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OPPOSITE-SIGN KLOOSTERMAN SUM ZETA FUNCTION

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Abstract. We study the meromorphic continuation and the spectral expansion of the opposite-sign Kloosterman sum zeta function

$$(2\pi\sqrt{mn})^{2s-1}\sum_{\ell=1}^{\infty}\frac{S(m,-n,\ell)}{\ell^{2s}}$$

for m, n positive integers, to all $s \in \mathbb{C}$. There are poles of the function corresponding to zeros of the Riemann zeta function and the spectral parameters of Maass forms. The analytic properties of this function are rather delicate. It turns out that the spectral expansion of the zeta function converges only in a left half-plane, disjoint from the region of absolute convergence of the Dirichlet series, even though they both are analytic expressions of the same meromorphic function on the entire complex plane.

§1. *Introduction.* Kloosterman sums play a central role in the spectral theory of modular forms, especially in regard to the Kuznetsov trace formula. One approach when studying Kloosterman sums and related spectral sums is to consider the Dirichlet series

$$Z_{m,n}(s) = (2\pi\sqrt{|mn|})^{2s-1}\sum_{\ell=1}^{\infty}\frac{S(m,n,\ell)}{\ell^{2s}}, \tag{1}$$

where

$$S(m,n,\ell) = \sum_{\substack{a,\bar{a} \pmod{\ell}\\ a\bar{a}\equiv 1 \pmod{\ell}}} \mathrm{e}^{2\pi\mathrm{i}(an+\bar{a}m)/\ell}$$

is the Kloosterman sum associated with the group $\mathrm{SL}(2,\mathbb{Z})$. In studying this Dirichlet series one may realize it as a Petersson inner product of two real analytic Poincaré series; this approach was first developed by Alte Selberg in [8]. The argument was also succinctly carried out in [5]. In the case where $mn > 0$, the Poincaré series involved are in L^2, and hence one may apply the spectral theory. In our case, to directly obtain the Dirichlet series we would need to use a Poincaré series of the form in [4], which is not in L^2, and hence a spectral expansion may be obtained only after great effort. One may see the strength of the L^2-theory for estimating sums of Kloosterman sums via a Dirichlet series such as (1) in [2].

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Another form of the Kuznetsov trace formula is a spectral expansion for the function

$$K(m,n,\phi)=\sum_{\ell=1}^{\infty}\frac{S(m,n,\ell)}{\ell}\phi\left(\frac{4\pi\sqrt{|mn|}}{\ell}\right),$$

with ϕ being a smooth function having sufficient decay conditions. One may relate one form to the other via the Mellin transform formula

$$K(m,n,\phi)=\frac{1}{2\pi\mathrm{i}}\int_{(1)}Z_{m,n}(s)\phi^*(s)\,ds, \tag{2}$$

where ϕ^* is related to the Mellin transform of ϕ by

$$\phi^*(s)=\int_0^{\infty}\phi(x)\left(\frac{x}{2}\right)^{-(2s-1)}\frac{dx}{x}.$$

Both forms of the Kuznetsov trace formula have been extensively studied by Motohashi (see [6] Lemma 2.5 and Theorem 2.3 for spectral expansions of either version and (2.4.6) for the necessary decay conditions on the function ϕ). However, in the case of the zeta function $Z_{m,n}(s)$, this was only done when $mn>0$. The case of m and n having opposite signs was left out. In fact, while deriving the opposite-sign Kuznetsov trace formula, Motohashi remarked in his book "Unfortunately we do not have an analogue of [the spectral expansion of Kloosterman sum zeta function] in our present situation, which would make our problem easier", while trying to employ a variant of (2) in the case $mn<0$. Instead, he obtains the Kuznetsov trace formula involving the smooth function ϕ with recourse to the function

$$\widetilde{Z}_{m,-n}(s)=(2\pi\sqrt{mn})^{2s-1}\sum_{\ell=1}^{\infty}\frac{S(m,-n;\ell)}{\ell^{2s}}\exp(-4\pi\sqrt{mn}/\ell) \tag{3}$$

and its spectral expansion. We show that we can obtain a spectral expansion of $Z_{m,-n}(s)$ using that of $\widetilde{Z}_{m.-n}(s)$.

We first introduce some notation on the spectral theory of $L^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$. Let u_j be L^2-normalized Maass forms with Laplacian eigenvalue $\lambda_j=\frac{1}{4}+t_j^2$ and let $\rho_j(n)$ be the nth Fourier coefficient of u_j, as normalized in [6, (1.1.41)]. Hence

$$u_j(z)=y^{1/2}\sum_{n\neq0}\rho_j(n)K_{\mathrm{i}t_j}(2\pi|n|y)\exp(2\pi\mathrm{i}nx).$$

Let $R\geqslant0$ be an integer. Given this normalization, the size of the Fourier coefficients are controlled by

$$\sum_{T/2\leqslant|t_j|\leqslant T}|\rho_j(n)|^2\mathrm{e}^{-\pi|t_j|}\ll T^2+m^{1/2+\varepsilon}$$

as in [6, Lemma 2.4]. Also, since Selberg's eigenvalue conjecture is valid in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ (i.e. since $\lambda_j\geqslant\frac{1}{4}$), we have all $t_j\in\mathbb{R}$. Finally, by Weyl's law we

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have that $\#\{t_j : t_j \leqslant T\} \sim cT^2$ for the number of eigenvalues less than a given quantity, counted with multiplicity.

The main goal of this paper is to prove the following theorem.

THEOREM 1. *Let $m, n > 0$. The function $Z_{m,-n}(s)$ defined in (1) has a meromorphic continuation to all $s \in \mathbb{C}$.*

For $s \in \mathbb{C}$ such that $\mathrm{Re}(s) < 0$ we have the spectral expansion

$$Z_{m,-n}(s) = Z_d(s) + Z_c(s).$$

Here

$$Z_d(s) = \frac{1}{2}\sum_{j=1}^{\infty} \overline{\rho_j(m)}\rho_j(-n)\Gamma\left(s - \frac{1}{2} + \mathrm{i}t_j\right)\Gamma\left(s - \frac{1}{2} - \mathrm{i}t_j\right)$$

is the discrete spectrum and, if $-R - \frac{1}{2} < \mathrm{Re}(s) < -R + \frac{1}{2}$,

$$Z_c(s) = I(s) + \sum_{r=0}^{R} R_r^-(s)$$

is the continuous spectrum. The continuous spectrum consists of the integral

$$I(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sigma_{2\mathrm{i}t}(m)\sigma_{2\mathrm{i}t}(n)}{(mn)^{\mathrm{i}t}|\zeta(1+2\mathrm{i}t)|^2}\cosh(\pi t)\Gamma\left(s - \frac{1}{2} + \mathrm{i}t\right)\Gamma\left(s - \frac{1}{2} - \mathrm{i}t\right)dt,$$

and the additional terms obtained as residues when analytically continuing the integral expression $I(s)$ successively into the strips $-r - \frac{1}{2} < \mathrm{Re}(s) < -r + \frac{1}{2}$, are defined as

$$R_r(s) = 2\frac{(-1)^r}{r!}\Gamma(2s + r - 1)\frac{\sigma_{2s+2r-1}(m)\sigma_{2s+2r-1}(n)}{(mn)^{s+r-1/2}\zeta^*(2s+2r)\zeta^*(2-2s-2r)}.$$

In the above notation, the values $m, -n$ have been suppressed. We may write $Z_d = Z^d_{m,-n}$ if we want to make the dependence explicit. We may also write $Z_d = Z_d^-$ or $R_r = R_r^-$ to indicate that the indices m and $-n$ are of opposite sign.

If $\mathrm{Re}(s) = -R + \frac{1}{2}$, then the continuous spectrum changes slightly and

$$Z_{m,-n}(s) = Z_d(s) + I(s, \mathcal{C}) + {\sum_{r=1}^{R}}' R_r(s).$$

Here $\sum'$ indicates that the $r = R$ term in the sum is halved and, given any contour of integration Ω from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$,

$$I(s, \Omega) = \frac{1}{2\pi\mathrm{i}}\int_{\Omega} \frac{\sigma_{2u}(m)\sigma_{2u}(n)\cos(\pi u)}{(mn)^u\zeta(1+2u)\zeta(1-2u)}\Gamma\left(s - \frac{1}{2} + u\right)\Gamma\left(s - \frac{1}{2} - u\right)du,$$

as long as the contour does not contain any poles of the integrand. The contour $\mathcal{C}$ is given as $\mathrm{Re}(u) = 1/(20\log|\mathrm{Im}(u)+2|)$ and, finally, $\zeta^*(s) = \zeta(s)\Gamma(s/2)\pi^{-s/2}$ is the completed Riemann zeta function.

Note that if (0) is the path $\mathrm{Re}(s) = 0$, then $I(s, (0)) = I(s)$.

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The sum over the discrete spectrum in $Z_d(s)$ converges absolutely for $\mathrm{Re}(s) < 0$, and the integral in the continuous spectrum converges for $\mathrm{Re}(s) < \frac{1}{2}$. This does not overlap with the region of absolute convergence of the Dirichlet series (1), and therefore any attempts to prove such a spectral expansion by trying to insert a "correct" test function into the formula [**6**, (2.5.6)] would fail, because that would have found a spectral expansion for the Dirichlet series and not its analytic continuation. This is the gist of the subtleties in dealing with $Z_{m,-n}(s)$, and is likely to be the reason that it is left out of the literature.

This phenomenon of a Dirichlet series converging on a right half-plane and its spectral expansion converging slowly on a disjoint left half-plane was also encountered in [**4**] with a shifted convolution series $\sum_{n=1}^{\infty} a(n)\overline{b(n+h)}n^{-s}$, where $a(n), b(n)$ are Fourier coefficients of modular forms. These two occurrences are related; the Fourier coefficients of the Poincaré series constructed by Jeffrey Hoffstein in order to obtain this Dirichlet series contains the Kloosterman sums $S(m, -h, c)$.

Although the two regions are disconnected, there is a separate expression, involving the auxiliary function $\widetilde{Z}_{m,-n}(s)$, which ensures meromorphic continuation of the opposite-sign Kloosterman sum zeta function to the entire complex plane. See the end of §2. We then prove the spectral expansion of $Z_{m,-n}(s)$ in §3.

The expression for the discrete spectrum suggests that there should be poles at $s = \frac{1}{2} + it_j$ arising from the gamma factors in $Z_d(s)$, as well as poles of $R_0^-(s)$ at $s = \rho/2$, where ρ is a critical zero of the Riemann zeta function. Unfortunately, these poles are not in the region of convergence of the spectral expansion, nor in the region of absolute convergence of the Dirichlet series expression. Yet, in §4, we show that there are poles where there should be and explicitly write the residues. The residues at these poles equal the apparent residues of the spectral expansion of $Z_{m,-n}(s)$.

In any application, it will be useful to have a bound of $Z_{m,-n}(s)$ in terms of its mn and $\mathrm{Im}(s)$ dependence. We establish an estimation in §5.

As an easy application of the analytic properties of $Z_{m,-n}(s)$, we find a bound for the sum of opposite-sign Kloosterman sums, which is uniform in all the parameters.

THEOREM 2. *Let $m, n > 0$, $\varepsilon > 0$ and $X \gg 1$ a large quantity. One has the bound*

$$\sum_{\ell \leqslant X} \frac{S(m, -n, \ell)}{\ell} \ll_{\varepsilon} X^{1/6+\varepsilon}((m,n)^{\varepsilon} + (mn)^{\theta}) + X^{\varepsilon}(mn)^{1/4+\varepsilon},$$

where the implied constant only depends on $\varepsilon > 0$, and θ is the best progress towards the Ramanujan–Petersson conjecture. We may take $\theta = 7/64$ according to the Kim–Sarnak result.

This sum has been studied in [**7**] for the case $mn > 0$. Sarnak and Tsimerman obtained strong estimates for this sum using a careful study of uniform asymptotic expansions of the Bessel functions which arise in the

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Kuznetsov trace formula. Possibly, their methods could be adapted to obtain a better bound in the opposite-sign case as well.

Let us also record the spectral expansion of the same-sign Kloosterman sum zeta function. The following spectral expansion on the Kloosterman sum zeta function is due to Kuznetsov and a reworking of it can be found in Motohashi's paper [5]. We also include the expression of the meromorphic continuation to all of $\mathbb{C}$, for completeness.

THEOREM 3 (Kuznetsov). *For* $\mathrm{Re}(s) > \frac{1}{2}$ *and for any positive integer* m *and* n, *one has the spectral expansion*

$$\begin{aligned} Z_{m,n}(s) &= \frac{1}{2}\sin(\pi s)\sum_{j=1}^{\infty}\frac{\rho_j(n)\rho_j(m)}{\cosh(\pi t_j)}\Gamma\left(s-\frac{1}{2}+\mathrm{i}t_j\right)\Gamma\left(s-\frac{1}{2}-\mathrm{i}t_j\right)\\ &\quad+\frac{1}{2\pi}\sin(\pi s)\int_{-\infty}^{\infty}\frac{\sigma_{2\mathrm{i}t}(m)\sigma_{2\mathrm{i}t}(n)}{(mn)^{\mathrm{i}t}|\zeta(1+2\mathrm{i}t)|^2}\\ &\quad\times\Gamma\left(s-\frac{1}{2}+\mathrm{i}t\right)\Gamma\left(s-\frac{1}{2}-\mathrm{i}t\right)dt\\ &\quad+\sum_{k=1}^{\infty}p_{m,n}(k)\frac{\Gamma(k-1+s)}{\Gamma(k+1-s)}-\frac{1}{2\pi}\delta_{m,n}\frac{\Gamma(s)}{\Gamma(1-s)}, \end{aligned}$$

where

$$\begin{aligned} p_{m,n}(k) &= (2k-1)\sum_{\ell=1}^{\infty}\frac{S(m,n;\ell)}{\ell}J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{\ell}\right)\\ &= (-1)^k\frac{\Gamma(2k)}{(4\pi\sqrt{mn})^{2k-1}}\sum_{f\in\mathcal{F}_{2k}}\frac{a_f(m)a_f(n)}{\langle f,f\rangle}-\frac{(-1)^k}{2\pi}\delta_{m,n}. \end{aligned}$$

Here, $\mathcal{F}_{2k}$ *is an orthogonal basis of cusp forms of weight* $2k$ *on* $\mathrm{SL}(2,\mathbb{Z})$, *and* $a_f(n)$ *is the* n*th Fourier coefficient of* f. *In order to refer to this symbol easily, let us denote the first second and third lines of the above formula by* $Z_d^+(s) = Z_{m,n}^d(s)$, $Z_c^+(s) = Z_{m,n}^+(s)$ *and* $Z_{hr}(s) = Z_{m,n}^{hr}(s)$, *respectively.*

This spectral expansion has a meromorphic continuation to all $s\in\mathbb{C}$. *However, the spectral expansion is not given by the same formula as above. The integral has an integrand which has a pole if* $\mathrm{Re}(s) = \frac{1}{2} - r$ *for* $r\in\mathbb{N}$. *We need to add correctional terms in order to obtain the meromorphic continuation. If* $R\geqslant 0$ *is an integer and* s *lies in the region* $R-\frac{1}{2} < \mathrm{Re}(s) < R+\frac{1}{2}$, *then additional terms* $\sum_{r=0}^{R}R_r^+(s)$ *are included, where*

$$R_r^+(s) = (-1)^r R_r^-(s).$$

Finally, in the case $\mathrm{Re}(s) = \frac{1}{2} - R$ *for some positive integer* R, *the integral in the "continuous spectrum" is replaced by the path integral*

$$\frac{\sin(\pi s)}{2\pi\mathrm{i}}\int_{\mathcal{C}}\frac{\sigma_{2u}(m)\sigma_{2u}(n)}{(mn)^u\zeta(1+2u)\zeta(1-2u)}\Gamma\left(s-\frac{1}{2}+u\right)\Gamma\left(s-\frac{1}{2}-u\right)dt,$$

where the contour $\mathcal{C}$ *is as above and we add half of the* $r = R$ *term.*

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In obtaining the additional terms R_r^+, we have used the formula

$$\sin(\pi s) = (-1)^r \sin(\pi(s+r)) = \frac{(-1)^r \pi}{\Gamma(s+r)\Gamma(1-s-r)},$$

changing the Riemann zeta functions in the denominator to completed ones.

§2. *Expressing $Z_{m,-n}(s)$ in terms of $\widetilde{Z}_{m,-n}(s)$ and* vice versa. The function $\widetilde{Z}_{m,-n}(s)$ contains all the necessary information about the unmodified Kloosterman sum zeta function $Z_{m,n}(s)$. In fact, we are going to extract the latter from a summation of the former. Let us express the exponential factor as a Taylor series around 0.

PROPOSITION 4. *For* $\mathrm{Re}(s) > 3/4$, *we have the equality*

$$\widetilde{Z}_{m,-n}(s) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} Z_{m,-n}(s + k/2).$$

Proof. Expand the exponential function in a Taylor series and exchange orders of summation.

$$\begin{aligned}
\widetilde{Z}_{m,-n}(s) &= (2\pi\sqrt{mn})^{2s-1} \sum_{\ell=1}^{\infty} \frac{S(m,-n;\ell)}{\ell^{2s}} \exp(-4\pi\sqrt{mn}/\ell) \\
&= (2\pi\sqrt{mn})^{2s-1} \sum_{\ell=1}^{\infty} \frac{S(m,-n;\ell)}{\ell^{2s}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{4\pi\sqrt{mn}}{\ell}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} (2\pi\sqrt{mn})^{2s+k-1} \sum_{\ell=1}^{\infty} \frac{S(m,-n;\ell)}{\ell^{2s+k}} \\
&= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} Z_{m,-n}\left(s + \frac{k}{2}\right).
\end{aligned}$$

□

Now let us write $\widetilde{Z}_{m,-n}(s + k/2)$ in terms of the $Z_{m,-n}$. We find a linear combination of the former that yields $Z_{m,-n}(s)$.

$$\begin{aligned}
\widetilde{Z}_{m,-n}(s) &= Z_{m,-n}(s) - \frac{2^1}{1!} Z_{m,-n}\left(s + \frac{1}{2}\right) + \frac{2^2}{2!} Z_{m,-n}(s+1) - \cdots \\
\widetilde{Z}_{m,-n}\left(s + \frac{1}{2}\right) &= Z_{m,-n}\left(s + \frac{1}{2}\right) - \frac{2^1}{1!} Z_{m,-n}(s+1) + \cdots \\
\widetilde{Z}_{m,-n}(s+1) &= Z_{m,-n}(s+1) - \cdots .
\end{aligned}$$

Notice that the combination $\widetilde{Z}_{m,-n}(s) + 2\widetilde{Z}_{m,-n}(s + \frac{1}{2})$ cancels the second column. The third column is also eliminated with the linear combination $\widetilde{Z}_{m,-n}(s) + 2\widetilde{Z}_{m,-n}(s + \frac{1}{2}) + 2^2/2!\widetilde{Z}_{m,-n}(s+1)$.

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PROPOSITION 5. *For* $\mathrm{Re}(s) > 3/4$, *we have the equality*

$$Z_{m,-n}(s) = \sum_{k=0}^{\infty} \frac{2^k}{k!} \widetilde{Z}_{m,-n}\left(s + \frac{k}{2}\right). \tag{4}$$

Proof. We express the sum (4) in terms of the columns (as we have written above) in their expansion of the $Z_{m,n}$ functions

$$\sum_{i=0}^{\infty} \frac{2^i}{i!} \widetilde{Z}_{m,-n}\left(s + \frac{i}{2}\right) = Z_{m,-n}(s) + \sum_{k=1}^{\infty} Z_{m,n}\left(s + \frac{k}{2}\right) \sum_{i+j=k} \frac{(2)^i}{i!} \frac{(-2)^j}{j!}.$$

The innermost summation simply vanishes, so

$$2^k \sum_{i+j=k} \frac{(-1)^i}{i!j!} = \frac{2^k}{k!} \sum_{i+j=k} \frac{k!}{i!j!} (-1)^i = \frac{2^k}{k!} (1-1)^k = 0.$$

This yields the formula. □

We have expressed the function $Z_{m,-n}(s)$ as an infinite linear combination of $\widetilde{Z}_{m,-n}(s)$. Motohashi, in [**6**], gives a spectral expansion of this latter modified Kloosterman sum zeta function for $\mathrm{Re}(s) > \frac{3}{4}$, as

$$\begin{aligned} \widetilde{Z}_{m,-n}(s) = &\sum_{j=1}^{\infty} \overline{\rho_j(m)} \rho_j(-n) 2^{3-4s} \sqrt{\pi} \frac{\Gamma(2s-1+2it_j)\Gamma(2s-1-2it_j)}{\Gamma(2s-1/2)} \\ &+ \frac{2^{3-4s}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sigma_{2it}(m)\sigma_{2it}(n)\cosh(\pi t)}{(mn)^{it}|\zeta(1+2it)|^2} \\ &\times \frac{\Gamma(2s-1+2it)\Gamma(2s-1-2it)}{\Gamma(2s-1/2)}\, dt. \end{aligned} \tag{5}$$

This spectral expansion gives an analytic continuation of $\widetilde{Z}_{m,-n}(s)$, initially to $\mathrm{Re}(s) > \frac{1}{2}$, and then a meromorphic continuation to all $s \in \mathbb{C}$. Notice that, although the infinite summation and the integral converges for all $s \in \mathbb{C}$ not on a pole of the summands or the integrand, this does not mean that the analytic continuation of $\widetilde{Z}_{m,-n}(s)$ is given by the same formula as in (5). The poles of the integrand of the continuous spectrum separate the regions in which s is allowed to be. So the analytic continuation of the expression will be given by another formula. That is done in Theorem 6 below.

Before that, let us see that (4) gives us a meromorphic continuation to all $s \in \mathbb{C}$.

Proof of meromorphic continuation of $Z_{m,-n}(s)$. For $\mathrm{Re}(s) > 1$, the function $\widetilde{Z}_{m,-n}(s)$ is given by its Dirichlet series (3), and we can bound it by

$$(2\pi\sqrt{mn})^{2\sigma-1} \zeta(\tfrac{3}{2}).$$

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Therefore the summation

$$\sum_{k=0}^{\infty} \frac{2^k}{k!} \widetilde{Z}_{m,-n}\left(s + \frac{k}{2}\right)$$

converges absolutely. Except for a finite number of terms, the terms of the summation are majorized by $\sum_{k \text{ s.t. } \mathrm{Re}(s+k/2)>1} ((2\sqrt{mn})^k/k!) < \infty$. In fact, this shows that the convergence is uniform in compact sets of $s \in \mathbb{C}$ away from the poles of $\widetilde{Z}_{m,-n}(s + k/2)$.

Furthermore, for $\mathrm{Re}(s) > 3/4$ (i.e. in the region of absolute convergence of these Dirichlet series), we have equality. So the infinite summation over k gives a meromorphic continuation of $Z_{m,-n}(s)$ to all s. □

§3. *Spectral expansions.* The above proof used that $\widetilde{Z}_{m,-n}$ had meromorphic continuation to all $s \in \mathbb{C}$. Below, we write the expression giving the function in various domains.

THEOREM 6. *Let*

$$\widetilde{Z}_d(s) = \sum_{j=1}^{\infty} \overline{\rho_j(m)}\rho_j(-n)2^{3-4s}\sqrt{\pi}\frac{\Gamma(2s-1+2it_j)\Gamma(2s-1-2it_j)}{\Gamma(2s-1/2)},$$

and

$$\widetilde{I}(s,\Omega) = \frac{2^{4-4s}\sqrt{\pi}}{2\pi i}\int_{\Omega} \frac{\sigma_{2u}(m)\sigma_{2u}(n)\cos(\pi u)}{(mn)^u\zeta(1+2u)\zeta(1-2u)} \times \frac{\Gamma(2s-1+2u)\Gamma(2s-1-2u)}{\Gamma(2s-1/2)}\,du,$$

where Ω is a path from $-i\infty$ to $i\infty$ on $\mathbb{C}$. Let (0) be the vertical path $\mathrm{Re}(u) = 0$, and $\mathcal{C}$ be the path $\mathrm{Re}(u) = 1/(20\log|\mathrm{Im}(u)+2|)$. This region is chosen so that there are no zeros of $\zeta(1-2u)$ between the curves (0) and $\mathcal{C}$. Let us also call, for brevity, $\widetilde{I}(s,(0)) = \widetilde{I}(s)$.

The spectral expansion of $\widetilde{Z}_{m,-n}(s)$ in the region $\mathrm{Re}(s) \geqslant \frac{1}{2}$ is given by

$$\widetilde{Z}_{m,-n}(s) = \widetilde{Z}_d(s) + \widetilde{I}(s,(0)).$$

Let us call

$$\widetilde{R}_r(s) := \frac{(-1)^r}{r!}\frac{2^{3-4s}\sqrt{\pi}\sigma_{2s-1+r}(m)\sigma_{2s-1+r}(n)}{(mn)^{s-1/2+r/2}\zeta^*(2s+r)\zeta^*(2-2s-r)}\frac{\Gamma(4s-2+r)}{\Gamma(2s-1/2)}. \quad (6)$$

Then for s, such that $-R/2 < \mathrm{Re}(s) < -R/2 + \frac{1}{2}$,

$$\widetilde{Z}_{m,-n}(s) = \widetilde{Z}_d(s) + \widetilde{I}(s) + \sum_{r=0}^{R} 2\widetilde{R}_r(s). \quad (7)$$

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If $\mathrm{Re}(s) = -R/2 + \frac{1}{2}$,

$$\widetilde{Z}_{m,-n}(s) = \widetilde{Z}_d(s) + \widetilde{I}(s,\mathcal{C}) + \sum_{r=0}^{R-1} 2\widetilde{R}_r(s) + \widetilde{R}_R(s). \tag{8}$$

The terms other than the discrete spectrum will be denoted by $\widetilde{Z}_c(s)$ *and are called the continuous part of* $\widetilde{Z}_{m,-n}(s)$.

Proof. The spectral expansion (5), given in [6], for $\widetilde{Z}_{m,-n}(s)$ actually converges in $\mathrm{Re}(s) > \frac{1}{2}$. In fact, the sum converges as long as s does not equal one of $r/2 + \frac{1}{2} \pm it_j$ for a non-negative integer r and spectral parameter t_j, and the integral converges as long as s does not lie on the lines $\mathrm{Re}(s) = \frac{1}{2} - r/2$, $r \geqslant 0$.

However, these regions of s are separated by the poles of the integrand as u varies in $i\mathbb{R}$, and hence the same expression $\widetilde{Z}_d(s) + \widetilde{I}(s)$ is not the analytic continuation of itself for $\mathrm{Re}(s) > \frac{1}{2}$ and $0 < \mathrm{Re}(s) < \frac{1}{2}$.

Now let $r \geqslant 0$, and s lie between the curves $(-r/2 + \frac{1}{2})$ and $\mathcal{C} - r/2 + \frac{1}{2}$. We may express

$$\begin{aligned}\widetilde{I}(s) &= \frac{(-1)^r}{r!} \frac{2^{3-4s}\sqrt{\pi}\sigma_{2s-1+r}(m)\sigma_{2s-1+r}(n)\cos(\pi(s-1/2+r/2))}{(mn)^{s-1/2+r/2}\zeta(2s+r)\zeta(2-2s-r)} \\ &\quad \times \frac{\Gamma(4s-2+r)}{\Gamma(2s-1/2)} + \widetilde{I}(s,\mathcal{C}) \\ &= \frac{(-1)^r}{r!} \frac{2^{3-4s}\sqrt{\pi}\sigma_{2s-1+r}(m)\sigma_{2s-1+r}(n)}{(mn)^{s-1/2+r/2}\zeta^*(2s+r)\zeta^*(2-2s-r)} \frac{\Gamma(4s-2+r)}{\Gamma(2s-1/2)} \\ &\quad + \widetilde{I}(s,\mathcal{C}),\end{aligned}$$

where, in the last line, we have turned the cosine function into sine and that into a product of reciprocal gamma functions. Notice that $\widetilde{I}(s,\mathcal{C})$ is an analytic function in the region where s is between $-\mathcal{C} - r/2 + \frac{1}{2}$ and $\mathcal{C} - r/2 + \frac{1}{2}$. Thus, we get an analytic continuation of $\widetilde{I}(s)$ into this domain, which includes the line $\mathrm{Re}(s) = -r/2 + \frac{1}{2}$. Now if we assume that s is to the left of that line, when we move the line of integration back to (0), we pass over the pole at $2s-1+2u = -r$, and the residue is the same, namely

$$\begin{aligned}\widetilde{I}(s) &= \frac{(-1)^r}{r!} \frac{2^{3-4s}\sqrt{\pi}\sigma_{2s-1+r}(m)\sigma_{2s-1+r}(n)}{(mn)^{s-1/2+r/2}\zeta^*(2s+r)\zeta^*(2-2s-r)} \frac{\Gamma(4s-2+r)}{\Gamma(2s-1/2)} \\ &\quad + \widetilde{I}(s,\mathcal{C}) \\ &= 2\frac{(-1)^r}{r!} \frac{2^{3-4s}\sqrt{\pi}\sigma_{2s-1+r}(m)\sigma_{2s-1+r}(n)}{(mn)^{s-1/2+r/2}\zeta^*(2s+r)\zeta^*(2-2s-r)} \frac{\Gamma(4s-2+r)}{\Gamma(2s-1/2)} \\ &\quad + \widetilde{I}(s).\end{aligned}$$

□

We are going to plug this spectral expansion into (4) and obtain a proof of Theorem 1.

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Proof of Theorem 1. We take the infinite linear combination (4) of $\widetilde{Z}_{m,-n}(s+k/2)$ expressing the function $Z_{m,-n}(s)$. This summation converges for all $s \in \mathbb{C}$ except for when $s+k/2$ lies on a pole of $\widetilde{Z}_{m,n}$ for some k. For each $\widetilde{Z}_{m,-n}(s+k/2)$ plug in the spectral expansion in Theorem 6. This will be a spectral expansion of $Z_{m,-n}(s)$. As it will turn out, although (4) converges for all $s \in \mathbb{C}$, the spectral expansion will only absolutely converge for $\mathrm{Re}(s) < 0$.

Assume that r is a non-negative integer and $-R/2 < \mathrm{Re}(s) < -R/2 + \frac{1}{2}$. Then

$$\begin{aligned} Z_{m,-n}(s) &= \sum_{k=0}^{\infty} \frac{2^k}{k!} \widetilde{Z}_{m,-n}\left(s+\frac{k}{2}\right) \\ &= \sum_{j=1}^{\infty} \overline{\rho_j(m)}\rho_j(-n) 2^{3-4s}\sqrt{\pi} \\ &\quad \times \sum_{k=0}^{\infty} \frac{2^{-k}}{k!} \frac{\Gamma(2s+k-1+2it_j)\Gamma(2s+k-1-2it_j)}{\Gamma(2s+k-1/2)} \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_{2it}(m)\sigma_{2it}(n)}{(mn)^{it}|\zeta(1+2it)|^2} 2^{3-4s}\sqrt{\pi}\cosh(\pi t) \\ &\quad \times \sum_{k=0}^{\infty} \frac{2^{-k}}{k!} \frac{\Gamma(2s+k-1+2it)\Gamma(2s+k-1-2it)}{\Gamma(2s+k-1/2)}\, dt \\ &\quad + \sum_{k=0}^{R}\sum_{j=0}^{R-k} \frac{2^k}{k!} \widetilde{R}_j\left(s+\frac{k}{2}\right). \end{aligned}$$

The summation over k in the discrete and continuous spectra can be explicitly evaluated. Denote the sum over these Gamma functions by $G(s)$, where

$$\begin{aligned} G(s) &= \sum_{k=0}^{\infty} \frac{2^{-k}}{k!} \frac{\Gamma(2s+k-1+2it)\Gamma(2s+k-1-2it)}{\Gamma(2s+k-1/2)} \\ &= \frac{\Gamma(2s-1+2it)\Gamma(2s-1-2it)}{\Gamma(2s-1/2)} {}_2F_1 \\ &\quad \times \left(2s-1+2it, 2s-1-2it, 2s-\frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

The Gauss hypergeometric function appears because the summation over k is exactly its Taylor series expansion. Here we have used the formula

$$ {}_2F_1\left(a, b; \frac{a+b}{2}+\frac{1}{2}; \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma((a+b)/2+1/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)} \tag{9}$$

to evaluate the Gauss hypergeometric function at $1/2$, which can be found at [3], # 9.136.1. Legendre's duplication formula of the gamma function simplifies this expression to give

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$$G(s) = \frac{\Gamma(2s-1+2it)\Gamma(2s-1-2it)}{\Gamma(2s-1/2)}\sqrt{\pi}\frac{\Gamma(2s-1/2)}{\Gamma(s+it)\Gamma(s-it)}$$
$$= \frac{2^{4s-4}}{\sqrt{\pi}}\Gamma\left(s-\frac{1}{2}+it\right)\Gamma\left(s-\frac{1}{2}-it\right).$$

Having established the summation over the discrete spectrum and the integral in the continuous spectrum, we now turn our attention to the additional residue terms that show themselves when studying the continuous spectrum with $\mathrm{Re}(s) < \frac{1}{2}$. We will make use of the following lemma.

LEMMA 7. *For $\widetilde{R}_j(s)$ defined as in (6),*

$$\widetilde{R}_j(s) = \frac{(-1)^j}{j!}4^j\frac{(2s-1/2)_j}{(4s-2+j)_j}\widetilde{R}_0\left(s+\frac{j}{2}\right),$$

where $(z)_n = \Gamma(z+n)/\Gamma(z)$ is the Pochhammer symbol, or the rising factorial.

Proof. We simply realize the shift in s caused by the j index in the definition of $\widetilde{R}_j(s)$,

$$\begin{aligned}\widetilde{R}_j(s) &= \frac{(-1)^j}{j!}\frac{2^{3-4s}\sigma_{2s-1+j}(m)\sigma_{2s-1+j}(n)}{(mn)^{s-1/2+j/2}\zeta^*(2s+j)\zeta^*(2-2s-j)}\frac{\Gamma(4s-2+j)}{\Gamma(2s-1/2)}\\
&= \frac{(-1)^j}{j!}2^{2j}\frac{\Gamma(4s-2+j)}{\Gamma(4s-2+2j)}\frac{\Gamma(2s-1/2+j)}{\Gamma(2s-1/2)}\\
&\quad\times\frac{2^{3-4(s+j/2)}\sigma_{2(s+j/2)-1}(m)\sigma_{2(s+j/2)-1}(n)}{(mn)^{(s+j/2)-1/2}\zeta^*(2(s+j/2))\zeta^*(2-2(s+j/2))}\\
&\quad\times\frac{\Gamma(4(s+j/2)-2)}{\Gamma(2(s+j/2)-1/2)}\\
&= \frac{(-1)^j}{j!}4^j\frac{(2s-1/2)_j}{(4s-2+j)_j}\widetilde{R}_0\left(s+\frac{j}{2}\right).\end{aligned}$$

□

We also record the following equality, for convenience,

$$\widetilde{R}_j\left(s+\frac{k}{2}\right) = \frac{(-1)^j}{j!}4^j\frac{(2s+k-1/2)_j}{(4s+2k-2+j)_j}\widetilde{R}_0\left(s+\frac{j+k}{2}\right). \tag{10}$$

Call $r = j+k$ and sum over k and j to obtain

$$\sum_{k=0}^{R}\frac{2^k}{k!}\sum_{j=0}^{R-k}\widetilde{R}_j\left(s+\frac{k}{2}\right) = \sum_{r=0}^{R}\widetilde{R}_0\left(s+\frac{r}{2}\right)\sum_{k+j=r}\frac{2^k}{k!}\frac{(-1)^j}{j!}4^j\frac{(2s+k-1/2)_j}{(4s+2k-2+j)_j}.$$

Denote each summand of the sum over r by $R_{r/2}(s)$, which is the analogue of $\widetilde{R}_r(s)$ in $Z_{m,-n}(s)$. Now we look at the inner sum (i.e. the coefficient of $\widetilde{R}_0(s+r/2)$ in this sum). Call it G_r, where

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$$\begin{aligned}G_r &= \sum_{k+j=r} \frac{2^k}{k!}\frac{(-1)^j 4^j}{j!}\frac{(2s+k-1/2)_j}{(4s+2k-2+j)_j}\\ &= \sum_{k+j=r} \frac{2^k}{k!}\frac{(-4)^j}{j!}\frac{\Gamma(2s+k-1/2+j)}{\Gamma(2s+k-1/2)}\frac{\Gamma(4s+2k-2+j)}{\Gamma(4s+2k-2+2j)}\\ &\quad\times\frac{\Gamma(2s+k-1+j)}{\Gamma(2s+k-1+j)}\\ &= \frac{1}{\Gamma(2s-1+r)}\sum_{k+j=r}\frac{2^k}{k!}\frac{(-4)^{r-k}}{(r-k)!}\sqrt{\pi}2^{3-4s-2r}\frac{\Gamma(4s+2k-2+j)}{\Gamma(2s+k-1/2)}.\end{aligned}$$

Here we have used the Legendre duplication formula. The sum over k gives a polynomial which can be viewed as a hypergeometric function

$$\begin{aligned}G_r &= \frac{(-1)^r\sqrt{\pi}2^{3-4s}}{r!\Gamma(2s-1+r)}\sum_{k=0}^{r}\frac{2^{-k}(-1)^k r!}{k!(r-k)!}\frac{\Gamma(4s-2+r+k)}{\Gamma(2s+k-1/2)}\\ &= \frac{(-1)^r\sqrt{\pi}2^{3-4s}}{r!\Gamma(2s-1+r)}\frac{\Gamma(4s-2+r)}{\Gamma(2s-1/2)}\sum_{k=0}^{r}\frac{(-r)_k(4s-2+r)_k}{(2s-1/2)_k}\frac{2^{-k}}{k!}\\ &= \frac{(-1)^r\sqrt{\pi}2^{3-4s}}{r!\Gamma(2s-1+r)}\frac{\Gamma(4s-2+r)}{\Gamma(2s-1/2)}{}_2F_1\left(-r,4s-2+r;2s-\frac{1}{2};\frac{1}{2}\right).\end{aligned}$$

We use (9) once again to evaluate the hypergeometric function. Since the hypergeometric function vanishes for odd r, we put $r=2r'$. Thus

$$\begin{aligned}\sum_{r=0}^{R}\widetilde{R}_{r/2}(s) &= \sum_{r=0}^{R}\widetilde{R}_0\left(s+\frac{r}{2}\right)\frac{(-1)^r\sqrt{\pi}2^{3-4s}}{r!\Gamma(2s-1+r)}\\ &\quad\times\frac{\Gamma(4s-2+r)}{\Gamma(2s-1/2)}\frac{\sqrt{\pi}\Gamma(2s-1/2)}{\Gamma((1-r)/2)\Gamma(2s-1/2+r/2)}\\ &= \sum_{r=0}^{R}\widetilde{R}_0\left(s+\frac{r}{2}\right)\frac{(-1)^r\sqrt{\pi}2^r\Gamma(2s-1+r/2)}{r!\Gamma(2s-1+r)\Gamma((1-r)/2)}\\ &= \sum_{r'=1}^{\lfloor R/2\rfloor}\frac{(-1)^{r'}}{(r')!}\frac{\Gamma(2s-1+r')}{\Gamma(2s-1+2r')}\widetilde{R}_0(s+r').\end{aligned}$$

Finally, substituting the expression for $\widetilde{R}_0$,

$$\begin{aligned}\sum_{r=0}^{\lfloor R/2\rfloor}\widetilde{R}_r(s) &= \sum_{r=0}^{\lfloor R/2\rfloor}\frac{(-1)^r}{r!}\frac{\Gamma(2s-1+r)}{\Gamma(2s-1+2r)}\frac{\Gamma(4(s+r)-2)}{\Gamma(2(s+r)-1/2)}\\ &\quad\times\frac{2^{3-4(s+r)}\sqrt{\pi}\sigma_{2(s+r)-1}(m)\sigma_{2(s+r)-1}(n)}{(mn)^{(s+r)-1/2}\zeta^*(2(s+r))\zeta^*(2-2(s+r))}\end{aligned}$$

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$$= \sum_{r=0}^{\lfloor R/2 \rfloor} \frac{(-1)^r}{r!} \Gamma(2s+r-1) \times \frac{\sigma_{2s+2r-1}(m)\sigma_{2s+2r-1}(n)}{(mn)^{s+r-1/2}\zeta^*(2s+2r)\zeta^*(2-2s-2r)}. \tag{11}$$

Since the odd terms vanish, we see that the only additional terms come when we pass $\mathrm{Re}(s) = -R + \frac{1}{2}$ for a non-negative integer R. If $\mathrm{Re}(s) = -R + \frac{1}{2}$, then the treatment is identical, except that (7) is replaced with (8). □

§4. *Poles and residues.* The opposite-sign Kloosterman sum zeta function has poles in the region $0 < \mathrm{Re}(s) < 1$, where the discrete part of the spectral expansion does not converge. Yet the expression (4) in Proposition 5 allows us to identify those poles as well.

PROPOSITION 8. *For $m, n > 0$, the function $Z_{m,-n}(s)$ is analytic in the region $\mathrm{Re}(s) > \frac{1}{2}$, has simple poles at $s = \frac{1}{2} + \mathrm{i}t_j$ where $\frac{1}{4} + t_j^2$ is the jth Laplace eigenvalue on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ (here we take both positive and negative values for t_j), and poles at $\rho/2$ where $\zeta(\rho) = 0$. The order of the pole is determined by the order of the zero of the Riemann zeta function.*

Consider an orthonormal basis for the space of Maass forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$. We may, without loss of generality, choose the basis to also be an eigenfunction of the Hecke operators, with the nth Hecke eigenvalue being $\lambda_u(n)$. The residues of the poles at $s = \frac{1}{2} + \mathrm{i}t_j$ are given as a sum over this Hecke-basis as

$$\operatorname*{Res}_{s=1/2+\mathrm{i}t_j} Z_{m,-n}(s) = \frac{1}{2} \sum_{u:\Delta u = \lambda_j u} \lambda_u(m)\lambda_u(n)\rho_u(1)\overline{\rho_u(-1)}\Gamma(2\mathrm{i}t_j).$$

The space of Maass forms corresponding to a fixed eigenvalue is generally considered (although not yet proven) to be one-dimensional, reducing the sum to a single term.

For the poles coming from the continuous contribution in the spectral expansion it would be caused by $\zeta^*(2s)$ in the denominator: i.e. the function

$$Z_{m,-n}(s) - \frac{2}{\zeta^*(2s)} \left(\frac{\sigma_{2s-1}(m)\sigma_{2s-1}(n)}{(mn)^{s-1/2}\zeta^*(2s-1)} \Gamma(2s-1) \right)$$

is regular at $s = \rho/2$. Notice that the expression inside the parentheses is a holomorphic function of s in $0 \leqslant \mathrm{Re}(s) \leqslant \frac{1}{2}$, except at $s = 0$, and that the simple pole is cancelled by the zero of $1/\zeta^*(2s)$ outside the parentheses.

Proof. We look at the convergent expression

$$Z_{m,-n}(s) = \sum_{k=0}^{\infty} \frac{2^k}{k!} \widetilde{Z}_{m,-n}\left(s + \frac{k}{2}\right)$$

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given in Proposition 5. The spectral expansion of $\widetilde{Z}_{m,-n}$ does not have any poles in the region $\mathrm{Re}(s) > \frac{1}{2}$ and therefore, in observing the poles of $Z_{m,-n}(s)$, we would only have to look at the $k = 0$ term in the above infinite linear combination: i.e.

$$\mathrm{PolarPart}_{s=\xi}\, Z_{m,-n}(s) = \mathrm{PolarPart}_{s=\xi}\, \widetilde{Z}_{m,-n}(s)$$

for s in the region $0 < \mathrm{Re}(s) < 1$.

The rest of the proof involves calculating the poles and residues of $\widetilde{Z}_d(s)$ and $\widetilde{R}_0(s)$ from the expressions given in Theorem 6. One needs to apply the duplication formula of the Gamma function in obtaining some of the simplification. □

The rest of the poles of $Z_{m,-n}(s)$ are in the region of absolute convergence for the spectral expansion of the function, and hence we can use the expression in Theorem 1. Those poles are at the locations $\frac{1}{2} + it_j - r$ for $r \geqslant 1$ integers. The residues at these poles are given by

$$\operatorname*{Res}_{s=1/2+it_j-r} Z_{m,-n}(s) = \frac{(-1)^r}{2r!}\lambda_j(m)\lambda_j(n)\rho_j(1)\overline{\rho_j(-1)}\Gamma(2it_j - r). \tag{12}$$

From the continuous spectrum (more precisely the terms $R_r(s)$ in Theorem 1), there are going to be poles at $\rho/2 - r$, where $r \geqslant 1$ is an integer and where ρ is a zero in the critical strip of the Riemann zeta function.

§5. *Growth in m, n, and* $\mathrm{Im}(s)$. We will investigate the growth of $Z_{m,n}(s)$ in m, n and the imaginary part of s. Call $\sigma = \mathrm{Re}(s)$. The aim of this section is to prove the following theorem.

THEOREM 9. *Let $m, n > 0$ be positive integers and let $Z_{m,-n}(s)$ be the opposite-sign Kloosterman sum zeta function. We have the following bound in the following regions. Let $\delta > 0$. Then*
for $\sigma > \frac{3}{4}$ and $\varepsilon > 0$ arbitrarily small

$$Z_{m,-n}(s) \ll_\varepsilon (mn)^{1/4+\varepsilon}, \tag{13}$$

for $\frac{1}{2} + \delta < \sigma < \frac{3}{4}$

$$Z_{m,-n}(s) \ll_\varepsilon (mn)^{\theta}(1+|t|)^{1/2} + (mn)^{1/4+\varepsilon}, \tag{14}$$

for $\sigma < 0$,

$$Z_{m,-n}(s) \ll_s (mn)^{1/2-s}, \tag{15}$$

and for $-R - \frac{1}{2} < \sigma < -R$ for $R \geqslant 0$ an integer

$$Z_{m,-n}(s) \ll_\sigma \begin{cases} (mn)^{\theta}(1+|t|)^{\sigma} + (mn)^{1/2-\sigma}(1+|t|)^{4\sigma-2} & \text{if } R = 0, \\ (mn)^{\theta}(1+|t|)^{\sigma} + (mn)^{1/2-\sigma}(1+|t|)^{\sigma-2} & \text{if } R \geqslant 1. \end{cases} \tag{16}$$

Here θ is the best progress towards the Ramanujan conjecture.

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In the strips $-R < \sigma < -R+\frac{1}{2}$ *for* $R \geqslant 1$, *the bound on the function* $Z_{m,-n}(s)$ *differs depending on whether we assume the Riemann hypothesis (RH) or not. If* $s = \sigma + it$ *is chosen so that*

$$\left| s + R - \frac{\rho}{2} \right| \gg \frac{1}{\log(1+|t|)}$$

for all zeros ρ *of the Riemann zeta function, then for some positive constant* A

$$Z_{m,-n}(s) \ll \begin{cases} (mn)^{1/2-\sigma}(1+|t|)^{\sigma+A} + (mn)^{\theta}(1+|t|)^{\sigma} & \text{assuming RH,} \\ (mn)^{1/2-\sigma}(1+|t|)^{A\log\log(10+|t|)} + (mn)^{\theta}(1+|t|)^{\sigma} & \text{unconditionally.} \end{cases} \tag{17}$$

Finally, for $0 < \sigma < \frac{1}{2}$,

$$Z_{m,-n}(s) \ll_{\varepsilon} \begin{cases} (mn)^{1/2-\sigma}(1+|t|)^{4\sigma+A} + (mn)^{\theta}(1+|t|)^{1/2} + (mn)^{1/4+\varepsilon} & RH, \\ (mn)^{1/2-\sigma}(1+|t|)^{A\log\log(10+|t|)} + (mn)^{\theta}(1+|t|)^{1/2} + (mn)^{1/4+\varepsilon} & unc., \end{cases} \tag{18}$$

under the same assumption as above.

Employing Phragmén Lindelöf Principle to the bounds (13) and (14) we obtain

$$Z_{m,-n}(s) \ll_{\varepsilon} (1+|t|)^{3/2-2\sigma+\varepsilon}(mn)^{1/4+\varepsilon} \tag{19}$$

in the region $\frac{1}{2} < \sigma < \frac{3}{4}$.

As a last subsection, we also state the growth of $Z_{m,n}(s)$ in terms of its mn dependence, for completeness.

Region of absolute convergence of Dirichlet series. Notice that, when $\sigma > 1$, we can trivially bound the Kloosterman sum in the Dirichlet series. Hence

$$|Z_{m,n}(s)| \leqslant (2\pi\sqrt{mn})^{2\sigma-1}\zeta(2\sigma-1) \ll (mn)^{\sigma-1/2}.$$

However, we can also use Weil's bound,

$$S(m,n,\ell) \leqslant \tau(\ell)(m,n,\ell)^{1/2}\ell^{1/2},$$

where τ is the divisor function.

LEMMA 10. *In the region* $\mathrm{Re}(s) = \sigma > \frac{3}{4}$, *we have the bounds*

$$Z_{m,n}(s) \ll_{\varepsilon} (mn)^{\sigma-1/2}(m,n)^{\varepsilon} \quad \text{and} \quad \widetilde{Z}_{m,n}(s) \ll_{\varepsilon} (mn)^{\sigma-1/2}(m,n)^{\varepsilon}.$$

Proof. Note that we can bound both of these functions with

$$(2\pi\sqrt{mn})^{2\sigma-1}\sum_{\ell=1}^{\infty}|S(m,n,\ell)|\ell^{-2\sigma}.$$

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Using Weil's bound and that $\tau(\ell) \ll_\varepsilon \ell^\varepsilon$, for any $\varepsilon > 0$

$$\sum_{\ell=1}^{\infty} \frac{S(m,n,\ell)}{\ell^{2s}} \ll_\varepsilon \sum_{\ell=1}^{\infty} \frac{1}{\ell^{2\sigma-1/2-\varepsilon}} \sum_{\substack{d|\ell \\ d|(m,n)}} d^{1/2}.$$

Here we overcount: each ℓ term with $(m,n,\ell) = d$ shows up. Exchange orders of summation and note that

$$\sum_{d|(m,n)} \frac{1}{d^{2\sigma-1-\varepsilon}} \leqslant \tau((m,n)) \ll_\varepsilon (m,n)^\varepsilon$$

in the region $\sigma > \frac{3}{4}$. In the same region, the sum over ℓ converges and we obtain the lemma. □

We can have a better bound using formula (4). Initially, we bound $\widetilde{Z}_{m,-n}(s)$ in the region $\mathrm{Re}(s) > 1$.

LEMMA 11. *Let $\varepsilon > 0$. One has the bound*

$$\widetilde{Z}_{m,-n}(s) \ll_\varepsilon (mn)^{1/4+\varepsilon}$$

for $\sigma > \frac{3}{4}$.

Proof. Assume that $\sigma > 1$. Split the sum over ℓ into dyadic segments,

$$\begin{aligned} |\widetilde{Z}_{m,-n}(s)| &= \left|(2\pi\sqrt{mn})^{2s-1} \sum_{\ell=1}^{\infty} \frac{S(m,-n,\ell)}{\ell^{2s}} \exp\left(-\frac{4\pi\sqrt{mn}}{\ell}\right)\right| \\ &\leqslant (2\pi\sqrt{mn})^{2\sigma-1} \sum_{k=-\log_2(\sqrt{mn})}^{\infty} \\ &\quad \times \sum_{2^{-(k+1)} \leqslant (2\pi\sqrt{mn})/\ell < 2^{-k}} \frac{1}{\ell^{2\sigma-1}} \exp\left(-\frac{1}{2^k}\right). \end{aligned}$$

Here, we have used the trivial bound $|S(m,n,\ell)| \leqslant \ell$. Next

$$\begin{aligned} |Z_{m,-n}(s)| &\leqslant (2\pi\sqrt{mn})^{2\sigma-1} \sum_{k=0}^{\infty} \frac{2^k 2\pi\sqrt{mn}}{(2^k 2\pi\sqrt{mn})^{2\sigma-1}} \exp\left(-\frac{1}{2^k}\right) \\ &\quad + (2\pi\sqrt{mn})^{2\sigma-1} \sum_{k=0}^{\log_2(\sqrt{mn})} \frac{2^k 2\pi\sqrt{mn}}{(2^k 2\pi\sqrt{mn})^{2\sigma-1}} \exp(-2^k) \\ &\leqslant 2\pi\sqrt{mn}\left(\sum_{k=0}^{\infty} \frac{1}{2^{k(2\sigma-2)}} + \sum_{k=0}^{\infty} \frac{1}{2^{k(2\sigma-2)}} \exp(-2^k)\right). \end{aligned}$$

The sums over k converge, and therefore one has

$$\widetilde{Z}_{m,-n}(s) \ll_\sigma \sqrt{mn}$$

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under the assumption that $\sigma > 1$. In fact, the bound is uniform in σ given that $\sigma > 1 + \delta$ for $\delta > 0$.

Combine this result with Lemma 10 and use the Phragmén Lindelöf principle with $\sigma = \frac{3}{4} + \delta$ and σ fixed but large. We then get the result. □

If we had used the Weil bound instead of the trivial one we would have achieved the $(mn)^{1/4}$ bound without recourse to Phragmén–Lindelöf.

After a use of (4) we obtain

$$Z_{m,-n}(s) = \sum_{k=0}^{\infty} \frac{2^k}{k!} \widetilde{Z}_{m,-n}\left(s + \frac{k}{2}\right) \ll_{\varepsilon} (mn)^{1/4+\varepsilon} \sum_{k=0}^{\infty} \frac{2^k}{k!} \ll (mn)^{1/4+\varepsilon}.$$

We have proved the following proposition.

PROPOSITION 12. *The opposite-sign Kloosterman sum zeta function can be bounded as*

$$Z_{m,-n}(s) \ll_{\varepsilon} (mn)^{1/4+\varepsilon},$$

where $\mathrm{Re}(s) = \sigma > \frac{3}{4}$.

In the next section we look at the spectral expansions to deduce the growth of these functions in other regions.

The region of spectral convergence. Given that $\sigma < 0$, we bound $Z_{m,-n}(s)$ using its spectral expression. However, because the continuous spectrum contains the additional terms $R_r^-(s)$ that involve reciprocal zeta functions, the $\mathrm{Im}(s) = t$ dependence will be very different in alternating strips of length $1/2$.

PROPOSITION 13. *Assume* $\mathrm{Re}(s) = \sigma < 0$. *Ignoring the* $\mathrm{Im}(s) = t$ *dependence, one has the bound*

$$Z_{m,-n}(s) \ll_s (mn)^{1/2-\sigma}.$$

If $R \geqslant 0$ *is an integer and* $-R - \frac{1}{2} < \sigma < R$, *then we can also bound the* t *dependence of the opposite-sign Kloosterman sum zeta function as*

$$Z_{m,-n}(s) \ll_{\sigma} \begin{cases} (mn)^{\theta}(1+|t|)^{\sigma} + (mn)^{1/2-\sigma}(1+|t|)^{4\sigma-2} & \text{if } R = 0, \\ (mn)^{\theta}(1+|t|)^{\sigma} + (mn)^{1/2-\sigma}(1+|t|)^{\sigma-2} & \text{if } R \geqslant 1. \end{cases}$$

And, finally, in the region $-R < \sigma < -R + \frac{1}{2}$, *for* $R \geqslant 1$ *an integer*

$$Z_{m,-n}(s) \ll \begin{cases} (mn)^{\theta}(1+|t|)^{\sigma} + (mn)^{1/2-\sigma}(1+|t|)^{\sigma+A} & \text{on RH,} \\ (mn)^{\theta}(1+|t|)^{\sigma} + (mn)^{1/2-\sigma}(1+|t|)^{A\log\log(10+|t|)} & \text{unconditionally,} \end{cases}$$

where s *is positioned so that its distance from all* $\rho/2$ *where* $\zeta(\rho) = 0$ *is at least* $1/\log(|t|)$ *and* A *is some positive constant.*

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Proof. From Stirling's formula

$$Z_d(s) \ll_\sigma \sum_{t_j} |\lambda_j(n)\lambda_j(m)||\rho_j(1)|^2 \left|\Gamma\left(s-\frac{1}{2}+\mathrm{i}t_j\right)\Gamma\left(s-\frac{1}{2}-\mathrm{i}t_j\right)\right|$$
$$\ll (mn)^\theta \sum_{t_j} \mathrm{e}^{-(\pi/2)(-2t_j+|t+t_j|+|t-t_j|)}(1+|t-t_j|)^{\sigma-1}(1+|t+t_j|)^{\sigma-1}.$$

The sum is symmetric under conjugation, so let us, without loss of generality, assume that $t > 0$. The exponential factor is 1 when $t_j > t$ and is equal to $\mathrm{e}^{\pi(t-t_j)}$ when $t_j < t$. So if t_j is much less than t, then the contribution from those terms will be negligible. Separate the summation over t_j into dyadic intervals. Let $T > 2t > T/2$ with $T = 2^k$ for some k. Given that $1 \ll \Delta \ll T$, we will look into the regions where $|t-t_j| \asymp \Delta$; the choice of $\Delta = 2^i$ with $i = 1, \dots, k$ covers $t_j \leqslant T$.

We also will look into the intervals $2^n T \leqslant |t_j| \leqslant 2^{n+1}T$, for $n \in \mathbb{N}$. Then

$$\begin{aligned} Z_d(s) &\ll_\sigma (mn)^\theta \sum_{\substack{i=1\\ \Delta=2^i}}^{k} \sum_{|t-t_j|\asymp\Delta} \mathrm{e}^{-\pi(t-t_j)}(1+T)^{\sigma-1}(1+\Delta)^{\sigma-1} \\ &\quad + (mn)^\theta \sum_{n=0}^{\infty} \sum_{2^nT<|t_j|<2^{n+1}T} (1+|t+t_j|)^{\sigma-1}(1+|t-t_j|)^{\sigma-1} \\ &\ll_\sigma (mn)^\theta \sum_{\substack{i=1\\ \Delta=2^i}}^{k} \mathrm{e}^{-\Delta}(1+T)^\sigma(1+\Delta)^\sigma + \sum_{n=0}^{\infty}(1+2^nT)^{2\sigma} \\ &\ll (mn)^\theta (1+T)^\sigma. \end{aligned}$$

The integral in the continuous spectrum can be similarly bounded. In fact, the result is better because the continuous spectral density is less than that of the discrete spectrum and bounding the divisor sum functions in m and n is easier than bounding Maass form coefficients. The integrand contains zeta functions in the denominator and one of them is on the edge of the critical strip. Since the reciprocal zeta function can be bounded by $O(\log(1+|t|))$, we have

$$I(s) \ll (1+|t|)^{\sigma-1}\log(1+|t|).$$

The other piece of the continuous spectrum (i.e. the additional terms that show up when the function is analytically continued past the lines $\sigma = \frac{1}{2} - r$ for $r \geqslant 0$ an integer) is given by

$$R_r^-(s) = 2\frac{(-1)^r}{r!}\Gamma(2s+r-1)\frac{\sigma_{2s+2r-1}(m)\sigma_{2s+2r-1}(n)}{(mn)^{s+r-1/2}\zeta^*(1-2s-2r)\zeta^*(2-2s-2r)}.$$

On the one hand, ignoring the t dependence

$$R_r^-(s) \ll_s (mn)^{1/2-\sigma-r}.$$

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This choice is maximal when $r = 0$. This yields the first part of the proposition.

On the other hand, if we care about the t dependence,

$$R_r^-(s) \ll_\sigma \frac{1}{r!}(mn)^{1/2-\sigma-r}\left|\frac{\Gamma(2s+r-1)}{\Gamma(1/2-s-r)\Gamma(1-s-r)\zeta(1-2s-2r)}\right| \ll (mn)^{1/2-\sigma-r}\frac{(1+|t|)^{4\sigma+3r-2}}{|\zeta(1-2s-2r)|}. \tag{20}$$

The problem with the t dependence is the existence of the zeta function in the denominator. Therefore, we first restrict our attention to $-R-\frac{1}{2} < \sigma < -R$, the left half of the strip, where the arguments of the zeta functions are in the region of absolute convergence. Furthermore, the subscripts of the divisor sums have real part less than zero. Hence, the largest growth in mn is caused by the $r = 0$ term, in which case it can be bounded by $(mn)^{1/2-\sigma}(1+|t|)^{4\sigma-2}$ and, when $r = R$, we have the largest contribution in the $|t|$ aspect. It is $(mn)^{1/2-\sigma-R}(1+|t|)^{4\sigma+3R-2}$, which still decays in $|t|$.

Using the location of R and σ we deduce the bound

$$Z_c(s) \ll \begin{cases} (1+|t|)^{\sigma-1+\varepsilon} + (mn)^{1/2-\sigma}(1+|t|)^{4\sigma-2} & \text{if } R = 0, \\ (1+|t|)^{\sigma-1+\varepsilon} + (mn)^{1/2-\sigma}(1+|t|)^{4\sigma-2} + (mn)(1+|t|)^{\sigma-2} & \text{if } R \geqslant 1. \end{cases}$$

Here $\varepsilon > 0$ is arbitrarily small and comes from the logarithmic term in bounding $I(s)$.

Now let us assume that we are in the other half of the strip (i.e. $-R < \sigma < -R+\frac{1}{2}$). The only problem is going to come from the $r = R$ term, because then the argument of the zeta function in the denominator is in the critical strip. We quote (9.6.3) from [9] which states that

$$\log\zeta(s) = \sum_{|t-\gamma|<1} \log(s-\rho) + O(\log t),$$

where the sum is over critical zeros $\rho = \beta + i\gamma$ of the Riemann zeta function. Together with the restriction $|s-\rho| > 1/\log t$ and density results on the zeros of the Riemann zeta function, we deduce that

$$|\log\zeta(s)| \ll \log t \log\log t.$$

Hence

$$\frac{1}{\zeta(s)} \ll (1+|t|)^{A\log\log t}$$

for some constant A. This yields the result.

To consider what happens when we assume the Riemann hypothesis, we then revert to [9, equation (14.15.1)] which states

$$\log\zeta(s) = \sum_{|t-\gamma|<1/(\log\log t)} \log(s-\rho) + O\left(\frac{\log t \log\log\log t}{\log\log t}\right)$$

for $\mathrm{Re}(s) \geqslant \frac{1}{2}$. For $\mathrm{Re}(s) < \frac{1}{2}$, we may use the functional equation.

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First note that, with the restriction of $|s-\rho| > 1/\log t$, we have that

$$|\log(s-\rho)| \leqslant \log\log t.$$

The number of ρs in the interval $|t-\gamma| \leqslant 1/\log\log t$ is bounded by $O(\log t/(\log\log t))$. Hence, with the above,

$$\frac{1}{\zeta(s)} \ll \mathrm{e}^{\log\log(1+|t|)((A\log(1+|t|))/(\log\log(1+|t|)))} + O(1) \ll (1+|t|)^A$$

for some positive constant A. □

One advantage of $Z_{m,-n}(s)$ to $\widetilde{Z}_{m,-n}(s)$ is that, in the spectral expansion of $\widetilde{Z}_{m,n}(s)$, there are additional terms $\widetilde{R}_r^-(s)$ obtained when σ passes every integer and half-integer, as opposed to just the latter. This happens because of the vanishing of the sum over k in (11). Hence, we are able to obtain a safe strip of width $\frac{1}{2}$, where we are away from the critical zeros of $\zeta(1-2s+r)$.

The region in-between. In the region $0 < \sigma < \frac{3}{4}$, we cannot use the spectral expansion of $Z_{m,-n}(s)$ because it simply does not converge absolutely. Hence we use (4), in order to bound the mn dependence, and for that we need the bounds on $\widetilde{Z}_{m,-n}(s)$ in that region. The proof is similar to that of Proposition 13 above. We split the spectral sum into dyadic intervals and estimate.

LEMMA 14. *Let $\delta > 0$. In the region $\mathrm{Re}(s) = \sigma > \frac{1}{2}+\delta$, we have the bound*

$$\widetilde{Z}_{m,-n}(s) \ll_\delta (mn)^\theta (1+|t|)^{1/2},$$

where θ is the best progress towards the Ramanujan–Petersson conjecture, and $t = \mathrm{Im}(s)$.

In the region $0 < \sigma < \frac{1}{2}$, the function is not bounded in terms of t due to the additional term $\widetilde{R}_0(s)$ in the continuous spectrum. It contains the reciprocal of the Riemann zeta function. However, we can still bound the m and the n dependence, not on the zeros of $\zeta(2s)$, by

$$\widetilde{Z}_{m,-n}(s) \ll_s (mn)^{1/2-\sigma} + (mn)^\theta.$$

The t dependence is given as

$$\widetilde{Z}_{m,-n}(s) \ll \begin{cases} (mn)^{1/2-\sigma}(1+|t|)^{4\sigma-2+(1/2\pi)} + (mn)^\theta(1+|t|)^{1/2} & \text{on RH,} \\ (mn)^{1/2-\sigma} t^{A\log\log(10+|t|)} + (mn)^\theta(1+|t|)^{1/2} & \text{unconditionally,} \end{cases} \tag{21}$$

if s is of distance $1/\log(1+|t|)$ away from all zeros of $\zeta(2s)$ in the critical strip and $A > 0$.

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Proof. The proof is analogous to Proposition 13. Apply Stirling's formula to the Gamma functions involved

$$\begin{aligned}\widetilde{Z}_d(s) &\ll_\sigma \sum_{t_j} \lambda_j(n)\lambda_j(m)\rho_j(1)\overline{\rho_j(-1)}\frac{\Gamma(2s-1+2it_j)\Gamma(2s-1-2it_j)}{\Gamma(2s-1/2)}\\ &\ll (mn)^\theta \sum_{t_j} e^{\pi(t_j-|t+t_j|-|t-t_j|+|t|)}((1+|t-t_j|)(1+|t+t_j|))^{2\sigma-3/2}\\ &\quad\times (1+|t|)^{1-2\sigma}.\end{aligned}$$

We again assume that $t > 0$, for convenience. Let us separate the regions of t_j into dyadic intervals. First, let $T > 2t$ with $T = 2^k$ for some k. Given that $1 \ll \Delta \ll T$, we will look into the regions where $|t - t_j| \asymp \Delta$; the choice of $\Delta = 2^i$ with j up to k covers $t_j \leqslant T$. We need to consider larger Δ for the rest of the t_j. Then

$$\begin{aligned}\widetilde{Z}_d(s) &\ll_\sigma \sum_{\substack{i=0\\ \Delta=2^i}}^{k} \sum_{|t-t_j|\asymp\Delta} e^{-\Delta}(1+T)^{-1/2}(1+\Delta)^{2\sigma-3/2}\\ &\quad+\sum_{\substack{i=k+1\\ \Delta=2^i}}^{\infty} \sum_{|t-t_j|\asymp\Delta} e^{-\Delta}(1+T)^{1-2\sigma}(1+\Delta)^{4\sigma-3}\\ &= (1+T)^{1/2}\sum_{\substack{i=1\\ \Delta=2^i}}^{k} e^{-\Delta}(1+\Delta)^{2\sigma-1/2}\\ &\quad+(1+T)^{1-2\sigma}\sum_{\substack{i=k+1\\ \Delta=2^i}}^{\infty} e^{-\Delta}(1+\Delta)^{4\sigma-1},\end{aligned}$$

where in the last line we have used Weyl's law

$$\#\{t_j : |t-t_j| \asymp \Delta\} \ll \begin{cases}\Delta T & \text{for } 1 \ll \Delta \ll T,\\ T^2 & \text{for } \Delta \gg T.\end{cases}$$

Note that the second sum goes to 0 as $T \to \infty$. In the first sum, the sum over Δ converges to a number (depending on σ), even if we complete the sum over i to infinity. Thus

$$\widetilde{Z}_d(s) \ll_\sigma (mn)^\theta(1+T)^{1/2}.$$

The integral in the continuous spectrum is majorized by this quantity.

The argument up to this point is equally valid for the regions $1/2 < \sigma < 3/4$ and $0 < \sigma < \frac{1}{2}$ as long as s is not on the zeros of $\zeta(2s)$. However, once s is past the line $\sigma = \frac{1}{2}$, the term $\widetilde{R}_0^-(s)$ shows up. Looking at (6) we are able to see that

$$\widetilde{R}_0(s) \ll_s (mn)^{1/2-\sigma}$$

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in this region. Once again, the greatest contribution is from the $r = 0$ term. If we do not ignore the s dependence, then the bound is given by

$$\widetilde{R}_r(s) \ll (mn)^{1/2-\sigma} \frac{(1+|t|)^{4\sigma-2}}{|\zeta(1-2s)|}.$$

The considerations about the size of the reciprocal zeta function at the end of Proposition 13 apply once again to give the lemma. □

Using this lemma and the fact that $Z_{m,-n}(s)$ can be written as an infinite linear combination, we deduce that, if $\frac{1}{2} < \sigma < \frac{3}{4}$, then

$$\begin{aligned} Z_{m,-n}(s) &= \sum_{k=0}^{\infty} \frac{2^k}{k!} \widetilde{Z}_{m,-n}\left(s+\frac{k}{2}\right) \\ &= \widetilde{Z}_{m,-n}(s) + O_\varepsilon((mn)^{1/4+\varepsilon}) \\ &\ll_\varepsilon (mn)^\theta (1+|t|)^{1/2} + (mn)^{1/4+\varepsilon}, \end{aligned}$$

and using Phragmén–Lindelöf we get (19). In the region $0 < \sigma < \frac{1}{2}$, if we ignore the t dependence,

$$\begin{aligned} Z_{m,-n}(s) &= \widetilde{Z}_{m,-n}(s) + \widetilde{Z}_{m,-n}\left(s+\frac{1}{2}\right) + O_\varepsilon((mn)^{1/4+\varepsilon}) \\ &\ll_{\varepsilon,s} (mn)^{\min\{1/2-\sigma,\,1/4+\varepsilon\}}. \end{aligned}$$

Otherwise, we get (18). This proves Theorem 9.

§6. *Sums of Kloosterman sums.* We are going to use the analytic properties of the zeta function $Z_{m,-n}(s)$ in order to prove a bound on

$$S_{m,-n}(X) = \sum_{\ell \leqslant X} \frac{S(m,-n,\ell)}{\ell}.$$

We find a bound that is uniform in m and n.

Proof of Theorem 2. The proof is a standard application of Perron's formula as in [1]. Let $\sigma_1 > 3/4$ and T be a large quantity. Let X be a half-integer. Set

$$I(X,T) = \int_{\sigma_1 - iT}^{\sigma_1 + iT} Z_{m,-n}(s) \left(\frac{X}{2\pi\sqrt{mn}}\right)^{2s-1} \frac{ds}{s}.$$

On the one hand, if we express $Z_{m,-n}(s)$ as a Dirichlet series and exchange the orders of integration and summation, we obtain

$$I(X,T) = S(X) + O\left(\sum_{\ell=1}^{\infty} \frac{S(m,-n,\ell)}{\ell} \left(\frac{X}{\ell}\right)^{2\sigma_1-1} \min\left\{1, \frac{1}{T|\log X/\ell|}\right\}\right).$$

Split the sum inside the big-oh into regions $\ell < 3X/4$, $3X/4 < \ell < X$, $X < \ell < 5X/4$ and $5X/4 < \ell$, calling them D_1, D_2, D_3 and D_4, respectively. Analyze

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each term separately. Firstly,

$$D_1 \leqslant X^{2\sigma_1-1} \sum_{\ell=1}^{(3X)/4} \frac{|S(m,-n,\ell)|}{\ell^{3/2}} \frac{1}{T|\log(3/4)|} \ll \frac{X^{2\sigma-1}\log^2 X}{T}(m,n)^{\varepsilon}.$$

Similarly, D_4 can be bounded.

In bounding D_2 we first realize that

$$\log X/\ell = -\log\left(1-\frac{X-\ell}{X}\right) \geqslant \frac{X-\ell}{\ell}.$$

Call $\nu = X-\ell$. Then separate the sum into the regions $\nu < X/T$ and $X/T < \nu < X/4$. We can also bound these sums by the term above.

Hence we have

$$I(X) = S(X) + O\left(\frac{X^{1/2}\log^2 X}{T}(m,n)^{\varepsilon}\right). \tag{22}$$

Let $\sigma_0 > \frac{1}{2}$. We will move the line of integration to the vertical line of length $2T$ and abscissa σ_1.

$$I(X) = \left(\int_{\sigma_0+\mathrm{i}T}^{\sigma_1+\mathrm{i}T} + \int_{\sigma_1-\mathrm{i}T}^{\sigma_0-\mathrm{i}T} + \int_{\sigma_0-\mathrm{i}T}^{\sigma_0+\mathrm{i}T}\right) Z_{m,-n}(s)\left(\frac{X}{2\pi\sqrt{mn}}\right)^{2s-1}\frac{ds}{s}.$$

In the previous section we have established bounds for $Z_{m,-n}(s)$ in each of these domains. For the top and bottom parts of the rectangle, we may use (19) and see that,

$$\begin{aligned}
&\int_{\sigma_0+\mathrm{i}T}^{\sigma_1+\mathrm{i}T} + \int_{\sigma_1-\mathrm{i}T}^{\sigma_0-\mathrm{i}T} Z_{m,-n}(s)\left(\frac{X}{2\pi\sqrt{mn}}\right)^{2s-1}\frac{ds}{s} \\
&\ll T^{1/2}(mn)^{1/4+\varepsilon}\frac{\sqrt{mn}}{X}\int_{\sigma_0}^{\sigma_1} T^{-2\sigma}\left(\frac{X}{\sqrt{mn}}\right)^{2\sigma} d\sigma \\
&\ll \frac{T^{1/2}}{X}(mn)^{3/4+\varepsilon}\frac{1}{|\log(X/(\sqrt{mn}T))|}\max\left\{\left(\frac{X}{\sqrt{mn}T}\right)^{2\sigma_1}, \left(\frac{X}{\sqrt{mn}T}\right)^{2\sigma_0}\right\}.
\end{aligned} \tag{23}$$

Note that, if $X \asymp \sqrt{mn}T$, then we can simply bound the integral in (23) with $O(1)$.

In the vertical line we use the bound (14)

$$\begin{aligned}
&\int_{\sigma_0-\mathrm{i}T}^{\sigma_0+\mathrm{i}T} Z_{m,-n}(s)\left(\frac{X}{2\pi\sqrt{mn}}\right)^{2s-1}\frac{ds}{s} \\
&\ll \left(\frac{X}{\sqrt{mn}}\right)^{2\sigma_0-1}\left(\int_1^T \frac{(mn)^{\theta}}{t^{1/2}} + \frac{(mn)^{1/4+\varepsilon}}{t}dt\right) \\
&\ll X^{2\sigma_0-1}((mn)^{\theta-(\sigma_0-1/2)}T^{1/2} + (mn)^{1/4+\varepsilon-(\sigma_0-1/2)}\log T).
\end{aligned}$$

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There are three cases, $X \leqslant \frac{1}{2}\sqrt{mn}T$, $X \geqslant 2\sqrt{mn}T$ and X in-between. In the second case, choose $T = X^{1/3}$; in the other cases choose $T = (mn)^{1/4}$.

Now choose $\sigma_0 = \frac{1}{2} + \varepsilon$ and $\sigma_1 = \frac{3}{4} + \varepsilon$. Combining everything we deduce that

$$S(X) \ll_\varepsilon X^{1/6+\varepsilon}((m,n)^\varepsilon + (mn)^{\theta+\varepsilon}) + X^\varepsilon (mn)^{1/4+\varepsilon}. \qquad \square$$

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