$$

which implies

$$\sqrt{\frac{\Delta t_n}{\tau_n}}|F^{n+1}| \le \sqrt{2K+1}|E^{n+1}| + M\Delta t_0. \tag{2.11}$$

Since $|F^{n+1}| \ge (|E^{n+1}| - |E^n|)/\Delta t_n$, it follows from (2.11)

$$\begin{aligned}
|E^{n+1}| \le &\frac{|E^n| + M\Delta t_0\Delta t_n\sqrt{\frac{\tau_n}{\Delta t_n}}}{1 - \sqrt{2K+1}\Delta t_n\sqrt{\frac{\tau_n}{\Delta t_n}}} \\
\le &\big(1 + 2\sqrt{2K+1}\sqrt{\tau_n\Delta t_n}\big)\big(|E^n| + M\Delta t_0\sqrt{\tau_n\Delta t_n}\big),
\end{aligned} \tag{2.12}$$

for $\tau$ sufficiently small. Let $G_n = \sqrt{2K+1}\sqrt{\tau_n\Delta t_n}$. Then,

$$|E^{n+1}| \quad \le \quad (1 + 2G_n)|E^n| + \frac{M\Delta t_0}{\sqrt{2K+1}}\big(G_n(1 + 2G_n)\big).$$

Note that

$$\sum_{k=1}^{n} G_k = \sqrt{2K+1}\sum_{k=1}^{n}\sqrt{\tau_k\Delta t_k} \le \sqrt{2K+1}\sum_{k=1}^{n}\Delta t_{k-1} \le T_0\sqrt{2K+1}.$$

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Now a standard argument yields

$$\begin{aligned}|E^{n+1}| \le & \exp\Big(2\sum_{k=1}^{n} G_k\Big)|E^1| \\ & + \frac{\Delta t_0 M}{\sqrt{2K+1}}\Big(G_n \exp(2G_n) + G_{n-1}\exp\big(2(G_n+G_{n-1})\big) \\ & + \cdots + G_1 \exp\Big(2\sum_{k=1}^{n} G_k\Big)\Big) \\ \le & \exp\Big(2\sum_{k=1}^{n} G_k\Big)|E^1| + \frac{\Delta t_0 M}{\sqrt{2K+1}}\Big(\sum_{k=1}^{n} G_k\Big)\cdot \exp\Big(2\sum_{k=1}^{n} G_k\Big) \\ \le & \Delta t_0 \exp\big(2T_0\sqrt{2K+1}\big)(MT_0+1), \end{aligned} \tag{2.13}$$

for sufficiently small $\tau$. Here, use has been made of the fact $|E^1| = O((\Delta t_0)^2)$. This, together with (2.11), implies

$$\sqrt{\frac{\Delta t_n}{\tau_n}}|F^{n+1}| \le \Delta t_0\big((\sqrt{2K+1}\exp(2T_0\sqrt{2K+1})(MT_0+1)+M\big). \tag{2.14}$$

Since $\Delta t_n/\tau_n \le 1$,

$$\frac{\Delta t_n}{\tau_n}|F^{n+1}| \le \sqrt{\frac{\Delta t_n}{\tau_n}}|F^{n+1}| < 1,$$

for $\tau$ sufficiently small. Now (2.13) and (2.14) guarantee

$$|E^{n+1}| < 1 \quad \text{and} \quad \frac{\Delta t_n}{\tau_n}|F^{n+1}| < 1,$$

for sufficiently small $\tau$ as long as $t_n \le T_0$. Thus, (2.13) and (2.14) hold for all $t_n \le T_0$. □

**Remark 2.1.** The assumption (2.2) is just a sufficient condition to guarantee the validity of (2.3) for $k = 1$. In fact, it is not difficult to show, for $0 < \gamma < 1/2$,

$$\lim_{k\to\infty} \frac{U^k}{U^{k+1}} = 0.$$

As a result, for large $a_1$ such that (2.2) fails, though the ratio $U^k/U^{k+1}$ might increase at first, once (2.3) holds for certain $k_0$, the induction argument given in the proof of Lemma 2.2 implies that (2.3) holds for all $k \ge k_0$. See Fig. 2. Keeping this in mind, one can prove that Theorem 1.1 holds for all positive $a_0, a_1$ by suitably modifying the proof given above.

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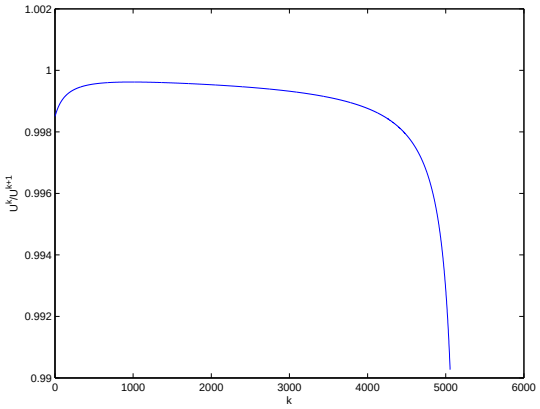



Figure 2: $k$ v.s. $U^k/U^{k+1}$. Here, $\alpha = 1$, $a_0 = 1$, $a_1 = 5$ and $\gamma = 0.3$.

## 3. Proof of Theorem 1.2

**Lemma 3.1.** *Let* $\{U^n\}$ *be the solution of* (1.4). *Then*

$$V^{n+1} \geq \frac{1}{\sqrt{2}} g(U^n), \tag{3.1}$$

*where* $g$ *is the function given in* (1.3) *with* $\alpha = 1$.

*Proof.* By virtue of (1.4) and the fact that $\tau_n \geq \Delta t_{n-1}/2$, we have

$$(V^{n+1})^2 = \left(V^n + \tau_n (U^n)^2\right)^2 \geq (V^n)^2 + 2\tau_n V^n (U^n)^2 \geq (V^n)^2 + (U^n - U^{n-1})(U^n)^2,$$

from which follows

$$(V^{n+1})^2 \geq (V^1)^2 + \sum_{k=1}^{n} (U^k - U^{k-1})(U^k)^2 \geq (V^1)^2 + \int_{U^0}^{U^n} s^2 ds \geq \frac{1}{2} \left(g(U^n)\right)^2.$$

□

***Proof of Theorem 1.2***

We complete the proof by showing that

$$T_* \equiv \liminf_{\tau \to 0} T(\tau) \geq T \quad \text{and} \quad T^* \equiv \limsup_{\tau \to 0} T(\tau) \leq T.$$

First, we assume that $T_* < T$. Then there exists a sequence $\{\tau_i\}$ with $\tau_i \to 0$ as $i \to \infty$ such that

$$T(\tau_i) \leq \frac{T + T_*}{2} < T.$$

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Let $\{U^n(\tau_i)\}$ be the solution corresponding to $\tau_i$. Then it follows from the finiteness of $T(\tau_i)$ that $|U^n(\tau_i)| \to \infty$ as $n \to \infty$ for all $i$. However, the solution of (1.1) remains bounded in $[0, (T + T_*)/2]$. This is a contradiction to Theorem 1.1. Hence, we have $T_* \geq T$.

Next, we assume that $T^* > T$. Take a $\tau$, smaller that any prescribed positive number, such that

$$T(\tau) \geq \frac{T^* + T}{2} > T. \tag{3.2}$$

Now, let $M > 0$ be given arbitrary. Then Theorem 1.1 guarantees, by choosing smaller $\tau$ if necessary, the existence of an $\tilde{n}$ such that

$$t_{\tilde{n}} < T, \quad U^n(\tau) \geq M \ \ (\forall n \geq \tilde{n}).$$

We then consider the finite difference equation

$$\frac{Z^{n+1} - Z^n}{\Delta s_n} = \frac{1}{\sqrt{2}} g(Z_n) \ \ (\forall n \geq \tilde{n}), \quad Z^{\tilde{n}} = U^{\tilde{n}}(\tau), \tag{3.3}$$

where $\Delta s_n$ is given by $\Delta s_n = \tau/(Z^n)^\gamma$. It can be easily shown that $Z^n \leq U^n(\tau)$ $(\forall n \geq \tilde{n})$. We then have

$$\begin{aligned} T(\tau) =& t_{\tilde{n}} + \sum_{k=\tilde{n}}^{\infty} \frac{\tau}{(U^k)^\gamma} \\ \leq& t_{\tilde{n}} + \sum_{k=\tilde{n}}^{\infty} \frac{\tau}{(Z^k)^\gamma} \\ <& T + \sqrt{2} \int_M^\infty \frac{ds}{g(s)} + \tau \int_M^\infty \frac{g'(s)}{g(s)s^\gamma} ds. \end{aligned}$$

Here, use has been made of Corollary 2.2 in Cho et al. (2007). Since $M$ can be chosen arbitrarily large and $\tau$ can be chosen so small so that

$$\sqrt{2} \int_M^\infty \frac{ds}{g(s)} + \tau \int_M^\infty \frac{g'(s)}{g(s)s^\gamma} ds < \frac{T^* - T}{2},$$

we have

$$T(\tau) < T + \frac{T^* - T}{2} = \frac{T^* + T}{2},$$

which contradicts (3.2). Thus, we are done. □

## 4. Numerical Results

Since a larger $\gamma$ defines a smaller grid size, which usually gives more accurate estimate in usual finite difference approximation, one might be tempted to conclude that a larger $\gamma$ is better. However, this issue should deserve more attention. In fact, the larger $\gamma$ is, the smaller the grid size becomes. As a result, the computation becomes unnecessarily slow.

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Table 1: $\gamma$ vs. $T(\tau)$ and $n_b$ with initial data $a_0 = a_1 = 1$. $\tau = 10^{-3}$.

| $\gamma$ | $T(\tau)$ | $n_b$ |
|---|---|---|
| 0.2 | 2.375862693250660 | 3988 |
| 0.3 | 2.375821199893985 | 5888 |
| 0.4 | 2.375814510940243 | 10220 |
| 0.5 | 2.375813912980349 | 22471 |
| 0.6 | 2.375813721091320 | 64935 |
| 0.7 | 2.375813669853368 | 237568 |
| 0.8 | 2.375813648488554 | 1021288 |
| 0.9 | 2.375727338959789 | 3808593 |

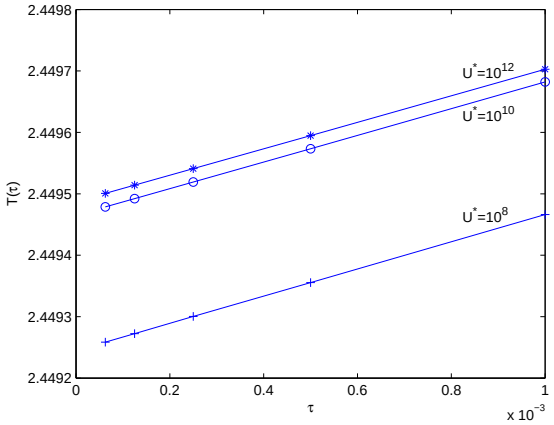



Figure 3: $\tau$ v.s. $T(\tau)$ with different $U^*$. $a_0 = 1$, $a_1 = \sqrt{2/3}$. $\tau = 10^{-3}$.

Let us show by an example. We put in this example $\alpha = 1$ and $a_0 = a_1 = 1 > 0$, which leads to finite-time blow-up for the solution of (1.1). We set a threshold $U^* = 10^{10}$ and stop our computation when $U^n > U^*$. The step $n$ where we stopped is denoted by $n_b$. Then our experiments gave Table 1. $n_b$, the number of steps required to achieve $U^n > U^*$, increases remarkably with $\gamma$, especially when $\gamma \geq 0.5$. Nevertheless, we remark that the difference between the numerical blow-up times for $\gamma < 0.5$ and $\gamma \geq 0.5$ is smaller than the prescribed parameter $\tau = 10^{-3}$. In other words, we still have a good estimate of the blow-up time even if a small $\gamma$ is used for computation.

Next, we illustrate the numerical blow-up time with different stop values $U^*$ in Fig. 3. Let $\alpha = 1$, and $a_0 = 1$, $a_1 = \sqrt{2/3}$. Then the blow-up time $T$ can be computed analytically and equals to $\sqrt{6} \approx 2.44949$. We set $\gamma = 0.3$ in our computation. Since the numerical blow-up time is defined by an infinite sum, we have to stop at a certain step. A larger $U^*$ seems to give a better approximation, but it should be noted that the computation cost and the rounding error also become larger at the same time.

We now turn our attention to the computation for the blow-up solutions of the nonlinear

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Table 2: $\gamma$ vs. $T(\tau,h)$ and $n_b$: $\alpha=1$, $p=\infty$ (see (1.9)) and $h=\tau=1/512$. The initial data are given by $u_0(x)=100(1-\cos(\pi x))$ and $u_1(x)=10(1-\cos(\pi x))$.

| $\gamma$ | $T(\tau,h)$ | $n_b$ |
|---|---|---|
| 0.3 | 0.217251237840345 | 1130 |
| 0.4 | 0.217159177954297 | 2912 |
| 0.5 | 0.217110279833605 | 8611 |
| 0.6 | 0.217081880672826 | 29605 |
| 0.7 | 0.217065307475655 | 117055 |

wave problem (1.7) under the zero Dirichlet boundary condition, that is, we consider the following initial-boundary value problem

$$\begin{cases} u_{tt}(t,x)=u_{xx}(t,x)+u^{1+\alpha}(t,x) & (0<x<1, \quad t>0) \\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x) & (0<x<1) \\ u(t,0)=u(t,1)=0 & (t>0) \end{cases} . \tag{4.1}$$

It should be noted that the solutions of (4.1) blow up in finite time with sufficiently large initial data. (See for instance [5, 14]). Now we compute the solution and an approximate blow-up time by the scheme (1.8). The discrete initial and boundary conditions are given respectively by

$$U_j^0=u_0(x_j), \quad U_j^1=U_j^0+\Delta t_0 u_1(x_j) \quad (j=1,\cdots,N-1)$$

and

$$U_0^n=U_N^n=0 \quad (\forall n\geq 0).$$

Here, $N$ is a given positive integer and $h=1/N$. Due to the discretization in space, the computational loads with respect to different temporal grid sizes become more remarkable than that for the ODE case. See Table 2. The computation is stopped when $\|U^n\|_\infty=\max_j|U_j^n|>10^8$. The step where we stop our computation is again denoted by $n_b$.

All our computations seem to proceed stably. See Figs. 4 and [5] for some computational results. If one can prove that $\Delta t_n/\Delta t_{n-1}$, or equivalently $\|U^{n-1}\|_\infty/\|U^n\|_\infty$, keeps its monotonicity after a certain time $t_0$ for all sufficiently small $\tau$, convergence of the scheme (1.8) can then be carried out in a standard way. Although our numerical experiments seem to suggest that $\Delta t_n/t_{n-1}$ becomes monotone after a certain time $t_0$ for all sufficiently small $\tau$, we could not prove this rigorously at the present stage. See Fig. 5.

## 5. Conclusion

In this paper, we considered the three-level finite difference scheme (1.4) with adaptively-defined grid size (1.5) for the second order nonlinear ODE blow-up problem (1.1) and proved the convergence for the numerical solutions and the numerical blow-up times. Note

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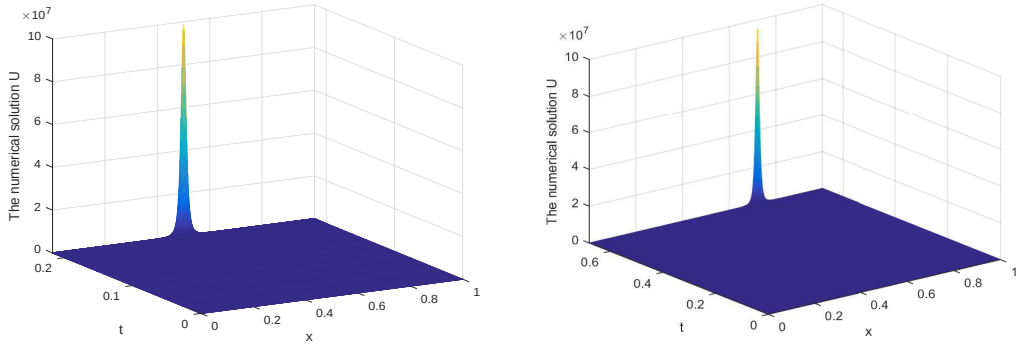



Figure 4: $N = 512$, $\tau = 1/N$, $\gamma$ and $p$ in (1.9) is chosen to be $\gamma = 0.5$ and $p = \infty$. Left: $u_0(x) = 100(1-\cos(2\pi x))$, $u_1(x) = 10(1-\cos(2\pi x))$. Right: Left: $u_0(x) = 10\sin(\pi x)$, $u_1(x) = -200\sin(2\pi x)$.

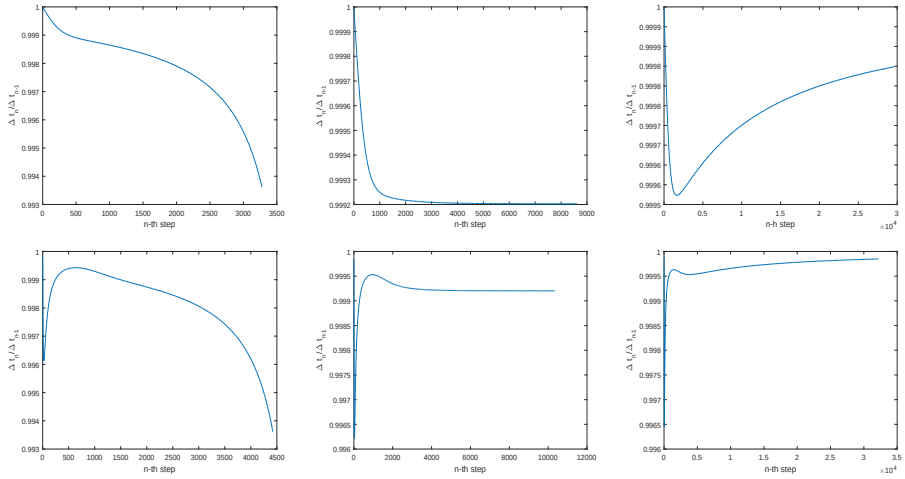



Figure 5: step $n$ v.s. $\Delta t_n/\Delta t_{n-1}$ with different $\gamma$. $\alpha = 1$. The results in the first row are computed with initial data $u_0(x) = 100(1-\cos(2\pi x))$, $u_1(x) = 10(1-\cos(2\pi x))$, while the second row are the results computed with $u_0(x) = 10\sin(\pi x)$, $u_1(x) = -200\sin(2\pi x)$. The first column is computed with $\gamma = 0.4$; the second column is computed with $\gamma = 0.5$; the third column is computed with $\gamma = 0.6$.

that it can be proved that

$$\lim_{n\to\infty} \frac{U^{n+1}}{U^n} \begin{cases} < \infty & , \text{if } \gamma \geq \frac{\alpha}{2} \\ = \infty & , \text{if } \gamma < \frac{\alpha}{2} \end{cases} .$$

Since the growth of the numerical solution at each step is at most at a constant rate in the cases of $\gamma \geq \alpha/2$ due to the smallness of the grid sizes, the proof for the convergence can be carried out by a comparison theorem. See [6] for the detail. However, for the case of $0 < \gamma < \alpha/2$, the numerical solution becomes unbounded too fast at the steps near the blow-up time so that a more careful analysis is required. In addition, if smaller grid sizes are applied, more iterations are needed to reach blow-up, which result in the increase of

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the computational loads and errors. This is why we study here the convergence for the case $\gamma < \alpha/2$.

On the other hand, the current study also provides an approach to show the convergence of three-level schemes with adaptive temporal meshes for blow-up problems. It should be noted that, to prove the convergence of a three-level scheme, we often need some a priori estimates or stability in some norms, which are usually available only when $\Delta t_n$ is nondecreasing. However, if we apply three-level adaptive-time-mesh schemes to compute blow-up solutions, $\Delta t_n$ will finally decrease to 0. This is the reason that we need an extra condition on the monotonicity of the ratio $\Delta t_n/\Delta t_{n-1}$ in the convergence analysis. See also [6, 16, 20] for discussion concerning the stability for three-level schemes with decreasing $\Delta t_n$.

We also reported some computational results for the blow-up problem (4.1). In [5], the author considered the three-level explicit scheme (1.8) for the computation of the blow-up solutions of (4.1) and showed the convergence of the numerical blow-up time under the assumption that the numerical solutions converge. But we failed in showing the convergence of the scheme. To solve this problem, one has to verify the monotonicity of the ratio $\Delta t_n/\Delta t_{n-1}$, which plays an important role in the stability and convergence analysis for a three-level scheme. Our numerical experiments confirmed this assumption, however, a rigorous proof still remains open.

Another interesting issue for the nonlinear blow-up problem (4.1) is the convergence order of the numerical blow-up time. Although our numerical results seem to suggest

$$|T(\tau,h)-T| < C\tau,$$

we do not have any mathematical proof yet. See also the discussion in [5, 19]. Here, $T$ denotes the blow-up time of the solution of (4.1). Similar difficulties for parabolic blow-up problems can be found in [8]. For the finite difference approximation of the nonlinear blow-up problem (4.1), there is still much room for improvement.

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