PERCOLATION PROCESSES
II. THE CONNECTIVE CONSTANT

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ABSTRACT. The previous paper (1) omitted the proof of the theorem on the connective constant of a crystal. This paper supplies the proof. For ease of reference, we first recall the relevant nomenclature and enunciate the theorem; to this extent the present paper is self-contained.

We deal with *atoms* and *bonds*: a bond is a (perhaps directed) path between two atoms. An *n-stepped walk* is an ordered connected path along n consecutive bonds, each step being in the permitted direction of a bond (if the bond in question is directed) and starting from the atom reached by the previous step (if any). $W_n(A)$ denotes a typical n-stepped walk starting from an atom A. A walk is *self-avoiding* if it visits no atom more than once. $S_n(A)$ denotes a self-avoiding $W_n(A)$. Two walks are *distinct* if they do not utilize the same set of bonds, with due regard to order. $f_A(n, r)$ denotes the number of distinct $W_n(A)$, each of which can be broken into r or fewer self-avoiding subwalks. In particular we write $f_A(n) = f_A(n, 1)$ for the number of distinct $S_n(A)$. Two atoms A and B are *outlike* if $f_A(n) = f_B(n)$ for all n. An *outlike class* is a class of pairwise outlike atoms. A *crystal* is an infinite set of atoms and bonds satisfying the three postulates:

P 1. Each atom belongs to just one of a *finite* number of outlike classes, denoted by $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$.

P 2. The number of bonds *from* (but not necessarily *to*) any atom is finite.

P 3. If a subset of atoms *either* (*a*) contains only finitely many atoms, *or* (*b*) does not contain any atoms of at least one outlike class, then this subset contains an atom from which a bond leads to some atom not in the subset.

In view of P 1 we may define

$$f_i(n) = f_A(n) \quad (A \in \Gamma_i;\ i = 1, 2, \ldots, k), \tag{1}$$

$$f(n) = \max_{1 \leqslant i \leqslant k} f_i(n). \tag{2}$$

Each crystal possesses a *connective constant* κ, defined by

$$\kappa = \inf_{n \geqslant 1} n^{-1} \log f(n). \tag{3}$$

Theorem 1 in (1) was the following:

THEOREM. *For any atom A of a crystal with a connective constant κ,*

$$0 \leqslant \kappa = \lim_{n \to \infty} n^{-1} \log f_A[n, r(n)] < \infty, \tag{4}$$

provided $\lim_{n \to \infty} r(n)/n = 0$.

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Counter examples in (1) showed that the postulates P 1, P 2 and P 3 were each necessary for the truth of this theorem. The following proof shows that they are also sufficient.

A single atom is a finite set of atoms; and, by P 3 (*a*), a bond leads from it to another atom. Similar considerations hold for this other atom, and so on. Hence, there exists at least one $W_n(A)$ for each prescribed n and A, i.e. $f_A(n, n) \geqslant 1$. The number of bonds leading from $A \in \Gamma_i$ is $f_A(1) = f_i(1) \leqslant f(1)$, which is finite by P 1 and P 2. Thus there are never more than $f(1)$ alternative paths at each step of a walk; and therefore

$$1 \leqslant f_A(n, n) \leqslant [f(1)]^n. \tag{5}$$

For a given atom A, let $\mathfrak{N}^n(A)$ denote the set of atoms consisting of A itself and of all atoms which can be reached from A in n or fewer steps. By (5), $\mathfrak{N}^n(A)$ is a finite non-empty set; and hence, by P 3 (*a*), there exist atoms $A' \in \mathfrak{N}^n(A)$ and $A'' \notin \mathfrak{N}^n(A)$ such that A'' can be reached from A' in a single step. Since $A' \in \mathfrak{N}^n(A)$, there exists a walk of n or fewer steps from A to A'. This walk must be an $S_n(A)$; for otherwise, by adding the step from A' to A'' and (perhaps) deleting a superfluous loop on the walk from A to A', we could reach A'' from A in n or fewer steps, in contradiction of $A'' \notin \mathfrak{N}^n(A)$. Hence, there exists at least one $S_n(A)$ for every prescribed n and A. Thus, by (1) and (2),

$$1 \leqslant f_A(n) \leqslant f(n). \tag{6}$$

Then (3) and (6) establish $0 \leqslant \kappa$, required in (4).

Let $\mathfrak{N}^\infty(A)$ denote the union of all sets $\mathfrak{N}^n(A)$ for $n = 1, 2, \ldots$. Suppose that $\mathfrak{N}^\infty(A)$ does not contain any atoms of at least one outlike class. Then, by P 3 (*b*), there exist atoms $A' \in \mathfrak{N}^\infty(A)$ and $A'' \notin \mathfrak{N}^\infty(A)$ and a bond from A' to A''. This is a contradiction; because the existence of the bond ensures that A'' can be reached by a walk from A whenever A' can, i.e. that $A' \in \mathfrak{N}^\infty(A)$ implies $A'' \in \mathfrak{N}^\infty(A)$. Hence, the union of all $\mathfrak{N}^n(A)$ for $n = 1, 2, \ldots$ contains at least one atom from each outlike class. By P 1, there must now exist a positive integer $Q = Q(A)$ such that $\mathfrak{N}^Q(A)$ contains at least one atom from each outlike class.

For a given atom A and given positive integers m and n, the set of all $S_{m+n}(A)$ is a subset of the $W_{m+n}(A)$, whose first m and last n steps separately constitute self-avoiding walks of m and n steps. Hence, from (6),

$$f_A(m+n) \leqslant f_A(m) f(n) \leqslant f(m) f(n). \tag{7}$$

Because the right-hand side of (7) is independent of A, we deduce

$$f(m+n) \leqslant f(m) f(n). \tag{8}$$

Now (8) states that the function $\log f(n)$ is subadditive on the positive integers; and hence, by a known theorem on subadditive functions (see, for example, Hille (2), Theorem 6·6·1),

$$\lim_{n \to \infty} n^{-1} \log f(n) = \inf_{n \geqslant 1} n^{-1} \log f(n) = \kappa, \tag{9}$$

by (3). Therefore, if λ is any constant greater than κ, there exists a constant

$$C = C(\lambda) \geqslant \tfrac{1}{2}(1 + e^\lambda) \geqslant 1, \tag{10}$$

such that

$$f_A(n) \leqslant f(n) \leqslant C e^{\lambda n} \tag{11}$$

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for all A and n. From (7) and (11)

$$f_A(m+n) \leqslant Ce^{\lambda n} f_A(m). \tag{12}$$

Let A and B be two atoms such that there is a bond from A to B. Call this particular bond β (if there are several bonds from A to B then β is any particular fixed one of them). Consider any $S_B(2n)$. If this walk does not pass through A, we can derive from it an $S_A(2n+1)$ by prefacing it by β. On the other hand, if this $S_B(2n)$ visits A after precisely p steps where $p < 2n$, we can preface it with β and remove the step leading to A, thus deriving two walks, an $S_A(p)$ and an $S_A(2n-p)$. Finally, if $p = 2n$, we omit the $S_A(2n-p)$. For each distinct $S_B(2n)$, there is a distinct derived walk or a distinct pair of derived walks. Hence,

$$f_B(2n) \leqslant f_A(2n+1) + \sum_{p=1}^{2n-1} f_A(p) f_A(2n-p) + f_A(2n). \tag{13}$$

Application of (11) and (12) to the right-hand side of (13) gives

$$\begin{aligned} f_B(2n) &\leqslant C e^{\lambda(n+1)} f_A(n) + \sum_{p=1}^{n} C e^{\lambda p} . C e^{\lambda(n-p)} f_A(n) \\ &\qquad + \sum_{p=n+1}^{2n-1} C e^{\lambda(p-n)} f_A(n) . C e^{\lambda(2n-p)} + C e^{\lambda n} f_A(n) \\ &\leqslant (2n+1) C^2 e^{\lambda n} f_A(n), \end{aligned} \tag{14}$$

by (10). If now there is a q-stepped walk from an atom A to another atom Z, we have by repeated application of (14) to the successive pairs of atoms $YZ, XY, \ldots, AB$ along the walk

$$\begin{aligned} f_Z(2^q n) &\leqslant [(2^q n+1) C^2 \exp(\lambda 2^{q-1} n)] [(2^{q-1} n+1) C^2 \exp(\lambda 2^{q-2} n)] \ldots \\ &\qquad \times [(2n+1) C^2 \exp(\lambda n)] f_A(n) \\ &\leqslant 3^{q(q+1)/2} C^{2q} n^q \exp[\lambda(2^q - 1) n] f_A(n). \end{aligned} \tag{15}$$

From (12) and (15) it follows that, if $q \leqslant Q = Q(A)$,

$$f_Z(2^Q n) \leqslant C \exp[\lambda(2^Q - 2^q) n] f_Z(2^q n) \leqslant D n^Q \exp[\lambda(2^Q - 1) n] f_A(n), \tag{16}$$

where D is a number depending only on $Q(A)$ and $C(\lambda)$. Since every atom of $\mathfrak{N}^Q(A)$ can be reached from A in Q or fewer steps, (16) holds for all atoms $Z \in \mathfrak{N}^Q(A)$. Because $\mathfrak{N}^Q(A)$ contains at least one atom of each outlike class, (2) and (16) yield

$$f(2^Q n) = \max_{1 \leqslant i \leqslant k} f_i(2^Q n) = \max_{Z \in \mathfrak{N}^Q(A)} f_Z(2^Q n) \leqslant D n^Q \exp[\lambda(2^Q - 1) n] f_A(n). \tag{17}$$

By (9) and (17)

$$\begin{aligned} \kappa &= \lim_{n \to \infty} 2^{-Q} n^{-1} \log f(2^Q n) \\ &\leqslant \liminf_{n \to \infty} 2^{-Q} n^{-1} [\log D + Q \log n + \lambda(2^Q - 1) n + \log f_A(n)] \\ &\leqslant \lambda + 2^{-Q} [\liminf_{n \to \infty} n^{-1} \log f_A(n) - \kappa]. \end{aligned} \tag{18}$$

Now Q is independent of λ. Hence, letting $\lambda \to \kappa + 0$ in (18) we derive

$$\liminf_{n \to \infty} n^{-1} \log f_A(n) \geqslant \kappa. \tag{19}$$

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The total number of ways of breaking an n-stepped walk into r pieces equals the number of ways of selecting $r-1$ division points from the $n-2$ atoms in the centre of the walk. This is $\binom{n-2}{r-1}$. Applying (11) to each of the r pieces, we have

$$f_A(n,r) \leqslant \binom{n-2}{r-1} C^r e^{\lambda n}. \tag{20}$$

We derive from (20) with the help of Stirling's formula

$$\limsup_{n\to\infty} n^{-1} \log f_A(n,r) \leqslant \lambda \quad (r/n \to 0). \tag{21}$$

Since (21) is true for all $\lambda > \kappa$, and $f_A(n) \leqslant f_A(n,r)$, the theorem follows from (19) and (21).

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