Student Problems

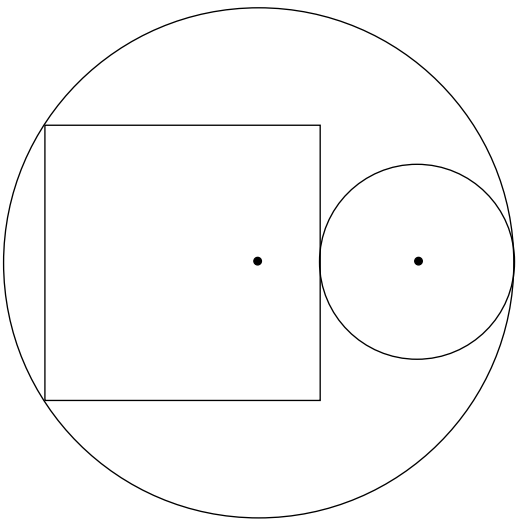
Students up to the age of 19 are invited to send solutions to either or both of the following problems to Tuya Sa, SCH.1.17, Schofield Building, Loughborough University, Loughborough, LE11 3TU. Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Solutions should arrive by 20th January 2023 and will be published in the March 2023 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions must be accompanied by the SPC permission form which is available on the Mathematical Association website

https:www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others****.*

Problem 2022.5 **(Paul Stephenson)**



A square and circle fit a circle of unit radius symmetrically as shown. Is it possible to adjust their sizes so that their combined area is under half of that unit circle?

Problem 2022.6 **(Paul Stephenson)**

The primes p and q ($p > q$) are greater than 3. They are respectively the hypotenuse and a side of a Pythagorean triangle. Show that 12 divides the other side, r.

https://doi.org/10.1017/mag.2022.149 Published online by Cambridge University Press

Solutions to 2022.3 and 2022.4

Both problems were solved by Soham Bhadra (Patha Bhavan, India), Aditya Gupta (Westminster School, London, UK) and Owen Mackenzie (King's College London Mathematics School, UK). Problem 2022.3 was also solved by Asmi Choudhary (Delhi Public School, Kuwait), Aditya Ranjan Patra (St. Xavier's High School, India) and Talaal Farooqi (Solihull School, UK). Problem 2022.4 was also solved by Aparup Roy (Zoom International School, India). It was hard to choose the prizewinners as all students gave impressive solutions.

Problem 2022.3 **(Mark Emson)**

Evaluate

$$\int_0^\infty \frac{1}{1+x^3}\,dx.$$

Solution by Owen Mackenzie

Let $I = \int_0^\infty \frac{1}{1+x^3}\,dx$. Using the substitution $y = \frac{1}{x}$ we get

$$\begin{aligned} I &= \int_0^\infty \frac{1}{1+x^3}\,dx \\ &= \int_\infty^0 \frac{y^3}{1+y^3}\,\frac{-1}{y^2}\,dy \\ &= \int_0^\infty \frac{y}{1+y^3}\,dy \\ &= \int_0^\infty \frac{x}{1+x^3}\,dx. \end{aligned}$$

This means we can write,

$$\begin{aligned} I &= \frac{1}{2}(I+I) \\ &= \frac{1}{2}\left(\int_0^\infty \frac{1}{1+x^3}\,dx + \int_0^\infty \frac{x}{1+x^3}\,dx\right) \\ &= \frac{1}{2}\int_0^\infty \frac{1+x}{1+x^3}\,dx \\ &= \frac{1}{2}\int_0^\infty \frac{1}{1-x+x^2}\,dx \\ &= \int_0^\infty \frac{2}{4x^2-4x+4}\,dx \\ &= \int_0^\infty \frac{2}{(2x-1)^2+3}\,dx \\ &= \lim_{N\to\infty}\int_0^N \frac{\frac{2}{3}}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}dx \\ &= \lim_{N\to\infty}\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^N \end{aligned}$$

https://doi.org/10.1017/mag.2022.149 Published online by Cambridge University Press

$$= \frac{1}{\sqrt{3}} \lim_{N \to \infty} \left(\arctan\left(\frac{2N - 1}{\sqrt{3}} \right) - \arctan\left(\frac{-1}{\sqrt{3}} \right) \right)$$
$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{2} + \frac{\pi}{6} \right) = \frac{2\sqrt{3}\pi}{9}.$$

Problem 2022.4 **(Geoffrey Strickland)**

A "nearly isosceles" right-angled triangle with integer side-lengths is defined as one in which the two sides adjacent to the right angle differ in length by just 1 unit. A triangle with side lengths 20, 21 and 29 is an example. Find, with proof, a method for generating a sequence of such triangles.

Solution by the proposer

A 3 - 4 - 5 triangle is another obvious example and a quick spreadsheet search for values of n such that $n^2 + (n + 1)^2$ is a perfect square yields 119 - 120 - 169 and 696 - 697 - 985.

No obvious pattern yet but it is known that all Pythagorean Triples can be obtained from $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$ where x and y are co-prime integers with one being even and the other odd. It is helpful to list the results obtained so far together with the associated values of x and y.

$a = (x^2 - y^2)$	$b = 2xy$	$c = (x^2 + y^2)$	x	y
3	4	5	2	1
21	20	29	5	2
119	120	169	12	5
697	696	985	29	12

The table above suggests the following hypothesis:

If x_1, y_1 are integers such that $2x_1y_1 - (x_1^2 - y_1^2) = 1$. (A)

Then $x_2 = (2x_1 + y_1)$ and $y_2 = x_1$ will satisfy $2x_2y_2 - (x_2 - y_2)^2 = -1$. (B)

Substituting these values into left-hand-side of B gives:

$$2(2x_1 + y_1)x_1 - ((2x_1 + y_1)^2 - x_1^2)$$
$$= (x_1^2 - y_1^2) - 2x_1y_1 = -1 \text{ (from A).}$$

It is clear that if the expression in A was equal to −1 then LHS(B) will equal 1. Which explains why in the table above $b - a$ is alternately +1 and −1. The procedure given above can obviously be continued indefinitely.

Prize Winners

The first prize of £25 is awarded to Owen Mackenzie. The second prize of £20 is awarded to Aditya Gupta.

BETH WOOLLACOTT and TUYA SA

10.1017/mag.2022.149 © The Authors, 2022
Published by Cambridge University Press on behalf of The Mathematical Association

https://doi.org/10.1017/mag.2022.149 Published online by Cambridge University Press