101.31 On series for the Euler constant

This Note features series for the Euler constant γ unencumbered by irrational numbers such as π, e, $\log 2$.

We will first derive a double series from the usual definition for γ, and follow this with a little-known series attributed to Mascheroni.

The double series

The well-known constant $\gamma (\approx 0.5772)$ is defined as the limit of the sequence $\{\gamma_n\}$, where

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \log n.$$

Then

$$\gamma_n - \gamma_{n-1} = \frac{1}{n} + \log\left(1 - \frac{1}{n}\right) = -\sum_{m=2}^{\infty} \frac{1}{mn^m},$$

leading to

$$\gamma = 1 - \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{mn^m} \tag{1}$$

after summation from 2 to ∞. Note that the double series may be interpreted as a single series of diagonal sums (given by $m + n = 4, 5, 6, \ldots$).

The integral

A graphing calculator would reveal that the function

$$y = \frac{1}{\log(1 - x)} + \frac{1}{x} \tag{2}$$

is continuous, bounded and increasing over the open interval $0 < x < 1$; in fact $\frac{1}{2} < y < 1$ with a cusp at $x = 1$.

Using the function (2), we will show that $\gamma = \int_0^1 y\,dx$ by invoking the integral ([1])

$$\gamma = \int_0^1 \left(\frac{1 - e^{-x} - e^{-1/x}}{x}\right) dx,$$

where the 'rogue' term $e^{-1/x}$ has no Maclaurin series!

We next establish equality for the two integrals, listing substitutions $t = f(x)$ with prompt return to x and using limits $a \to 0$, $b \to \infty$ to contain implicit singularities.

Thus

$$\int_0^1 \frac{1 - e^{-x}}{x}\,dx - \int_0^1 \frac{e^{-1/x}}{x}\,dx$$

$$= \int_0^1 \frac{1 - e^{-x}}{x}\,dx - \int_1^{\infty} \frac{e^{-t}}{t}\,dt, \left(t = \frac{1}{x}\right)$$

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$$= \lim\left(\int_a^1 \frac{1 - e^{-x}}{x}\,dx - \int_1^b \frac{e^{-x}}{x}\,dx\right)$$

$$= \lim\left(-\log a - \int_a^b \frac{e^{-x}}{x}\,dx\right)$$

$$= \lim\left(-\log a + \int_{1-e^{-a}}^{1-e^{-b}} \frac{1}{\log(1-t)}\,dt\right), \quad (t = 1 - e^{-x})$$

$$= \lim\left(\int_{1-e^{-a}}^{1-e^{-b}} \left(\frac{1}{x} + \frac{1}{\log(1-x)}\right)dx + \log\left(\frac{1 - e^{-a}}{a(1 - e^{-b})}\right)\right)$$

$$= \int_0^1 \left(\frac{1}{x} + \frac{1}{\log(1-x)}\right)dx.$$

Next, the following arguments provided by the referee are undoubtedly superior to a Leibnitz-Maclaurin treatment for the function y in (2).

The use of undetermined coefficients

Writing $y = \sum_{n=0}^{\infty} a_n x^n$, we have

$$1 + \frac{\log(1-x)}{x} = \log(1-x) \sum_{n=0}^{\infty} a_n x^n,$$

providing the successive recurrences

$$a_0 = \tfrac{1}{2}, \quad a_1 + \tfrac{1}{2}a_0 = \tfrac{1}{3}, \quad a_2 + \tfrac{1}{2}a_1 + \tfrac{1}{3}a_0 = \tfrac{1}{4}, \quad \ldots$$

and generating

$$a_0 = \frac{1}{2}, \qquad a_1 = \frac{1}{12}, \qquad a_2 = \frac{1}{24}, \qquad \ldots .$$

The use of a related integral

Writing $1 - x = \exp\big(\log(1-x)\big)$, we have

$$y = \frac{1}{x}\left(1 - \int_0^1 (1-x)^t\,dt\right)$$

$$= \frac{1}{x}\left(1 - \int_0^1 \big(1 - tx + \tfrac{1}{2}t(t-1)x^2 - \ldots\big)\,dt\right)$$

$$= \frac{1}{2} + \frac{x}{12} + \frac{x^2}{24} + \ldots,$$

where

$$a_n = \frac{(-1)^n}{(n+1)!} \int_0^1 t(t-1)\ldots(t-n)\,dt.$$

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Hence in both cases we have the Mascheroni series

$$\gamma = \int_0^1 y(x)\,dx = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \ldots . \qquad (3)$$

The series of diagonal sums gleaned from the double series (1) is slower to converge to γ than the series (3) for comparable labour.

Reference

1. J. A. Scott, Evaluating the Euler constant, *Math. Gaz.* **94** (July 2010) pp. 304-306.

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J. A. SCOTT

1 Shiptons Lane, Great Somerford, Chippenham SN15 5EJ

101.32 Yet another Cantor series

With the product of the first n prime numbers aptly termed the nth primorial, the series of reciprocal primorials may be converted to a Cantor series with irrational sum ([1]).

In this *Gazette* Note we will discover a natural Cantor series with irrational sum by replacing the primes by the corresponding Fibonacci numbers.

Fibonacci prerequisites

The well-known Fibonacci sequence of positive integers

$$1, 1, 2, 3, 5, 8, \ldots$$

may be denoted (u_n) where

$$u_1 = u_2 = 1, \qquad u_n = u_{n-1} + u_{n-2}\ (n > 2).$$

We also need to define $u_0 = u_2 - u_1 = 0$ to be consistent with the recurrence relation in order to establish the following lemma ([2]).

Each $k \in N$ divides u_n for some n.

Now for $k \geqslant 2$ we consider the $k^2 + 1$ pairs $(u_i, u_{i+1}) \pmod k$ for $0 \leqslant i \leqslant k^2$. Then since only k^2 pairs are possible modulo k, two pairs must coincide whence there exist $i \geqslant 0$ and $j \geqslant 1$ such that $u_i \equiv u_{i+j}$ and $u_{i+1} \equiv u_{i+j+1} \pmod k$.

The use of the recurrence relation (both forwards and backwards) leads easily to $u_{n+j} \equiv u_n \pmod k$ for all n, that is, the Fibonacci sequence (u_n) is periodic modulo k.

Furthermore, since $u_j \equiv u_0 \pmod k$ in particular, u_j is a multiple of k.

The Cantor theorem

For the Cantor series

$$a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \ldots$$

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