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Periodic points and transitivity on dendrites

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Abstract. We study relations between transitivity, mixing and periodic points on dendrites. We prove that, when there is a point with dense orbit which is a cutpoint, periodic points are dense and there is a terminal periodic decomposition. We also show that it is possible that all periodic points except one (and points with dense orbit) are contained in the (dense) set of endpoints. It is also possible that a dynamical system is transitive but there is a unique periodic point which, in fact, is the unique fixed point. We also prove that on almost meshed continua (a class of continua containing topological graphs and dendrites with closed or countable set of endpoints), periodic points are dense if and only if they are dense for the map induced on the hyperspace of all non-empty compact subsets.

1. *Introduction*

Topological transitivity is a global property of dynamical systems. Under the assumption of transitivity, the whole space becomes a global attractor, making the dynamics indivisible in some sense. It also represents a minimal requirement for being able to speak about mixing

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in the dynamics. Hence, it is not surprising that transitivity is an important ingredient of global (and topological) definitions of chaos [**5**, **7**, **9**, **21**]. One of the most well-known definitions of chaos was introduced by Devaney in his book [**16**]. In this definition, a map is chaotic when it is transitive, sensitive and contains a dense subset of periodic points. It was soon noticed that when periodic points are dense and the space is infinite (in particular, when the space is a non-degenerate continuum), transitivity implies sensitivity (see [**6**] or [**20**]). It is well known that, for dynamical systems on the unit interval, transitivity implies that periodic points are dense (see, for example, [**13**, §VI]). In other words, each transitive dynamical system on the unit interval is chaotic in the sense of Devaney. It is also known that if the dynamical system ($[0, 1]$, f) is transitive but not mixing, then $[0, 1]$ decomposes into two proper subintervals $I_0 = [0, p]$, $I_1 = [p, 1]$ such that p is a fixed point of f, $f(I_i) = I_{i+1 \pmod 2}$ and $f^2|_{I_i}$ is mixing, where $i = 0, 1$ (see, for example, [**13**, §VI]). Furthermore, on the unit interval, the closure of the set of periodic points coincides with the closure of the set of recurrent points (so-called $\overline{R} = \overline{P}$ property first proved by Sharkovsky [**34**]; see also [**15**]). In general, dynamics on the unit interval are well understood and many beautiful and surprising results can be proved in that context (for example, the celebrated Sharkovsky theorem [**33**] and many other surprising results [**3**, **13**]). An interesting exposition on transitivity and its connection to other notions, with special emphasis on one-dimensional dynamics, can be found in the survey paper by Kolyada and Snoha [**28**].

A natural question that comes to mind when considering one-dimensional dynamics is to what extent these results can be generalized beyond the unit interval and what are the minimal assumptions that suffice for these results to hold. A natural direction of research is to consider a class of 'similar' one-dimensional continua and, actually, this is how it was done in the literature. Many authors investigated dynamical properties (related to density of periodic points) of topological trees, the circle and then of topological graphs. In many cases (after an appropriate adjustment) this direction was successful (see, for example, the appendix in [**3**]). A good example of necessary adjustments is the $\overline{R} = \overline{P}$ property. It is still true when we consider topological trees, but for topological graphs it is no longer valid (e.g. consider an irrational rotation of the unit circle). However, it is valid in the case in which $\overline{\mathrm{Rec}(f)} = \mathrm{Rec}(f) \cup \overline{\mathrm{Per}(f)}$, where $\mathrm{Per}(f)$ and $\mathrm{Rec}(f)$ denote the set of periodic and recurrent points of f, respectively (see [**11**] or the more recent paper [**31**]). Of course, while the result looks similar, a modification of the arguments in the proof was essential. In some other cases only partial answers are known (e.g. a characterization of the structure of periods of maps on topological graphs is far from being complete). In spite of technical difficulties, in the case of topological graphs, dynamical properties remain to a large extent 'similar' to those known from the unit interval (e.g. see [**22**] and references therein).

The first serious problems arise when we consider dendrites. On these (locally connected one-dimensional) continua, various counterexamples to facts known for the unit interval (or topological trees) can be constructed, and concrete properties hold only in special cases. For example, if a dendrite has a countable set of endpoints, then the $\overline{R} = \overline{P}$ property holds (see [**29**]), but when the set of endpoints is uncountable, then a counterexample can be constructed [**26**]. It was recently proved by Dirbák *et al* in [**17**] that, in the case of dynamical systems on continua which are not homeomorphic to the

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unit circle and with a free interval (in particular, on dendrites whose set of endpoints is not dense), a transitive map always has a dense set of periodic points and has a terminal periodic decomposition (i.e. a regular periodic decomposition of maximal length; see Definitions 4.1 and 4.2). Furthermore, an appropriate iterate restricted to any of the sets in the decomposition is mixing. On the other hand, recently, Hoehn and Mouron constructed, in [**24**], an example of a dendrite and a weakly mixing map on it which is not mixing. It is also known that there are dynamical systems on dendrites without a maximal ω-limit set [**27**], while, for a dynamical system on a topological graph, every ω-limit set is contained in a maximal one [**11**] (this can be also deduced from the fact that the space of all ω-limit sets endowed with the Hausdorff metric is compact [**30**]).

Motivated by the results given above for topological trees (and difficulties in the case of dendrites) we state the following questions for future research.

Question 1.1. Does every transitive dynamical system on a dendrite admit a terminal periodic decomposition?

Question 1.2. Can we provide a complete characterization of transitive dynamical systems on dendrites with dense periodic points?

While we do not know complete answers to the above questions, we are able to provide some partial answers. In §3, in Theorem 3.5, we prove that if a dynamical system (D, f) on a dendrite D has at least one point with dense orbit which is not an endpoint, then the set of periodic points of f is dense in D. We also prove that it is possible that there is a unique periodic point (and thus a fixed point) but the map is transitive (construction from [**24**] serves as an example), or the set of periodic points (as well as the set of points with dense orbit), while dense, is contained in the set of endpoints. We also provide an example of a map (and a dendrite with a dense set of endpoints) which satisfies the assumptions of Theorem 3.5. This leads us to another open question.

Question 1.3. Does there exist a dendrite D and a mixing dynamical system on it which does not have dense periodic points?

In §4 we prove that a totally transitive dynamical system on a dendrite which has periodic points that are cutpoints must be mixing. We also prove that at least one point with dense orbit which is a cutpoint is sufficient for admitting a terminal periodic decomposition.

Finally, in §6, we prove that on almost meshed continua, periodic points of (X, f) are dense if and only if they are dense for the induced dynamical system $(2^X, 2^f)$ on the space of compact subsets. The class of almost meshed continua, among others, contains all topological graphs and dendrites with closed set of endpoints. It is worth mentioning here, that the dendrite presented by Kato [**26**] is an almost meshed continuum such that (X, f) has a unique periodic point, while $(2^X, 2^f)$ has two periodic points.

2. *Preliminaries*

A *map* is a continuous function. Let (X, d) be a compact metric space. The space of all maps $f : X \to X$ will be always endowed with the complete metric $\rho(f, g) =$

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$\sup_{x\in X} d(f(x), g(x))$. The closure of a set $A \subset X$ is denoted either by $\overline{A}$ or by $\mathrm{Cl}_X(A)$, the interior by $\mathrm{Int}_X(A)$ and the boundary of A is denoted by $\mathrm{Bd}_X(A)$.

For a set Y, the identity on Y is denoted by id_Y. The cardinality of Y is denoted by $\#Y$. A *continuum* is a compact, connected and non-empty metric space. An arc is a space homeomorphic to the interval $[0, 1]$. If A is an arc in X with endpoints a and b ($a \neq b$), then A is said to be a *free arc* in X if the set $A\backslash\{a, b\}$ is open in X.

2.1. *Continua and hyperspaces.* A continuum S is a *star* if there is a point $a \in S$ such that S can be represented as the countable union $S = \bigcup_{n=1}^{+\infty} B_n$ of arcs B_n, each having a as an endpoint and satisfying $\lim_{n\to+\infty} \mathrm{diam}(B_n) = 0$ and $B_m \cap B_n = \{a\}$ when $m \neq n$. We will say that each B_n is a *beam* of the star S.

A (*topological*) *graph* is a continuum that can be written as the union of a finite number of arcs, any two of which are either disjoint or intersect only at one of their endpoints. A *tree* is any topological graph without cycles, that is, homeomorphic copies of the unit circle.

A continuum D is a *dendrite* if any two distinct points in D can be separated by a third point in D. An accessible presentation of dendrites and their basic properties can be found in [**32**]. Let D be a dendrite, let $p \in D$ and let β be a cardinal number. We say that p is of order β, denoted by $\mathrm{ord}_D(p) = \beta$, if, for each open neighborhood U of p and each cardinal number $\alpha < \beta$, there is a neighborhood $V \subset U \subset D$ of p such that $\alpha < \#\mathrm{Bd}_X(V) \leq \beta$. We say that p is an *endpoint* of D if $\mathrm{ord}_D(p) = 1$, a *cutpoint* of D if $\mathrm{ord}_D(p) > 1$ and a *branch point* of D when $\mathrm{ord}_D(p) > 2$ (cf. [**32**, Theorem 10.7]). If D is a dendrite, then we denote by $\mathrm{End}(D)$ the set of *endpoints* of D.

Let D be a dendrite and $a, b \in D$. We will denote the unique arc between a and b by $[a, b]$. We define

$$[a, b) = [a, b] - \{b\}, \quad (a, b] = [a, b] - \{a\} \quad \text{and} \quad (a, b) = [a, b] - \{a, b\}.$$

If D is a dendrite, we will write $x \leq y \leq z$ if $x, y, z \in D$ and $y \in [x, z]$. This relation can be extended to any number of points contained in an arc. Clearly, if $a < p < b$, $b < q < c$ and $[a, b] \cap [b, c] = \{b\}$, then $a < p < b < q < c$.

For any subcontinuum Y of a dendrite D we define $r : D \to Y$ by letting $r(x)$ be a unique common point of all arcs $[x, y]$ with $y \in Y$. It is known that the function r is well defined and continuous (and, hence, it is a retraction from D onto Y). We say that r is the *first point map for* Y and denote it, for convenience, by $\pi_Y = r$. By Theorem 10.27 in [**32**], for every dendrite D and every $\varepsilon > 0$ there exists a tree $Y_\varepsilon \subset D$ such that the first point map π_{Y_ε} satisfies $\rho(\pi_{Y_\varepsilon}, \mathrm{id}_D) < \varepsilon$.

Given a continuum X let $\mathcal{FA}(X) = \bigcup\{\mathrm{Int}_X(J) : J \text{ is a free arc in } X\}$. In [**23**], X is defined to be *almost meshed* if the set $\mathcal{FA}(X)$ is dense in X. The class of almost meshed continua contains several interesting examples, among them dendrites with their set of endpoints closed as well as dendrites with a countable set of endpoints.

If X and Y are continua and $f : X \to Y$ is a map, then f is said to be *weakly confluent* if, for every subcontinuum Q of Y, there exists a continuum C in X such that $f(C) = Q$.

For a continuum X we consider the hyperspace

$$2^X = \{A \subset X : A \text{ is non-empty and closed in } X\}$$

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endowed with the Hausdorff metric. If $A \subset X$, we define $\langle A\rangle = \{B \in 2^X : B \subset A\}$. It is known that if A is an open subset of X, then $\langle A\rangle$ is open in 2^X. If $f : X \to Y$ is a map, then we can consider the function $2^f : 2^X \to 2^Y$ defined, for $A \in 2^X$, by $2^f(A) = f(A)$. It is known that 2^f is continuous (see [**25**] for more details on properties of the hyperspace 2^X).

2.2. *Dynamical systems.* A *dynamical system* is a pair (X, f) consisting of a compact metric space (X, d) and a map $f : X \to X$. For such system we define $f^1 = f$ and $f^{n+1} = f^n \circ f$, for each $n \in \mathbb{N}$. The *orbit of* $x \in X$ is the set $\mathrm{Orb}^+(x) = \{f^n(x) : n \in \mathbb{N}\}$, and the *ω-limit set of* x *in* f is the set

$$\omega_f(x) = \left\{ y \in X : \liminf_{n\to+\infty} d(f^n(x),\, y) = 0 \right\}.$$

We say that f is *minimal* if the orbit of every $x \in X$ is a dense subset of X. A closed subset $M \subset X$ is a *minimal* set if and only if $f(M) \subset M$ and if $(M,\, f|_M)$ is a minimal dynamical system.

A point x is *periodic* if $f^n(x) = x$ for some $n > 0$. When $n = 1$, we say that x is a *fixed point*. The *period* of p is the least natural number m such that $f^m(p) = p$. If (X, f) is a dynamical system, we denote by $\mathrm{Per}(f)$ the set of periodic points of f.

A dynamical system (X, f) is *transitive* (respectively, *mixing*) if, for any pair of non-empty open sets $U, V \subset X$, there exists $n > 0$ ($N > 0$) such that $f^n(U) \cap V \neq \emptyset$ (for all $n \geq N$, respectively), *totally transitive* if f^n is transitive for all $n \geq 1$, *weakly mixing* (*topologically*) if $f \times f$ is transitive on $X \times X$, and *topologically exact* if for any non-empty open subset U of X there is $N > 0$ such that $f^N(U) = X$. It is well known that if (X, f) is transitive, then the set $\{x \in X : \overline{\mathrm{Orb}^+(x)} = X\}$ is *residual* in X (i.e. it is the intersection of a countable number of sets with dense interiors). An interesting exposition on transitivity, including dynamical systems in dimension one, can by found in the survey paper by Kolyada and Snoha [**28**].

3. *Transitivity implies dense periodic points in some cases*

The following fact is well known.

LEMMA 3.1. *Let D be a dendrite. If* $\mathrm{End}(D)$ *is dense in D, then* $\mathrm{End}(D)$ *is residual in D.*

This immediately leads to the following corollary.

COROLLARY 3.2. *Let D be a dendrite and assume that $f : D \to D$ is transitive. If* $\mathrm{End}(D)$ *is dense in D, then* $\{x \in \mathrm{End}(D) : \overline{\mathrm{Orb}^+(x)} = D\}$ *is residual in D (in particular, it is dense in D).*

Note that it may happen that $\{x \in D : \overline{\mathrm{Orb}^+(x)} = D\} \subset \mathrm{End}(D)$. An example of such a dendrite and a transitive map acting on it is presented in §5. In what follows, we will be concerned with a special class of maps on dendrites with dense endpoints.

The following fact is stated in [**29**, Lemma 2.3] and is an easy consequence of [**32**, Theorem 10.27(5)].

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LEMMA 3.3. *Let $(C_i)_{i\in\mathbb{N}}$ be a sequence of connected (not necessarily closed) subsets of a dendrite D. If $C_i \cap C_j = \emptyset$ for all $i \neq j$, then*

$$\lim_{n\to+\infty} \operatorname{diam}(C_n) = 0.$$

We will also need another result by the same authors, which is [**29**, Theorem 2.13].

LEMMA 3.4. *Let D be a dendrite and let (D, f) be a dynamical system. Let $[x, y]$ be an arc in D, and let U be the connected component of $D\backslash\{x, y\}$ containing the open arc (x, y). If there exist $m, n > 0$ such that $\{f^m(x), f^n(y)\} \subset U$, then U contains a periodic point of f.*

Now we are ready to prove the main theorem of this section.

THEOREM 3.5. *Let D be a dendrite and let (D, f) be a dynamical system. If there is $x \in D\backslash \operatorname{End}(D)$ such that $\overline{\operatorname{Orb}^+(x)} = X$, then the set of periodic points of f is dense in D.*

Proof. First, note that, since x has dense orbit, it is enough to prove that every open neighborhood of x contains a periodic point.

Fix any $\varepsilon > 0$ and let p be a fixed point of f (see [**32**, Theorem 10.31]). Since $\operatorname{ord}_D(x) > 1$, we can consider two connected components C and U of $D\backslash\{x\}$ such that $p \in C$. Note that $p \notin U$ and, since connected components of open subsets of D are open, both C and U are open in D.

Fix any endpoint $e \in U \cap \operatorname{End}(D)$ and observe that $e \neq x$. Since the set of branch points is at most countable in each dendrite (see, for example, [**32**, Theorem 10.23]), the arc $[e, x]$ contains at most a countable number of such points. In particular, the set $D\backslash[e, x]$ has at most a countable number of connected components and, by Lemma 3.3, all but a finite number of them have diameter bounded by $\varepsilon/2$. Hence, there is $q \in U$ such that, if we denote by V the connected component of $D\backslash\{x, q\}$ that contains the open arc (x, q), then $\operatorname{diam}(V) < \varepsilon$. Clearly, $V \subset U$ so, in particular, $p \notin V$. Note that V is open in D.

The orbit of x is dense, and thus there exists an integer $N > 0$ such that $f^N(x) \in V$ and, because p is a fixed point and $p \in \omega_f(x)$, we also have $p \in \omega_{f^N}(x)$. Hence there is $m \geq 2$ such that $f^{Nm}(x) \in C$. This implies that there is $k \geq 1$ such that $f^{kN}(x) \in U$ and $f^{(k+1)N}(x) \notin U$.

Since f^{kN} is continuous and x is a cutpoint,

$$x \in [f^{kN}(x),\ f^{(k+1)N}(x)] \subset f^{kN}([x,\ f^N(x)])$$

and, since x is not periodic, there is $y \in (x,\ f^N(x))$ such that $f^{kN}(y) = x$. Note that $y \in V$, and there are the following two possibilities.

(i) If $y \in [q, x]$ (see Figure 1), we denote by W the connected component of $D\backslash\{y, x\}$ containing (y, x). Then W is contained in V and W is an open subset of D. By the fact that both x and y have dense orbits, there are $m, n > 0$ such that $f^n(x),\ f^m(y) \in W$. Lemma 3.4 implies that there is a periodic point in $W \subset V$.

(ii) If $y \notin [q, x]$ (see Figure 2), then y and $f^N(x)$ are contained in the same connected component of $D\backslash[q, x]$ and, in particular, the connected component W' of $D\backslash\{y, f^N(x)\}$ containing $(y, f^N(x))$ is contained in V. Applying Lemma 3.4, we obtain a periodic point in $W' \subset V$.

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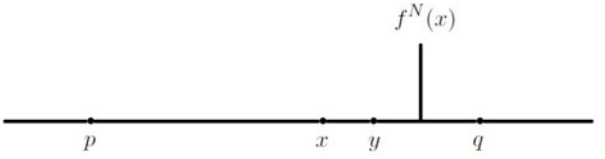


FIGURE 1. Case (i) in Theorem 3.5.

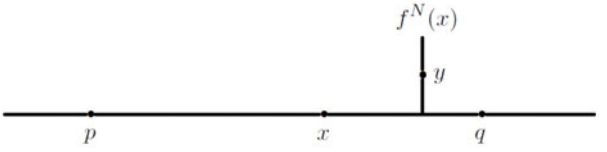


FIGURE 2. Case (ii) in Theorem 3.5.

We have proved that, for each $\varepsilon > 0$, there is a periodic point z with $d(x, z) \leq \varepsilon$ and hence the proof is complete. □

As we will see later in §5, there is a transitive dendrite map with only one periodic point and hence, by Theorem 3.5, any point with dense orbit in this example must be an endpoint.

COROLLARY 3.6. *Let D be a dendrite and let (D, f) be a transitive dynamical system. If the set of periodic points of f is not dense, then any point x with dense orbit is an endpoint of D.*

Remark 3.7. Theorem 3.5 simplifies a necessary condition for transitivity on dendrites presented in [**12**, Theorem 1.9]. We do not, however, have an example of a dendrite D, a map $f : D \to D$ and a cutpoint $x \in D$ with dense orbit which, in a finite number of iterations, enters the set of endpoints.

4. *Transitivity and periodic decompositions on dendrites*

In this section we follow the approach to periodic decompositions introduced by Banks in [**4**]. It is worth mentioning that decompositions on various one-dimensional continua were considered by numerous authors. In this context, it is especially appropriate to mention the work by Alsedá *et al* [**1**] on periodic decompositions of topological graphs (see also [**8**, **10**]). The situation that is known for topological graphs (trees) is probably the best possible situation that can be expected for dendrites.

A set $D \subset X$ is *regular closed* if it is the closure of its interior.

Definition 4.1. A *regular periodic decomposition (of length m)* for a dynamical system (X, f) is a finite sequence $\mathcal{D} = (D_0, \ldots, D_{m-1})$ such that:

(4.1) $\mathcal{D}$ is a cover of X consisting of regular closed sets;

(4.2) $f(D_i) = D_{i+1 (\mathrm{mod}\, m)}$ for $i = 0, \ldots, m-1$; and

(4.3) $D_i \cap D_j$ is nowhere dense in X for $i \neq j$.

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Notice that, for any dynamical system (X, f), the space X forms a trivial regular periodic decomposition of length 1. Using (4.3), we easily get that:

(i) the boundary of each D_i is nowhere dense; and

(ii) $\mathrm{Int}_X(D_i) \cap D_j = \emptyset$ for $i \neq j$.

If $\mathcal{D} = (D_0, \ldots, D_{m-1})$ is a regular periodic decomposition for a transitive dynamical system (X, f), then (see [**4**, Lemma 2.1 and Theorem 2.1]):

(iii) $f^l(D_i) = D_{i+l(\mathrm{mod}\ m)}$ for $i = 0, 1, \ldots, m-1$ and $l \geq 0$;

(iv) $f^{-l}(\mathrm{Int}_X(D_i)) \subset \mathrm{Int}_X(D_{i-l\ (\mathrm{mod}\ m)})$ for $i = 0, 1, \ldots, m-1$ and $l \geq 0$;

(v) $E = \bigcup_{i \neq j}(D_i \cap D_j)$ is closed, invariant and nowhere dense; and

(vi) f^m is transitive on each D_i.

If all the sets of $\mathcal{D}$ are connected, we say that $\mathcal{D}$ is *connected*. By (iii) and the fact that the continuous image of a connected set is connected, we get that $\mathcal{D}$ is connected if and only if at least one of the sets D_i is connected. By [**4**, Theorems 2.3 and 2.4], if (X, f) is transitive but (X, f^n) is not transitive for some $n > 0$, then (X, f) has a regular periodic decomposition of length $p > 1$, where p is a prime number dividing n.

Assume now that $\mathcal{C} = (C_0, \ldots, C_{n-1})$ and $\mathcal{D} = (D_0, \ldots, D_{m-1})$ are regular periodic decompositions for (X, f). We say that $\mathcal{C}$ *refines* (or is a *refinement* of) $\mathcal{D}$ if every C_i is contained in some D_j. Note that, in such a case, each element of $\mathcal{D}$ contains the same number of elements of $\mathcal{C}$, so n is a multiple of m. It was proved in [**4**, Theorem 6.1] that if X is locally connected, then every regular periodic decomposition for (X, f) has a connected refinement. By [**4**, Lemma 3.2], if (X, f) has regular periodic decompositions $\mathcal{C}$ of length m and $\mathcal{D}$ of length n, then (X, f) has a regular periodic decomposition of length k, where k is lowest common multiple of m and n, which is a refinement of both $\mathcal{C}$ and $\mathcal{D}$. Hence we can consider the following definition.

Definition 4.2. A *terminal periodic decomposition* for a dynamical system (X, f) is a regular periodic decomposition of (X, f) whose length is maximal.

There are situations where terminal periodic decomposition exists. In such a case we have the following characterization (see [**4**, Theorem 3.1]). Recall that (X, f) is totally transitive if (X, f^n) is transitive for every integer $n \geq 1$.

THEOREM 4.3. *A regular periodic decomposition $\mathcal{D} = (D_0, \ldots, D_{n-1})$ for a transitive dynamical system (X, f) is terminal if and only if f^n is totally transitive on each D_i.*

Note that every connected subset of a dendrite is arcwise connected and every subcontinuum of a dendrite is a dendrite (for a useful summary of properties of dendrites and their equivalent characterizations the reader is referred to [**14**]).

THEOREM 4.4. *If D is a dendrite, (D, f) is transitive and there is no terminal periodic decomposition for f, then, for any $q \in D \setminus \mathrm{End}(D)$, there is a connected regular periodic decomposition $\mathcal{D} = (D_0, \ldots, D_{m-1})$ such that $q \in D_i \cap D_j$ for some $i \neq j$.*

Proof. Since there is no terminal periodic decomposition, there are numbers $n_1, n_2, \ldots$, where each $n_j \geq 2$, and connected regular periodic decompositions

$$\mathcal{D}_k = \{D_{i_1, i_2, \ldots, i_k} : i_j \leq n_j\}$$

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of lengths $n_1 n_2 \cdots n_k$ such that $\mathcal{D}_{k+1}$ is a refinement of $\mathcal{D}_k$ and $D_{i_1,i_2,\ldots,i_k,i_{k+1}} \subset D_{i_1,i_2,\ldots,i_k}$. Fix a cutpoint $q \in D$, that is, $q \notin \mathrm{End}(D)$. We will prove that there is k such that $q \in D_{i_1,i_2,\ldots,i_k} \cap D_{j_1,j_2,\ldots,j_k}$ for some $(i_1, i_2, \ldots, i_k) \neq (j_1, j_2, \ldots, j_k)$. Assume, on the contrary, that there are indices $r_1, r_2, \ldots$ such that $q \in \mathrm{Int}_D(D_{r_1,r_2,\ldots,r_k})$ for every k. We shall arrive at a contradiction.

Every dendrite has the fixed point property, so there exists a fixed point $p \in D$. For every k and $j_i \in \{0, \ldots, n_i - 1\}$, where $1 \leq i \leq k$, we cannot have $p \in \mathrm{Int}_D(D_{j_1,j_2,\ldots,j_k})$ because then

$$p \in \mathrm{Int}_D(D_{j_1,j_2,\ldots,j_k}) \cap f(D_{j_1+1(\mathrm{mod}\, n_1)}, j_2, \ldots, j_k) = \emptyset.$$

So the only possibility is that $p \in \bigcap D_{j_1,j_2,\ldots,j_k}$. Clearly, $p \neq q$. Let V be a connected component of $D\backslash\{q\}$ which does not contain p. Note that V is open in D. Since q is a cutpoint, such a component clearly exists.

We fix any k and claim that $V \subset D_{r_1,r_2,\ldots,r_k}$. Assume that this is not the case, so there is $z \in V \cap D_{i_1,i_2,\ldots,i_k}$ where $(i_1, i_2, \ldots, i_k) \neq (r_1, r_2, \ldots, r_k)$. Since $p, z \in D_{i_1,i_2,\ldots,i_k}$ and $D_{i_1,i_2,\ldots,i_k}$ is arcwise connected, $[p, z] \subset D_{i_1,i_2,\ldots,i_k}$. Since $q \in [p, z]$, $q \in D_{i_1,i_2,\ldots,i_k}$, which is impossible because we are assuming that $q \in \mathrm{Int}_D(D_{r_1,r_2,\ldots,r_k})$ and this set is disjoint from $D_{i_1,i_2,\ldots,i_k}$. Indeed, the claim holds.

As V is open in D, $V \subset \mathrm{Int}_D(D_{r_1,r_2,\ldots,r_k})$ for any k. By transitivity, there is $m > 0$ such that $f^m(V) \cap V \neq \emptyset$. Choose $k \in \mathbb{N}$ such that $n_1 n_2 \cdots n_k > m$. The number m is smaller than the length of the decomposition $\mathcal{D}_k$, and hence

$$f^m(V) \cap V \subset f^m(D_{r_1,r_2,\ldots,r_k}) \cap \mathrm{Int}_D(D_{r_1,r_2,\ldots,r_k}) = \emptyset,$$

which is a contradiction. □

Remark 4.5. Assume that $\mathcal{C} = (C_0, \ldots, C_{n-1})$ and $\mathcal{D} = (D_0, \ldots, D_{m-1})$ are regular periodic decompositions for a dynamical system (D, f) on a dendrite D, and assume that $\mathcal{C}$ refines $\mathcal{D}$. If i, j, k are such that $D_i \cap D_j \neq \emptyset$ and $C_k \subset D_i$, then $C_k \cap D_j \neq \emptyset$ (see [**4**, Theorem 6.4]). So if there is a dendrite without terminal periodic decomposition, then, in view of Theorem 4.4, it seems to have a very special structure.

COROLLARY 4.6. *If (D, f) is a transitive dynamical system on a dendrite D and there is a point $q \in D\backslash \mathrm{End}(D)$ with dense orbit, then there is a terminal periodic decomposition for (D, f).*

Proof. For any regular periodic decomposition $\mathcal{D} = (D_0, \ldots, D_{m-1})$ the set $E = \bigcup_{i \neq j}(D_i \cap D_j)$ is invariant and nowhere dense; in particular, $q \notin E$. By Theorem 4.4, there is a terminal periodic decomposition. □

We also present an interesting property of totally transitive systems, which was first proved by Banks in [**4**, Theorem 1.1].

LEMMA 4.7. *If (X, f) is a totally transitive dynamical system with dense set of periodic points, then (X, f) is weakly mixing.*

Now, we have all the tools to prove the following fact.

THEOREM 4.8. *Let D be a dendrite and let $f : D \to D$ be a totally transitive map. Suppose that the set* $\mathrm{Per}(f)\backslash \mathrm{End}(D)$ *is dense in D. Then f is mixing.*

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Proof. Take any two non-empty open sets U, V in D. Taking a smaller set, if necessary, we may assume that U is connected. There is a periodic point $p \in V \setminus \mathrm{End}(D)$. Denote the period of p by n. Since p is a cutpoint of D, there are two open sets V_1, V_2 contained in two different connected components of $D\setminus\{p\}$. By Lemma 4.7, f is weakly mixing, and hence the $(2n+2)$-times Cartesian product of copies of f is also transitive (see [**18**]). Then there is $N > 0$ such that

$$(\Pi_{1\le i\le 2n+2} f^N)(\Pi_{1\le i\le 2n+2} U) \cap (\Pi_{0\le i\le n} f^{-i}(V_1) \times \Pi_{0\le i\le n} f^{-i}(V_2)) \neq \emptyset.$$

So $f^{N+i}(U) \cap V_1 \neq \emptyset$ and $f^{N+i}(U) \cap V_2 \neq \emptyset$ for $i = 0, 1, \ldots, n$. But since U is connected, we obtain that $p \in f^{N+i}(U)$ for $i = 0, \ldots, n$. Now, if we fix any $k \ge N$, then $k = N + i + jn$ for some $j \ge 0$ and $0 \le i < n$. Therefore

$$p = f^{jn}(p) \in f^{jn}(f^{N+i}(U)) = f^k(U),$$

which gives $f^k(U) \cap V \neq \emptyset$ and completes the proof. □

If there exists a regular periodic decomposition $\mathcal{D} = (D_0, \ldots, D_{n-1})$ for a dynamical system (X, f) such that $f^n|_{D_i}$ is mixing (weakly mixing) for every i, then we say that (X, f) is *relatively mixing* (respectively, *relatively weakly mixing*).

COROLLARY 4.9. *Let (D, f) be dynamical system on a dendrite D and assume that there is $x \in D\setminus \mathrm{End}(D)$ such that $\overline{\mathrm{Orb}^+(x)} = X$. Then (D, f) is relatively weakly mixing.*

Proof. Since D does not have isolated points, (D, f) is transitive and, by Theorem 3.5, periodic points are dense in D. By Corollary 4.6, there is terminal periodic decomposition $\mathcal{D} = (D_0, \ldots, D_{n-1})$, and $f^n|_{D_i}$ is totally transitive for each i, by Theorem 4.3. Then each $f^n|_{D_i}$ is weakly mixing, by Lemma 4.7, because sets D_i are regular closed. □

Remark 4.10. By the method of the proof in Theorem 3.5, we may not guarantee that there are periodic points which are cutpoints, and hence we cannot apply Theorem 4.8 to obtain relative mixing in Corollary 4.9.

Remark 4.11. In Theorem 1.11 in [**12**], Blokh provided a method of construction of invariant dendrites in topological Julia sets, that contain dense periodic points which are cutpoints. By Theorem 4.8 these dynamical systems are relatively mixing, when transitive.

5. *Endpoints and transitivity*

In this section we will present a construction of a dendrite and transitive map on it, where periodic points are dense and all belong to the set of endpoints of the dendrite, except for only one.

Let S be a star with center $b_\emptyset$ and beams B_n, for $n \in \mathbb{N}$. In the interior of each beam B_n, we locate a dense sequence of points $(b_{(n,i)})_{i>0}$.

In every point $b_{(n,i)}$, we attach a star $S_{(n,i)}$ with center $b_{(n,i)}$ such that $S_{(n,i)} \cap S = b_{(n,i)}$, $\mathrm{diam}(S_{(n,i)}) < 2^{-(n+i)}$ and $S_{(n,i)} \cap S_{(m,j)} = \emptyset$, whenever $(n, i) \neq (m, j)$. Let us call the beams of the star $S_{(n,i)}$ by $B_m^{(n,i)}$ for $m \in \mathbb{N}$. In the interior of every beam $B_m^{(n,i)}$, we locate, similarly to the method used above, a dense sequence of points $(b_{(n,i)\times(m,j)})_{j>0}$.

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We follow this process by induction. For any $k \in \mathbb{N}$ and for any $\alpha \in (\mathbb{N}^2)^k$, we attach a star S_α in the point b_α such that $\operatorname{diam}(S_\alpha) < 2^{-(\sum_{i=1}^{k} n_i + j_i)}$, where $\alpha = (n_1, j_1) \dots (n_k, j_k)$. We denote the beams of S_α by B_m^α, $m \in \mathbb{N}$. In the interior of every beam B_m^α, we locate, similarly, a dense sequence of points $(b_{\alpha\times(m,j)})_{j>0}$. In each point $b_{\alpha\times(m,j)}$, we attach a star $S_{\alpha\times(m,j)}$ such that $S_{\alpha\times(m,j)} \cap S_\alpha = b_{\alpha\times(m,j)}$ and $S_{\alpha\times(m,j)} \cap S_{\alpha\times(m',j')} = \emptyset$ when $(m, i) \neq (m', j')$.

Then, we let $X := \overline{S \cup \bigcup_{k>0}(\bigcup_{\alpha\in(\mathbb{N}^2)^k} S_\alpha)}$. The set of branch points of D is $\{b_\emptyset\} \cup \bigcup_{k\in\mathbb{N}}\{b_\alpha, \alpha \in (\mathbb{N}^2)^k\}$. It is clear, from the construction of X, that the set of branch point is dense and so is the set of endpoints.

For $k \in \mathbb{N} \cup \{+\infty\}$ and $i \in \{1, 2, \dots, k\}$, if $\alpha \in (\mathbb{N}^2)^k$ we denote by α_i the pair in $\mathbb{N}^2$ that occur in the position i in the sequence α. Hence we can write $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$ if $k \neq +\infty$. For $n, k \in \mathbb{N}$ and $\alpha \in (\mathbb{N}^2)^k$, the concatenation of α for n times, denoted by α^n, is the sequence $\beta \in (\mathbb{N}^2)^{nk}$ given by: $\beta_i = \alpha_i$ for each $i \in \{1, \dots, k\}$ and $\beta_j = \beta_{j-k}$ for each $j \in \{k+1, \dots, nk\}$. For $k \in \mathbb{N}$ and $k' \in \mathbb{N} \cup \{+\infty\}$ such that $k < k'$, if $\alpha \in (\mathbb{N}^2)^k$ and $\alpha' \in (\mathbb{N}^2)^{k'}$, we write $\alpha \subset \alpha'$ if $\alpha_i = \alpha'_i$ for $1 \leq i \leq k$. Let us denote this by $\Lambda = \{\emptyset\} \cup \bigcup_{k\in\mathbb{N}}(\mathbb{N}^2)^k$.

We define the function $\sigma : \Lambda \to \Lambda$, which will be used later to describe the dynamics of the map F on the set of branch points: for an $\alpha \in \Lambda$, let us denote $\alpha_i = (n_i, j_i)$ for any $1 \leq i \leq k$. Then we define $\sigma(\alpha) = \beta$ in the following way.

(i) If $n_1 = 1$ and $k > 1$, then $\beta_1 = (j_1 + n_2 - 1, j_2)$ and $\beta_i = \alpha_{i+1}$ for all $1 < i \leq k$.

(ii) If $n_1 = 1$ and $k = 1$, then $\beta = \emptyset$.

(iii) If $n_1 > 1$, then $\beta_1 = (n_1 - 1, j_1)$ and $\beta_i = \alpha_i$ for all $1 < i \leq k$.

For $k \in \mathbb{N}$ and $\alpha \in (\mathbb{N}^2)^k$, we denote by X_α the union of the closure of connected components of $X\backslash\{b_\alpha\}$ that contains all the beams B_m^α, for $m \in \mathbb{N}$. Then, for any non-empty open subset U of X, there is $\alpha \in (\mathbb{N}^2)^k$ for some $k \in \mathbb{N}$ such that $X_\alpha \subset U$.

For any $n \in \mathbb{N}$, let us denote the closure of the connected components of $D\backslash\{b_\emptyset\}$ that contains the beam B_n by D_n.

Now we are ready to define a map $F : X \to X$. We let $F|_{D_{n+1}} : D_{n+1} \to D_n$ be a homeomorphism which maps linearly each arc

$$[b_\emptyset, b_{(n+1,j_1)(n_2,j_2)\cdots(n_k,j_k)}]$$

to the arc

$$[b_\emptyset, b_{(n,j_1)(n_2,j_2)\cdots(n_k,j_k)}],$$

see Figure 3.

Now we will define the map F on the dendrite D_1. First, we let $F(b_\emptyset) = b_\emptyset$. Then we let $F|_{D_{(1,n)}} : D_{(1,n)} \to \bigcup_{i\geq n} D_i$ be a homeomorphism which maps linearly any arc

$$[b_{(1,n)}, b_{(1,n)(n_1,j_1)\cdots(n_k,j_k)}]$$

to the arc

$$[b_\emptyset, b_{(n+n_1-1,j_1)\cdots(n_k,j_k)}],$$

where $n, k \in \mathbb{N}$ and $(n_1, j_1)(n_2, j_2) \cdots (n_k, j_k) \in (\mathbb{N}^2)^k$ (see Figure 4).

In this way, the beam B_1 is collapsed to the fixed point $b_\emptyset$ and $F(D_{(1,n)}) = \bigcup_{i\geq n} D_i$.

Hence, by using the map σ, we have, for any $\alpha \in \Lambda$,

$$F(b_\alpha) = b_{\sigma(\alpha)}. \tag{5.1}$$

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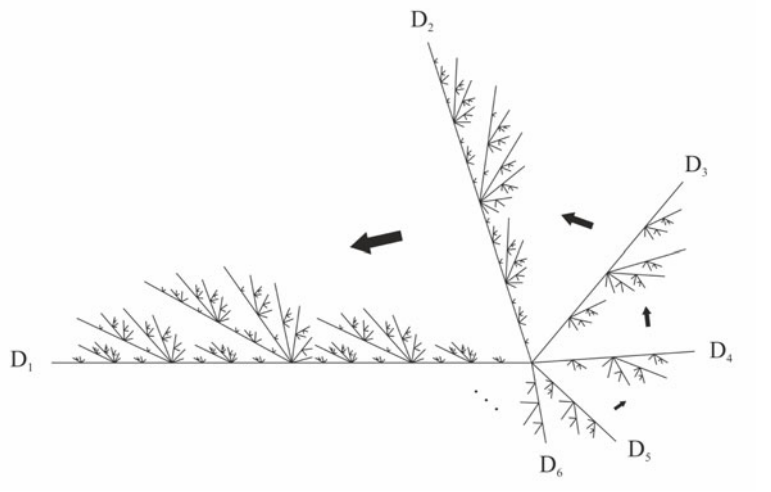


FIGURE 3. Dendrite D with a sketch of action of $F|_{D_{n+1}} : D_{n+1} \to D_n$.

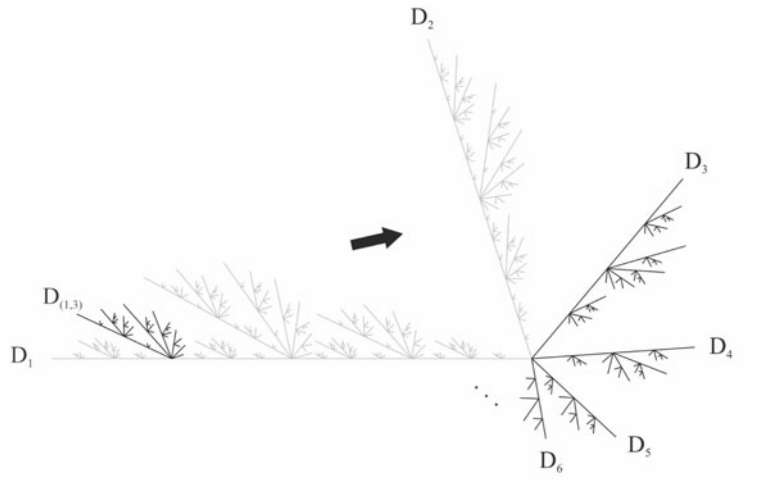


FIGURE 4. Dendrite D with a sketch of action of $F|_{D_{(1,3)}} : D_{(1,3)} \to \bigcup_{i\geq 3} D_i$.

Notice that if $1 < k \in \mathbb{N}$, $(n_1, j_1)(n_2, j_2) \cdots (n_k, j_k) \in \mathbb{N}^{2k}$ and $m = (\sum_{i=1}^{k} n_i + j_i) - k$, then

$$F^m([b_\emptyset, b_{(n_1,j_1)(n_2,j_2)\ldots(n_k,j_k)}]) = b_\emptyset. \tag{5.2}$$

THEOREM 5.1. *Let $F : X \to X$ be the map constructed in this section. The following conditions are satisfied.*

(i) *The dynamical system (X, F) is topologically exact.*

(ii) *The set of periodic points of F is dense in X and it is included in the set of endpoints of X except the point $b_\emptyset$.*

(iii) *If $x \in X$ is such that $\overline{\mathrm{Orb}^+(x)} = X$, then $x \in \mathrm{End}(X)$.*

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Proof. In order to prove (i) let us fix any non-empty open set $U \subset X$. Then there exist $k \in \mathbb{N}$ and $\alpha \in (\mathbb{N}^2)^k$ such that $X_\alpha \subset U$. Let $m = (\sum_{i=1}^{k} n_i + j_i) - k$, where $(n_i, j_i) = \alpha_i$ for $i = 1, \ldots, k$. By the construction of F (see, for example, (5.1)), we easily obtain that $F^m(X_\alpha) = X$. This completes the proof of (i).

To prove the density of the set of periodic points, it is sufficient to prove the existence of a periodic point in each X_α, for any $\alpha \in (\mathbb{N}^2)^k$ and for any $k \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have $X_{\alpha^{n+1}} \subset X_{\alpha^n}$.

Note that $\lim_{n\to+\infty} \operatorname{diam}(X_{\alpha^n}) = 0$ and hence $\bigcap_{n\geq 1} X_{\alpha^n}$ is a single point, say, $z \in \bigcap_{n\geq 1} X_{\alpha^n}$. We claim that z is an endpoint and $z = \lim_{n\to+\infty} b_{\alpha^n}$. Indeed, we have for any $n \geq 1$, $[b_{\alpha^n}, b_{\alpha^{n+1}}) \cap X_{\alpha^{n+1}} = \emptyset$ and we also have $b_{\alpha^{n+2}} \in X_{\alpha^{n+1}}$: hence $[b_{\alpha^n}, b_{\alpha^{n+1}}] \cup [b_{\alpha^{n+1}}, b_{\alpha^{n+2}}]$ is an arc. So, by induction, for any $n \geq 1$, $[b_\alpha, b_{\alpha^n}]$ is an arc, which implies that the sequence $(b_{\alpha^n})_{n\in\mathbb{N}}$ converges and hence its limit must be z. If $z \notin \operatorname{End}(X)$ then there is an endpoint $e \in X$ such that $z \in [b_\alpha, e]$ and so, for each $n \in \mathbb{N}$, $e \in X_{\alpha^n}$ since z and e are in the same connected component of $X\backslash\{b_{\alpha^n}\}$. This shows that $e, z \in \bigcap_{n\geq 1} X_{\alpha^n}$, which is a contradiction, since $z \neq e$. We conclude that z is an endpoint of X.

Now, we are going to prove that z is a fixed point of F^m, where $m = (\sum_{i=1}^{k} n_i + j_i) - k$. Fix any $\varepsilon > 0$ and let $0 < \delta < \varepsilon/2$ be such that $d(F^m(p), F^m(q)) < \varepsilon/2$, provided that $d(p, q) < \delta$. There is $N > 0$ such that, for each $n > N$, $d(z, b_{\alpha^n}) < \delta$ and so, for every $n > N$,

$$d(F^m(z), z) \leq d(z, b_{\alpha^n}) + d(F^m(z), F^m(b_{\alpha^{n+1}})) < \varepsilon.$$

Since ε was arbitrary, z is a fixed point of F^m. By (5.2), all cut points except $b_\emptyset$ are not periodic (but eventually fixed, since they are eventually sent by F to $b_\emptyset$). □

In next theorem, we will analyze the construction presented by Hoehn and Mouron in [**24**, Example 17]. Recall that a pair (x, y) is *proximal* in a dynamical system (X, f) if $\liminf_{n\to+\infty} d(f^n(x), f^n(y)) = 0$, and (X, f) is *proximal* if any pair of points in X is a proximal pair. It is known (e.g. see [**2**]) that a dynamical system (X, f) is proximal if and only if it has a fixed point which is the unique minimal subset of X. We will show that [**24**, Example 17] gives an example of a transitive dendrite map with a unique periodic point and hence without a dense set of periodic points.

THEOREM 5.2. *There is a dendrite D and a map $F : D \to D$ such that:*

(i) *the dynamical system (D, F) is weakly mixing but not mixing; and*

(ii) *the dynamical system (D, F) is proximal (in particular it has a unique periodic point which is the fixed point).*

Proof. Let D be the Ważewski universal dendrite (see [**32**, 10.37, p. 181]). It was proved in [**24**, Example 17] that if we fix two sequences N_k, M_k, $N_{k+1} = M_k + k$ where M_k is increasing sufficiently fast, and put $Z = \bigcup_k \{N_k, N_k + 1, \ldots, M_k\}$, then there exist subdendrites $\{E_j\}_{j=0}^{\infty}$, a point $o \in D$ and a map $F : D \to D$ such that:

(5.1) $D = \bigcup_{j=0}^{\infty} E_j$ and $E_i \cap E_j = \{o\}$ for each $i \neq j$;

(5.2) (D, F) is weakly mixing but not mixing;

(5.3) $F(o) = o$ and $f(E_j) = E_{j-1}$ for $j > 0$; and

(5.4) $f^n(E_0) \subset \bigcup_{n+j-1\in Z} E_j$ for every $n > 0$.

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Let M be a minimal subset of D, fix any $x \in M$ and suppose that $x \neq o$. Then there exist $s \geq 0$ and an open neighborhood U of x such that $U \subset E_s$. Since x belongs to a minimal set, there exists $K > 0$ such that, for every $n > 0$, there is $0 \leq j < K$ such that $f^{n+j}(x) \in U$.

Note that if we put $k = M_{s+K+3} + 1$, then $\{k, k+1, \ldots, k+s+K+1\} \cap Z = \emptyset$. Take any integer $0 \leq i \leq K$ and observe that

$$f^{i+k+s}(x) \in f^{k+i}(E_0) \subset \bigcup_{k+i+j-1 \in Z} E_j \subset \bigcup_{j>s} E_j.$$

Hence $f^{i+k+s}(x) \notin U$ for $i = 0, 1, \ldots, K$, which is a contradiction. This proves that $M = \{o\}$ is the unique minimal set for the dynamical system (D, F). The proof is complete. □

6. *Periodic points on dendrites and other continua*

It is easy to verify that if f has dense periodic points, then so does the induced map 2^f. It is also not surprising that the converse implication does not hold, in general. In this section, we will show that on some dendrites such a converse implication is valid (in fact, as we will see, it is valid on a wider class of continua containing some dendrites).

For the convenience of the reader, we present here a standard, still very useful fact (see [**32**, Theorems 8.16 and 8.23]).

LEMMA 6.1. *Let X be a continuum, let $\alpha \subset X$ be an arc and let $g : \alpha \to X$ be continuous. Then $g(\alpha)$ is arcwise connected and locally connected.*

Now we are ready to prove the following theorem.

THEOREM 6.2. *Let X be a continuum and $f : X \to X$ be a map such that* $\mathrm{Per}(2^f)$ *is dense in 2^X. Let $[a, b]$ be a free arc in X with $a \neq b$. Then* $\mathrm{Per}(f) \cap (a, b) \neq \emptyset$.

Proof. Fix $p \in (a, b)$. Let us consider that in the natural order $<$ of the arc $[a, b]$ from a to b, we have $a < p < b$. Take $a_1, b_1 \in [a, b]$ so that $a < a_1 < p < b_1 < b$. Since the open arcs (a_1, p) and (p, b_1) are open sets in X and $\mathrm{Per}(2^f)$ is dense in 2^X, there exist $A_1 \in \langle (a_1, p) \rangle \cap \mathrm{Per}(2^f)$ and $A_2 \in \langle (p, b_1) \rangle \cap \mathrm{Per}(2^f)$. Then $A_1, A_2 \in 2^X$, $A_1 \subset (a_1, p) \subset [a_1, b_1] \subset (a, b)$, $A_2 \subset (p, b_1) \subset [a_1, b_1] \subset (a, b)$ and there exist $n_1, n_2 \in \mathbb{N}$ such that

$$(2^f)^{n_1}(A_1) = f^{n_1}(A_1) = A_1 \quad \text{and} \quad (2^f)^{n_2}(A_2) = f^{n_2}(A_2) = A_2.$$

Let n be the lowest common multiple of n_1 and n_2. Then $f^n(A_1) = A_1$ and $f^n(A_2) = A_2$. Note that if A_1 is finite, then $(f^{n_1})|_{A_1} : A_1 \to A_1$ is a permutation of the elements of A_1 and, since permutations of finite sets have finite order, there exists $k \in \mathbb{N}$ such that $(f^{n_1}|_{A_1})^k = \mathrm{id}_{A_1}$. Hence $A_1 \subset \mathrm{Per}(f)$ and then $\mathrm{Per}(f) \cap (a, b) \neq \emptyset$. Similarly, if A_2 is finite then, proceeding as before, it follows that $\mathrm{Per}(f) \cap (a, b) \neq \emptyset$. Hence we may assume that both sets A_1 and A_2 are infinite. Define

$$c_1 = \min(A_1), \quad d_1 = \max(A_1), \quad c_2 = \min(A_2) \quad \text{and} \quad d_2 = \max(A_2).$$

Then $c_1, d_1 \in A_1 \subset [c_1, d_1] \subset (a_1, p)$, $c_2, d_2 \in A_2 \subset [c_2, d_2] \subset (p, b_1)$ and $c_1 < d_1 < p < c_2 < d_2$.

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We claim that

(6.1) there exist $c_1', d_1' \in [c_1, d_1]$ such that $(f^n)(c_1') = c_1$, $(f^n)(d_1') = d_1$ and $(c_1', d_1') \cap (f^n)^{-1}(\{c_1, d_1\}) = \emptyset$.

To show (6.1) let

$$x = \max((f^n)^{-1}(\{c_1\}) \cap [c_1, d_1]) \quad \text{and} \quad y = \min((f^n)^{-1}(\{d_1\}) \cap [c_1, d_1]).$$

Then $x \neq y$. If $x < y$, we define $c_1' = x$ and $d_1' = y$. Then $c_1' < d_1'$ and, by the definitions of x and y, the points c_1' and d_1' satisfy (6.1). Now assume that $y < x$. Define

$$c_1' = \min((f^n)^{-1}(\{c_1\}) \cap [y, x]) \quad \text{and} \quad d_1' = \max((f^n)^{-1}(\{d_1\}) \cap [y, c_1']).$$

Then $d_1' < c_1'$ and the points c_1', d_1' satisfy (6.1). This completes the proof of (6.1).

Proceeding as in the proof of (6.1) we can show that

(6.2) there exist $c_2', d_2' \in [c_2, d_2]$ such that $(f^n)(c_2') = c_2$, $(f^n)(d_2') = d_2$ and $(c_2', d_2') \cap (f^n)^{-1}(\{c_2, d_2\}) = \emptyset$.

By the properties of the points c_1' and d_1' given in (6.1), Lemma 6.1 and the fact that $[c_1, d_1]$ is a free arc in X, the set $f^n([c_1', d_1'])$ satisfies one of the following two conditions. Either:

(6.3) $f^n([c_1', d_1']) = [c_1, d_1]$; or

(6.4) $f^n([c_1', d_1'])$ is a locally connected subcontinuum of X such that $c_1, d_1 \in f^n([c_1', d_1'])$ and $f^n([c_1', d_1']) \cap (c_1, d_1) = \emptyset$.

Similarly, the set $f^n([c_2', d_2']) = [c_2, d_2]$ satisfies one of the following two conditions. Either:

(6.5) $f^n([c_2', d_2']) = [c_2, d_2]$; or

(6.6) $f^n([c_2', d_2'])$ is a locally connected subcontinuum of X such that $c_2, d_2 \in f^n([c_2', d_2'])$ and $f^n([c_2', d_2']) \cap (c_2, d_2) = \emptyset$.

Assume that $f^n([c_1', d_1']) = [c_1, d_1]$. Since $[c_1', d_1']$ is an arc contained in the arc $[c_1, d_1]$ such that $(f^n)(c_1') = c_1$ and $(f^n)(d_1') = d_1$, there exists a fixed point of f^n in $[c_1', d_1']$. This implies that $\mathrm{Per}(f) \cap (a, b) \neq \emptyset$. If $f^n([c_2', d_2']) = [c_2, d_2]$ then, proceeding as before, there exists a fixed point of f^n in $[c_2', d_2']$ and then $\mathrm{Per}(f) \cap (a, b) \neq \emptyset$.

Now let us assume now remaining case, that is, that conditions (6.4) and (6.6) hold. By (6.4) and Lemma 6.1, the set $f^n([c_1', d_1'])$ contains an arc B_1 with endpoints c_1 and d_1 so that $B_1 \subset X \setminus (c_1, d_1)$. Since $[a, b]$ is a free arc in X and the arcs in $[a, b]$ are unique, the subarc $[d_1, b]$ is contained in B_1. In particular,

$$[c_2', d_2'] \subset [d_1, b] \subset B_1 \subset f^n([c_1', d_1']).$$

Proceeding as before, and now using (6.6) we infer that there exists an arc B_2 with endpoints c_2 and d_2 so that $B_2 \subset X \setminus (c_2, d_2)$. Hence

$$[c_1', d_1'] \subset [a, c_2] \subset B_2 \subset f^n([c_2', d_2']).$$

It is not hard to see that there exist $c_1^\star, d_1^\star \in [c_1', d_1']$ such that $f^n([c_1^\star, d_1^\star]) = [c_2', d_2']$. There also exist $c_2^\star, d_2^\star \in [c_2', d_2']$ so that $f^n([c_2^\star, d_2^\star]) = [c_1^\star, d_1^\star]$. Hence

$$[c_1^\star, d_1^\star] = f^n([c_2^\star, d_2^\star]) \subset f^n([c_2', d_2']) = f^n(f^n([c_1^\star, d_1^\star])) = f^{2n}([c_1^\star, d_1^\star]) \subset (a, b).$$

From this, it follows that there is a fixed point of f^{2n} in $[c_1^\star, d_1^\star]$. Hence $\mathrm{Per}(f) \cap (a, b) \neq \emptyset$. □

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COROLLARY 6.3. *Let X be a continuum and $f : X \to X$ be a transitive map such that* $\mathrm{Per}(2^f)$ *is dense in* 2^X. *If X contains a free arc, then* $\mathrm{Per}(f)$ *is dense in X.*

Proof. Let $[a, b]$ be a free arc in X with $a \neq b$. By Theorem 6.2, $\mathrm{Per}(f) \cap (a, b) \neq \emptyset$. Then, by [**17**, Lemma 21], $\mathrm{Per}(f)$ is dense in X. □

THEOREM 6.4. *Let X be an almost meshed continuum and $f : X \to X$ be a map. If* $\mathrm{Per}(2^f)$ *is dense in* 2^X, *then* $\mathrm{Per}(f)$ *is dense in X.*

Proof. Let U be a non-empty open subset of X. Since X is almost meshed, the set $\mathcal{FA}(X)$ is dense in X. Then there exists $p \in U \cap \mathcal{FA}(X)$. Let J be a free arc in X such that $p \in \mathrm{Int}_X(J)$. Then $p \in U \cap \mathrm{Int}_X(J)$, so there exist $a, b \in U \cap \mathrm{Int}_X(J)$ such that $p \in (a, b) \subset [a, b] \subset U \cap \mathrm{Int}_X(J)$. Then $a \neq b$ and, by Theorem 6.2, $\mathrm{Per}(f) \cap (a, b) \neq \emptyset$. This implies that $\mathrm{Per}(f) \cap U \neq \emptyset$, so $\mathrm{Per}(f)$ is dense in X. □

Since topological graphs are almost meshed continua, as a corollary of Theorem 6.4, we obtain the following result.

COROLLARY 6.5. *Let X be a topological graph and $f : X \to X$ be a map. If* $\mathrm{Per}(2^f)$ *is dense in* 2^X, *then* $\mathrm{Per}(f)$ *is dense in X.*

Corollary 6.5 appeared first in [**19**, Theorem 3.3, p. 474]. However, the proof seems to be incomplete. As it is presented in [**19**], it works fine for trees.

COROLLARY 6.6. *Let (D, f) be a dynamical system on a dendrite D with either a countable or closed set of endpoints (i.e.* $\mathrm{End}(D) = \mathrm{Cl}_X(\mathrm{End}(D))$*). If* $\mathrm{Per}(2^f)$ *is dense in* 2^D, *then* $\mathrm{Per}(f)$ *is dense in D.*

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