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SOME GENERALIZED SHEFFER FUNCTIONS

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It was shown by Sheffer that all functions of the two-valued propositional calculus can be defined in terms of a single primitive. This result has been extended by Post (3) and Webb (5) to the functionally complete m-valued propositional calculus. We shall consider here similar extensions for the extended propositional calculus and for the (functionally incomplete) systems of Łukasiewicz (1).

We first show that the extended Łukasiewicz propositional calculus is functionally complete, provided that $m < \aleph_0$. Łukasiewicz defined the truth-tables of his primitives, implication and negation, so that if $p, q, \bar{p}, p \to q$ have the truth-values $x, y, f(x), g(x,y)$ respectively, then

$$f(x) = 1 - x$$

and

$$g(x,y) = \min(1, 1-x+y).$$

It has been shown by Rosser and Turquette (4) that in terms of these primitives we can define functions $J_i(p)$ $(i = 0, 1/(m-1), \ldots, 1)$ such that $J_i(p)$ takes the truth-value 1 when p takes the truth-value i, $J_i(p)$ otherwise taking the truth-value 0. Łukasiewicz defined conjunction by

$$P \,\&\, Q =_{\mathrm{df}} \overline{(\overline{P} \to \overline{Q}) \to \overline{Q}},$$

so if $p \,\&\, q$ has the truth-value $h(x,y)$ then

$$h(x,y) = \min(x,y).$$

Thus we can define the logical constant $(m-2)/(m-1)$ by

$$(m-2)/(m-1) =_{\mathrm{df}} (\exists p)(p \,\&\, J_{(m-2)/(m-1)}(p)).$$

It follows at once from a theorem of Rosser and Turquette (4) that the extended Łukasiewicz propositional calculus is functionally complete.

We now show that a single primitive is sufficient for the m-valued extended propositional calculus. Instead of applying quantifiers only to single propositions we apply them also to ordered triples of propositions. The definition of a formula of the system is now:

(i) A propositional variable is a formula.

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(ii) If P, Q, R are formulae and $\mathbf{p}$ is a propositional variable* which occurs as a free variable in at least one of P, Q, R and $\mathbf{p}$ does not occur as a bound variable in any of P, Q, R, then $\{\mathbf{p}\}\{P, Q, R\}$ is a formula.

(iii) An expression can only be a formula in virtue of (i) and (ii).

Post (3) defined the truth-tables of alternation and negation so that if $\sim p, p \vee q$ have the truth-values $n(x)$, $a(x, y)$ respectively, then

$$n(0) = 1,$$

$$n(x) = x - 1/(m-1) \quad (x \neq 0),$$

and

$$a(x, y) = \max(x, y).$$

$\{p\}\{P, Q, R\}$ has the same meaning as $(p)(\sim P \vee \sim Q \vee \sim R)$. In terms of our primitive we define Post's negation function by

$$\sim P =_{\mathrm{df}} \{\mathbf{p}\}\{P, P, \mathbf{p}\},$$

where $\mathbf{p}$ is a propositional variable which does not occur in P. We then define alternation by

$$P \vee Q =_{\mathrm{df}} \{\mathbf{p}\}\{\sim^{m-1} P, \sim^{m-1} Q, \mathbf{p}\},$$

where $\mathbf{p}$ is a propositional variable which does not occur in P or in Q and $\sim^{m-1}$ denotes $m-1$ consecutive negation symbols. It follows at once from a theorem of Post (3) that we can define all functions of the propositional calculus in terms of the primitive.

We define the logical constants 0 and $1/(m-1)$ by

$$0 =_{\mathrm{df}} \{p\}\{p, p, p\}$$

and

$$1/(m-1) =_{\mathrm{df}} \sim^{m-1} 0.$$

Thus we can define the universal quantifier by

$$(q)\, Q =_{\mathrm{df}} \{q\}\{1/(m-1), \quad 1/(m-1), \quad \sim^{m-1} Q\}.$$

Hence we can define all functions of the extended propositional calculus in terms of the primitive.

We now consider the m-valued Łukasiewicz propositional calculus without quantifiers. We define alternation by

$$P \vee Q =_{\mathrm{df}} \overline{P} \to Q,$$

so $p \vee q$ has the truth-value $\min(1, x+y)$. We now define the function

$$\langle P_1, P_2, \ldots, P_\alpha \rangle =_{\mathrm{df}} \sum_{r=1}^{\alpha} \overline{P}_r,$$

where Σ denotes summation by $\vee$.

We shall show that if α is the smallest integer $\geqslant 2$ such that $m-1$ is not divisible by $\alpha+1$, then $\langle p_1, p_2, \ldots, p_\alpha \rangle$ can be taken as a single primitive for the m-valued Łukasiewicz propositional calculus, provided that $m < \aleph_0$. We define the propositional functions $\Phi_i(P)$ as follows:

$$\Phi_1(P) =_{\mathrm{df}} \langle P, P, \ldots, P \rangle,$$

$$\Phi_{i+1}(P) =_{\mathrm{df}} \langle \Phi_i(P), \Phi_i(P), \ldots, \Phi_i(P) \rangle.$$

* We use lower-case bold-face letters to denote unspecified propositional variables. Thus (ii) holds for any propositional variable satisfying the conditions stated in (ii).

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LEMMA. *To each number* a $(a = 0, 1/(m-1), \ldots, 1)$ *there corresponds a number* $j(a)$ *such that if* p *takes the truth-value* a *and* $\Phi_i(p)$ *takes the truth-value* u_i, *then whenever* $i \geqslant j, u_i(u_i - 1) = 0$.

It follows from the truth-table of our primitive and the definition of the Φ_i that

$$u_{i+1} = \min(1, \alpha(1-u_i)).$$

Let us suppose that for all $i, u_i \neq 1$. Hence

$$u_{i+1} = \alpha(1-u_i),$$
$$u_{i+1} - \alpha/(\alpha+1) = \alpha(\alpha/(\alpha+1) - u_i),$$
$$u_i = \alpha/(\alpha+1) + (-\alpha)^i(a - \alpha/(\alpha+1)).$$

Since $m-1$ is not divisible by $\alpha+1$

$$a - \alpha/(\alpha+1) \neq 0.$$

Hence, since $\alpha \geqslant 2$, $|u_i|$ tends to ∞ as i tends to ∞. This is clearly impossible. Hence, for some $j, u_j = 1$. Hence $u_{j+1} = \min(1, \alpha(1-1)) = 0$

and $u_{j+2} = \min(1, \alpha(1-0)) = 1.$

It follows that $u_{j+l} = \frac{1}{2}(1 + (-1)^l).$

Thus, whenever $i \geqslant j$, $u_i(u_i - 1) = 0.$

Let the greatest value of j's needed for any of the a's be k. Then, if the truth-value of $\Phi_k(p)$ is $v(a)$, $v(v-1) = 0.$

It follows from the above that if $\Phi_{k+1}(p)$ takes the truth-value $w(a)$ then $w = 0$ when $v = 1$ and $w = 1$ when $v = 0$. Hence

$$\langle \Phi_{k+1}(p), \Phi_k(p), \Phi_k(p), \ldots, \Phi_k(p) \rangle$$

always takes the truth-value 1. Let us denote this formula by $\Psi(p)$. Then we can define negation by $\bar{P} =_{\mathrm{df}} \langle P, \Psi(P), \Psi(P), \ldots, \Psi(P) \rangle.$

We then define alternation by

$$P \vee Q =_{\mathrm{df}} \langle \bar{P}, \bar{Q}, \Psi(P), \Psi(P), \ldots, \Psi(P) \rangle.$$

Thus we can define implication by

$$P \to Q =_{\mathrm{df}} \bar{P} \vee Q.$$

Hence we can define all functions of the Łukasiewicz propositional calculus in terms of the primitive.

We now consider the $\aleph_0$-valued case. We shall show that if a function can be defined in terms of the primitives of Łukasiewicz, then this function cannot be taken as a single primitive for this system. Suppose that the function $\Phi(p_1, p_2, \ldots, p_n)$ could be so taken. Let us denote the function $\Phi(p, p, \ldots, p)$ by $\Psi(p)$.

LEMMA. *There exists* i *such that when* p *takes the truth-value* i, $\Psi(p)$ *takes the truth-value* i.

Let us denote the truth-value of $\Psi(p)$ by $\psi(x)$, where x is the truth-value of p. Suppose that, for all i, $\psi(i) \neq i$. Since $f(x)$ and $g(x, y)$ cannot take truth-values other than 0 and 1 when x and y take these truth-values, it follows that $\psi(0) = 1$ and

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$\psi(1) = 0$. Since $1-x$ and $\min(1, 1-x+y)$ are continuous if x and y are real variables, $\psi(x)$ is continuous if x is a real variable. Hence there exists a real number y such that $0 < y < 1$ and $\psi(y) = y$. We now show that y is rational.

By a theorem of McNaughton (2) there exist integers a, b such that

$$\psi(y) = a + by.$$

Hence

$$a + by = y.$$

It follows at once that y is rational. Hence y is a truth-value, contrary to our hypothesis. Thus the lemma is proved.

We now show that if $\Xi(p)$ is a propositional function of p which can be defined in terms of the primitive, then when p takes the truth-value i, where i is as in the lemma, $\Xi(p)$ takes the truth-value i. We prove this by strong induction on the number m of occurrences of p in Ξ.

If $m = 1$ the result is trivial. We now assume the result for $1, 2, \ldots, m$ and prove it for $m+1$. Then $\Xi(p)$ is of the form $\Phi(P_1, P_2, \ldots, P_n)$, where none of $P_1, P_2, \ldots, P_n$ contains more than m occurrences of p. Then by our induction hypothesis, when p takes the truth-value i, $P_1, P_2, \ldots, P_n$ all take the truth-value i. Hence $\Xi(p)$ takes the same truth-value as $\Phi(p, p, \ldots, p)$, i.e. i. Thus the result is proved.

It follows at once that we cannot define in terms of the primitive any function which takes a truth-value different from i when all propositional variables occurring in it take the truth-value i. Thus if $i = 1$ we cannot define negation since $f(1) = 0$, and if $i \neq 1$ we cannot define implication since $g(i, i) = 1$. Hence the theorem is proved.

However, if to the primitives of Łukasiewicz we add the strong equivalence function $p \equiv q$ which takes the truth-value 1 when the truth-values of p and q are the same and which takes the truth-value 0 otherwise, then we can define in terms of our primitives a function in terms of which all three primitives can be defined. After defining $\vee$ and $\&$ in terms of $\rightarrow$ and $\overline{\ \ }$ as above, we define the function $\langle p, q, r, s\rangle$ by

$$\langle P, Q, R, S\rangle =_{\mathrm{df}}$$
$$((P \rightarrow Q) \,\&\, \overline{(R \equiv S)}) \vee (\overline{P} \,\&\, (R \equiv S)) \vee ((P \equiv \overline{P}) \,\&\, (Q \equiv \overline{Q}) \,\&\, (R \equiv \overline{R}) \,\&\, (S \equiv \overline{S})).$$

We now take $\langle p, q, r, s\rangle$ as a single primitive. Then $\langle P, P, P, P\rangle$ takes the same truth-value as $\overline{P}$ except when P takes the truth-value $\frac{1}{2}$, in which case $\langle P, P, P, P\rangle$ takes the truth-value 1. Hence the function $\overline{\wedge}(P)$ defined by

$$\overline{\wedge}(P) =_{\mathrm{df}} \langle P, P, P, \langle P, P, P, P\rangle\rangle$$

always takes the truth-value 1. Hence the function $\odot(P)$ defined by

$$\odot(P) =_{\mathrm{df}} \langle \overline{\wedge}(P), P, \overline{\wedge}(P), \overline{\wedge}(P)\rangle$$

always takes the truth-value 0. Hence we can define the original three primitives by

$$P \rightarrow Q =_{\mathrm{df}} \langle P, Q, \overline{\wedge}(P), \odot(P)\rangle$$
$$\overline{P} =_{\mathrm{df}} \langle P, P, \overline{\wedge}(P), \overline{\wedge}(P)\rangle$$
$$P \equiv Q =_{\mathrm{df}} \overline{\langle \overline{\wedge}(P), \overline{\wedge}(P), P, Q\rangle}.$$

We now consider the extended $\aleph_0$-valued propositional calculus of Łukasiewicz (1). We use the same primitive as before, but $\{p\}\{P, Q, R\}$ now has the same meaning as $(p)(\overline{P} \vee \overline{Q} \vee \overline{R})$, where alternation and negation are the functions used immediately above

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and the universal quantifier has the same meaning as before. We define the logical constant 0 by

$$0 =_{\mathrm{df}} \{p\}\{p, p, p\}.$$

We define negation by

$$\bar{P} =_{\mathrm{df}} \{\mathbf{p}\}\{P, \mathbf{p}, \mathbf{p}\},$$

where $\mathbf{p}$ is a propositional variable which does not occur in P. We then define alternation by

$$P \vee Q =_{\mathrm{df}} \{\mathbf{p}\}\{\bar{P}, \bar{Q}, \mathbf{p}\},$$

where $\mathbf{p}$ is a propositional variable which does not occur in P or in Q. We can now define implication by

$$P \to Q =_{\mathrm{df}} \bar{P} \vee Q,$$

and we can define the logical constant 1 by

$$1 =_{\mathrm{df}} \bar{0}.$$

Finally, we define the universal quantifier by

$$(q)\, Q =_{\mathrm{df}} \{q\}\{\bar{Q}, 1, 1\}.$$

Thus we can define all primitive functions of the extended $\aleph_0$-valued Łukasiewicz propositional calculus in terms of the primitive.

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