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On transverse rigidity for singular foliations in $(\mathbb{C}^2, 0)$

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Abstract. This note is a companion paper to 'Monodromie et classification topologique des germes de feuilletages holomorphes' (in French) by Marin and Mattei. In their paper the authors develop a very complete theory about topologically conjugate singularities of foliations. However, their techniques require the topological conjugacies to be transversely holomorphic. The purpose of the present note is to show that this assumption can be made without restricting the generality of their results.

1. *Introduction*

The purpose of this note is to provide a complement to the work of Marin and Mattei [**M-M-2**] by proving Theorems A and B below. Before stating these theorems it is convenient to recall some basic facts from the theory of singularities of holomorphic foliations. For further information and background material, the reader is referred to [**M-M-1, M-M-2**]. Consider a singular holomorphic foliation $\mathcal{F}$ defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$. This foliation can then be given by holomorphic 1-form ω having an isolated singularity at the origin. This form is clearly unique up to an invertible factor. Recall that the singularity of $\mathcal{F}$ at the origin is said to be *reduced* if one of the following holds.

- ω has two eigenvalues λ_1, λ_2 different from zero at the origin and, besides, neither $-\lambda_1/\lambda_2$ nor $-\lambda_2/\lambda_1$ belongs to $\mathbb{N}$.
- ω has exactly one eigenvalue $\lambda \neq 0$ at the origin. In this case (the germ of) $\mathcal{F}$ is called a *saddle-node*.

In the first case above, there are local coordinates (x, y) where ω becomes

$$\omega = \lambda_1 x(1 + \cdots)\, dy + \lambda_2 y(1 + \cdots)\, dx.$$

Singularities as above for which λ_1/λ_2 belongs to $\mathbb{R}_-$ will be called nodes. On the other hand, when $\mathcal{F}$ is a saddle-node, then the 1-form ω can be chosen as

$$\omega = x(1 + R(x, y))\, dy + y^{p+1}\, dx,$$

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where $R(0, 0) = 0$ and $p \geq 1$. This form is called the Dulac normal form of the saddle-node. The Dulac normal form shows in particular that $\mathcal{F}$ possesses a smooth invariant curve which is associated with the direction of its eigenvalue different from zero. This curve is usually referred to as the *strong invariant manifold* of $\mathcal{F}$ (or of the saddle-node).

Next recall that a Seidenberg procedure consists of a finite sequence of blow-up maps and corresponding induced foliations

$$\mathcal{F} = \mathcal{F}^0 \xleftarrow{\pi_1} \mathcal{F}^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} \mathcal{F}^r$$

such that each blow-up map π_i is centered at a (non-reduced) singularity of $\mathcal{F}^{i-1}$. Given a singular foliation $\mathcal{F}$ as above, there always exists a Seidenberg procedure leading to a foliation $\mathcal{F}^r$ all of whose singularities are reduced [**S**]. The resulting foliation $\mathcal{F}^r$ is then called a Seidenberg resolution of $\mathcal{F}$. Furthermore, $\mathcal{F}$ is said to be *non-dicritical* if the exceptional divisor $(\pi_1 \circ \cdots \circ \pi_r)^{-1}(0)$ is totally invariant for $\mathcal{F}^r$. Note that when this property holds for one Seidenberg resolution of $\mathcal{F}$ it will automatically hold for every Seidenberg procedure.

For a non-dicritical foliation $\mathcal{F}$, its *projective holonomy* is defined as follows. Let $\widetilde{\mathcal{F}}$ denote the blow-up of $\mathcal{F}$. Then the exceptional divisor $\pi^{-1}(0)$ is invariant for $\widetilde{\mathcal{F}}$ so that its regular part, $\pi^{-1}(0)\backslash\mathrm{Sing}(\widetilde{\mathcal{F}})$, yields a regular leaf of $\widetilde{\mathcal{F}}$. The natural holonomy representation from the fundamental group of $\pi^{-1}(0)\backslash\mathrm{Sing}(\widetilde{\mathcal{F}})$ to $\mathrm{Diff}(\mathbb{C}, 0)$ is called the *projective holonomy* of $\mathcal{F}$.

In the following let us consider non-dicritical singular foliations $\mathcal{F}_1$, $\mathcal{F}_2$. Let us assume that the blow-up $\widetilde{\mathcal{F}}^1$ ($\widetilde{\mathcal{F}}^2$) of $\mathcal{F}_1$ ($\mathcal{F}_2$) has only reduced singularities. We suppose in addition that none of these singularities is a node. Besides, saddle-node singularities are supposed to have their strong invariant manifold contained in $\pi^{-1}(0)$. We are now finally able to state Theorem A in a form that is prototypical for the study carried out in [**M-M-2**].

THEOREM A. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two non-dicritical singular holomorphic foliations defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$ and satisfying the above conditions. Suppose that the projective holonomy of $\mathcal{F}_1$ is not solvable. Then every (local) homeomorphism conjugating $\mathcal{F}_1$, $\mathcal{F}_2$ is transversely conformal.*

Foliations satisfying the conditions of Theorem A are, in fact, rather 'generic' in a suitable sense. Also they contain most of the essential difficulties discussed in [**M-M-2**]. The generalization of this theorem to the other cases treated in [**M-M-2**] is not hard. It depends on Seidenberg's theorem, mentioned above, on the so-called Dulac transforms discussed in details in [**M-M-2**] and on our Theorem C concerning topological rigidity for certain pseudogroups. For example, we have the following statement that is well adapted to the study carried out in [**M-M-2**].

THEOREM B. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two non-dicritical singular holomorphic foliations defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$. Suppose that the Seidenberg resolutions of $\mathcal{F}_1$, $\mathcal{F}_2$ contain neither singularities of type node nor singularities of type saddle-node. Suppose also the existence of a component C_1 of the total exceptional divisor associated with the resolution of $\mathcal{F}_1$ possessing a non-solvable holonomy group. Then every (local) homeomorphism conjugating $\mathcal{F}_1$, $\mathcal{F}_2$ is transversely conformal.*

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We shall not elaborate on further generalizations of the above statements (e.g. toward codimension-one foliations), instead we devote the remainder of this introduction to explaining the role of Theorems A and B in the theory of Marin and Mattei. This will be done by putting these theorems in perspective with previous rigidity results.

First let us briefly recall the context of [**M-M-2**]. Consider again non-dicritical singular foliations $\mathcal{F}_1, \mathcal{F}_2$ defined around $(0, 0) \in \mathbb{C}^2$ along with their Seidenberg resolutions $\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2$. Let Π_1, Π_2 denote the respective resolution maps. Here we consider the general case where Π_1, Π_2 consist of finite compositions of blow-up maps. Since $\mathcal{F}_1, \mathcal{F}_2$ are non-dicritical, every irreducible component $C_{1,i}$ $(C_{2,j})$ of the exceptional divisor $\Pi_1^{-1}(0)$ $(\Pi_2^{-1}(0))$ is invariant by $\widetilde{\mathcal{F}}_1$ $(\widetilde{\mathcal{F}}_2)$. Denote by $\mathrm{Sing}(\widetilde{\mathcal{F}}_1)$ $(\mathrm{Sing}(\widetilde{\mathcal{F}}_2))$ the finite set consisting of the singular points of $\widetilde{\mathcal{F}}_1$ $(\widetilde{\mathcal{F}}_2)$. For every component $C_{1,i}$ as above, $L_{1,i} = C_{1,i} \backslash \mathrm{Sing}(\widetilde{\mathcal{F}}_1)$ is a regular leaf of $\widetilde{\mathcal{F}}_1$. Hence it is natural to consider the induced holonomy representation

$$\rho_{1,i} : \pi_1(L_{1,i}) \longrightarrow \mathrm{Diff}(\mathbb{C}, 0).$$

In the following we suppose the existence of $L_{1,i}$ such that, at the level of groups of germs, the image $\rho_{1,i}(\pi_1(L_{1,i})) \subset \mathrm{Diff}(\mathbb{C}, 0)$ is not solvable. This very general condition will be assumed without further comment throughout this note. The irreducible component of $\Pi_1^{-1}(0)$ mentioned above will simply be denoted by C_1. Similarly, we shall write ρ_1, L_1 instead of $\rho_{1,i}$ $L_{1,i}$.

Suppose now the existence of a topological conjugacy Ξ between $\widetilde{\mathcal{F}}_1$, $\widetilde{\mathcal{F}}_2$. Then Ξ lifts to a homeomorphism $\widetilde{\Xi} : \widetilde{U}_1 \backslash \Pi_1^{-1}(0) \to \widetilde{U}_2 \backslash \Pi_2^{-1}(0)$ where $\widetilde{U}_1$ $(\widetilde{U}_2)$ is a small neighborhood of $\Pi_1^{-1}(0)$ $(\Pi_2^{-1}(0))$. However, a central difficulty in the study of these topological conjugacies lies precisely in the fact that $\widetilde{\Xi}$ cannot *a priori* be extended to the exceptional divisors $\Pi_1^{-1}(0)$, $\Pi_2^{-1}(0)$. In particular, we cannot ensure the existence of a topological conjugacy between the holonomy groups $\rho_{1,i}(\pi_1(L_{1,i}))$ and $\rho_{2,j}(\pi_1(L_{2,j}))$ for 'corresponding components' $C_{1,i} \subseteq \Pi_1^{-1}(0)$ and $C_{2,j} \subseteq \Pi_2^{-1}(0)$. Indeed the notion of 'corresponding components' is already unclear from our assumption. If this 'extended conjugacy' were to exist, then Shcherbakov–Nakai rigidity theorem [**Na, Sh1, Sh2**] would immediately imply the conformal character of the conjugacy in question.

To remedy the above difficulty, consider a small section Σ_1 transverse to C_1 so as to be able to identify the holonomy pseudogroup of $L_1 = C_1 \backslash \mathrm{Sing}(\widetilde{\mathcal{F}}_1)$ with a pseudogroup of holomorphic diffeomorphisms defined on a neighborhood V_1 of $0 \in \mathbb{C}$. Here we recall that C_1 is an irreducible component of $\Pi_1^{-1}(0)$ having non-solvable holonomy in the sense already explained. This pseudogroup will be denoted by Γ_1. For the applications to foliations, it is convenient to make clear what is meant by the pseudogroup Γ_1. First Γ_1 is generated by finitely many local holomorphic diffeomorphisms $f_1, \ldots, f_s$ along with their inverses $f_1^{-1}, \ldots, f_s^{-1}$. Then every element $f \in \{f_1, f_1^{-1}, \ldots, f_s, f_s^{-1}\}$ is defined on a common neighborhood U of $0 \in \mathbb{C}$. Next fix a 'word' $g = f_{i_k} \circ \cdots \circ f_{i_1}$ on the generators $\{f_1, f_1^{-1}, \ldots, f_s, f_s^{-1}\}$. Given a subset $V \subset U$, the corresponding holomorphic diffeomorphism $g = f_{i_k} \circ \cdots \circ f_{i_1} : V \to \mathbb{C}$ belongs to the pseudogroup Γ_1 *generated by* $\{f_1, f_1^{-1}, \ldots, f_s, f_s^{-1}\}$ *on* U if and only if for every $1 \leq l < k$ the diffeomorphism $f_{i_l} \circ \cdots \circ f_{i_1}$ satisfies $f_{i_k} \circ \cdots \circ f_{i_1}(V) \subset U$. Note in particular that some words $f_{i_k} \circ \cdots \circ f_{i_1}$ may have empty domain of definition. Also note that the maximal domain of definition of the word $f_{i_k} \circ \cdots \circ f_{i_1}$ as an element of Γ_1 is given by

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the intersection $U \cap \bigcap_{j=1}^{k-1}(f_{i_s} \circ \cdots \circ f_{i_1})^{-1}(U)$. In particular, if $f_{i_k} \circ \cdots \circ f_{i_1}$ happens to have a holomorphic extension to a set $\widetilde{V}$ which is not contained in $U \cap \bigcap_{j=1}^{k-1}(f_{i_s} \circ \cdots \circ f_{i_1})^{-1}(U)$ this extended diffeomorphism is not considered as belonging to Γ_1.

Now considering Γ_1 as above, we note that this pseudogroup is non-solvable in the sense that its germ at $0 \in \mathbb{C}$ is a non-solvable group (by assumption). Given an open set $W_1 \subset V_1$ whose closure $\overline{W}_1$ does not contain $0 \in \mathbb{C}$, let $\Gamma_{1|W}$ denote the pseudogroup of mappings from W_1 to $\mathbb{C}$ induced by restriction of the elements in Γ_1. This pseudogroup is defined analogously to Γ_1 itself as follows: again we can suppose that W belongs to the common domain of definition of the generators $\{f_1, f_1^{-1}, \ldots, f_s, f_s^{-1}\}$. Then given a word $f_{i_k} \circ \cdots \circ f_{i_1}$ on these generators, the 'maximal' domain of definition for the corresponding element of $\Gamma_{1|W}$ is given by the intersection $W \cap \bigcap_{j=1}^{k-1}(f_{i_s} \circ \cdots \circ f_{i_1})^{-1}(W)$. Through $\widetilde{\Xi}$, we obtain a homeomorphism $\xi : W_1 \to W_2 = \xi(W_1) \subset \mathbb{C}$ conjugating Γ_1 to a certain pseudogroup Γ_2 consisting of holomorphic mappings from subsets of W_2 to $\mathbb{C}$. Here it is to be noted that 0 does not belong to the closure of W_2, and similarly that elements of Γ_2 need not be defined at $0 \in \mathbb{C}$. However, the pseudogroup Γ_2 is naturally contained in the holonomy groupoid of $\widetilde{\mathcal{F}}_2$ on $\widetilde{U}_2$. Then we shall prove the following theorem.

THEOREM C. *Let Γ_1, Γ_2 and $\xi : W_1 \to W_2$ be as above. Then ξ is conformal. In particular, ξ is a holomorphic diffeomorphism provided that it preserves orientation.*

Note that for 'most' non-solvable groups of Diff($\mathbb{C}$, 0) the stabilizer of every point $p \neq 0$ is either trivial or cyclic. In other words, although we can find non-trivial elements of Γ_1 having fixed points in W_1, we cannot in general find a non-solvable pseudogroup with a common fixed point. This means that the proof of Theorem C cannot be reduced to an application of Shcherbakov–Nakai theorem. However, a similar method introduced in [**R**] allows us to deal with pseudogroups as above, i.e. with those pseudogroups for which every point has cyclic (possibly trivial) stabilizer. It should be mentioned that many ideas in the mentioned papers [**Gh, Na, R, Sh1, Sh2**] were greatly expanded in [**L-R**] to handle a different problem concerning holomorphic foliations in higher dimensions. Independently of [**L-R**], Belliart has also further developed these ideas in [**Be**]. Essentially the proofs of the theorems stated above amount to an appropriate combination of ideas/methods appearing in the preceding works, most notably in [**B-L-L, Gh, L-R, Na**]. However, since the corresponding material is spread over a few papers, I have tried to make the discussion reasonably self-contained. This is also convenient because these techniques differ markedly from those of [**M-M-2**], naturally reducing the number of readers comfortable with the two types of arguments. Yet, while attempting to make this paper self-contained, I have included precise references to the papers mentioned, mostly related to elementary estimates, so as to avoid the pointless duplication of published work.

In closing this section, let us also mention that Theorem C admits versions involving pseudogroups that need not arise from a foliation (or from the holonomy group of a leaf of a foliation). An example of these versions is Theorem 4.3 stated at the end of §4.

2. *Commutators and non-solvable groups in* Diff($\mathbb{C}$, 0)

Consider an open set $W \subset \mathbb{C}$ and finitely many diffeomorphisms $f_1, \ldots, f_s$ together with their inverses $f_1^{-1}, \ldots, f_s^{-1}$ such that all the elements in $\{f_1, f_1^{-1}, \ldots, f_s, f_s^{-1}\}$ are

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defined on the whole of W. Within the framework of the introduction, throughout this work by the *pseudogroup generated by* $\{f_1, f_1^{-1}, \ldots, f_s, f_s^{-1}\}$ is meant the pseudogroup Γ for which an element $g = f_{i_k} \circ \cdots \circ f_{i_1} : V \to \mathbb{C}$ on the generators $\{f_1, f_1^{-1}, \ldots, f_s, f_s^{-1}\}$ belongs to Γ if and only if $V \subseteq W \cap \bigcap_{j=1}^{k-1} (f_{i_s} \circ \cdots \circ f_{i_1})^{-1}(W)$.

Consider a subgroup G of $\mathrm{Diff}(\mathbb{C}, 0)$ generated by finitely many local diffeomorphisms. We say that G is non-solvable if its germ at $0 \in \mathbb{C}$ is a non-solvable group. In particular, the first derived group D^1G, i.e. the subgroup generated by the elements of G having the form $[g_1, g_2] = g_1 \circ g_2 \circ g_1^{-1} \circ g_2^{-1}$, $g_1, g_2 \in G$, is itself non-solvable. Since every element g in D^1G is *tangent to the identity* i.e. it satisfies $g'(0) = 1$, we can assume without loss of generality that G is entirely constituted by elements tangent to the identity. Given an element g of $\mathrm{Diff}(\mathbb{C}, 0)$ having the form $g(z) = z + c_z^{k+1} + \cdots$ with $c \neq 0$ and $k \geq 1$, the number $k + 1$ is called the (order of the) contact of g with the identity.

On the other hand, suppose that we are given local diffeomorphisms g_1, g_2 whose Taylor series satisfy

$$g_1(z) = z + az^{r+1} + \cdots, \tag{1}$$

$$g_2(z) = z + bz^{s+1} + \cdots, \tag{2}$$

with $ab \neq 0$. Then the commutator $[g_1, g_2]$ has the form

$$[g_1, g_2](z) = z + (s - r)abz^{r+s+1} + \cdots. \tag{3}$$

In particular, we can find elements $g_1, g_2 \in G$ as in equations (1), (2) for which $s > r > 0$ (strictly). From now on fix $g_1, g_2 \in G$ satisfying all the conditions above.

LEMMA 2.1. *We can assume without loss of generality that* g_1, g_2 *are defined on the unit disk* D *of* $\mathbb{C}$. *Furthermore, given* $\varepsilon > 0$, *we can also assume that*

$$\max\{\|g_1 - \mathrm{id}\|_D, \|g_2 - \mathrm{id}\|_D\} < \varepsilon$$

where

$$\|g_1 - \mathrm{id}\|_D = \sup_{z \in D} |g_1(z) - z| \quad \textit{and} \quad \|g_2 - \mathrm{id}\|_D = \sup_{z \in D} |g_2(z) - z|.$$

Proof. For $0 < \lambda < 1$, consider the change of variable $z \mapsto \lambda z$. If λ is less than the minimum between the convergence radii of the Taylor series (1), (2) of g_1, g_2, then g_1, g_2 in the new coordinates have the form

$$g_1(z) = z + a\lambda^r z^{r+1} + \lambda^{r+1} O(z^{r+2}),$$
$$g_2(z) = z + b\lambda^s z^{s+1} + \lambda^{s+1} O(z^{s+2}),$$

these formulas being valid on the disk D with uniform constants. The statement then follows from letting $\lambda \to 0$. □

In the following let us consider diffeomorphisms g_1, g_2 as in equations (1), (2) satisfying the condition of Lemma 2.1 for some $\varepsilon > 0$ to be fixed later. Denote by Γ the pseudogroup generated by g_1, g_2 on D. The next step follows an idea of Ghys [**Gh**] whose aim is to construct a non-trivial sequence of elements in Γ converging to the identity on a uniform domain. The corresponding adaptations needed for the case in hand are simple and detailed below. To begin with, let us inductively define sets $S(i)$ by the rule

$$S(1) = \{[g_i^{\pm 1}, g_j^{\pm 1}] : i, j \in \{1, 2\}\}, \quad S(2) = \{[g_i^{\pm 1}, g_{S(1)}] : i \in \{1, 2\} \text{ and } g_{S(1)} \in S(1)\}.$$

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Finally for $k \geq 2$, we set

$$S(k) = \{[g_{S(k-2)}, g_{S(k-1)}] : g_{S(k-2)} \in S(k-2) \text{ and } g_{S(k-1)} \in S(k-1)\}.$$

LEMMA 2.2. *None of the sets $S(k)$ is reduced to* {id}.

Proof. This amounts to an application of formula (3). Indeed, non-trivial elements in $S(1)$ have contact with identity of order exactly equal to $r+s+1$. In particular, this order of contact is strictly larger than the corresponding contacts of g_1, g_2. Thus, by again using formula (3), the existence of a non-trivial element in $S(2)$ whose contact with the identity is strictly greater than the contacts of all elements in $S(1)$ follows. Denote by $g_{S(1),\max}$ ($g_{S(2),\max}$) the element of $S(1)$ ($S(2)$) having maximal contact order with identity among all non-trivial elements in $S(1)$ ($S(2)$). A repetition of the preceding argument then shows that $[g_{S(1),\max}, g_{S(2),\max}]$ is different from the identity and, moreover, has maximal contact order with the identity among elements in $S(3)$. The rest of the proof follows by induction. □

Now suppose that we are given $r, \varepsilon, \tau > 0$ so that $4\varepsilon + \tau < r$. Consider also local diffeomorphisms f_1, f_2 defined on the disk D_r (of radius r about $0 \in \mathbb{C}$) satisfying

$$\max\left\{\sup_{z\in D_r} |f_1(z) - z|, \sup_{z\in D_r} |f_2(z) - z|\right\} < \varepsilon.$$

With the above conditions, it is proved in [**Gh**] (see also [**L-R**, p. 159]) that the commutator $[f_1, f_2] = f_1 \circ f_2 \circ f_1^{-1} \circ f_2^{-1}$ is defined on $D_{r-4\varepsilon-\tau}$ *as an element of the pseudogroup* Γ generated by g_1, g_2 on D. Furthermore, this commutator satisfies the estimate

$$\begin{aligned} \|[f_1, f_2] - \mathrm{id}\|_{D_{r-4\varepsilon-\tau}} &= \sup_{z\in D_{r-4\varepsilon-\tau}} |[f_1, f_2](z) - z| \\ &\leq \frac{2}{\tau}\|f_1 - \mathrm{id}\|_{D_r} \cdot \|f_2 - \mathrm{id}\|_{D_r}. \end{aligned} \tag{4}$$

In particular, taking $\tau = 4\varepsilon$ and ε sufficiently small, we obtain

$$\|[f_1, f_2] - \mathrm{id}\|_{D_{r-8\varepsilon}} \leq \frac{2}{4\varepsilon}\varepsilon \cdot \varepsilon = \frac{\varepsilon}{2}.$$

Therefore, for ε small, it follows that elements in $S(1)$ are defined on the disk of radius $1 - 8\varepsilon$. Now, if we reset r as $1 - 8\varepsilon$ and the new ε as $\varepsilon/2$, the foregoing allows us to conclude that elements in $S(2)$ are defined on the disk of radius $1 - 8\varepsilon - (8\varepsilon/2)$. Furthermore, on this disk, their distance to the identity is bounded by $\varepsilon/4$. Choosing $\varepsilon < 1/32$, a simple induction argument shows that elements on $S(k)$ are always defined on the disk $D_{1/2}$ of radius $1/2$. In addition, on $D_{1/2}$, the distance to the identity of an element of $S(k)$ is bounded by $\varepsilon/2^k$. Hence the combination of these estimates with Lemma 2.2 leads to the following proposition.

PROPOSITION 2.3. *Let g_1, g_2 and Γ be as above. Then the pseudogroup Γ contains a sequence $\{h_k\}_{k\in\mathbb{N}}$ of elements satisfying the following conditions.*

(1) *For every $k \in \mathbb{N}$, h_k is defined on the disk $D_{1/2}$ and $h_k \neq \mathrm{id}$.*
(2) *The sequence $\{h_k\}$ converges uniformly to the identity on $D_{1/2}$.* □

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3. *Vector fields approximating the dynamics of pseudogroups*

In this section we shall construct a special type of vector fields which will play the key role in the proof of our rigidity results. These vector fields have properties similar to those of the vector fields constructed by Shcherbakov and Nakai. However, they have the advantage that, in their construction, we do not need to control the behavior of maps in arbitrarily small neighborhoods of the origin. Thanks to this fact, they will be better suited to establishing the rigidity theorems in the introduction.

In what follows we retain the notation of §2. Let us begin with a brief review of the topological dynamics associated with a pseudogroup Γ as before.

3.1. *Topological dynamics.* The material presented below is borrowed from [**B-L-L, Na, Sh1, Sh2**]. Strictly speaking, most of the material discussed below relative to Shcherbakov–Nakai vector fields will not be necessary in what follows. The indispensable parts of what follows are a rough discussion of the topological dynamics of non-solvable subgroups of $\mathrm{Diff}(\mathbb{C}, 0)$, as in [**HV**], along with the theorem of [**B-L-L**] stating the existence of hyperbolic fixed points for certain elements of the group in question. I believe, however, that the details included below will be useful to the reader unfamiliar with Shcherbakov–Nakai theory while also preparing the way for the rest of the section. Finally, it should be mentioned that these vector fields played a large role in [**B-L-L**].

Once again consider local diffeomorphisms g_1, g_2 as in equations (1), (2) that are defined on the unit disk D and let Γ denote the pseudogroup they generate on D. Next, given $f \in \mathrm{Diff}(\mathbb{C}, 0)$ defined on D, let $\mathrm{Bas}(f)$ denote the *basin* of f, namely the set constituted by those points $z \in D$ for which $\lim_{n\to+\infty} f^n(z) = 0$, with $f^n(z)$ being defined for all $n \in \mathbb{N}$. If, moreover, f is tangent to the identity at $0 \in \mathbb{C}$, then it is well known that the union $\mathrm{Bas}(f) \cup \mathrm{Bas}(f^{-1})$ contains a punctured neighborhood of $0 \in \mathbb{C}$. If we consider the diffeomorphism g_1, then we can suppose without loss of generality that $\mathrm{Bas}(g_1) \cup \mathrm{Bas}(g_1^{-1}) = D\backslash\{0\}$. According to Nakai, there is an analytic vector field X defined on $\mathrm{Bas}(g_1)$ and satisfying all the conditions below.

(1) $X(z) \neq 0$ for every $z \in \mathrm{Bas}(g_1)$.

(2) Denote by ϕ_X^t the local flow of X. Then given a relatively compact open set $V \subset \mathrm{Bas}(g_1)$ and $t_0 > 0$ such that ϕ_X^t is defined on V for every $0 \leq t \leq t_0$, the mapping $\phi_X^{t_0} : V \to \mathrm{Bas}(g_1)$ is the uniform limit of a sequence of mappings $g_1^{-n} \circ g_2^{k(n)} \circ g_1^n$ belonging to Γ.

(3) Item (2) above holds for every sequence $k(n)$ satisfying $k(n) \cdot n^{r-s} \to t_0$.

Vector fields whose local flow can be approximated, in the sense of item (2), by elements belonging to Γ will be said *to belong to the closure of* Γ. Note that the pseudogroup naturally acts by pull-back on the collection of the vector fields belonging to its own closure. Therefore the existence of one non-trivial vector field in the closure of Γ usually yields the existence of several others. By exploiting the construction/existence of these vector fields, Shcherbakov and Nakai have independently given a clear picture of the dynamics associated to the pseudogroup Γ. We shall present this picture in its more accurate form, which is due to Nakai. He establishes the existence of an analytic set $\Sigma(\Gamma)$ which is a *separatrix* for Γ. The structure of the set $\Sigma(\Gamma)$ is easy to describe: $\Sigma(\Gamma)$ either is reduced to a single point (i.e. to $0 \in \mathbb{C}$) or, in suitable coordinates, is given by the

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equation $\mathrm{Im}(z^N) = 0$ for some $N \in \mathbb{N}$. By definition, the condition of being a separatrix means that the set possesses the following two properties. First, the restriction of the (pseudoaction) of Γ to every connected component of $D \setminus \Sigma(\Gamma)$ has all orbits dense in this component (i.e. it is minimal). Second, the restriction of the (pseudoaction) of Γ to each connected component of $\Sigma(\Gamma)\setminus\{0\}$ is also minimal in the component in question. In particular, if $\Sigma(\Gamma)$ is reduced to $0 \in \mathbb{C}$, then the pseudoaction of Γ is minimal on a punctured neighborhood of $0 \in \mathbb{C}$.

Clearly, a point at which the closure of Γ contains two linearly independent vector fields has locally dense orbit. Hence, away from $0 \in \mathbb{C}$, a separatrix must be contained in the set of points at which all vector fields lying in the closure of Γ are linearly dependent. This easily establishes that a separatrix must be a (proper) analytic set away from $0 \in \mathbb{C}$. However, ensuring that this set is still analytic at $0 \in \mathbb{C}$, and that each of its branches is smooth, requires a detailed asymptotic analysis which is one of the improvements of Nakai's work when compared to the preceding papers [**Sh1, Sh2**].

Later it was shown in [**B-L-L**] that Γ also contains elements with hyperbolic fixed points. To make the statement precise, let $\mathcal{A}$ ($\mathcal{S}$) denote a connected component of $D\setminus\Sigma(\Gamma)$ ($\Sigma(\Gamma)\setminus\{0\}$). Then the set

$$\mathrm{Hyp}_{\mathcal{A}} = \{p \in \mathcal{A};\ \text{there is } f \in \Gamma \text{ such that } f(p) = p \text{ and } |f'(p)| \neq 1\}$$

is dense in $\mathcal{A}$, and

$$\mathrm{Hyp}_{\mathcal{S}} = \{p \in \mathcal{S};\ \text{there is } \ f \in \Gamma \text{ such that } f(p) = p \text{ and } |f'(p)| \neq 1\}$$

is dense in $\mathcal{S}$.

3.2. *Constructing vector fields in the closure of pseudogroups.* After this brief detour on the topological dynamics of a non-solvable subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$, we can start our approach to the proofs of the theorems stated in the introduction. Naturally the central point of the discussion will be the proof of Theorem C, to be completed in the next section. The main point of this section will be the construction of non-trivial vector fields contained in the closure of certain pseudogroups which differ from Nakai's vector field in the sense that their construction does not depend on the existence of a 'common fixed point' for the whole pseudogroup. These vector fields will quickly lead to the proofs of the desired rigidity results in §4.

Let us then consider a finitely generated non-solvable pseudogroup Γ_1. Modulo rescaling coordinates, Γ_1 contains elements g_1, g_2 and h_k, $k \in \mathbb{N}$, as in the statement of Proposition 2.3. Furthermore, we can suppose that g_1, g_2 also satisfy the normalizations considered above. In particular, every point in $D\setminus\{0\}$ is contained in the domain of definition of at least one Nakai vector field for Γ_1.

Next consider an open set W_1 whose closure is contained in $D_{1/2}\setminus\{0\}$ along with a homeomorphism $\xi : W_1 \to W_2 = \xi(W_1)$ where the closure of W_2 is also contained in $D_{1/2}\setminus\{0\}$. For every element $f \in \Gamma_1$ defined on W_1, we suppose that $\xi \circ f \circ \xi^{-1} : W_2 \to W_2$ is holomorphic on W_2. Finally, let Γ_2 denote the pseudogroup generated on W_2 by the mappings $\xi \circ f \circ \xi^{-1}$ for all elements $f \in \Gamma_1$ that are defined on W_1. Theorem C is now reduced to the following.

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THEOREM 3.1. *With the previous notation, ξ is, in fact, a holomorphic diffeomorphism provided that it preserves orientation.*

The standard technique for promoting a topological conjugacy into a differential (or holomorphic) conjugacy relies on constructing non-trivial vector fields which are conjugate by the homeomorphism in question. Here *conjugate* is to be understood as a condition stronger than *topologically equivalent*: in the latter case we require the homeomorphism to send orbits of one vector field to the orbits of the other. For a topological conjugacy, we require, in addition, that the parameterizations of these orbits by the corresponding local flows are also preserved. Further details will be given in the next section.

Let us begin with a simple reduction of the statement.

LEMMA 3.2. *It suffices to prove the statement of Theorem 3.1 on a small neighborhood of an arbitrary point in $\mathcal{A} \cup \mathcal{S}$.*

Proof. With the previous notation, suppose that $z_0 \neq 0$ belongs to $D_{1/2}$. Suppose that $z_0 \notin \Sigma(\Gamma_1)$ and let $\mathcal{A}$ denote the connected component of $D_{1/2} \backslash \Sigma(\Gamma_1)$ to which z_0 belongs. Since $\mathrm{Hyp}_{\mathcal{A}}$ is not empty, for fixed $p \in \mathrm{Hyp}_{\mathcal{A}}$ and a small neighborhood $B_p(\epsilon)$ of p, there exists an element $g \in \Gamma$ taking z_0 into $B_p(\epsilon)$. The holomorphic character of ξ at z_0 then results from its holomorphic character on $B_p(\epsilon)$. The argument for the case in which z_0 belongs to $\Sigma(\Gamma_1)$ is identical and left to the reader. □

Consider a point $p \in \mathrm{Hyp}_{\mathcal{A}} \cup \mathrm{Hyp}_{\mathcal{S}}$. Take an element $f \neq \mathrm{id}$ in Γ_1, defined on a neighborhood of p, and satisfying $f(p) = p$, $|f'(p)| \neq 1$. Modulo replacing f by f^{-1} we can also assume that $|f'(p)| < 1$. Next choose also a sequence $\{h_k\}_{k \in \mathbb{N}} \subset \Gamma_1$, $h_k \neq \mathrm{id}$ for every k, such that h_k is defined on a uniform ball $B_p(\epsilon)$ about p and, in fact, converges uniformly to the identity on this ball. In fact, Proposition 2.3 allows us to choose h_k defined and converging to the identity on the whole of $D_{1/2}$. Another consequence of this proposition along with Lemma 2.2 is that the h_k can be chosen so as to not commute, i.e. whenever $i \neq j$, one has $h_i \circ h_j \circ h_i^{-1} \circ h_j^{-1} \neq \mathrm{id}$. Finally, since it was seen above that p is not an isolated point of its own orbit, modulo conjugating the h_i by a suitable element of Γ_1, we can suppose in addition that $h_i(p) \neq p$ for every i.

The rest of this section is devoted to constructing a non-trivial vector field, defined on a neighborhood of $p \in \mathrm{Hyp}_{\mathcal{A}} \cup \mathrm{Hyp}_{\mathcal{S}}$, that is contained in the closure of Γ_1.

For the sake of notation, let us choose coordinates where p is identified with $0 \in \mathbb{C}$. Furthermore, we can suppose that $f(z) = \lambda z$ with $|\lambda| < 1$. To clarify the argument, let us make explicit the assumptions that Γ_1 is supposed to satisfy in this local setting for the remainder of the section.

(1) Γ_1 contains a hyperbolic contraction $z \mapsto \lambda z$, $|\lambda| < 1$.
(2) Γ_1 contains a sequence of h_k, $h_k \neq \mathrm{id}$ for every k, such that each h_k is defined on a uniform neighborhood of $0 \in \mathbb{C}$. Furthermore, the sequence $\{h_k\}$ converges uniformly to the identity on the mentioned neighborhood and satisfies $h_k(0) \neq 0$ for every k.
(3) For every $i \neq j$, one has $h_i \circ h_j \circ h_i^{-1} \circ h_j^{-1} \neq \mathrm{id}$.

Remark 3.3. With the above notation, Theorem 3.1 is then reduced to proving that the homeomorphism ξ is conformal on a small neighborhood of $0 \in \mathbb{C}$.

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In the rest of this section we shall construct a non-trivial vector field Y defined on a neighborhood of $0 \in \mathbb{C}$ and contained in the closure of Γ_1. Non-trivial vector fields contained in the closure of pseudogroups as above were first constructed in [**R**]. In what follows we give a direct argument.

LEMMA 3.4. *There exists a sequence of maps g_k, $k \geq 3$, contained in Γ_1 together with $\tau > 0$ satisfying the following conditions.*

(1) *g_k is defined on the disk $D_\tau(p)$ of radius τ about p.*

(2) *There are strictly positive constants* Const_1, Const_2, Const_3 *such that*

$$\mathrm{Const}_1 > \mathrm{Const}_2 \sup_{z \in D_{\tau/2}(p)} |g_k(z) - z| > \sup_{z \in D_\tau(p)} |g_k(z) - z| > \mathrm{Const}_3 > 0.$$

Proof. Choose $\tau > 0$ so that, for all $k \geq 2$, the Taylor series h_k about p converges on $D_\tau(p) \simeq D_\tau$ (since p is identified with $0 \in \mathbb{C}$). With these identifications we set

$$h_k(z) = h_k(0) + D_0 h_k \cdot z + r_k(z).$$

Clearly, $r_k(z)$ converges uniformly to zero. Given $i \in \mathbb{N}$, consider the diffeomorphism $\lambda^{-i} h_k(\lambda^i z)$. This diffeomorphism is clearly defined on $D_\tau(p) \simeq D_\tau$ and, in addition, belongs to the pseudogroup Γ_1. Next note that, on $D_\tau(p) \simeq D_\tau$, we have

$$\lambda^{-i} h_k(\lambda^i z) = \lambda^{-i} h_k(0) + D_0 h_k \cdot z + r_k^i(z)$$

where $r_k^i(z) < C \cdot \lambda^i r_k(z)$ for a uniform constant C. Now, since $h_k(0) \neq 0$, we choose $i(k)$ so that $\lambda^{-i(k)} h_k(0)$ is larger than $\tau/10$. Actually we choose the smallest positive $i(k)$ satisfying this inequality. Since $h_k(0) \to 0$, it follows that $i(k)$ converges to infinity. Then, for k large enough, $r_k^{i(k)} < C \cdot \lambda^i r_k(z)$ converges uniformly to *zero*. Since $|D_0 h_k \cdot z - z|$ converges uniformly to *zero* as well, the sequence formed by the elements $g_k = \lambda^{-i(k)} h_k(\lambda^{i(k)} z)$ fulfills the conditions above. The lemma is proved. □

PROPOSITION 3.5. *There exists a non-singular vector field Y defined on $D_{\tau/2}$ and contained in the closure of Γ_1.*

Proof. Let C_k be defined by $C_k = \sup_{z \in D_\tau} |g_k(z) - z|$. Note that C_k is uniformly bounded by Const_1. Next for k large enough and for $z \in D_\tau$, we define a vector field Y_k by

$$Y_k(z) = \frac{1}{C_k} \cdot (g_k(z) - z) \frac{\partial}{\partial z}.$$

Clearly, $\sup_{z \in D_\tau} |Y_k(z)| = 1$. Hence, Montel's theorem allows us to suppose that Y_k converges uniformly to a vector field Y on $D_{\tau/2}$. Furthermore, Lemma 3.4 ensures that $\sup_{z \in D_{\tau/2}} |Y_k(z)| > 1/C_k \mathrm{Const}_1 \geq 1/(\mathrm{Const}_1 \cdot \mathrm{Const}_2) > 0$ so that Y is not identically *zero*.

The next step of the proof consists of showing that Y belongs to the closure of Γ_1. For this let ϕ_Y denote the local flow of Y. We need to show that, for fixed t, the induced mapping ϕ_Y^t is a uniform limit of mappings contained in Γ_1 (on the domain of definition of ϕ_Y^t). To show this it suffices to check that ϕ_Y^t is, indeed, uniformly approximated by the sequence of mappings $g_k^{[t/C_k]}$, where $[\cdot]$ stands for the integer part. The verification of

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the last assertion amounts to a straightforward application of the classical Euler 'polygonal approximation' and is therefore left to the reader.

Finally, it remains to show that the vector field Y can be chosen non-singular on $D_{\tau/2}$. Using again the fact that the local orbit of $p \simeq 0$ is non-discrete, modulo conjugating Y by a suitable element of Γ_1, we can assume without loss of generality that $Y(0) \neq 0$. Now to guarantee that Y does not vanish on $D_{\tau/2}$ we only need to replace Y by its pull-back by the homothety $z \mapsto \lambda^n z$ for some n sufficiently large. The proof of the proposition is complete. □

4. *Topological rigidity*

In this final section we shall provide the proofs of the main statements in the introduction. Let us retain the notation of §3. In particular, we have pseudogroups Γ_1, Γ_2 that are conjugate by the homeomorphism ξ. More precisely, recall that Γ_1 was defined on $B_p(\epsilon)$ so that, on $\xi(B_p(\epsilon))$ we consider the pseudogroup Γ_2 of holomorphic mappings induced from Γ_1 by means of the conjugacy ξ. In particular, this pseudogroup contains the elements $f_\xi = \xi \circ f \circ \xi^{-1}$ and $h_{k,\xi} = \xi \circ h_k \circ \xi^{-1}$. Again we consider local coordinates identifying p and $\xi(p)$ with $0 \in \mathbb{C}$. In these coordinates, we have constructed a non-singular vector field Y on $D_{\tau/2}$ that is contained in the closure of Γ_1 (for some small $\tau > 0$). Next we shall prove the existence of another vector field Y_ξ contained in the closure of Γ_2 such that Y, Y_ξ are *synchronized* by ξ in the sense that the equation

$$\psi^t_{Y_\xi}(z) = \xi \circ \phi^t_Y \circ \xi^{-1}(z) \tag{5}$$

holds for every t, x as long as both sides are defined, where $\psi^t_{Y_\xi}$ stands for the local flow of Y_ξ. For reference, let us state the following proposition.

PROPOSITION 4.1. *Let Γ_1, Γ_2 be as above. Then there are analytic vector fields Y, Y_ξ, defined on a neighborhood of $0 \in \mathbb{C}$ and contained respectively in the closures of Γ_1, Γ_2, that are conjugate by ξ in the sense of equation (5). Besides these vector fields are non-singular in the neighborhood in question.*

Proof. Consider the non-singular vector field Y constructed in Proposition 3.5. As observed, its local flow ϕ^t_Y is the uniform limit of the sequence of mappings $g_k^{[t/C_k]}$. Consider now the homeomorphism $\xi \circ \phi^t_Y \circ \xi^{-1}$. Since ξ conjugates Γ_1, Γ_2, the mappings $\xi \circ g_k^{[t/C_k]} \circ \xi^{-1}$ belong to Γ_2 and therefore are holomorphic. It follows that the homeomorphism $\xi \circ \phi^t_Y \circ \xi^{-1}$ is, in fact, a holomorphic diffeomorphism as the uniform limit of holomorphic mappings.

Set $\psi^t_{Y_\xi} = \xi \circ \phi^t_Y \circ \xi^{-1}$. The assignment $t \mapsto \psi^t_{Y_\xi}$ yields a *pseudo-homomorphism* from a neighborhood of $0 \in \mathbb{R}$ to the (pseudo)group of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$. More precisely, for a small neighborhood U of $0 \in \mathbb{C}$ and small $t_0 > 0$, we define a mapping $\Psi : U \times [-t_0, t_0] \to \mathbb{C}$ by letting

$$\Psi(z, t) = \psi^t_{Y_\xi}(z) = \xi \circ \phi^t_Y \circ \xi^{-1}(z).$$

Clearly, Ψ is uniformly continuous on $U \times [-t_0, t_0]$. For t fixed, Ψ is holomorphic on z and, furthermore, $\Psi^{t+s}(z) = \Psi^t \circ \Psi^s(z)$ provided that both sides are defined. It is then a

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classical fact that Ψ must be induced by the local flow of some vector field Y_ξ which, by construction, satisfies equation (5). In fact, this classical phenomenon appears for example in the problems discussed in [**M-Z**, Ch. 5]. An elementary proof of our assertion adapted to the present context can be found in [**L-R**, Proposition 4.8]. Finally, since Y is non-singular, equation (5) implies that Y_ξ is also non-singular. □

Let us now show the existence of another vector field Z in the closure of Γ_1 and linearly independent of Y at 'generic' points. Here let us return to the initial coordinates of §2 as used, for example, in Proposition 2.3. In particular, $p \neq 0$ and the vector field Y is defined on the disk $D_\tau(p)$ of sufficiently small radius $\tau > 0$ about p. Now we have the following lemma.

LEMMA 4.2. *Modulo reducing τ, there is a non-singular analytic vector field Z defined on $D_\tau(p)$ which is contained in the closure of Γ_1 and is not parallel to Y at generic points.*

Proof. Suppose for a contradiction that the statement is false. Denote by $\mathcal{F}$ the real one-dimensional foliation associated with Y. Since the pseudogroup Γ naturally acts on the vector fields contained in its own closure, the fact that the statement is false means that every element $g \in \Gamma_1$ such that $g(D_\tau(p)) \cap D_\tau(p) \neq \emptyset$ must preserve $\mathcal{F}$ on the corresponding domains. Therefore we can extend $\mathcal{F}$ as a non-singular real one-dimensional foliation to a whole 'Nakai sector' (in particular, to a punctured neighborhood of $0 \in \mathbb{C}$ if the separatrix $\Sigma(\Gamma)$ is reduced to a single point). The extended foliation is still invariant under Γ_1.

In particular, $\mathcal{F}$ must be preserved by the elements $h_k \in \Gamma_1$ given by Proposition 2.3 (cf. also Lemma 2.2). These h_k are defined on the disk of radius $1/2$ and, by construction, their order of contact with the identity at 0 forms a sequence going to infinity.

Now consider a diffeomorphism h_k for k fixed. Since h_k is tangent to the identity, it is well known that the only one-dimensional foliation, satisfying the above conditions, that is preserved by h_k is precisely the foliation associated with the *unique* formal vector field that contains h_k as a time-one map, cf. for example [**M-R-2**]. The order of this vector field at the origin is the contact order of h_k with the identity minus 1. Since the contact orders of the h_k constructed above are not always the same, there cannot exist a foliation as above simultaneously preserved by all these maps. The resulting contradiction proves the lemma. □

We are finally able to prove Theorem 3.1 which implies Theorem C in the introduction.

Proof of Theorem 3.1. Consider non-singular vector fields Y, Z defined on a neighborhood of p and contained in the closure of Γ_1. We note that Y is not parallel to Z everywhere. The set formed by those points at which all vector fields contained in the closure of Γ_1 are parallel is clearly invariant by Γ_1. In particular, it must be invariant by Y, Z. Therefore, on a sufficiently small neighborhood of p, this set is either empty or consists of a smooth (irreducible) analytic curve (an orbit of Y) through p.

Let us first consider the case in which the above mentioned set does not intersect a neighborhood of p. In this case we can suppose that $Y(p)$ and $Z(p)$ are linearly independent. As before, denote by ϕ_Y^t, ϕ_Z^t the local flows of Y, Z. According to Proposition 4.1, there are vector fields Y_ξ, Z_ξ contained in the closure of Γ_2 and defined

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on a neighborhood of $\xi(p)$. Furthermore, these vector fields are synchronized with Y, Z in the sense that equation (5) is satisfied for the corresponding local flows $\psi^t_{Y_\xi}$, $\psi^t_{Z_\xi}$.

Now using the fact that $Y(p)$ and $Z(p)$ are linearly independent, we parameterize a small neighborhood of p by the real flows of Y, Z. That is, to every point q sufficiently close to p we assign a pair (s_q, t_q), for uniquely defined small real s_q, t_q, such that $q = \phi_Y^{s_q} \circ \phi_Z^{t_q}(p)$. The assignment $q \mapsto (s_q, t_q)$ is clearly a C^∞-diffeomorphism. Now equation (5) yields

$$\xi(q) = \xi(\phi_Y^{s_q} \circ \phi_Z^{t_q}(p)) = (\psi_{Y_\xi}^{s_q} \circ \psi_{Z_\xi}^{t_q})(\xi(p))$$

so that ξ is a C^∞-mapping. To conclude that ξ is a conformal diffeomorphism it is therefore sufficient to check that its derivative is conformal. Indeed, a holomorphic homeomorphism is necessarily a holomorphic diffeomorphism. To see that $D\xi$ is conformal we proceed as follows.

First recall that Γ_1 contains a local diffeomorphism f such that $f(p) = p$ and $f'(p) = \lambda$ with $|\lambda| < 1$. The element $f_\xi = \xi \circ f \circ \xi^{-1}$ of Γ_2 has a (contractive) hyperbolic fixed point at $\xi(p)$. Indeed, this holomorphic diffeomorphism must take a sufficiently small disk about $\xi(p)$ inside itself. Thus we can consider suitable renormalizations of the pull-backs $(f^N)^*(Y)$ $((f^N)^*(Z))$. It is promptly seen that these renormalizations can be arranged to converge to the constant vector fields $Y(p)\partial/\partial z$ and $Z(p)\partial/\partial z$. Thus we can suppose that Y, Z are constant. Furthermore, since we can multiply vector fields in the closure of a pseudogroup by real constants, we can assume also that $|Y(p)| = |Z(p)|$. With these vector fields the assignment $q \mapsto (s_q, t_q)$ becomes conformal. The vector fields Y, Z are now synchronized with vector fields that are limits of similar renormalizations of the pull-backs $(f_\xi^N)^*(Y_\xi)$ $((f_\xi^N)^*(Z_\xi))$. Because f_ξ has a hyperbolic fixed point at $\xi(p)$, it follows that these vector fields are also constant. Their flows are therefore translations. Now, considering ξ in the conformal coordinates (s, t), equation (5) says that $\xi(s, t)$ is a composition of translations and therefore conformal.

Finally, suppose that p is contained in the smooth analytic curve where all vector fields contained in the closure of Γ_1 are parallel. Since the foregoing shows that ξ is conformal away from this curve, the conformal character of ξ must extend over the curve thanks to the Painlevé theorem. This completes our proof. □

As promised in the introduction, let us state a version of Theorem C that makes no reference to foliations. Consider an open set $W \subset \mathbb{C}$ and let Γ stand for the pseudogroup generated on W by diffeomorphisms $\{f_1, f_1^{-1}, \ldots, f_s, f_s^{-1}\}$ (all of them defined on the whole of W). Suppose that Γ satisfies the following conditions.

(1) Γ contains an element g possessing a hyperbolic fixed point $p_0 \in W$. Furthermore, W is contained in the basin of g at p_0.

(2) Γ is not discrete in the sense that Γ contains a non-trivial sequence of elements $\{h_j\}$ converging uniformly to the identity on W (in particular, h_j is defined on the whole of W for every j).

(3) Γ has no finite orbit and the pseudogroup Γ cannot be identified with a pseudogroup generated by (restrictions of) elements in the affine group $\mathrm{Aff}(\mathbb{C})$ of $\mathbb{C}$.

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THEOREM 4.3. *Suppose that $\xi : W \to \xi(W) \subseteq \mathbb{C}$ is a homeomorphism such that for every element $h : V \subset W \to \mathbb{C}$ of Γ, the corresponding homeomorphism $\xi \circ h \circ \xi^{-1} : \xi(V) \to \mathbb{C}$ is indeed a holomorphic diffeomorphism. Then ξ is itself a holomorphic diffeomorphism.*

Proof. Note that $\xi \circ g \circ \xi^{-1}$ has a hyperbolic fixed point at $\xi(p_0)$ since a small disk about $\xi^{-1}(p_0)$ is sent inside itself by $\xi \circ g \circ \xi^{-1}$. Let Γ_ξ denote the pseudogroup on $\xi(W)$ induced by Γ and by ξ. The preceding construction allows us to find a non-trivial vector field Y (Y_ξ) defined on a neighborhood of p_0 ($\xi(p_0)$) which is contained in the closure of Γ (Γ_ξ). Furthermore, the vector fields Y, Y_ξ are synchronized in the sense given by equation (5). The argument employed in the proof of Theorem 3.1 will then ensure the conformal character of ξ provided that we can find vector fields Z, Z_ξ in the closure of Γ, Γ_ξ respectively, synchronized by ξ and, in addition, such that Y, Z are linearly independent at generic points.

To produce the vector field Z we proceed as follows. Since Γ has no finite orbits, we can assume that $Y(p_0) \neq 0$. Next, modulo acting upon X by pull-backs of the form $(g^n)^* X$, we can assume without loss of generality that X is constant on a suitable local coordinate z, i.e. we have $X = \text{cte}\, \partial/\partial z$ for some $\text{cte} \neq 0$. To complete the proof we just need to check that the foliation associated with the orbits of X cannot be preserved by Γ. Given an element h of Γ, we have $h^* X = \text{cte}(h'(z))^{-1} \partial/\partial z$. Thus if X is parallel to $h^* X$, then $h'(z)$ must be constant. Hence the pseudogroup Γ embeds into a subgroup of $\text{Aff}(\mathbb{C})$, which is impossible. The theorem is proved. □

To conclude this paper, let us provide proofs for Theorems A and B.

Proof of Theorem A. Let $\mathcal{F}_1, \mathcal{F}_2$ be as in the statement and denote by $\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2$ their respective blow-ups. The homeomorphism ξ conjugating $\mathcal{F}_1$, $\mathcal{F}_2$ can then be lifted to a Ξ homeomorphism from $\widetilde{U} \backslash \pi^{-1}(0) \to \widetilde{\mathbb{C}}^2$ conjugating $\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2$, where $\widetilde{U}$ stands for a neighborhood of $\pi^{-1}(0)$ in the blow-up $\widetilde{\mathbb{C}}^2$ of $\mathbb{C}^2$. Consider a local section Σ through a point $P \in \pi^{-1}(0) \backslash (\text{Sing}(\widetilde{\mathcal{F}}_1) \cup \text{Sing}(\widetilde{\mathcal{F}}_2))$ that is transverse to $\mathcal{F}_1, \mathcal{F}_2$. Note that each singular point p of $\widetilde{\mathcal{F}}_1$ ($\widetilde{\mathcal{F}}_2$) may or may not have a (unique) separatrix which is not contained in $\pi^{-1}(0)$. This separatrix, if it exists, is necessarily smooth and transverse to $\pi^{-1}(0)$. Denote by $p_1^1, \ldots, p_s^1$ ($p_1^2, \ldots, p_r^2$) the singularities of $\widetilde{\mathcal{F}}_1$ ($\widetilde{\mathcal{F}}_2$) in $\pi^{-1}(0)$. Accordingly, for $i = 1, \ldots, s$ ($j = 1, \ldots, r$), let S_i^1 (S_j^2) denote the separatrix of $\widetilde{\mathcal{F}}_1$ ($\widetilde{\mathcal{F}}_2$) at p_i^1 (p_j^2) that is transverse to $\pi^{-1}(0)$, where S_i^1, S_j^2 may be empty. We make the following claim.

CLAIM. *The union of $\bigcup_{i=1}^s S_i^1$ ($\bigcup_{j=1}^r S_j^2$) with the saturated $\widetilde{\mathcal{F}}_1^\Sigma$ ($\widetilde{\mathcal{F}}_2^\Sigma$) of Σ by $\widetilde{\mathcal{F}}_1$ ($\widetilde{\mathcal{F}}_2$) contains a neighborhood of $\pi^{-1}(0)$.*

Proof of the Claim. It suffices to prove the statement for $\widetilde{\mathcal{F}}_1$. Similarly, we only need to check what happens around the singular points of $\widetilde{\mathcal{F}}_1$. In other words, on a neighborhood of a singular point p_i^1 we consider a local section Σ_i that is transverse to one of its separatrixes. We need to see that the saturated set of Σ_i by the foliation restricted to a neighborhood of p_i^1 contains a neighborhood of p_i^1 modulo adding to it the other separatrix of $\widetilde{\mathcal{F}}_1$ at p_i^1 (if this separatrix exists). First consider the case where $\widetilde{\mathcal{F}}_1$ has two eigenvalues λ_1, λ_2 different from zero at p_i^1. If $\lambda_1/\lambda_2 \in \mathbb{C} \backslash \mathbb{R}$ then the statement is clear since the

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singularity is conjugate to its linear part (Poincaré's theorem). Since, by assumption, λ_1/λ_2 is not real negative, there remains only the case $\lambda_1/\lambda_2 \in \mathbb{R}_+$. In this case the statement follows from estimates in [**M-Mo**] or, more explicitly, in [**Re**]. Finally, considering the case where p_i^1 is a saddle-node, $\pi^{-1}(0)$ must coincide with its strong invariant manifold (by assumption). The fact that the saturated set of a local section transverse to the strong invariant manifold of a saddle-node contains a neighborhood of it (modulo adding another possible separatrix) is then an easy consequence of sectorial normalization as explained in [**M-R-1**]. The claim is proved. □

Note that ξ must take separatrixes of $\mathcal{F}_1$ to separatrixes of $\mathcal{F}_2$ so that the analogous phenomenon happens with Ξ. Combining this remark with the above claim, it follows that the restriction of Ξ to $\widetilde{U}\backslash\bigcup_{i=1}^{s} S_i^1$ induces a homeomorphism defined on $\Sigma\backslash\{P\}$ which must be transversely holomorphic thanks to Theorem C. Since the separatrixes S_i^1 are sent by Ξ to separatrixes S_j^2, the transversely holomorphic character of Ξ extends to these separatrixes thanks to the Riemann extension theorem. By collapsing Ξ back into ξ, we conclude the theorem. □

Proof of Theorem B. This is analogous to the proof of Theorem A. Let $\Pi^{-1}(0)$ be the total exceptional divisor associated with the Seidenberg resolution $\widetilde{\mathcal{F}}_1$ of $\mathcal{F}_1$. Next recall that C_1 is a component of $\Pi^{-1}(0)$ possessing a non-solvable holonomy group. Thus it suffices to show that the saturated set by $\widetilde{\mathcal{F}}_1$ of a local transverse section Σ through a point of C_1 that is regular for $\widetilde{\mathcal{F}}_1$ still contains a neighborhood of $\Pi^{-1}(0)$ (except for finitely many separatrixes at the singularities of $\widetilde{\mathcal{F}}_1$). In view of the preceding discussion, the only difficulty occurs at those points where two irreducible (and necessarily invariant) components C_{i_1}, C_{i_2} of $\Pi^{-1}(0)$ intersect each other. These points are singular points for $\widetilde{\mathcal{F}}_1$ where $\widetilde{\mathcal{F}}_1$ has two eigenvalues λ_1, λ_2 different from zero. The passage from a section transverse to the component C_{i_1} to the component C_{i_2} is then performed by using the corresponding Dulac transform which is well defined since $\lambda_1/\lambda_2 \in \mathbb{C}\backslash\mathbb{R}_-$ (see, for example, [**M-M-1**]). This completes the proof of Theorem B. □

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References

[Be] M. Belliart. Sur certains pseudogroupes de biholomorphismes locaux de $(\mathbb{C}^n, 0)$. *Bull. Soc. Math. France* **129** (2001), 259–284.

[B-L-L] M. Belliart, I. Liousse and F. Loray. Sur l'existence de points fixes attractifs pour les sous-groupes de Aut$(\mathbb{C}, 0)$. *C. R. Acad. Sci. Paris, Sér. I, Math.* **324** (1997), 443–446.

[HV] M. O. Huddai-Verenov. A property of the solutions of a differential equation. *Mat. Sb.* **56**(98) (1962), 301–308 (in Russian).

[Gh] E. Ghys. Sur les groupes engendrés par des difféomorphismes proches de l'identité. *Bol. Soc. Bras. Mat. Nova Sér.* **24** (1993), 137–178.

https://doi.org/10.1017/S0143385710000180 Published online by Cambridge University Press

[L-R] F. Loray and J. C. Rebelo. Minimal, rigid foliations by curves on $\mathbb{CP}(n)$. *J. Eur. Math. Soc.* **5**(2) (2003), 147–201.

[M-M-1] D. Marin and J.-F. Mattei. Incompressibilité des feuilles de germes de feuilletages holomorphes singuliers. *Ann. Sci. École Norm. Sup.* (4) **41** (2008), 855–903.

[M-M-2] D. Marin and J.-F. Mattei. Monodromie et classification topologique des germes de feuilletages holomorphes. *Preprint*.

[M-Mo] J.-F. Mattei and R. Moussu. Holonomie et intégrales premières. *Ann. Sci. École Norm. Sup.* **16** (1983), 469–523.

[M-R-1] J. Martinet and J.-P. Ramis. Problèmes de modules pour des équations différentielles non linéaires du premier ordre. *Publ. Math. Inst Hautes Études Sci.* **55** (1982), 63–164.

[M-R-2] J. Martinet and J.-P. Ramis. Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. *Ann. Sci. École Norm. Sup.* (4) **16**(4) (1983), 571–621.

[M-Z] D. Montgomery and L. Zippin. *Topological Transformation Groups*, 3rd edn. *(Tracts in Pure and Applied Mathematics, 1)*. Interscience Publishers, New York, 1965.

[Na] I. Nakai. Separatrices for non-solvable dynamics on $(\mathbb{C}, 0)$. *Ann. Inst. Fourier* **44** (1994), 569–599.

[R] J. C. Rebelo. Ergodicity and rigidity for certain subgroups of $\mathrm{Diff}^{\omega}(S^1)$. *Ann. Sci. École Norm. Sup.* (4) **32** (1999), 433–453.

[Re] H. Reis. Equivalence and semi-completude of foliations. *Nonlinear Anal.* **64**(8) (2006), 1654–1665.

[S] A. Seidenberg. Reduction of singularities of the differentiable equation $Ady = Bdx$. *Amer. J. Math.* **90** (1968), 248–269.

[Sh1] A. A. Shcherbakov. On the density of an orbit of a pseudogroup of conformal mappings and a generalization of the Hudai–Verenov theorem. *Vestn. Mosk. Univ.* **31**(Ser. I) (1982), 10–15.

[Sh2] A. A. Shcherbakov. Topological and analytic conjugation of non-commutative groups of conformal mappings. *Tr. Semin. Petrovsk* **10** (1984), 170–192.

https://doi.org/10.1017/S0143385710000180 Published online by Cambridge University Press