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On multivariate minimal polynomials

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Abstract

Given compact sets E and F in $\mathbb{C}^n$ ($n \geqslant 1$) related by $F = q^{-1}(E)$ where q is a polynomial map, we are interested in the general problem of comparing minimal polynomials for E with minimal polynomials for F. Let α be an n-multi-index of length d. We define the classes of polynomials $\mathbb{P}(\alpha) := z^\alpha + \mathbb{C}_{d-1}[z]$ and $\mathscr{P}(\alpha) := \{p\colon p(z) = z^\alpha + \sum_{\beta \prec \alpha} a_\beta z^\beta\}$ where $\prec$ denotes the usual graded lexicographic order. Polynomials in $\mathbb{P}(\alpha)$ or in $\mathscr{P}(\alpha)$ of least deviation from zero on E (with respect to the supremum norm) are called *minimal polynomials* for E. We prove that if q is a simple (i.e. $\hat{q}_i(z) = z_i^m$) polynomial mapping of degree m and if p is minimal polynomial for E then $p \circ q$ is a minimal polynomial for $F = q^{-1}(E)$ and, using some algebraic machinery, we can also construct minimal polynomials for E from minimal polynomials for F. The result seems to be new even in the one-dimensional case.

1. *Introduction*

1. Let d be a positive integer and K be a compact set in the complex plane that contains at least $d+1$ points. It is a classical theorem of Tonelli that among the monic polynomials of degree d there exists one and only one polynomial T_d that minimizes the supremum norm on K, that is $||T_d||_K = \inf\{||z^d + \sum_{i=0}^{d-1} a_i z^i||_K\}$ where the infimum is taken with respect to $(a_0, a_1, \ldots, a_{d-1}) \in \mathbb{C}^d$ (see [**7**, p. 143]). In other words $T_d - z^d$ is the polynomial (of degree at most $d-1$) of best approximation to z^d on K. This polynomial is called *the minimal* (*monic*) *polynomial of degree d*. Very few explicit minimal polynomials are known. The most important comes from the Chebyshev polynomials of first kind (those defined by $\nu_d(\cos\theta) = \cos(d\theta), \theta \in [0, 2\pi]$). Indeed $2^{d-1}\nu_d$ is minimal of degree d for $K = [-1, 1]$. In fact minimal polynomials are often called *Chebyshev polynomials*. Apart from their intrinsic interest, they are closely related to the main objects of potential theory. For example, if $\tau_d(K) := ||T_d||_K$ (τ_d is called the dth Chebyshev constant) then, as d goes to ∞, $\tau_d(K)^{1/d}$ converges to the

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logarithmic capacity $\operatorname{cap}(K)$ (or the transfinite diameter) of K. The principal topic of this paper is the *construction* of *multivariate minimal polynomials* (the definitions are given below) although we shall also say something apparently new about the one variable theory. Our main motivation comes from pluripotential theory, see below, where different kind of multivariate minimal polynomials are naturally involved.

2. We first establish some notation. We work in $\mathbb{C}^n$ (the letter 'n' will be kept for the dimension of our ambient space) and let $\mathbb{C}[z] = \mathbb{C}[z_1, z_2, \dots, z_n]$ denote the space of polynomials and $\mathbb{C}_m[z]$ its subspace of polynomials of (total) degree at most m, that is

$$\mathbb{C}_m[z] = \{p\colon z \to p(z) = \sum_{|\alpha|\leqslant m} a_\alpha z^\alpha\},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| \coloneqq \alpha_1+\alpha_2+\cdots+\alpha_n$ is its length and $z^\alpha \coloneqq z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$.

It will be necessary to order the multi-indices (the monomials). We shall use the *graded lexicographic* order $\prec$ which is defined as follows. If $|\alpha| < |\alpha'|$ then $\alpha \prec \alpha'$ ($z^\alpha \prec z^{\alpha'}$) while if $|\alpha| = |\alpha'|$ then $\alpha \prec \alpha'$ ($z^\alpha \prec z^{\alpha'}$) if the first (starting from the left) non zero coefficient of $\alpha' - \alpha$ is positive. This order satisfies the following three properties:

$$z_1 \succ z_2 \succ \cdots \succ z_n, \tag{2·1}$$

$$|\alpha| < |\alpha'| \Longrightarrow \alpha \prec \alpha' \tag{2·2}$$

and

$$\left.\begin{matrix}\alpha_1 \prec \alpha_2 \\ \beta_1 \preccurlyeq \beta_2\end{matrix}\right\} \Longrightarrow \alpha_1 + \beta_1 \prec \alpha_2 + \beta_2. \tag{2·3}$$

Here $\beta_1 \preccurlyeq \beta_2$ means $\beta_1 \prec \beta_2$ or $\beta_1 = \beta_2$. Thus if, for example, $n = 2$ we have

$$\cdots z_1^4 \succ z_1^3 z_2 \succ z_1^2 z_2^2 \succ z_1 z_2^3 \succ z_2^4 \succ z_1^3 \succ z_1^2 z_2 \cdots.$$

If the coefficients a and b are both different from zero the relation $az^\alpha \prec bz^\beta$ just means $\alpha \prec \beta$.

When $p(z) = a_\alpha z^\alpha + \sum_{\beta\prec\alpha} a_\beta z^\beta$ with $a_\alpha \neq 0$, the monomial $a_\alpha z^\alpha$ is called the *leading term* or *monomial* of p and is denoted by $\operatorname{Lead}(p)$. For example, if $p(z) = 3z_1^3 z_2 + z_1 z_2^2 + z_2^2 + 1$ then $\operatorname{Lead}(p)(z) = 3z_1^3 z_2$. It is convenient to set $\operatorname{Lead}(0) = 0$. A polynomial is said to be *monic* if the coefficient of its leading monomial is 1. The following remarks help computing leading terms:

$$\operatorname{Lead}(pq) = \operatorname{Lead}(p) \cdot \operatorname{Lead}(q) \tag{2·4}$$

and

$$\operatorname{Lead}(p_i) \prec \operatorname{Lead}(q) (i \in I) \Longrightarrow \operatorname{Lead}\left(\sum_{i\in I} p_i + q\right) = \operatorname{Lead}(q). \tag{2·5}$$

Conditions (2·2) and (2·3) are natural for they just mean that the order is compatible with the degree and multiplication of monomials.

3. We can now state the general multivariate problems. As usual, there is no unique analogue to the one-dimensional case.

Let E be a compact set in $\mathbb{C}^n$ and α a multi-index of length d. We define the

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following two sets of polynomials:

$$\mathbb{P}(\alpha) \coloneqq z^\alpha + \mathbb{C}_{d-1}[z] \tag{3·1}$$

and

$$\mathscr{P}(\alpha) \coloneqq \{p : p(z) = z^\alpha + \sum_{\beta \prec \alpha} a_\beta z^\beta,\ a_\beta \in \mathbb{C}\}, \tag{3·2}$$

that is, $\mathscr{P}(\alpha)$ is the set of all polynomials whose leading term is z^α.

Problems. Find *a* polynomial of minimal supremum norm on E in $\mathbb{P}(\alpha)$ (resp. $\mathscr{P}(\alpha)$) and (or) the corresponding Chebyshev constant $\mathbb{T}(\alpha, E)$ (resp. $\mathscr{T}(\alpha, E)$) that is

$$\mathbb{T}(\alpha, E) \coloneqq \inf\{||p||_E,\ p \in \mathbb{P}(\alpha)\} \quad \text{and} \quad \mathscr{T}(\alpha, E) \coloneqq \inf\{||p||_E,\ p \in \mathscr{P}(\alpha)\}. \tag{3·3}$$

We shall speak of a $\mathbb{P}(\alpha)$-minimal polynomial (resp. $\mathscr{P}(\alpha)$-minimal polynomial) for E. Note that $\mathbb{P}(\alpha) \subset \mathscr{P}(\alpha)$ (so that $\mathbb{T}(\alpha, E) \geqslant \mathscr{T}(\alpha, E)$) but $\mathbb{P}(\alpha)$ does not depend on the ordering $\prec$ while $\mathscr{P}(\alpha)$ does.

Of course both problems coincide in case $n = 1$. The first one is the most natural to consider for an approximation theorist. It is the problem of finding a best approximation of z^α on E by polynomials of smaller degree. The second is motivated by the study of the multivariate transfinite diameter (see [**4**]). Both problems are difficult and very few explicit minimal polynomials are currently known. Available (non-trivial) examples are due to Gearhart [**8**], Newman and Xu [**12**] and Reimer [**15**, **16**]. We shall consider them in the text. The referee also pointed out a work by Brass [**6**]. Older (related) references can be found in these papers. (In fact, these authors studied minimality only among polynomials with *real* coefficients. But it is easy to see that if E is a real set that is included in $(\mathbb{R} + i0)^n \subset \mathbb{C}^n$ then every minimal polynomial, as defined in this paper, must have real coefficients.) It is important to note that in general *uniqueness* does not hold in the multivariate case and this is of course in contrast to the one variable theory.

The main result of this paper is to provide a new general construction of minimal polynomials. Roughly speaking, we shall prove, under suitable conditions, that *if q is a polynomial mapping, $F = q^{-1}(E)$ and p is a minimal polynomial for E then $p \circ q$ is a minimal polynomial for F*. This very natural result seems to be new (for general q and E) even in the one variable case.

4. Here is an outline of the paper. In Section 2, we prove and illustrate our result in dimension one. The proofs are simple and they should illuminate the more technical sequel. Section 3 is devoted to the so called $q_\star$ map. We provide there the small necessary background on proper polynomial maps. Our main theorem is proved in Section 4. In the final section, we briefly show how our technique can be used in other situations.

4. *The one variable case*

5. We will consider a polynomial mapping defined by a *monic* polynomial in the one variable case ($n = 1$). (This does not restrict generality.) Throughout this section, given an integer d and a compact set K in $\mathbb{C}$ containing at least $d + 1$ points, $T_{d,K}$ will denote the unique monic minimal polynomial for K of degree d and $\tau_d(K)$ will be its uniform norm on K, i.e. $\tau_d(K) \coloneqq ||T_{d,K}||$.

The following theorem seems to be known only in particular cases with *ad hoc*

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proofs. For example, it is a simple consequence of the maximum principle when E is a lemmniscate and the case $E = [-1, 1]$ has recently been treated in [**13**].

THEOREM 1. *Let E be a compact set in $\mathbb{C}$ containing at least $d+1$ points, q a monic polynomial of degree $m \geqslant 1$ and $F = q^{-1}(E)$. Then, if $T = T_{d,E}$ is the minimal polynomial of degree d for E then $T \circ q$ is the minimal polynomial of degree md for $F = q^{-1}(E)$, that is, $T_{md,F} = T_{d,E} \circ q$.*

Note. We shall use the fact that F contains at least $md+1$ points (this is proved below) and this implies, by Tonelli's theorem, that F admits only one minimal polynomial of degree md.

Proof of Theorem 1. We first prove that $\tau_{md}(F) \leqslant \tau_d(E)$. Since q is monic of degree m, $T \circ q$ is monic of degree md and is therefore a competitor for $\tau_{md}(F)$. Thus $\tau_d(E) = ||T||_E = ||T \circ q||_F \geqslant \tau_{md}(F)$. We now establish the reverse inequality. Let $r = T_{md,F}$. We shall consider the function

$$q_\star(r)(w) := \frac{1}{m} \sum_{z \in q^{-1}(w)} r(z). \tag{5·1}$$

We take multiplicity into account so that there are exactly m terms in this sum. Assume for the moment that $q_\star(r)$ is a monic polynomial of degree d. Then for every $w \in E$, the numbers z involved in the above sum lie in F so that $|r(z)| \leqslant ||r||_F = \tau_{md,F}$. Hence

$$|q_\star(r)(w)| \leqslant \frac{1}{m} \sum_{z \in q^{-1}(w)} \tau_{md}(F) = \tau_{md}(F)$$

and $\tau_d(E) \leqslant \tau_{md}(F)$. Thus we have $\tau_{md}(F) = \tau_d(E)$. But $||T \circ q||_F = \tau_d(E)$ and a minimal polynomial of degree md for F is unique thus $T \circ q = r$. So the proof will be completed as soon as the next theorem is proved.

6. Note that the map $q_\star$ is well defined on $\mathbb{C}[z]$ by (5·1) and is linear. We shall give a purely algebraic (and elementary) proof.

THEOREM 2. *Let q be a monic polynomial of degree m. If p is a polynomial of degree k then $q_\star(p)$ is a polynomial of degree at most $[k/m]$ where $[\delta]$ denotes the greatest integer in δ. Furthermore if p is monic of degree lm then $q_\star(p)$ is monic of degree l.*

Proof. Write $k = lm + \rho$ with $0 \leqslant \rho < m$. First, by performing sufficiently many Euclidean divisions, we can find polynomials p_i $(i = 0, \ldots, l)$, each of degree at most $m-1$, such that $p = \sum_{i=0}^{l} p_i q^i$. Since $q_\star(p_i q^i)(w) = q_\star(p_i)(w)\,.\,w^i$ we obtain

$$q_\star(p)(w) = \sum_{i=0}^{l} q_\star(p_i)(w) w^i$$

and, in order to prove $q_\star(\mathbb{C}_k[z]) \subset \mathbb{C}_l[w]$, it suffices therefore to verify that $q_\star(z^s)$ is a constant polynomial for $1 \leqslant s \leqslant m-1$. (We shall systematically use the notation $q_\star(z^s)$ to denote $q_\star(z \to z^s)$. This abuse should not give rise to confusion.) Now the polynomial $q_\star(z^s)$ is only the sum π_s of the sth power of the roots of the polynomial $q(z) - w := z^m + a_1 z^{m-1} + \cdots + \cdots a_{m-1} z + a_m$. Here the only coefficient that depends

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on w (as a polynomial of degree 1) is a_m. But, by Newton's formulas of elementary algebra, we have

$$\pi_s + \pi_{s-1}a_1 + \pi_{s-2}a_2 + \cdots + \pi_1 a_{s-1} + sa_s = 0 \quad (1 \leqslant s \leqslant m-1)$$

from which we conclude immediatly that $q_\star(z^s)$ does not depend on w, i.e. is a constant polynomial. This concludes the proof of the first claim. As for the second, if p is monic of degree lm then $p - q^l \in \mathbb{C}_{lm-1}[z]$. Therefore, by the first result, $q_\star(p - q^l) \in \mathbb{C}_{l-1}[z]$ that is $q_\star(p)(w) - w^l$ is at most of degree $l-1$ and this proves that $q_\star(p)(w)$ is monic of degree l. This finishes the proof of both Theorem 2 and Theorem 1.

Corollary 1. *If E contains at least $d+1$ points and if q is a monic polynomial of degree m then $q^{-1}(E)$ contains at least $md+1$ points.*

Proof. Suppose, to the contrary, that $q^{-1}(E)$ contains at most md points. Then one can find a monic polynomial p_0 of degree md which vanishes on F. It follows from the definition of $q_\star$ and F that $q_\star(p_0)$ vanishes on E. But, by Theorem 2, it is a monic polynomial of degree d. This is a contradiction for there is no monic polynomial of degree d that vanishes on a set containing at least $d+1$ points.

We will have to work with a multivariate counterpart of the map $q_\star$. This will require some more sophisticated tools but all the basic ideas have been already used. Before doing that, we shall make some comments on Theorem 1.

7. Peherstorfer [**13**] considered the problem of finding polynomials that are minimal not only for a compact set E but also for all the equipotential sets of E, that is, all the sets $E_\rho := \{G_E \leqslant \rho\}$ $(\rho > 1)$ where G_E is the Green function of $\mathbb{C}\setminus E$ with pole at infinity. He proves (in a more complicated way) the result below in the case $E = [-1, 1]$.

Corollary 2. *Let E, q and F as above. We suppose further that G_E is continuous in the plane (in particular E is not polar). If we have $T_{d,E} = T_{d,E_\rho}(\rho > 1)$ then $T_{md,F} = T_{md,F_\rho}(\rho > 1)$.*

Proof. The Green function of F is $G_F = (1/m)G_E \circ q$. This implies that $F_\rho = q^{-1}(E_{m\rho})$ $(\rho > 1)$. Consequently, in view of the hypothesis and Theorem 1, we have for $\rho > 1$

$$T_{md,F_\rho} = T_{d,E_{m\rho}} \circ q = T_{d,E} \circ q = T_{md,F}$$

and the corollary is proved.

We refer to [**13**] for motivation, some consequences and more information on this problem.

Theorem 1 leads also to a new proof of a basic property of the transfinite diameter. Indeed it follows from Theorem 1 that $\sqrt[d]{\tau_d(E)} = (\sqrt[md]{\tau_{md}(F)})^m$ whence, letting $d \to \infty$, we get (see Introduction)

$$\operatorname{cap}(F) = \sqrt[m]{\operatorname{cap}(E)}. \tag{7·1}$$

This result of course is well known. The classical proof is due to Fekete and can be found in [**9**, p. 299]. A direct potential theoretic proof is also available (see [**14**]) but our (new) approach is the starting point of a multivariate generalization.

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3. *Polynomial mappings*

8. Here and in the sequel $n \geqslant 2$. We shall consider polynomial mappings of degree m, that is to say, mappings $q = (q_1, q_2, \ldots, q_n)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ such that $\deg q_i = m$ for every i. We will assume that

$$\hat{q}_1^{-1}(0) \bigcap \hat{q}_2{}^{-1}(0) \bigcap \cdots \bigcap \hat{q}_n{}^{-1}(0) = 0 \tag{8·1}$$

where $\hat{q}_i$ is the homogeneous part of degree m in q_i. By [**11**, theorem 5·3·1] this is equivalent to

$$\varliminf_{|z|\to\infty} \frac{|q(z)|}{|z|^m} > 0$$

where $|\,.\,|$ denotes the Euclidean norm on $\mathbb{C}^n$. Such polynomial mappings are proper although not all proper polynomial mappings satisfy (8·1) (see [**2**, p. 113]). Thus if E is a compact set then $F := q^{-1}(E)$ is a also a compact set and, in view of the previous section, it is reasonable to try to relate minimal polynomials of E composed with q to minimal polynomials of F.

The first step consists in checking whether the composition by q sends the sets $\mathbb{P}(\alpha)$ or $\mathscr{P}(\alpha)$ into sets of the same type. Not surprisingly, a positive answer requires further restrictions on q.

LEMMA 1. *In order that for every multi-index α, there exists a multi-index $\alpha(q)$ such that $p \in \mathbb{P}(\alpha) \Longrightarrow p \circ q \in \mathbb{P}(\alpha(q))$ it is necessary and sufficient that, for some $\sigma \in S_n$ (the group of permutations of $\{1, \ldots, n\}$), $\hat{q}_i(z) = z^m_{\sigma(i)}$, $i = 1, 2, \ldots, n$. Furthermore, in that case $\alpha(q) = m(\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})$.*

Proof. The sufficiency and the form of $\alpha(q)$ are clear. Let us prove the necessity. It follows easily from the relation

$$z^{\alpha(q)} = \hat{q}_1{}^{\alpha_1}(z)\hat{q}_2{}^{\alpha_2}(z) \cdots \hat{q}_n{}^{\alpha_n}(z)$$

that $\hat{q}_i$ must be a monomial for every i, that is $\hat{q}_i(z) = a_i z^{\beta^i}$ where $a_i \neq 0$. (Since $\deg q_i = m$ we have also $|\beta^i| = m$.) But, since q satisfies (8·1), this monomial must be of the form $a_i z^m_{\sigma(i)}$. Finally the a_is also have to be equal to 1 for $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} = 1$ for every α. The Lemma is proved.

In fact, permutations of variables cause no essential changes for, if $\sigma \in S_n$ and M_σ is the automorphism defined by $M_\sigma(z) = (z_{\sigma(1)}, \ldots, z_{\sigma(n)})$, then $(M_\sigma \circ q)^{-1}(E) = q^{-1}(M_{\sigma^{-1}}(E))$ and if p is a $\mathbb{P}(\alpha)$-minimal polynomial for E then $p \circ M_\sigma$ is a $\mathbb{P}(\alpha')$-minimal polynomial for $M_{\sigma^{-1}}(E)$ with $\alpha' = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})$.

We may therefore restrict our attention to *simple* polynomials mappings defined as follows.

Definition. A polynomial mapping q of degree m on $\mathbb{C}^n$ is said to be *simple* if, for every $i = 1, 2, \ldots, n$, $\hat{q}_i(z) = z_i^m$. Every simple polynomial mapping is proper.

Simple polynomial mappings and the associated systems of equations arise, for example, in the study of general systems of algebraic equations (see e.g. [**1**, chapter 21] or [**2**, section 2·5]).

We shall also study the $\mathscr{P}(\alpha)$-problems with simple polynomial mappings although

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the use of other orders would permit to consider them under more general hypotheses on q.

LEMMA 2. *If q is a simple polynomial mapping of degree m then for every multi-index α we have $p \in \mathscr{P}(\alpha) \Longrightarrow p \circ q \in \mathscr{P}(m\alpha)$.*

Proof. If $p \in \mathscr{P}(\alpha)$ then $p(z) = z^\alpha + \sum_{\beta \prec \alpha} a_\beta z^\beta$ so that $p \circ q = q^\alpha + \sum_{\beta \prec \alpha} \alpha_\beta q^\beta$. We want to show that $\mathrm{Lead}(p \circ q) = z^{m\alpha}$. It suffices to note that $\mathrm{Lead}(q^\beta) = \mathrm{Lead}^{\beta_1}(q_1) \ldots \mathrm{Lead}^{\beta_n}(q_n) = z_1^{m\beta_1} \ldots z_n^{m\beta_n} = z^{m\beta}$ while $\mathrm{Lead}(q^\alpha) = z^{m\alpha} \succ z^{m\beta}$ since $\beta \prec \alpha \Longrightarrow m\beta \prec m\alpha$. In view of (2.5) we deduce the required result.

9. We now turn to define the multivariate anlogue of the $q_\star$ map introduced in the previous section. We shall work with a general (i.e not necessarily simple) proper polynomial mapping of degree m satisfying (8·1). Such a mapping q has only finitely many zeroes and, taking multiplicity into account, by Bezout's theorem (see [**1**, p. 180] or [**18**, p. 141]) exactly m^n. (A definition of multiplicity can be found in [**1**, p. 18]). In any case, for almost all values of w, the equation $q(z) = w$ has exactly m^n distinct solutions.

Definition. Given a proper polynomial map q of degree m as above, the $q_\star$ map is defined on the space of polynomials by the formula

$$q_\star(p)(w) = \frac{1}{m^n} \sum_{z \in q^{-1}(w)} p(z) \quad (z \in \mathbb{C}^n). \tag{9·1}$$

This sum has been investigated by Aizenberg. (The reader should note that he worked with $w = 0$.) Using multivariate residues, he has been able to establish some explicit formulas which we shall use later on. These formulas however are complicated and we shall just state explicitly one of them in Section 4. The basic result is the following theorem.

THEOREM 3. *If p is a polynomial of degree k then $q_\star(p)$ is a polynomial of degree at most $[k/m]$.*

Proof. That $q_\star(p)$ is a polynomial follows from [**1**, corollary 21·2, p. 162] but a use of [**10**, theorem 5, p. 32] implies (abstractly) that it must be an entire function (see also [**18**, p. 25]). Then, in order to prove that $q_\star(p)$ is a polynomial of degree $[k/m]$ it suffices to show that, for some constant C, one has $|q_\star(p)(w)| \leqslant C|w|^{k/m}$ for $w \in \mathbb{C}^n$, $|w|$ large. First, since p is a polynomial of degree k, there exists C_1 such that

$$|p(z)| \leqslant C_1|z|^k, \quad \text{for } |z| \text{ large.} \tag{9·2}$$

On the other hand, since q is proper of degree m satisfying (8·1), there exists a constant $C_2 > 0$, see [**11**, theorem 5·3·1] such that $|q(z)| \geqslant C_2|z|^m$ (for $|z|$ large) which gives,

$$\text{for } |w| \text{ large}, \quad z \in q^{-1}(w) \Longrightarrow |z| \leqslant C_2^{-1/m}|w|^{1/m}. \tag{9·3}$$

Hence, using (9·2) and (9·3) we get for $|w|$ large,

$$z \in q^{-1}(w) \Longrightarrow |p(z)| \leqslant C_1|z|^k \leqslant C_1 C_2^{-k/m}|w|^{k/m} \Longrightarrow |q_\star(p)(w)| = O(|w|^{k/m})$$

from which the conclusion follows.

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Remark. Let $q^\star$ denote the map $\mathbb{C}[w] \to \mathbb{C}[z]$ given by $q^\star(p) = p \circ q$. Then $q^\star$ is a ring homomorphism which is an injection of $\mathbb{C}[w]$ into $\mathbb{C}[z]$. This gives $\mathbb{C}[z]$ the structure of a finite $\mathbb{C}[w]$-module. The map $q_\star\colon \mathbb{C}[z] \to \mathbb{C}[w]$ is a $\mathbb{C}[w]$-module homomorphism and $q_\star q^\star$ is the identity on $\mathbb{C}[w]$. In particular, $q_\star q^\star(w^\alpha) = w^\alpha$.

10. *The several variables case*

10. We are now able to prove the multivariate results corresponding to Theorem 1.

THEOREM 4. *Let E be a compact set in $\mathbb{C}^n$, q a simple polynomial mapping of degree m on $\mathbb{C}^n$ and α a multi-index of length $d \geqslant 1$.*

(i) *If T_E is a $\mathbb{P}(\alpha)$-minimal polynomial for E then $T_E \circ q$ is a $\mathbb{P}(m\alpha)$-minimal polynomial for $F = q^{-1}(E)$, in particular $\mathbb{T}(\alpha, E) = \mathbb{T}(m\alpha, F)$.*

(ii) *Conversely, if T_F is a $\mathbb{P}(m\alpha)$-minimal polynomial for $F = q^{-1}(E)$ then $q_\star(T_F)$ is a $\mathbb{P}(\alpha)$-minimal polynomial for E.*

(iii) *Both assertions remain true for if we replace $\mathbb{P}(\alpha)$ by $\mathscr{P}(\alpha)$ and $\mathbb{P}(m\alpha)$ by $\mathscr{P}(m\alpha)$, in particular $\mathscr{T}(\alpha, E) = \mathscr{T}(m\alpha, F)$*

(iv) *It follows that if a minimal polynomial T for F, either in $\mathbb{P}(\alpha)$ or $\mathscr{P}(\alpha)$, is of the form $T = r \circ q$ then r is a minimal polynomial for E of the corresponding type.*

Proof. Since for every polynomial p one has $||p \circ q||_F = ||p||_E$ and $||q_\star(p)||_E \leqslant ||p||_F$, propositions (i) and (ii) can be derived (see the proof of Theorem 1) from the following two facts: (a) $T_E \circ q \in \mathbb{P}(m\alpha)$ which implies $\mathbb{T}(\alpha, E) \geqslant \mathbb{T}(m\alpha, F)$ and (b) $q_\star(T_F) \in \mathbb{P}(\alpha)$ which implies $\mathbb{T}(\alpha, E) \leqslant \mathbb{T}(m\alpha, F)$. (a) is a consequence of Lemma 1. We just prove $p \in \mathbb{P}(m\alpha) \Longrightarrow q_\star(p) \in \mathbb{P}(\alpha)$. Indeed if $p \in \mathbb{P}(m\alpha)$ then $p - q^\alpha \in \mathbb{C}_{md-1}[z]$ hence, in view of Theorem 3, $q_\star(p - q^\alpha) \in \mathbb{C}_{d-1}[w]$. This means that $q_\star(p)(w) \in w^\alpha + \mathbb{C}_{d-1}[w] = \mathbb{P}(\alpha)$. This implies fact (*b*) and finishes the proof of the first two statements.

Proposition (iii) can be established in the same way. We use Lemma 2 to prove the corresponding part (*a*) and it remains only to show that $p \in \mathscr{P}(m\alpha) \Longrightarrow q_\star(p) \in \mathscr{P}(\alpha)$. This is somewhat more difficult and we shall need the next two lemmas.

LEMMA 3. *Let q be a simple polynomial map of degree m and z^β be a monomial such that*

(i) $\beta_i \leqslant m - 1$, $(i = 1, 2, \ldots, n)$

(ii) $|\beta| = lm$ *with* $l \geqslant 1$.

Then $q_\star(z^\beta) \subset \mathbb{C}_{l-1}[w]$.

Proof. We shall use a formula of Aizenberg and write the components of our simple map q as

$$q_i(z) = z_i^m + r_i(z) \quad (i = 1, 2, \ldots, n), \tag{10·1}$$

where $r_i \in \mathbb{C}_{m-1}[z]$.

First, if $\mathscr{L}[z]$ denotes the algebra of rational fractions of the form $h(z) = p(z)/z^\beta$ where p is a polynomial and β a multi-index then $\mathscr{N}$ is the linear functional that assignes to every $r \in \mathscr{L}[z]$ its constant term. It follows from Aizenberg's Theorem [**1**, theorem 21·3, p. 162] that for every polynomial p of degree d, we have

$$q_\star(p)(w) = \mathscr{N}\left(p \cdot \operatorname{Jac}(q) \cdot \frac{1}{z^{(m-1)\mathbf{1}}} \sum_{|\alpha| \leqslant d} (-1)^{|\alpha|} \frac{(r - w)^\alpha}{z^{m\alpha}}\right) \tag{10·2}$$

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where

$$\mathbf{1} = (1, 1, \ldots, 1), \quad r := (r_1, r_2, \ldots, r_n)$$

and

$$\mathrm{Jac}\,(q)(z) = \det\left(\frac{\partial q_i(z)}{\partial z_j}\right)_{i,j=1}^{n}. \tag{10·3}$$

Let us apply this formula to $p(z) = z^\beta$ with β as in the hypothesis. According to Theorem 3, we already know that $q_\star(z^\beta)$ is at most of degree l therefore it suffices to prove that the homogeneous part of degree l in $q_\star(z^\beta)(w)$ is zero. In view of formula (10·2) this part is

$$\sum_{|\mu|=l} a_\mu w^\mu,$$

where – using the notation $\alpha \geqslant \mu$ to indicate $\alpha_i \geqslant \mu_i$ for every i – the coefficients a_μ are given by

$$a_\mu = \sum_{\alpha \geqslant \mu} (-1)^{|\alpha|+|\mu|} \binom{\alpha_1}{\mu_1} \cdots \binom{\alpha_n}{\mu_n} \mathscr{N}\left(\frac{\mathrm{Jac}\, q}{z^{(m-1)\mathbf{1}-\beta}} \frac{r^{\alpha-\mu}}{z^{m\alpha}}\right). \tag{10·4}$$

Note that, due to hypothesis (i), $(m-1)\mathbf{1}-\beta$ is a multi-index of non-negative integers so that

$$D(z) = z^{(m-1)\mathbf{1}-\beta+m\alpha} \tag{10·5}$$

is really the denominator of the fraction in (10·4). We shall prove that, whatever α is in (10·5), we have

$$\mathscr{N}\left(\frac{\mathrm{Jac}\, q}{z^{(m-1)\mathbf{1}-\beta}} \frac{r^{\alpha-\mu}}{z^{m\alpha}}\right) = 0. \tag{10·6}$$

First, we remark that the Jacobian of q is of the form

$$\mathrm{Jac}\, q(z) = m^n z_1^{m-1} \cdots z_n^{m-1} + p_0(z),$$

where p_0 is of degree at most $(n-1)(m-1) + (m-2)$. Consequently, since $|\mu| = l$,

$$\deg(p_0 r^{\alpha-\mu}) \leqslant (n-1)(m-1) + (m-2) + (m-1)(|\alpha| - l)$$

while

$$\deg(D) = m|\alpha| + n(m-1) - |\beta| = m(|\alpha| - l) + n(m-1).$$

Since $|\alpha| - l \geqslant 0$, one checks immediately that

$$\deg(p_0 r^{\alpha-\mu}) < \deg(D).$$

Therefore there is no constant term in the fraction $(p_0/D)r^{\alpha-\mu}$ and this implies

$$\mathscr{N}\left(\frac{p_0}{z^{(m-1)\mathbf{1}-\beta}} \frac{r^{\alpha-\mu}}{z^{m\alpha}}\right) = 0. \tag{10·7}$$

Now, one easily checks that

$$\deg(m^n z_1^{m-1} \cdots z_n^{m-1}) \geqslant \deg D \tag{10·8}$$

if and only if $|\alpha| - |\mu| = 0$ and in this case (10·8) is in fact an equality. But, since we

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know already $\alpha \geqslant \mu$, the condition $|\alpha| - |\mu| = 0$ implies $\alpha = \mu$. In this later case the computation reduces to

$$\mathscr{N}\left(\frac{m^n z_1^{m-1}\cdots z_n^{m-1}}{z^{(m-1)\mathbf{1}-\beta}}\frac{1}{z^{m\mu}}\right).$$

The only possibility for this quantity to be not zero would be that $m\mu = \beta$. However $m\mu = \beta$ cannot occur for $|\mu| = l \geqslant 1$ implies for some i, $\mu_i \geqslant 1$ or $m\mu_i \geqslant m$ while, by hypothesis, $\beta_i \leqslant m-1$. Thus, in any case,

$$\mathscr{N}\left(\frac{m^n z_1^{m-1}\cdots z_n^{m-1}}{z^{(m-1)\mathbf{1}-\beta}}\frac{r^{\alpha-\mu}}{z^{m\alpha}}\right) = 0. \tag{10·9}$$

We conclude from (10·9) and (10·7) that (10·6) is true so that a_μ is equal to zero for every μ of length l and this means that $q_\star(z^\beta)$ is at most of degree $l-1$. This finishes the proof of the lemma.

LEMMA 4. *Let q be a simple polynomial map of degree m on $\mathbb{C}^n$ and α a multi-index of length $d \geqslant 1$. We have $q_\star(\mathscr{P}(m\alpha)) \subset \mathscr{P}(\alpha)$.*

Proof. Recall that if $p \in \mathscr{P}(m\alpha)$ then $p(z) = z^{m\alpha} + \sum_{\beta \prec m\alpha} a_\beta z^\beta$. We shall discuss the form of $q_\star(z^\beta)$ for every monomial z^β appearing in this sum.

Step 1. *The monomial $z^{m\alpha}$.* Since q is simple, we have $z^{m\alpha} - q^\alpha \in \mathbb{C}_{md-1}[z]$. Therefore, by Theorem 3, $q_\star(z^{m\alpha} - q^\alpha)(w) \in \mathbb{C}_{d-1}[w]$, that is

$$q_\star(z^{m\alpha})(w) = w^\alpha + (\text{lower degree terms}). \tag{10·10}$$

Step 2. *The monomials z^β where $|\beta| = m|\alpha|$.* In this case, one can find an index $k \in \{1,\dots,n\}$ such that

$$\beta = (m\alpha_1,\dots,m\alpha_{k-1}, ms_k + l_k,\dots,ms_n + l_n) \tag{10·11}$$

where the s_js and the l_js are non-negative integers such that

$$0 \leqslant s_k < \alpha_k \tag{10·12}$$

and

$$0 \leqslant l_j < m \quad (j = k, k+1,\dots,n). \tag{10·13}$$

Next, we set

$$\tilde{l} = (0,\dots,0,l_k,l_{k+1},\dots,l_n)$$

and

$$\tilde{s} = (\alpha_1,\dots,\alpha_{k-1},s_k,s_{k+1},\dots,s_n).$$

Note that

$$|\tilde{l}| + m|\tilde{s}| = m|\alpha|.$$

Since q is a simple map, we deduce from (10·11) that

$$z^\beta - z^{\tilde{l}} q^{\tilde{s}} \in \mathbb{C}_{md-1}[z]. \tag{10·14}$$

Here two subcases need to considered.

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(i) If $|\tilde{l}| = 0$ then (10·14) reduces to $z^\beta - q^{\tilde{s}} \in \mathbb{C}_{md-1}[z]$ which, in view of Theorem 3, gives in turn

$$q_\star(z^\beta)(w) = w^{\tilde{s}} + \text{(lower degree terms)}. \tag{10·15}$$

Furthermore, in that case, we have $|\tilde{s}| = |\alpha|$ with $\tilde{s}_i = \alpha_i$ for $i = 1, 2, \ldots, k-1$ but $\tilde{s}_k < \alpha_k$ (see (10·12)), so that

$$\tilde{s} \prec \alpha. \tag{10·16}$$

(ii) On the other hand if $|\tilde{l}| \neq 0$ then $|\tilde{l}| = m(|\alpha| - |\tilde{s}|) > 0$. This implies, see also (10·13), that the monomial $z^{\tilde{l}}$ satisfies the full hypothesis of Lemma 3. Therefore $q_\star(z^{\tilde{l}})(w)$ is a polynomial of degree at most $|\alpha| - |\tilde{s}| - 1$. Consequently $q_\star(z^{\tilde{l}} q^{\tilde{s}})(w) = w^{\tilde{s}} q_\star(z^{\tilde{l}})(w)$ is a polynomial of degree at most $|\alpha| - 1$, hence

$$q_\star(z^\beta) \in \mathbb{C}_{d-1}[w]. \tag{10·17}$$

Step 3. *The monomials z^β with $|\beta| < m|\alpha|$.* Theorem 3 implies again that

$$q_\star(z^\beta)(w) \in \mathbb{C}_{d-1}[w]. \tag{10·18}$$

We conclude from the three steps ((10·16) is essential here) that $q_\star(\mathscr{P}(m\alpha)) \subset \mathscr{P}(\alpha)$. This finishes the proof of Lemma 4 and point (iii) of Theorem 4.

11. Let us first give a few examples of $\mathbb{P}(\alpha)$-minimal polynomials. Recall that a compact set E is said to be *circular* of centre a if $|\lambda| \leqslant 1$ and $z \in E$ implies $a + \lambda(z-a) \in E$.

Proposition 1. *Let E be a circular compact of center a in $\mathbb{C}^n$ and let q be a simple polynomial mapping of degree m. Then $(q-a)^{m\alpha}$ is a $\mathbb{P}(m\alpha)$-minimal polynomial for $q^{-1}(E)$.*

Proof. The result follows from Theorem 4 for $p(z) := (z-a)^\alpha$ is a $\mathbb{P}(\alpha)$-minimal polynomial for E. We recall the proof of this known fact. First $p \in \mathbb{P}(\alpha)$ and every $r \in \mathbb{P}(\alpha)$ can be written as $r(z) = (z-a)^\alpha + \sum_{|\beta| \leqslant d-1} a_\beta (z-a)^\beta$. Take $z_0 \in E$ such that $|(z_0-a)^\alpha| = ||(z-a)^\alpha||_E$ and consider the one variable polynomial $s(\lambda) = r(a + \lambda(z_0-a))$ then, using Cauchy's inequalities,

$$||r||_E \geqslant ||s||_{\{|z| \leqslant 1\}} \geqslant |(z_0-a)^\alpha| = ||(z-a)^\alpha||_E.$$

12. The first class of $\mathbb{P}(\alpha)$-minimal polynomials for $\mathbf{D} := \{x_1^2 + x_2^2 \leqslant 1\}$ the unit Euclidean disc in $\mathbb{R}^2 = (\mathbb{R} + i0) \times (\mathbb{R} + i0) \subset \mathbb{C}^2$ has been found by Gearhart in [**8**, theorem 2·1]. His polynomials are defined, for $\alpha = (\alpha_1, \alpha_2)$, by

$$G_\alpha(x_1, x_2) = \frac{1}{2^{|\alpha|}} [U_{\alpha_1}(x_1) U_{\alpha_2}(x_2) + U_{\alpha_1 - 2}(x_1) U_{\alpha_2 - 2}(x_2)] \tag{12·1}$$

where U_m is the classical Chebyshev polynomial of second kind defined in term of those of first kind by

$$U_m = \frac{1}{m+1} \nu'_{m+1}.$$

Moreover, $\mathbb{T}(\alpha, \mathbf{D}) = 2^{1-|\alpha|}$. Since U_n is an even polynomial when n is even, there exists polynomials B_α such that

$$G_{2\alpha}(z_1, z_2) = B_\alpha(z_1^2, z_2^2) \quad (|\alpha| \geqslant 1). \tag{12·2}$$

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PROPOSITION 2. *For every nonzero α, the polynomial B_α is a $\mathbb{P}(\alpha)$-minimal polynomial for the triangle* $\mathbf{T} := \{(x_1, x_2) : x_1 \geqslant 0, x_2 \geqslant 0, x_1 + x_2 \leqslant 1\}$ *and* $\mathbb{T}(\alpha, \mathbf{T}) = 2^{1-2|\alpha|}$.

Proof. It follows from Theorem 4 (see point (iv)) once we have remarked that $\mathbf{D} = q^{-1}(\mathbf{T})$ with $q(z_1, z_2) = (z_1^2, z_2^2)$.

Reimer subsequently introduced another interesting family of $\mathbb{P}(\alpha)$-minimal polynomials for the unit disc. These can be defined (see [**15**, theorem 1]), by the relation

$$\sum_{|\alpha|=m} t^\alpha R_\alpha(z) = |t|^m \nu_m \left(\frac{\langle t, x \rangle}{|t|} \right), \quad (m = 0, 1, 2, \ldots), \tag{12·3}$$

where $\langle ., . \rangle$ is the Euclidean scalar product in $\mathbb{R}^2$ and $|.|$ the corresponding norm. Later Bos [**5**] showed that

$$2^{1-|\alpha|} R_\alpha = z^\alpha - K[r_0, \ldots, r_{|\alpha|-1}; z^\alpha],$$

where $K[r_0, \ldots, r_n; .]$ is the Kergin interpolation operator at the nodes

$$r_k = \left(\cos \frac{2k\pi}{|\alpha|}, \sin \frac{2k\pi}{|\alpha|} \right) \quad (k = 0, 1, \ldots, |\alpha| - 1).$$

Using (12·3), it is easy to see that there exist polynomials C_α such that

$$R_{2\alpha}(z_1, z_2) = C_\alpha(z_1^2, z_2^2) \quad (|\alpha| \geqslant 1)$$

and we obtain as the previous proof the following proposition.

PROPOSITION 3. *Every C_α is a $\mathbb{P}(\alpha)$-minimal polynomial for* $\mathbf{T}$.

These results are to be compared with those of Newman and Xu in [**12**] who first exhibited minimal polynomials for the triangle $\mathbf{T}$ and computed $\mathbb{T}(\alpha, \mathbf{T})$. Of course, we can work in the reverse sense and construct minimal polynomials on the disc using theirs.

13. Our main application of Theorem 4(iii) will be presented in [**4**]: we prove a generalization of formula (7·1) to the multivariate (Leja) transfinite diameter using the same approach we explained above in the one variable case.

We shall give here two simpler illustrations. Recall that a compact set E is said to be polycircular of centre a if $|\lambda_i| \leqslant 1$ for $i = 1, 2, \ldots, n$ and $z \in E$ implies $(a_1 + \lambda_1(z_1 - a_1), \ldots, a_n + \lambda_n(z_n - a_n)) \in E$.

PROPOSITION 4. *Let E be a polycircular compact set of centre a. Let q be a simple polynomial mapping of degree m. Then $(q - a)^\alpha$ is a $\mathscr{P}(\alpha)$-minimal polynomial for $F = q^{-1}(E)$.*

Proof. It is similar to that of Proposition 1. We use Cauchy's inequalities on a polydisc to show that $(z - a)^\alpha$ is a $\mathscr{P}(\alpha)$-minimal polynomial for E. We omit the details.

PROPOSITION 5. *Let q be a simple polynomial map of degree m. Define $F = q^{-1}([-1, 1]^n)$. Then the polynomial D_α defined by*

$$D_\alpha(z) := \frac{1}{2^{|\alpha|-n}} \left[\nu_{\alpha_1}(q_1(z)) \cdots \nu_{\alpha_n}(q_n(z)) \right],$$

is a $\mathscr{P}(\alpha)$-minimal polynomial for F and, when $\alpha_i \geqslant 1$, $(1 \leqslant i \leqslant n)$, $\mathscr{T}(\alpha, F) = 2^{|\alpha|-n}$.

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Proof. This a consequence of Theorem 4(iii) for it follows immediately from another result of Reimer (see [**16**]), that if c_{α_i} is the leading coefficient of ν_{α_i} then the polynomial $c_{\alpha_1}^{-1}\nu_{\alpha_1}(z_1)\cdots c_{\alpha_n}^{-1}\nu_{\alpha_n}(z_n)$ is a $\mathscr{P}(\alpha)$-minimal for $[-1,1]^n$. It is even minimal for a larger space than $\mathscr{P}(\alpha)$.

13. *Further approximation problems*

There are probably many other problems of extremal polynomials to which our method can be applied. We shall present two of them.

14. The first one is a generalization of the $\mathbb{P}(\alpha)$-minimal polynomials problem.

Let h be an *homogeneous* polynomial of degree d. A h-minimal polynomial for a compact set E is a polynomial of the form $h+p$ where $p \in \mathbb{C}_{d-1}[z]$ which is of least deviation from 0 on E, that is

$$||h+p||_E = \inf\{||h+r||_E, r \in \mathbb{C}_{d-1}[z]\}.$$

Thus, when $h(z) = z^\alpha$, a h-minimal polynomial is only a $\mathbb{P}(\alpha)$-minimal polynomial. We shall denote by $\mathbf{Che}_E(h)$ the (non empty) set of all the h-minimal polynomials for E while $||\mathbf{Che}_E(h)||$ will denote the supremun norm of any polynomial in $\mathbf{Che}_E(h)$. These h-minimal polynomials arise naturally in multivariate complex approximation and in pluripotential theory. We refer the interested reader to [**3**, **4**, **17**]. The following Theorem is a generalization of Theorem 4(i) and has the advantage of not requiring a special form for q. Given a set U of polynomials, $U \circ q$ denotes the set of the polynomials $p \circ q$ when p runs over U and, as usual, $\hat{p}$ is the highest homogeneous part of p. This definition extends obviously to a polynomial map: $\hat{q} = (\hat{q_1}, \ldots, \hat{q_n})$.

THEOREM 5. *Let E be a compact set in $\mathbb{C}^n$ and q a proper polynomial map of degree m satisfying* (8·1). *We set as usual $F = q^{-1}(E)$. For every homogeneous polynomial h we have*

$$\mathbf{Che}_E(h) \circ q \subset \mathbf{Che}_F(h \circ \hat{q}) \quad \text{and} \quad q_\star(\mathbf{Che}_F(h \circ \hat{q})) \subset \mathbf{Che}_E(h), \tag{14·1}$$

so that $||\mathbf{Che}_E(h)|| = ||\mathbf{Che}_F(h \circ \hat{q})||$.

Proof. This is the same idea as in the proof of Theorem 4. Take $p_1 \in \mathbf{Che}_E(h)$. We have

$$\widehat{p_1 \circ q} = \hat{p}_1 \circ \hat{q} = h \circ \hat{q}.$$

Therefore $p_1 \circ q$ is a competitor to $\mathbf{Che}_F(h \circ \hat{q})$ so that

$$||\mathbf{Che}_E(h)|| = ||p_1||_E = ||p_1 \circ q||_F \geqslant ||\mathbf{Che}_F(h \circ \hat{q})||_F. \tag{14·2}$$

Now, consider $p_2 \in \mathbf{Che}_F(h \circ \hat{q})$. For some $r \in \mathbb{C}_{md-1}$ where $d = \deg h$ we have

$$p_2 = h \circ \hat{q} + r = h \circ q + (h \circ \hat{q} - h \circ q) + r.$$

Note that the polynomial $s := (h \circ \hat{q} - h \circ q) = \widehat{h \circ q} - h \circ q$ is of degree at most $md-1$ for we remove its highest homogeneous part to a polynomial of degree md. Now, from the relation

$$p_2 = h \circ q + r + s,$$

we obtain by applying $q_\star$ and using the remark following Theorem 3:

$$q_\star(p_2) = h + q_\star(s+r).$$

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Since $s + r$ is of degree at most $md - 1$, it follows from Theorem 3 that $q_\star(s)$ is at most of degree $d - 1$ so that $q_\star(p_2)$ is competitor to $\mathbf{Che}_E(h)$. Whence it follows

$$||\,\mathbf{Che}_E(h)|| \leqslant ||q_\star(p_2)||_E \leqslant ||p_2||_F = ||\,\mathbf{Che}_F(h \circ \hat{q})||_F. \tag{14·3}$$

The conclusion follows from (14·2) together with (14·3).

Here is a simple illustration. We use the notation of **12** above. Consider for $\alpha = (\alpha_1, \alpha_2)$ the homogeneous polynomial

$$h(z_1, z_2) = z_1^{2\alpha_1}(z_1^2 + z_2^2)^{\alpha_2},$$

then, according to [**8**, theorem 3·2], the polynomials

$$(-1)^{\alpha_2} G_{2\alpha}(z_1, (1 - z_1^2 - z_2^2)^{\frac{1}{2}}) = (-1)^{\alpha_2} B_\alpha(z_1^2, 1 - z_1^2 - z_2^2)$$

are h-minimal polynomial for the disc $\mathbf{D}$. Proceeding as in **12**, it follows from Theorem 5 that the polynomials

$$(-1)^{\alpha_2} B_\alpha(z_1, 1 - z_1 - z_2)$$

are $z_1^{\alpha_1}(z_1 + z_2)^{\alpha_2}$-minimal for the triangle $\mathbf{T}$.

15. We shall conclude this paper with a rather different approximation problem. Details will be omitted. We first state it in the case of one variable.

Given a finite set A containing say k (distinct) elements and an integer $d > k$, we denote by $U_A(d)$ the set of all the polynomials p of degree at most d satisfying $a \in A \Longrightarrow p(a) = 1$. Thus a member of $U_A(d)$ is of the form $p(z) = 1 + r(z)s(z)$ where $\deg r \leqslant d - k$ and $s(z) = \prod_{z \in A}(z - a)$. Given a compact set E we may look for a polynomial in $U_A(d)$ of least supremum norm on E. This is the problem of approximating the constant function 1 by a polynomial of the prescribed form $r(z)s(z)$. If q is a polynomial of degree m then

$$U_A(d) \circ q \subset U_{q^{-1}(A)}(md)$$

and

$$q_\star(U_{q^{-1}(A)})(md) \subset U_A(d)$$

which would enable us to use our method.

For simplicity, let us just state a very particular case of the corresponding multivariate situation. We set now $U_0(d) = \{p \in \mathbb{C}_d[z]\colon p(0) = 1\}$. If q is a proper and homogeneous polynomial map of degree m then $q^{-1}(0) = \{0\}$. The relations $U_0(d) \circ q \subset U_0(md)$ and $q_\star(U_0)(md) \subset U_0(d)$ could again be used in the corresponding extremal problem.

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