9

Euclidean solutions

It is easy to understand why static solutions or solutions that oscillate in real time should be relevant in quantum field theory, since these describe objects that are already present in the classical theory. It is less obvious that Euclidean solutions—solutions in spacetimes with imaginary times and hence Euclidean metrics—should be physically interesting. Their importance arises from their connection with a purely quantum mechanical effect, barrier penetration. In this chapter I will focus on developing the Euclidean formalism for systems with a finite number of degrees of freedom (and often just one). The next three chapters will explore the application of these methods to field theories, systems with an infinite number of degrees of freedom.

9.1 Tunneling in one dimension

Consider a quantum mechanical particle in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(q)\,, \tag{9.1}$$

where the potential energy V has a barrier as shown in Fig. 9.1. If a right-moving wave with an energy E that is less than the value of V at the top of the barrier is incident on the potential energy barrier, most of the wave is reflected, but there is also a small transmitted wave with an amplitude proportional to $e^{-B/2}$, where the WKB approximation gives B as an integral

$$B = 2\int_{q_1}^{q_2} dq\sqrt{2m[V(q) - E]} \tag{9.2}$$

over the classically forbidden region $q_1 < q < q_2$.

In the region inside the potential energy barrier the total energy is less than the potential energy. If we were to try to interpret the difference as a negative kinetic energy, we would conclude that the velocity was imaginary, just as if it were the

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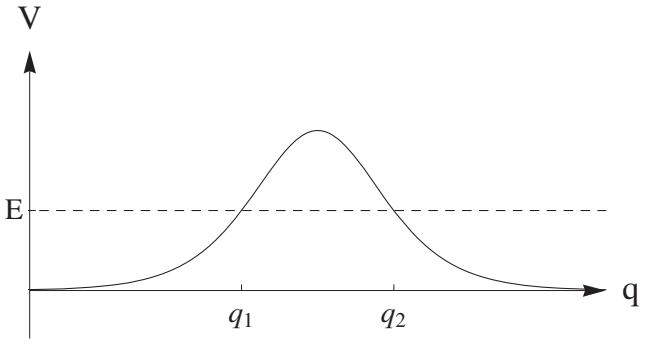


Fig. 9.1. A potential energy barrier in one dimension. The classically forbidden region is the region $q_1 < q < q_2$, where the turning points are defined by $V(q_1) = V(q_2) = E$.

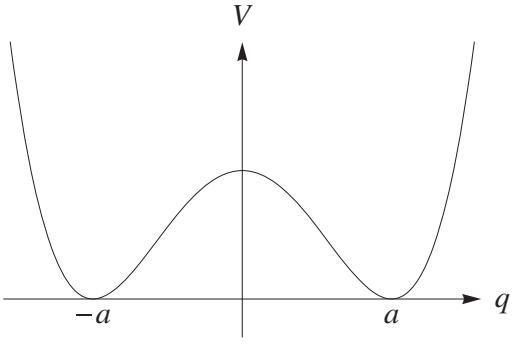


Fig. 9.2. A double-well potential.

derivative of q with respect to an imaginary time. This observation is the fundamental motivation for applying a Euclidean analysis to the problem. Although somewhat trivial in the case with only one degree of freedom, Euclidean methods lead to great simplifications when multiple degrees of freedom are involved.

The field theory applications that will be of primary interest to us are associated with two types of phenomenon that are already seen in quantum mechanical systems with one degree of freedom. The first is the mixing of two or more states that would be degenerate in the absence of tunneling. The simplest example of this occurs in a symmetric double-well potential, such as that shown in Fig. 9.2. If the barrier were infinitely high, and hence impenetrable, there would be a tower of energy eigenstates confined to the left side of the barrier, and a similar tower of states confined to the right side. The two ground states, $|L\rangle$ and $|R\rangle$, would each have an energy E_0. With a finite barrier neither of these is an energy eigenstate. Instead, assuming the barrier to be high relative to E_0, the two lowest eigenstates are given by the symmetric and antisymmetric linear combinations

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$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|L\rangle \pm |R\rangle\right) \tag{9.3}$$

with energies

$$E_\pm = E_0 \mp K\, e^{-B/2} \equiv E_0 \mp \Delta/2\,, \tag{9.4}$$

where K is a constant whose calculation will be discussed later.

Now consider a particle in a linear combination of these two states. Let us suppose that at $t = 0$ its wavefunction is localized on the left side of the barrier. Taking $|\Psi(0)\rangle = |L\rangle$, we have

$$\begin{aligned} |\Psi(t)\rangle &= \frac{1}{\sqrt{2}}\, e^{-iE_+ t}\left(|+\rangle + e^{-it\Delta}|-\rangle\right) \\ &= \frac{1}{2}\, e^{-iE_+ t}\left[\left(1 + e^{-it\Delta}\right)|L\rangle + \left(1 - e^{-it\Delta}\right)|R\rangle\right]\,. \end{aligned} \tag{9.5}$$

Thus, the system oscillates back and forth with a frequency Δ.

It is important here that the potential be symmetric, so that the energy levels in the two wells, before tunneling is taken into account, are the same. If the two wells had been different, with their respective energy levels differing by amounts large compared to Δ, then the true energy eigenstates of the full system would be concentrated on one side of the barrier or the other, with only exponentially small contributions from the opposite side. If the particle's wavefunction was initially on one side of the barrier, it would remain concentrated on that side, with only a small probability of the particle ever being found on the opposite side.

The second phenomenon of interest to us is the decay of a metastable state. This can be illustrated by the potentials in Fig. 9.3. The one on the left has a narrow local minimum of V on one side of the barrier and a broad lower minimum on the other side. Without barrier penetration there would be a tower of discrete energy levels on the left side of the barrier, the lowest of which can again be labeled $|L\rangle$, and a much denser spectrum of states on the right side. In contrast with the previous example, where there was a single right-hand state

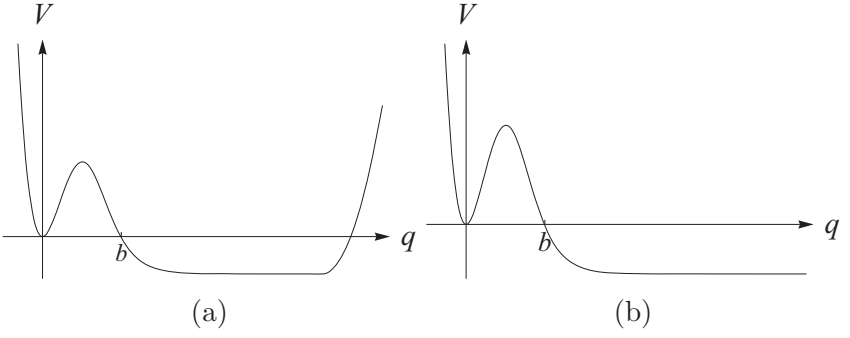


Fig. 9.3. (a) A potential with a metastable minimum at $q = 0$ and a very wide well with a lower energy. (b) The limiting case in which the width of the right-hand well becomes infinite.

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close in energy with $|L\rangle$, there are now many such states. With barrier penetration restored the energy eigenstates are again mixtures of left and right states, but with the contribution of $|L\rangle$ to any given eigenstate being small and tending toward zero as the width of the right-hand well increases.

Now consider the limit in which the width of the right-hand well goes to infinity, as shown in Fig. 9.3b. Let us again consider a state where $|\Psi(0)\rangle = |L\rangle$. Rather than finding oscillation, we now find that the magnitude of the overlap $\langle L|\Psi(t)\rangle$ falls exponentially with time, with the exponent itself being proportional to $e^{-B/2}$. Instead of expanding $|L\rangle$ as a linear combination of true energy eigenstates, it is more convenient to view it as a metastable state with a complex energy whose imaginary part is related to the decay width Γ by

$$\operatorname{Im} E = -\frac{\Gamma}{2}\,. \tag{9.6}$$

Alternatively, one can view the potential in Fig. 9.3b as being obtained by analytic continuation of a potential with only a single minimum, at $q = 0$. The complex energy of the metastable state in the former potential is then the analytic continuation of the real energy of the stable ground state of the original potential.

9.2 WKB tunneling with many degrees of freedom

The results of the previous section are familiar from elementary quantum mechanics. What we need to do is to generalize these to the case of a system with many degrees of freedom q^1, q^2, ..., q^N. It will be convenient to assemble these into an N-component vector $\mathbf{q}$.

Thus, suppose that we have a system described by the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}\sum_{j=1}^{N}\left(\frac{dq^j}{dt}\right)^2 - V(q^1, q^2, \ldots q^N) \\ &= \frac{1}{2}\left(\frac{d\mathbf{q}}{dt}\right)^2 - V(\mathbf{q}) \end{aligned} \tag{9.7}$$

and are interested in tunneling from an initial point that is a local minimum of the potential energy. With one degree of freedom, all that we were interested in was the amplitude for reaching the turning point on the far side of the barrier. With many degrees of freedom the starting point is surrounded by a barrier on all sides, so we will want to know not only the amplitude for passing through the barrier, but also the most likely place to emerge from the barrier into the classically allowed region on the other side.

The extension of the WKB approximation to address this problem was carried out by Banks, Bender, and Wu [224, 225]. Their method is based on finding the most probable escape path (MPEP). Any path P through the barrier can be specified as a trajectory $\mathbf{q}(s)$, where the parameter s along the path is defined by

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$$(ds)^2 = \sum_{j=1}^{N} (dq^j)^2 \equiv (d\mathbf{q})^2 \,, \tag{9.8}$$

with the initial condition that $\mathbf{q}(0) = \mathbf{q}_0$. [Note that the final point of the path, $\mathbf{q}(s_f) \equiv \mathbf{q}_f$, is not specified in advance.] Treating this path as a one-dimensional system, we can define a path-dependent barrier penetration integral

$$B[P] = 2 \int_0^{s_f} ds \, \sqrt{2[V(\mathbf{q}(s)) - E]} \,, \tag{9.9}$$

with $E = V(\mathbf{q}_0)$. The MPEP is the path that minimizes $B[P]$. Its end point is the most probable escape point from the barrier, and the leading WKB approximation for the tunneling amplitude is $Ae^{-B/2}$, with B evaluated on the MPEP and the prefactor A still to be determined

The task of minimizing B turns out to be most conveniently carried out in a Lagrangian framework, which also has the advantage of being easily carried over to the field theory context [226]. It is here that the advantages of the Euclidean approach become apparent. In classical mechanics Jacobi's principle tells us that, for a system described by a Lagrangian of the form of Eq. (9.7), a path from $\mathbf{q}_0$ to $\mathbf{q}_f$ that minimizes

$$I = \int_0^{s_f} ds \, \sqrt{2[E - V(\mathbf{q}(s))]} \tag{9.10}$$

gives a solution of the equations of motion whose time evolution is determined by

$$\frac{1}{2} \left(\frac{d\mathbf{q}}{dt} \right)^2 = E - V(\mathbf{q}) \,. \tag{9.11}$$

Alternatively, Hamilton's principle tells us that the same solution can be found by looking for a stationary point of the action

$$S = \int_{t_0}^{t_f} dt \, L(\mathbf{q}, \dot{\mathbf{q}}) \,, \tag{9.12}$$

where $\mathbf{q}(t_0) = \mathbf{q}_0$ and $\mathbf{q}(t_f) = \mathbf{q}_f$.

With appropriate sign changes, these results translate to the statement that a path that minimizes the $B[P]$ defined in Eq. (9.9) corresponds to a stationary point of the Euclidean action

$$S_E = \int_{\tau_0}^{\tau_f} d\tau \, \left[\frac{1}{2} \left(\frac{d\mathbf{q}}{d\tau} \right)^2 + V(\mathbf{q}) \right] \tag{9.13}$$

and thus is a solution of the Euclidean equation of motion

$$\frac{d^2 q_j}{d\tau^2} = \frac{\partial V}{\partial q_j} \,. \tag{9.14}$$

Let us denote this solution by $\bar{\mathbf{q}}(\tau)$.

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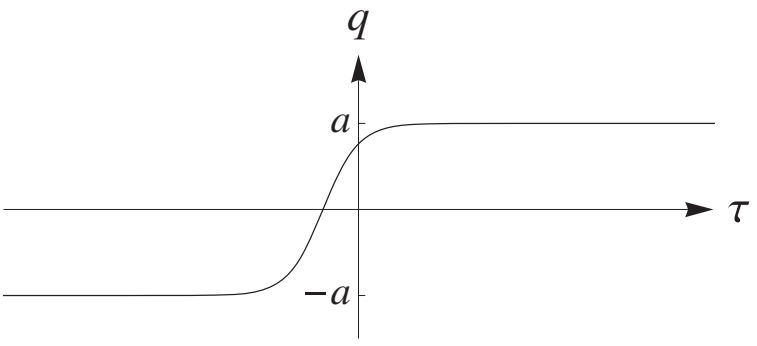


Fig. 9.4. The instanton corresponding to tunneling between the two minima of a potential such as that in Fig. 9.2.

For this solution the motion along the path satisfies

$$\frac{1}{2}\left(\frac{d\bar{\mathbf{q}}}{d\tau}\right)^2 = V(\bar{\mathbf{q}}) - E = V(\bar{\mathbf{q}}) - V(\mathbf{q}_0)\,. \tag{9.15}$$

This, together with Eq. (9.8), implies that

$$\begin{aligned} S_E[\bar{\mathbf{q}}] &= \int_{\tau_0}^{\tau_f} d\tau\, 2[V(\bar{\mathbf{q}}) - V(\mathbf{q}_0)] + \int_{\tau_0}^{\tau_f} d\tau V(\mathbf{q}_0) \\ &= \int_{\tau_0}^{\tau_f} d\tau \sqrt{\left(\frac{d\bar{\mathbf{q}}}{d\tau}\right)^2}\, \sqrt{2[V(\bar{\mathbf{q}}) - V(\mathbf{q}_0)]} + \int_{\tau_0}^{\tau_f} d\tau V(\mathbf{q}_0) \\ &= \int_0^{s_f} ds\, \sqrt{2[V(\bar{\mathbf{q}}) - V(\mathbf{q}_0)]} + \int_{\tau_0}^{\tau_f} d\tau V(\mathbf{q}_0)\,. \end{aligned} \tag{9.16}$$

This result gives us a relation between the tunneling exponent B and the Euclidean action, provided that we are careful in specifying the limits of the integration.

When the tunneling is between two degenerate minima of V, the approach of $\bar{\mathbf{q}}(\tau)$ to the end points of the trajectory is exponentially slow. In this case, τ runs from $-\infty$ to ∞ and $\bar{\mathbf{q}}(\tau)$ behaves as shown in Fig. 9.4. (Note the similarity with the one-dimensional kink solitons of Chap. 2.) Euclidean time translation invariance tells us that this solution is not unique, with any value allowed for the point where the solution crosses the middle of the barrier. Solutions of this type are referred to as instantons, with the term originating from the "time" at the center of the solution.[1] It is well to keep in mind, however, that τ is not a time,

[1] In the original paper on the Yang–Mills case [227], the solutions were referred to as pseudoparticles. Although most researchers in the field soon adopted the term instanton, "pseudoparticle" appears in much of the early literature on the subject because of the reluctance of some journals to accept neologisms.

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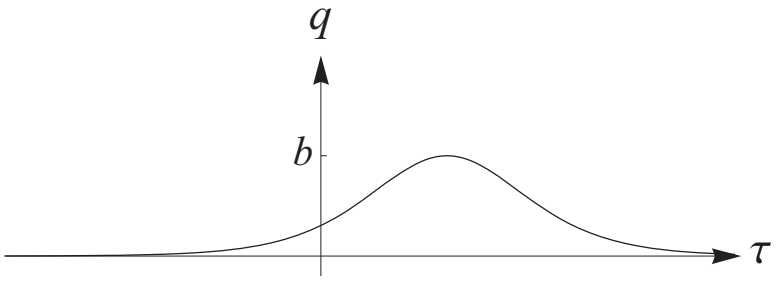


Fig. 9.5. The bounce solution corresponding to decay out of a metastable minimum such as that in Fig. 9.3b. The classical turning point is at $q = b$.

but just a particular parameterization of the favored tunneling path through configuration space.

We see from Eqs. (9.9) and (9.16) that for these instanton solutions

$$\frac{B}{2} = S_E[\bar{\mathbf{q}}] - S_E[\mathbf{q}_0] \qquad \text{(instanton)}\,, \tag{9.17}$$

where $S_E[\mathbf{q}_0]$, given by the last integral in Eq. (9.16), is the Euclidean action of the trivial constant solution $\mathbf{q}(\tau) = \mathbf{q}_0$. Note that this relation between B and S_E only holds at their stationary points.

We will also be interested in the case where the tunneling is between a local minimum and a turning point $\mathbf{q}_f$ that is not a minimum of V, such as the decay of a bound state in a potential like that in Fig. 9.3b. In this case, the solution begins at $\tau = -\infty$, but reaches $\mathbf{q}_f$ at a finite value of τ, at which point $d\bar{\mathbf{q}}/d\tau = 0$. Because the Lagrangian is invariant under time reversal, this solution can be continued back to the initial point $\mathbf{q}_i$, which is reached at $\tau = \infty$. This doubles the Euclidean action, so that for the full solution

$$B = S_E[\bar{\mathbf{q}}] - S_E[\mathbf{q}_0] \qquad \text{(bounce)}\,. \tag{9.18}$$

For obvious reasons this solution is called a bounce.[2] The bounce corresponding to a potential like that in Fig. 9.3b is shown in Fig. 9.5.

9.3 Path integral approach to tunneling: instantons

As discussed in the previous section, the exponent in the tunneling amplitude is readily obtained by WKB methods. This approach has the advantage of clarifying the physical significance of the Euclidean solution. However, path integral methods turn out to be more convenient for calculating the pre-exponential factor.

[2] Terminology varies in the literature, with some authors including bounces within the term instanton.

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In this section I will describe the application of these methods to the case of an instanton in a system with one degree of freedom. The analysis for the case of a bounce will be discussed in the next section.

The symmetric double-well potential provides a good illustration of the application of these methods to tunneling [228]. Let $\pm a$ be the values of q at the two minima, and $|\pm a\rangle$ be the corresponding position eigenstates. To simplify matters, let us take the value of the potential at these minima to be zero, so that B is obtained directly from the Euclidean action without any subtraction. The matrix elements

$$\langle a|e^{-HT}|a\rangle = \langle -a|e^{-HT}|-a\rangle \tag{9.19}$$

and

$$\langle a|e^{-HT}|-a\rangle = \langle -a|e^{-HT}|a\rangle \tag{9.20}$$

can be expressed as path integrals

$$\langle \pm a|e^{-HT}|a\rangle = \int [dq(\tau)]e^{-S_E[q]}, \tag{9.21}$$

where the integration is over paths such that $q(-T/2) = a$ and $q(T/2) = \pm a$. (To simplify notation, the subscript E on the action will be omitted for the remainder of this chapter, with all actions understood to be Euclidean.)

The matrix element on the left-hand side can be expanded in terms of energy eigenstates to give

$$\langle \pm a|e^{-HT}|a\rangle = \sum_n e^{-E_n T}\langle \pm a|n\rangle\langle n|a\rangle\,. \tag{9.22}$$

In the limit of large T this is dominated by the contributions from the states with lowest energy. Thus we can find these energies by evaluating the path integral for large T and picking out the dominant exponentials.

In particular, let us denote the lowest even and odd energy eigenstates by $|+\rangle$ and $|-\rangle$, respectively, and assume that T is large enough that the contributions from all higher states can be ignored. We then have

$$\langle a|e^{-HT}|a\rangle = |\langle a|+\rangle|^2 e^{-E_+ T} + |\langle a|-\rangle|^2 e^{-E_- T} \tag{9.23}$$

and

$$\langle -a|e^{-HT}|a\rangle = \langle -a|+\rangle\langle +|a\rangle e^{-E_+ T} + \langle -a|-\rangle\langle -|a\rangle e^{-E_- T}\,. \tag{9.24}$$

Clearly $\langle a|\pm\rangle = \pm\langle -a|\pm\rangle$. If these two lowest states are well separated from all the others, we also have $|\langle a|+\rangle| = |\langle a|-\rangle|$. In the limit of large T, we will then have

$$\frac{\langle a|e^{-HT}|a\rangle + \langle -a|e^{-HT}|a\rangle}{\langle a|e^{-HT}|a\rangle - \langle -a|e^{-HT}|a\rangle} = e^{(E_- - E_+)T}\,. \tag{9.25}$$

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We now use path integrals to evaluate the matrix elements in this expression, approximating each of the path integrals by a sum of Gaussian integrals about their stationary points. Given a Euclidean solution $\bar{q}(\tau)$ we write

$$q(\tau) = \bar{q}(\tau) + \sum_n c_n \psi_n(\tau)\,, \tag{9.26}$$

where $\psi_n(\tau)$ is an eigenmode with eigenvalue λ_n of

$$\left.\frac{\delta^2 S}{\delta q(\tau)\delta q(\tau')}\right|_{q=\bar{q}(\tau)} = -\frac{d^2}{d\tau^2} + V''(\bar{q}(\tau)) \equiv S''(\bar{q})\,. \tag{9.27}$$

We then change integration variables from $q(\tau)$ to the mode coefficients and write

$$[dq] = \prod_n \frac{dc_n}{\sqrt{2\pi}}\,. \tag{9.28}$$

(The factors of 2π are for later convenience. Our results will be insensitive to an overall—and generally divergent—normalization factor for the path integral.)

The contribution to the path integral from this stationary point then becomes

$$I = \int \prod_n \frac{dc_n}{\sqrt{2\pi}}\, e^{-[S(\bar{q}) + \frac{1}{2}\sum_k \lambda_k c_k^2 + \cdots]}\,, \tag{9.29}$$

where the ellipsis denotes terms that are cubic and higher order in the deviation from $\bar{q}$. Treating these as higher-order perturbations then gives

$$\begin{aligned} I &= e^{-S(\bar{q})} \prod \lambda_n^{-1/2}\,[1 + \cdots] \\ &= e^{-S(\bar{q})}\,[\det S''(\bar{q})]^{-1/2}\,[1 + \cdots]\,. \end{aligned} \tag{9.30}$$

The ellipsis again denotes higher-order corrections; these will not be shown explicitly in the following equations, but should be understood.

Let us now enumerate the relevant stationary points. For $\langle a|e^{-HT}|a\rangle$ we have the trivial constant solution $q_0(\tau) = a$. With our convention that $V(q)$ vanishes at the minima, $S(q_0) = 0$ and the contribution to the path integral is simply

$$I_0 = [\det S''(q_0)]^{-1/2}\,. \tag{9.31}$$

For $\langle -a|e^{-HT}|a\rangle$ we have the instanton solution, modified slightly so that q_1 runs from $-a$ at $\tau = -T/2$ to a at $\tau = T/2$. According to our formula, this should give a contribution

$$e^{-S_1}\,[\det S''(q_1)]^{-1/2}\,, \tag{9.32}$$

where S_1 is the Euclidean action of the instanton. However, there is a problem. There is a zero mode of S'',

$$\psi_0(\tau) = N^{-1/2}\,\frac{dq_1}{d\tau}\,, \tag{9.33}$$

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reflecting the broken τ-translation symmetry.[3] Because of the zero eigenvalue, $\det S''$ vanishes and the pre-exponential factor diverges.

This divergence is closely related to the infrared divergences associated with the zero mode that we encountered when calculating the quantum corrections to the kink mass in Sec. 2.3. The solution is again to replace the coefficient of the zero mode by a collective coordinate z that specifies the location of the center of the instanton, and to include only the nonzero modes in the determinant. Instead of a Gaussian integral over the zero-mode coefficient we must integrate over the full range of z from $-T/2$ to $T/2$. Because

$$\psi_0 \, dc_0 = \frac{dq}{d\tau} \, dz \tag{9.35}$$

we have

$$(2\pi)^{-1/2} dc_0 = \left(\frac{N}{2\pi}\right)^{1/2} dz \, . \tag{9.36}$$

The net contribution of the one-instanton stationary points is then

$$I_1 = e^{-S_1} \left(\frac{N}{2\pi}\right)^{1/2} T \, [\det{}' \, S''(q_1)]^{-1/2} \, , \tag{9.37}$$

with the prime on the determinant indicating that only the nonzero modes are to be included. Finally, let us define

$$K = \left(\frac{N}{2\pi}\right)^{1/2} \left[\frac{\det{}' \, S''(q_1)}{\det S''(q_0)}\right]^{-1/2} \, , \tag{9.38}$$

so that

$$I_1 = e^{-S_1} \, [\det S''(q_0)]^{-1/2} \, KT \, . \tag{9.39}$$

In addition to these, there are approximate stationary points that must also be considered. Although a static configuration with an instanton and an anti-instanton separated by a τ interval much greater than their width is not quite a solution of the Euclidean equations, it is close enough to being a stationary point of the action that it must be included, as must configurations with larger numbers of (necessarily alternating) instantons and anti-instantons. Let us focus on a configuration q_n with a total of n instantons and anti-instantons. Its action is just n times that of a single instanton, $S(q_n) = nS_1$. It has n approximate zero modes corresponding to independent τ-translations of its component instantons.

[3] The normalization factor is

$$N = \int_{-T/2}^{T/2} d\tau \left(\frac{dq_1}{d\tau}\right)^2 \, . \tag{9.34}$$

By virial arguments that parallel those leading to Eq. (2.79), one can show that for the example at hand $N = S_1$. Strictly speaking, this is exactly a zero mode only in the limit of infinite T. However, for large T the eigenvalue and the corrections to the mode are exponentially small, and in the end we are only interested in the large T limit.

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A collective coordinate z_j must be introduced for each of these, leading to a total Jacobian factor $(N/2\pi)^{n/2}$. Integrating over the instanton locations, which satisfy the constraint $z_1 < z_2 < \cdots < z_n$ gives a factor of

$$\int_{-T/2}^{T/2} dz_1 \int_{z_1}^{T/2} dz_2 \cdots \int_{z_{n-1}}^{T/2} dz_n = \frac{T^n}{n!} \,. \tag{9.40}$$

The final factor to be included is the functional determinant (with zero modes excluded) of $S''(q_n)$. The key point here is that the effect of an instanton on the spectrum of fluctuations is localized around the position of the instanton; i.e., the ratio of determinants that appears in Eq. (9.38) is independent of T for large T. Hence, the determinant factor for a configuration with n well-separated instantons and anti-instantons can be written as

$$[\det S''(q_0)]^{-1/2} \left[\frac{\det{}' S''(q_n)}{\det S''(q_0)}\right]^{1/2} = [\det S''(q_0)]^{-1/2} \left[\frac{\det{}' S''(q_1)}{\det S''(q_0)}\right]^{n/2} . \tag{9.41}$$

Putting all the factors together, we find that the contribution from all configurations with n instantons and anti-instantons is

$$I_n = e^{-nS_1} [\det S''(q_0)]^{-1/2} K^n \frac{T^n}{n!} \,. \tag{9.42}$$

Adding together the contributions from all of the stationary and approximately stationary points we have

$$\begin{aligned} \langle a|e^{-HT}|a\rangle &= \sum_{\text{even } n} I_n \\ &= [\det S''(q_0)]^{-1/2} \sum_{\text{even } n} \frac{\left[e^{-S_1} KT\right]^n}{n!} \\ &= [\det S''(q_0)]^{-1/2} \cosh\left[e^{-S_1} KT\right] \end{aligned} \tag{9.43}$$

and

$$\begin{aligned} \langle -a|e^{-HT}|a\rangle &= \sum_{\text{odd } n} I_n \\ &= [\det S''(q_0)]^{-1/2} \sum_{\text{odd } n} \frac{\left[e^{-S_1} KT\right]^n}{n!} \\ &= [\det S''(q_0)]^{-1/2} \sinh\left[e^{-S_1} KT\right] \,. \end{aligned} \tag{9.44}$$

Recalling Eq. (9.25), we have

$$e^{(E_- - E_+)T} = \exp\left[2KTe^{-S_1}\right] \,, \tag{9.45}$$

so that the splitting of the two lowest levels is

$$\Delta = E_- - E_+ = 2Ke^{-S_1}. \tag{9.46}$$

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The exponent is the standard WKB result. What has been gained by the path integral calculation is an expression for the pre-exponential factor, Eq. (9.38).

We could also have extracted the absolute energies of the two lowest states from Eqs. (9.43) and (9.44), obtaining

$$E_0 = \frac{1}{2}(E_+ + E_-) = -\lim_{T\to\infty} \frac{1}{2T} \ln \det S''(q_0)\,, \tag{9.47}$$

which turns out to be equal to $\frac{1}{2}\sqrt{V''(a)}$ [228, 232]. This is just the result one would obtain by neglecting tunneling and approximating each well as a simple harmonic oscillator. The perturbative corrections arising because the wells are anharmonic are much larger than the exponentially small instanton corrections we have calculated. However, these corrections add equally to both the even and the odd states, so the splitting is correctly given by the instanton result.

A note of caution. Our treatment of the n-instanton stationary points assumed that the instantons were well separated. This is known as the dilute-gas approximation. To check its validity we must verify that the dominant contributions to our sums are from the terms where the instanton density n/T is low enough for this assumption to be valid. In an exponential sum $\sum y^n/n!$, the dominant contribution is from the terms with $n \approx y$, so the dominant contribution to Eqs. (9.43) and (9.44) is from the terms with

$$\frac{n}{T} \approx K e^{-S_1}\,. \tag{9.48}$$

If $\delta\tau$ is the characteristic width of the instanton, we need that

$$(\delta\tau) K e^{-S_1} \ll 1\,. \tag{9.49}$$

We haven't actually evaluated the determinants entering K. However, dimensional arguments suggest that $K\delta\tau$ should be roughly of order unity, so that the dilute-gas approximation will be valid as long as the instanton action is large enough that $e^{-S_1} \ll 1$.

9.4 Path integral approach to tunneling: bounces

Let us now turn to the case of a state that decays by tunneling [229, 230]. Once again, for convenience, the value of the potential energy at the metastable minimum $q = a$ is taken to be zero. To start, let us consider the matrix element

$$\langle a|e^{-HT}|a\rangle = \sum_n e^{-E_n T} |\langle a|n\rangle|^2 = \int [dq(\tau)] e^{-S[q]}\,. \tag{9.50}$$

Arguments along the lines of the last section suggest that we can extract the energy E_0 of the lowest state in the well on the left from the large-time behavior of this matrix element. However, as remarked previously, a state with wavefunction

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concentrated in this well is not an energy eigenstate. Instead, it is a metastable state that can be described as having a complex energy whose imaginary part is related to the decay rate.

Nevertheless, let us proceed as before and evaluate the path integral by summing the contributions from the stationary points and approximate stationary points of the action: the trivial configuration $q_0(\tau) = a$, the bounce solution $q_b(\tau)$, and all possible multibounce solutions [230]. Just like the instanton, the bounce has a zero mode, proportional to its τ-derivative, that must be exchanged for a collective coordinate. There is no constraint that the number of bounces be even or odd, so summing over all numbers of bounces gives an exponential rather than a hyperbolic function, with the result being

$$\int [dq(\tau)] e^{-S[q]} = [\det S''(q_0)]^{-1/2} \exp\left[K T\, e^{-S(q_b)}\right] , \tag{9.51}$$

with

$$K = \left(\frac{N}{2\pi}\right)^{1/2} \left[\frac{\det{}' S''(q_b)}{\det S''(q_0)}\right]^{-1/2} , \qquad (\textbf{incorrect}) \tag{9.52}$$

where the normalization constant for the zero mode is $N = S(q_b)$.

Arguing as before, we can extract E_0 from the coefficient of T in the dominant exponential at large T, obtaining

$$E_0 = -\left[\lim_{T\to\infty} \frac{1}{2T} \ln\det S''(q_0)\right] - K e^{-S(q_b)} = \frac{1}{2}\sqrt{V''(a)} - K e^{-S(q_b)} . \tag{9.53}$$

As we will see, $S''(q_b)$ has a mode with negative eigenvalue. This makes the determinant factor in K, and thus the entire second term, imaginary. If it were not for this fact, it would be pointless to have calculated the exponentially small bounce contributions, because they are much smaller in magnitude than the (real) perturbative corrections to the contribution from the trivial solution q_0. This complex result for E_0 is not a problem, since we expected to find an imaginary part from which the decay rate could be extracted. What is a problem, however, is that we have missed a factor of $1/2$.

We will come back to this factor shortly, but first let us discuss the negative mode. The existence of such a mode can be inferred from the fact that the zero mode, $\psi_0 = N^{-1/2} dq_b/d\tau$, has a node located at the center of the bounce. Recall from Eq. (9.27) that S'' has the form of a one-dimensional Schrödinger operator. For such a potential the lowest eigenstate has no nodes, the next one node, and so on. Since the zero mode has one node, there must be a lower mode (and only one such mode) that has no nodes.

The origin of this negative mode can be understood by considering the series of configurations shown in Fig. 9.6 [230]. These can be parameterized by c, the extremum value of $q(\tau)$. The curve with $c = b$ is the bounce configuration of Fig. 9.5, and that with $c = 0$ (i.e., the x-axis) is the trivial solution $q(\tau) = 0$. The action of these configurations as a function of c is illustrated in Fig. 9.7.

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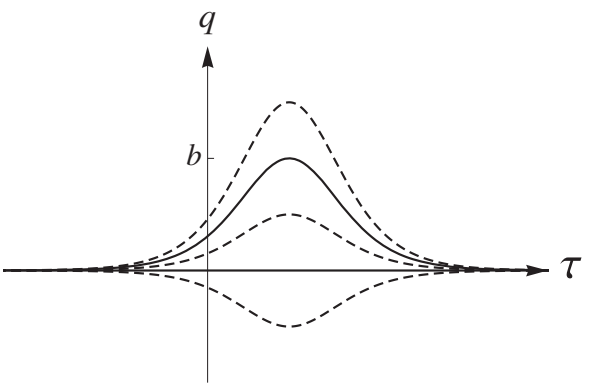


Fig. 9.6. A series of configurations in Euclidean space. The solid line with the maximum value of q equal to b is the bounce solution shown in Fig. 9.5. The dashed curve above this is a configuration that extends into the classically allowed region beyond the barrier, while the dashed curves below it are configurations that remain within the potential barrier.

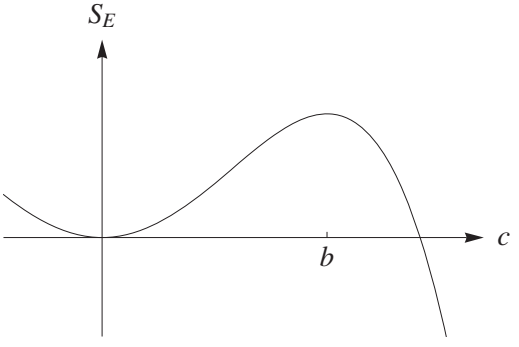


Fig. 9.7. The Euclidean action for a family of configurations such as those shown in Fig. 9.6. The maximum at $c = b$ corresponds to the bounce.

The trivial solution is clearly a local minimum, since any small deviation gives positive contributions to both the potential energy and the τ-derivative term. As c increases from 0 both of these terms, and thus the total action, increase monotonically until the bounce is reached. Since this is the only other stationary point, it must be a local maximum. As c increases further, the configuration spends more and more time in the region with negative potential energy, so the action decreases monotonically.

Thus the bounce, which is a local minimum of the barrier penetration integral $B[P]$, is only a saddle point of the Euclidean action. To understand this, recall that $B[P]$ is defined by an integral over a path from an initial point $\mathbf{q}_0$ to a point $\mathbf{q}_f$ lying on an equipotential surface Σ defined by $V(\mathbf{q}) = V(\mathbf{q}_0)$, and then back to $\mathbf{q}_0$. For the instanton the boundary conditions on S_E restrict it to the same

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class of paths, so a minimum of $B[P]$ is also a minimum of S_E. For the bounce, on the other hand, one calculates the Euclidean action over a path that starts at $\mathbf{q}_0$, tunnels through the barrier, and then tunnels back through the barrier to return to $\mathbf{q}_0$. Some of these are paths that reach Σ and then turn back; among this subset of paths, S_E is a minimum.

The negative mode arises from paths that do not turn back at Σ. The paths in Fig. 9.6 with $c < b$ never reach Σ, and remain within the potential barrier, while those with $c > b$ go beyond Σ and into the classically allowed region. Thus the negative mode corresponds to paths that are possible paths for the Euclidean action, but not for $B[P]$ [231].

Our path integral is an infinite-dimensional integral, but it is only the integration over the coefficient of the one negative mode that is problematic. We can see how to handle this integration by considering the analogous integral

$$J = \int_{-\infty}^{\infty} dc\, (2\pi)^{-1/2} e^{-S(c)}\,, \tag{9.54}$$

where $S(c)$ is the function plotted in Fig. 9.7. As $c \to -\infty$ the action increases and the integral converges. On the other hand, as $c \to \infty$ the action tends toward ∞ and the integral diverges. However, we can relate it to a finite integral by analytic continuation. As noted in Sec. 9.1, the potential with the metastable minimum at $q = 0$ can be viewed as the analytic continuation of a potential with a global minimum at $q = 0$. For the latter potential, the minimum corresponds to a stable state with a well-defined real energy, and the integration over c gives a finite integral. In our case, there is no stable state localized around $q = 0$ and the integral over c is ill-defined. However, we can make it well-defined by deforming the contour of integration from the real axis as shown in Fig. 9.8; this gives the integral a small imaginary part. The integration from $-\infty$ to b is clearly real. The imaginary part comes from the remainder of the contour, and in the steepest descent approximation is given by

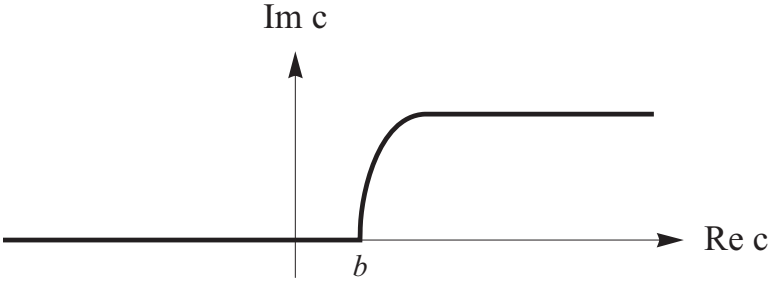


Fig. 9.8. The contour of integration that makes the integral in Eq. (9.54) well-defined.

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$$\begin{aligned}\operatorname{Im} J &= \operatorname{Im} \int_b^{b+i\infty} dc\,(2\pi)^{-1/2} e^{-S(b)} e^{-\frac{1}{2}S''(b)(c-b)^2} \\ &= \frac{1}{2}\, e^{-S(b)} |S''(b)|^{-1/2}\,, \end{aligned} \tag{9.55}$$

with the factor of $\frac{1}{2}$ being due to the fact that the contour only goes through half of the Gaussian peak.

With this factor taken into account, the incorrect Eq. (9.52) is replaced by

$$K = \frac{i}{2}\left(\frac{N}{2\pi}\right)^{1/2} \left|\frac{\det{}' S''(q_b)}{\det S''(q_0)}\right|^{-1/2}. \tag{9.56}$$

Finally, substituting the value of N, we obtain the decay width

$$\Gamma = -2 \operatorname{Im} E_0 = \left(\frac{S(q_b)}{2\pi}\right)^{1/2} \left|\frac{\det{}' S''(q_b)}{\det S''(q_0)}\right|^{-1/2} e^{-S(q_b)}\,. \tag{9.57}$$

As in the previous section, this result was obtained under the assumption that $V(a) = 0$. If $V(a) \neq 0$, we obtain instead

$$\Gamma = \left(\frac{B}{2\pi}\right)^{1/2} \left|\frac{\det{}' S''(q_b)}{\det S''(q_0)}\right|^{-1/2} e^{-B}\,, \tag{9.58}$$

where, as in Eq. (9.18), B is the difference between the bounce action and that of the trivial solution.

9.5 Field theory

The formalism developed in the previous sections is readily carried over to field theory [226, 230], a system with a (continuous) infinity of degrees of freedom. This is largely a matter of translating notation. For a theory with a single scalar field ϕ in D spatial dimensions, the N-dimensional configuration space of the q^j is replaced by the infinite-dimensional space of field configurations $\phi(\mathbf{x})$. For the tunneling path we have the replacement

$$\mathbf{q}(\tau) \longrightarrow \phi(\mathbf{x}, \tau)\,, \tag{9.59}$$

while for the potential energy the correspondence is

$$V(\mathbf{q}) \longrightarrow U[\phi(\mathbf{x})] = \int d^D x \left[\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + V(\phi)\right]. \tag{9.60}$$

The details of this generalization will be explored in the next few chapters, but it is worth commenting on a few points here.

To have a nonzero amplitude for a tunneling process, the barrier penetration integral must be finite, which in turn requires that the deviation of the fields from their value in the initial configuration must be localized not just in Euclidean

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time, but also in space. As a result, there are additional zero modes, corresponding to translation of the instanton or bounce in the various spatial directions. A position collective coordinate must be introduced for each such mode. There are also Jacobian factors related to the normalization constants of these modes; in many cases virial theorems relate these to the Euclidean action. We must integrate over the full range of the collective coordinates. This gives a factor of the volume of space, $\mathcal{V}$; we will see in the next few chapters how $\mathcal{V}$ factors out when one looks at physically measurable quantities. The Euclidean solution may also break other symmetries, leading to additional zero modes and collective coordinates, which must also be integrated over.

Although additional zero modes are acceptable, and indeed required, we cannot allow additional negative modes. In our examples with one degree of freedom, it was easy to see that the instanton had no negative modes and that the bounce had precisely one, which was associated with the possibility of extending the path into the classically allowed region. With more degrees of freedom, and in field theory in particular, there is another way in which negative modes can arise. Our method is based in finding the MPEP, the tunneling path that minimizes the barrier penetration integral B. However, finding a stationary point of the Euclidean action only guarantees that we have a stationary point of B. It is quite possible for the Euclidean solution to be a saddle point of B, with some variations of the path lowering the value of B. Such variations would lead to additional negative modes of S''. If these exist, then the Euclidean solution is not an acceptable bounce or instanton.

Finally, the usual short-distance divergences of field theory are inevitably encountered in the calculation of the functional determinants. The worst of these are canceled when the determinant about the instanton or bounce is divided by that about the trivial solution. The remaining divergences disappear by a cancellation between divergences in the functional determinant and those arising from evaluating the divergent counterterms for the Euclidean solution. This closely parallels the treatment of the divergences in the kink mass in Sec. 2.2, with the quotient of determinants being analogous to the difference of vacuum zero-point energies.

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