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CONSTRUCTION OF COMMUTATOR BLUEPRINTS

SEBASTIAN BISCHOF

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Abstract

Commutator blueprints can be seen as blueprints for constructing RGD systems over $\mathbb{F}_2$ with prescribed commutation relations. In this paper, we construct several families of Weyl-invariant commutator blueprints, mostly of universal type. Also applying another result of the author, we obtain new examples of exotic RGD systems of universal type over $\mathbb{F}_2$. In particular, we generalize Tits' construction of uncountably many trivalent Moufang twin trees to higher rank, we obtain an example of an RGD system of rank 3 such that the nilpotency degree of the groups U_w is unbounded, and we construct a commutator blueprint of type $(4, 4, 4)$ that is used to answer a question of Tits from the late 1980s about twin buildings.

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1. Introduction

To any generalized Cartan matrix A and any field $\mathbb{F}$ corresponds an associated Kac–Moody group $\mathbf{G}_A(\mathbb{F})$ (see [Tit87]). This group comes equipped with a distinguished family of subgroups $(U_\alpha)_{\alpha\in\Phi}$ called *root subgroups* indexed by the set Φ of real roots of A. The generalized Cartan matrix A provides also a *Weyl group* and hence a Coxeter system $(W(A), S(A))$. The real roots of A correspond to the roots of $(W(A), S(A))$ (viewed as half-spaces). The pair $(\mathbf{G}_A(\mathbb{F}), (U_\alpha)_{\alpha\in\Phi})$ satisfies certain properties, for instance, a commutation relation between root groups corresponding to

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prenilpotent pairs of roots. The combinatorics of such pairs has been formalized by Tits in the general framework of RGD systems (see [Tit92]).

Let (W, S) be a Coxeter system and let Φ be its associated set of roots. An *RGD system of type* (W, S) is a pair $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ consisting of a group G and a family of subgroups $(U_\alpha)_{\alpha\in\Phi}$ satisfying some axioms that are motivated by the theory of Kac–Moody groups. It is natural to ask whether any RGD system is of *Kac–Moody origin*. In the 2-spherical case, it follows from the main result of [Müh99] that any RGD system of irreducible and *crystallographic* type (that is, $o(st) \in \{2, 3, 4, 6\}$ for $s \neq t \in S$) having finite root groups of order > 3 that generate the group G must be a split or almost split Kac–Moody group in the sense of [Rém02]. In the right-angled case, we have more flexibility. In [RR06], Rémy and Ronan have constructed *exotic* RGD systems with prescribed isomorphism types of root groups. For mixed ground fields, these groups are not of Kac–Moody origin. One main aspect in their construction is that all root groups corresponding to prenilpotent pairs of roots commute. The main goal of this paper is to establish the existence of new examples of exotic RGD systems both of 2-spherical and right-angled type having nontrivial commutation relations. In all examples, the root groups have cardinality 2.

In [Bis25b], we have introduced the notion of a commutator blueprint; we refer to Section 3 for more details. These purely combinatorial objects can be seen as a blueprint for constructing RGD systems *over* $\mathbb{F}_2$ (that is, each root subgroup has cardinality 2) with prescribed commutation relations. To each RGD system over $\mathbb{F}_2$, one can associate a commutator blueprint. Such blueprints are called *integrable*. A necessary condition for integrability is for the commutator blueprint to be *Weyl-invariant* (roughly speaking: the commutation relations are Weyl-invariant). For this reason, we focus on constructing Weyl-invariant commutator blueprints. Using the main result of [Bis25b], one can show that Weyl-invariant commutator blueprints of *universal* type (that is, $o(st) = \infty$ for $s \neq t \in S$) are integrable (see [Bis25b, Introduction] for more details).

1.1. Generalizing Tits' construction of uncountably many trivalent Moufang twin trees. In [Tit13, Section 5.4 in 95/96], Tits constructed uncountably many examples of trivalent *Moufang twin trees*, which are essentially the same as (centre-free) RGD systems of type $\tilde{A}_1$. In the case $\tilde{A}_1$, we have $\Phi = \{+, -\} \times \mathbb{Z}$ and all the examples constructed by Tits satisfy the following commutation relations, where $\varepsilon \in \{+, -\}$:

$$\text{for all } z, z' \in \mathbb{Z} : [U_{\varepsilon,2z}, U_{\varepsilon,z'}] = 1 \quad \text{and} \quad [U_{\varepsilon,2z+1}, U_{\varepsilon,2z'+1}] \leq \langle U_{\varepsilon,2i} \mid i \in \mathbb{Z}\rangle. \quad (1\text{-}1)$$

The following theorem is a consequence of Theorem 4.6 and [Bis25b, Theorem A].

THEOREM A. *Suppose (W, S) is universal and of rank at least 2. Then, there exists an integrable commutator blueprint $\mathcal{M}$ and $s \neq t \in S$ such that the restriction of $\mathcal{M}$ to the $\{s, t\}$-residue coincides with the commutator blueprint coming from Tits' construction of trivalent Moufang twin trees.*

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REMARK 1. Suppose that (W, S) has rank 2. Then, Theorem A is just the commutator blueprint obtained from Tits' construction in [Tit13, Section 5.4 in 95/96]. Unfortunately, there is no proof available in the literature on the existence of these Moufang twin trees. Theorem A provides a different approach to the results about the trivalent Moufang twin trees constructed by Tits.

REMARK 2. Again, we suppose that (W, S) has rank 2. The RGD systems corresponding to the commutator blueprints in Theorem A yield Moufang twin trees of order 2 and the precise commutation relations can be found in Theorem 4.6. The group G from the RGD systems, which coincides with the automorphism of the corresponding trivalent Moufang twin tree, is virtually simple in many cases. To see this, we choose in Theorem 4.6 K to be finite and $J_k \neq \emptyset$ for some $k \in K$. Then, the automorphism group of these Moufang twin trees is virtually simple by [CR16, Theorem 1]. If we assume additionally $1 \in K$ and $J_1 := \{1\}$, then we can apply [CR16, Lemma 6(i′)]. This guarantees that the commutator subgroup of the automorphism group of the Moufang twin tree associated with the RGD system of type $\tilde{A}_1$ is simple and has finite index in the automorphism group. We note that there are elaborated results available that show that the automorphism group of almost all Moufang twin trees of order 2 is virtually simple (see [GHM]). The proof is based on the idea given in [CR16].

REMARK 3. Grüninger, Horn and Mühlherr have shown in [GHM16] that in the case $\tilde{A}_1$ over $\mathbb{F}_2$, the commutation relations are generally very restrictive and they cannot be significantly more complicated than those in (1-1). Thus, it is natural to ask whether the commutation relations of any trivalent Moufang twin tree are as in (1-1). In [GHM16, Introduction], Grüninger, Horn and Mühlherr announced that they constructed new examples of Moufang twin trees of order 2 having different commutation relations than those in (1-1). In the present paper, we provide independently such an example (see Theorem 4.8 for the precise commutation relations).

1.2. The nilpotency class of the groups U_w. We now turn our attention to the nilpotency class of the groups U_w. These groups appear naturally as subgroups of RGD systems and are generated by suitable root subgroups. It is a consequence of [GHM16, Theorem A] that in the case $\tilde{A}_1$, the group U_w is nilpotent of class at most 2 for all $w \in W$, provided that all root groups are isomorphic to $(\mathbb{F}_p, +)$ for a fixed prime p. This result was generalized in [Par21] and includes the cases where $U_{\alpha_s} \cong (\mathbb{K}_s, +)$ with $\mathbb{K}_s$ a field of characteristic different from 2 (see also [SW08, Seg09]).

In [Cap07, Theorem 1.2], Caprace has shown that the nilpotency class of the groups U_w in Kac–Moody groups is bounded above by a constant only depending on the generalized Cartan matrix A and not on w. We see that the general situation is very different and the results about Kac–Moody groups do not generalize to arbitrary RGD systems. Even more, we construct examples of any rank greater than or equal to 3 such that the nilpotency class of the groups U_w can be arbitrarily large (see Theorem 4.13(i) and [Bis25b, Theorem A]). To make the statement precise, for an RGD system $\mathcal{D}$, we

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define $\mathrm{ndeg}(\mathcal{D})$ to be the supremum of the nilpotency classes of the subgroups U_w for all $w \in W$.

THEOREM B. *For each* $(m, n) \in \mathbb{N}_{\geq 3} \times \mathbb{N}_{\geq 3}$, *there exists an RGD system* $\mathcal{D}_{(m,n)}$ *of rank* m *with* $\mathrm{ndeg}(\mathcal{D}_n) = n - 1$.

Theorem B implies that a generalization of [GHM16, Theorem A] to higher rank is not possible. At present, it is unclear whether for a fixed RGD system $\mathcal{D}$, the nilpotency class of the groups U_w is bounded above by a constant depending only on $\mathcal{D}$ and not on w. The next theorem, which is a consequence of Theorem 4.13(ii) together with [Bis25b, Theorem A], shows that such a constant does not exist in general.

THEOREM C. *For each* $m \in \mathbb{N}_{\geq 3}$, *there exists an RGD system* $\mathcal{D}_m$ *of rank* m *with* $\mathrm{ndeg}(\mathcal{D}) = \infty$.

The commutator blueprints that are used to prove Theorems B and C are of universal type. Opposite to Coxeter systems of universal type are the Coxeter systems of 2-spherical type. We have also constructed commutator blueprints of the 2-spherical *type* $(4, 4, 4)$, that is, the Coxeter system (W, S) is of rank 3 and $o(st) = 4$ for all $s \neq t \in S$.

1.3. Commutator blueprints of type (4, 4, 4). One of the most celebrated results in the theory of abstract buildings is Tits' classification of (irreducible) spherical buildings of rank at least 3 in [Tit74]. It relies on a local-to-global result for isometries of spherical buildings. Inspired by [Tit87], Ronan and Tits introduced twin buildings. It turns out that twin buildings are natural generalizations of spherical buildings. Tits conjectured right from the beginning that the methods proving the local-to-global result for isometries of spherical buildings can be carried over to 2-spherical twin buildings (see [Tit92, Remark 5.9(f) and Conjectures 1 and 1′]). This conjecture was confirmed by Mühlherr and Ronan in [MR95] under a mild condition excluding only a very short list of small residues of rank 2. In [BM23], Mühlherr and the author confirmed the conjecture for twin buildings of affine type. It is still an open question whether Tits' conjecture holds for twin buildings of type $(4, 4, 4)$ where each panel contains exactly 3 chambers.

We aim to construct a counterexample to Tits' conjecture. Our strategy is to construct two integrable commutator blueprints of type $(4, 4, 4)$ with different commutation relations. By [Bis25b, Theorem A] together with the main result of [Bis23], we know that Weyl-invariant commutator blueprints of type $(4, 4, 4)$ are integrable. This motivates the following result (see Lemma 4.24).

THEOREM D. *There exist uncountably many Weyl-invariant commutator blueprints of type* $(4, 4, 4)$.

1.4. Overview. In this paper, we construct several families of Weyl-invariant commutator blueprints. In Section 2, we fix notation and prove some elementary results about Coxeter systems. In Section 3, we recall the definition of commutator blueprints

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and we introduce the notion of a *pre-commutator blueprint*. These objects are weaker versions of commutator blueprints that only depend on combinatorial properties. In Lemma 3.5, we have worked out precise conditions on the commutation relations that guarantee $|U_w| = 2^{\ell(w)}$ and, in particular, that a pre-commutator blueprint is already a commutator blueprint. These conditions can be weakened and it turns out that these weaker conditions imply that the groups U_w are nilpotent of class at most 2 (see Theorem 3.8). Our construction of commutator blueprints (which is done in Section 4) is always by constructing pre-commutator blueprints, which satisfy the conditions of either Lemma 3.5 or of Theorem 3.8.

2. Preliminaries

2.1. Coxeter systems. Let (W, S) be a Coxeter system and let ℓ denote the corresponding length function. For $s, t \in S$, we denote the order of st in W by m_{st}. The *Coxeter diagram* corresponding to (W, S) is the labelled graph $(S, E(S))$, where $E(S) = \{\{s, t\} \mid m_{st} > 2\}$ and where each edge $\{s, t\}$ is labelled by m_{st} for all $s, t \in S$. The *rank* of the Coxeter system is the cardinality of the set S. Let (W, S) be of rank 3 and let $S = \{r, s, t\}$. Sometimes, we also call (m_{rs}, m_{rt}, m_{st}) the *type* of (W, S). If $3 \leq m_{rs}, m_{rt}, m_{st}$ and $(m_{rs}, m_{rt}, m_{st}) \neq (3, 3, 3)$, we call (W, S) *cyclic hyperbolic*.

It is well known that for each $J \subseteq S$, the pair $(\langle J\rangle, J)$ is a Coxeter system (see [Bou02, Ch. IV, Section 1, Theorem 2]). A subset $J \subseteq S$ is called *spherical* if $\langle J\rangle$ is finite. The Coxeter system is called 2-*spherical* if $\langle J\rangle$ is finite for each $J \subseteq S$ with $|J| \leq 2$ (that is, $m_{st} < \infty$); it is called *spherical* if S is spherical. Given a spherical subset J of S, there exists a unique element of maximal length in $\langle J\rangle$, which we denote by r_J (see [AB08, Corollary 2.19]).

2.2. The chamber system $\Sigma(W, S)$. Let (W, S) be a Coxeter system. Defining $w \sim_s w'$ if and only if $w^{-1}w' \in \langle s\rangle$, we obtain a chamber system with chamber set W and equivalence relations $\sim_s$ for $s \in S$, which we denote by $\Sigma(W, S)$. We call two chambers w, w' s-*adjacent* if $w \sim_s w'$ and *adjacent* if they are s-adjacent for some $s \in S$. A *gallery of length* n from w_0 to w_n is a sequence $(w_0, \ldots, w_n)$ of chambers where w_i and w_{i+1} are adjacent for each $0 \leq i < n$. A gallery $(w_0, \ldots, w_n)$ is called *minimal* if there exists no gallery from w_0 to w_n of length $k < n$ and we denote the length of a minimal gallery from w_0 to w_n by $\ell(w_0, w_n)$. Let $G = (w_0, \ldots, w_n)$ be a minimal gallery. Then, we call $(s_1, \ldots, s_n)$ the *type* of G, where $s_i := w_{i-1}^{-1} w_i \in S$.

For $J \subseteq S$, we define the J-*residue* of a chamber $c \in W$ to be the set $R_J(c) := c\langle J\rangle$. A *residue* R is a J-residue for some $J \subseteq S$; we call J the *type* of R and the cardinality of J is called the *rank* of R. A residue is called *spherical* if its type is a spherical subset of S. Let R be a spherical J-residue. Two chambers $x, y \in R$ are called *opposite in* R if $\delta(x, y) = r_J$. Two residues $P, Q \subseteq R$ are called *opposite in* R if for each $p \in P$, there exists $q \in Q$ such that p, q are opposite in R and if for each $q' \in Q$, there exists $p' \in P$ such that q', p' are opposite in R. A *panel* is a residue of rank 1. It is a fact that for every chamber $x \in W$ and every residue R, there exists a unique chamber $z \in R$ such

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that $\ell(x,y) = \ell(x,z) + \ell(z,y)$ holds for every chamber $y \in R$. The chamber z is called the *projection* of x onto R and is denoted by $z = \mathrm{proj}_R x$.

EXAMPLE 2.1. The group W acts on $\Sigma(W,S)$ by multiplication from the left (see [Wei09, 29.2]).

A subset $\Sigma \subseteq W$ is called *convex* if for any two chambers $c, d \in \Sigma$ and any minimal gallery $(c_0 = c, \dots, c_k = d)$, we have $c_i \in \Sigma$ for all $0 \le i \le k$. Note that residues are convex by [AB08, Example 5.44(b)].

For two residues R and T, we define $\mathrm{proj}_T R := \{\mathrm{proj}_T r \mid r \in R\}$. By [AB08, Lemma 5.36(2)], $\mathrm{proj}_T R$ is a residue contained in T. The residues R and T are called *parallel* if $\mathrm{proj}_T R = T$ and $\mathrm{proj}_R T = R$.

LEMMA 2.2. *Let R be a spherical residue of rank 2 and let $P \neq Q$ be two parallel panels contained in R. Then, P and Q are opposite in R.*

PROOF. This is a consequence of [DMV12, Lemma 18] and [AB08, Lemma 5.107]. □

2.3. Roots and walls. Let (W,S) be a Coxeter system. A *reflection* is an element of W that is conjugate to an element of S. For $s \in S$, we let $\alpha_s := \{w \in W \mid \ell(sw) > \ell(w)\}$ be the *simple root* corresponding to s. A *root* is a subset $\alpha \subseteq W$ such that $\alpha = v\alpha_s$ for some $v \in W$ and $s \in S$. We denote the set of all roots by $\Phi(W,S)$. We note that roots are convex (see [AB08, Proposition 5.81]). The set $\Phi(W,S)_+ := \{\alpha \in \Phi(W,S) \mid 1_W \in \alpha\}$ is the set of all *positive roots* and $\Phi(W,S)_- := \{\alpha \in \Phi(W,S) \mid 1_W \notin \alpha\}$ is the set of all *negative roots*. For each root $\alpha \in \Phi(W,S)$, we denote its *opposite root* by $-\alpha$ and we denote the unique reflection that interchanges these two roots by r_α. For $\alpha \in \Phi(W,S)$, we denote by $\partial\alpha$ (respectively $\partial^2\alpha$) the set of all panels (respectively spherical residues of rank 2) stabilized by r_α. Furthermore, we define $C(\partial\alpha) := \bigcup_{P \in \partial\alpha} P$ and $C(\partial^2\alpha) := \bigcup_{R \in \partial^2\alpha} R$. The set $\partial\alpha$ is called the *wall* associated with α. Let $G = (c_0, \dots, c_k)$ be a gallery. We say that G *crosses the wall* $\partial\alpha$ if there exists $1 \le i \le k$ such that $\{c_{i-1}, c_i\} \in \partial\alpha$. It is a basic fact that a minimal gallery crosses a wall at most once (see [AB08, Lemma 3.69]).

CONVENTION 2.3. For the rest of this paper, we let (W,S) be a Coxeter system of finite rank and we define $\Phi := \Phi(W,S)$ (respectively $\Phi_+ := \Phi(W,S)_+$ and $\Phi_- := \Phi(W,S)_-$). Moreover, we assume that $m_{st} \in \{2,3,4,6,8,\infty\}$ for all $s \neq t \in S$.

A pair $\{\alpha,\beta\} \subseteq \Phi$ of roots is called *prenilpotent* if $\alpha \cap \beta \neq \emptyset \neq (-\alpha) \cap (-\beta)$. For a prenilpotent pair $\{\alpha,\beta\}$ of roots, we write

$$[\alpha,\beta] := \{\gamma \in \Phi \mid \alpha \cap \beta \subseteq \gamma \text{ and } (-\alpha) \cap (-\beta) \subseteq (-\gamma)\}$$

and $(\alpha,\beta) := [\alpha,\beta] \backslash \{\alpha,\beta\}$. A pair $\{\alpha,\beta\}$ of roots is called *nested* if $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Let $(c_0, \dots, c_k)$ and $(d_0 = c_0, \dots, d_k = c_k)$ be two minimal galleries from c_0 to c_k and let $\alpha \in \Phi$. Then, $\partial\alpha$ is crossed by the minimal gallery $(c_0, \dots, c_k)$ if and only if it is crossed by the minimal gallery $(d_0, \dots, d_k)$. For a minimal gallery $G = (c_0, \dots, c_k)$, the

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sequence of roots crossed by G is the unique sequence of roots $(\alpha_1, \dots, \alpha_k)$ such that $c_{i-1} \in \alpha_i$ and $c_i \notin \alpha_i$ for all $1 \leq i \leq k$. In this case, we say that G is a minimal gallery *between* α_1 and α_k.

We denote the set of all minimal galleries $(c_0 = 1_W, \dots, c_k)$ starting at 1_W by Min. For $w \in W$, we denote the set of all $G \in \mathrm{Min}$ of type $(s_1, \dots, s_k)$ with $w = s_1 \cdots s_k$ by $\mathrm{Min}(w)$. For $w \in W$ with $\ell(sw) = \ell(w) - 1$, we let $\mathrm{Min}_s(w)$ be the set of all $G \in \mathrm{Min}(w)$ of type $(s, s_2, \dots, s_k)$. We extend this notion to the case $\ell(sw) = \ell(w) + 1$ by defining $\mathrm{Min}_s(w) := \mathrm{Min}(w)$. Let $w \in W, s \in S$ and $G = (c_0, \dots, c_k) \in \mathrm{Min}_s(w)$. If $\ell(sw) = \ell(w) - 1$, then $c_1 = s$ and we define $sG := (sc_1 = 1_W, \dots, sc_k) \in \mathrm{Min}(sw)$. If $\ell(sw) = \ell(w) + 1$, we define $sG := (1_W, sc_0 = s, \dots, sc_k) \in \mathrm{Min}(sw)$.

LEMMA 2.4 [Bis25a, Lemma 2.1]. *Let R be a spherical residue of $\Sigma(W, S)$ of rank 2 and let $\alpha \in \Phi$. Then, exactly one of the following hold:*

(a) $R \subseteq \alpha$;
(b) $R \subseteq (-\alpha)$;
(c) $R \in \partial^2\alpha$.

LEMMA 2.5 [CM06, Proposition 2.7]. *Let R and T be two spherical residues of $\Sigma(W, S)$. Then, the following are equivalent:*

(i) *R and T are parallel;*
(ii) *a reflection of $\Sigma(W, S)$ stabilizes R if and only if it stabilizes T;*
(iii) *there exist two sequences $R_0 = R, \dots, R_n = T$ and $T_1, \dots, T_n$ of residues of spherical type such that for each $1 \leq i \leq n$, the rank of T_i is equal to $1 + \mathrm{rank}(R)$, the residues R_{i-1}, R_i are contained and opposite in T_i, and, moreover, we have $\mathrm{proj}_{T_i} R = R_{i-1}$ and $\mathrm{proj}_{T_i} T = R_i$.*

LEMMA 2.6. *Let $\alpha \in \Phi$ be a root and let $x, y \in \alpha \cap C(\partial\alpha)$. Then, there exists a minimal gallery $(c_0 = x, \dots, c_k = y)$ such that $c_i \in C(\partial^2\alpha)$ for each $0 \leq i \leq k$. Moreover, for every $1 \leq i \leq k$, there exists $L_i \in \partial^2\alpha$ with $\{c_{i-1}, c_i\} \subseteq L_i$.*

PROOF. This is a consequence of [CM05, Lemma 2.3] and its proof. □

LEMMA 2.7 [Bis25a, Lemma 2.4]. *Let $\alpha, \beta \in \Phi, \alpha \neq \pm\beta$ be two roots and let $R, T \in \partial^2\alpha \cap \partial^2\beta$.*

(a) *The residues R and T are parallel.*
(b) *If $|\langle J \rangle| = \infty$ holds for all $J \subseteq S$ containing three elements, then $R = T$.*

LEMMA 2.8. *Let $\alpha, \beta \in \Phi, \alpha \neq \pm\beta$, be two roots.*

(a) *The following are equivalent:*

  (i) *either $\{\alpha, \beta\}$ is nested or $\{-\alpha, \beta\}$ is nested;*
  (ii) *we have $o(r_\alpha r_\beta) = \infty$;*
  (iii) *we have $\partial^2\alpha \cap \partial^2\beta = \emptyset$.*

  Moreover, if $o(r_\alpha r_\beta) = \infty$ and $\{\alpha, \beta\}$ is a prenilpotent pair, then $\{\alpha, \beta\}$ is nested.

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(b) *If* $o(r_\alpha r_\beta) < \infty$ *and* $\gamma \in (\alpha, \beta)$, *then* $\partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma \neq \emptyset$ *and* $o(r_\alpha r_\gamma)$, $o(r_\beta r_\gamma) < \infty$.

PROOF. The implication (i) $\Rightarrow$ (ii) follows exactly as in [AB08, Proposition 3.165]. Now, suppose item (ii) and assume that there exists $R \in \partial^2\alpha \cap \partial^2\beta$. As R is finite, there exists $k \in \mathbb{N}$ such that $(r_\alpha r_\beta)^k$ fixes a chamber in R, that is, $(r_\alpha r_\beta)^k w = (r_\alpha r_\beta)^k(w) = w$ for some $w \in R$. However, this implies $(r_\alpha r_\beta)^k = 1$. As $o(r_\alpha r_\beta) = \infty$, we obtain a contradiction. Now, suppose that none of $\{\alpha, \beta\}, \{-\alpha, \beta\}$ is nested. Then, we have $\alpha \nsubseteq \beta, \beta \nsubseteq \alpha$ and $(-\alpha) \nsubseteq \beta, \beta \nsubseteq (-\alpha)$. This implies that none of $\alpha \cap (-\beta), \beta \cap (-\alpha), (-\alpha) \cap (-\beta), \beta \cap \alpha$ is the empty set. Arguing as in the proof of [Wei09, Proposition 29.24] and using Lemma 2.5, there exists $R \in \partial^2\alpha \cap \partial^2\beta$ and we are done. The second part of (a) follows from the first part of (a) and [AB08, Lemma 8.42(*c*)].

To show assertion (b), we note that by assertion (a), there exists $R \in \partial^2\alpha \cap \partial^2\beta$. We deduce $\emptyset \neq R \cap \alpha \cap \beta \subseteq \gamma$ and $\emptyset \neq R \cap (-\alpha) \cap (-\beta) \subseteq (-\gamma)$. If follows from Lemma 2.4 that $R \in \partial^2\gamma$. In particular, $R \in \partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma$. We deduce $o(r_\alpha r_\gamma) < \infty$ and $o(r_\beta r_\gamma) < \infty$ from assertion (a). □

LEMMA 2.9. *Let* $\alpha, \beta, \gamma \in \Phi$ *be three pairwise distinct and pairwise nonopposite roots such that* $\partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma \neq \emptyset$ *(for example,* $\alpha \neq \pm\beta, o(r_\alpha r_\beta) < \infty, \gamma \in (\alpha, \beta)$*). Then, the following hold:*

(a) $\partial^2\alpha \cap \partial^2\beta = \partial^2\alpha \cap \partial^2\gamma$;
(b) $((\alpha, \beta) \cup (-\alpha, \beta)) \cap \{\gamma, -\gamma\} \neq \emptyset$.

PROOF. Let $R \in \partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma$ be a residue, let $\delta \in \{\beta, \gamma\}$ and let $T \in \partial^2\alpha \cap \partial^2\delta$. It suffices to show that $T \in \partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma$. Using Lemma 2.7, we deduce that R and T are parallel. Then, Lemma 2.5 implies that a reflection of $\Sigma(W, S)$ stabilizes R if and only if it stabilizes T. As $r_\alpha, r_\beta, r_\gamma$ stabilize R, they also stabilize T and assertion (a) follows. Before we show assertion (b), we prove the following claim:

Claim: $\gamma \notin (\alpha, \beta) \Rightarrow \alpha \cap \beta \cap (-\gamma) \cap R \neq \emptyset$.

We have $\alpha \cap \beta \nsubseteq \gamma$ or $(-\alpha) \cap (-\beta) \nsubseteq (-\gamma)$. In the first case, we have $\alpha \cap \beta \cap (-\gamma) \neq \emptyset$ and, as roots are convex, the claim follows from [AB08, Lemma 5.45]. Thus, we can assume $(-\alpha) \cap (-\beta) \nsubseteq (-\gamma)$. As roots are convex, [AB08, Lemma 5.45] implies $(-\alpha) \cap (-\beta) \cap \gamma \cap R \neq \emptyset$. Let x be contained in this set and let $y \in R$ be opposite to x in R. Then, $y \in \alpha \cap \beta \cap (-\gamma) \cap R$ and the claim follows.

We are now in the position to prove assertion (b). We assume $(\alpha, \beta) \cap \{\gamma, -\gamma\} = \emptyset = (-\alpha, \beta) \cap \{\gamma, -\gamma\}$. By the above, we deduce the following:

$$x \in \alpha \cap \beta \cap (-\gamma) \cap R \quad \text{and} \quad x' \in \alpha \cap \beta \cap \gamma \cap R,$$
$$y \in (-\alpha) \cap \beta \cap (-\gamma) \cap R \quad \text{and} \quad y' \in (-\alpha) \cap \beta \cap \gamma \cap R.$$

As residues and roots are convex, there exist $P, Q \in \partial\gamma$ such that $P \subseteq \alpha \cap \beta \cap R$ and $Q \subseteq (-\alpha) \cap \beta \cap R$. As $P \subseteq \alpha$ and $Q \subseteq (-\alpha)$, we have $P \neq Q$ and Lemma 2.2 implies that there exist $p \in P, q \in Q$ that are opposite in R. Using [Wei03, Proposition 5.4],

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every chamber in R lies on a minimal gallery from p to q. As roots are convex and $p, q \in \beta$, we infer $R \subseteq \beta$, which is a contradiction to $R \in \partial^2\beta$. □

LEMMA 2.10. *Suppose $\alpha \neq \pm\beta \in \Phi$ with $o(r_\alpha r_\beta) < \infty$ and let $\gamma \in (\alpha, \beta)$. Then, $\{-\alpha, \gamma\}$ is a prenilpotent pair and we have $\beta \in (-\alpha, \gamma)$.*

PROOF. Note that by Lemma 2.8 and [AB08, Lemma 8.42(3)], the pair $\{-\alpha, \gamma\}$ is prenilpotent. Thus, we have to show that $(-\alpha) \cap \gamma \subseteq \beta$ and $\alpha \cap (-\gamma) \subseteq (-\beta)$. As $\gamma \in (\alpha, \beta)$, we have $\alpha \cap \beta \subseteq \gamma$ and $(-\alpha) \cap (-\beta) \subseteq (-\gamma)$. In particular, we have $(-\gamma) \subseteq (-\alpha) \cup (-\beta)$ and $\gamma \subseteq \alpha \cup \beta$. We compute

$$(-\alpha) \cap \gamma \subseteq (-\alpha) \cap (\alpha \cup \beta) \subseteq (-\alpha) \cap \beta \subseteq \beta,$$
$$\alpha \cap (-\gamma) \subseteq \alpha \cap ((-\alpha) \cup (-\beta)) \subseteq \alpha \cap (-\beta) \subseteq (-\beta).$$ □

2.4. Reflection and combinatorial triangles in $\Sigma(W, S)$. A *reflection triangle* is a set R of three reflections such that the order of tt' is finite for all $t, t' \in R$ and such that $\bigcap_{t \in R} \partial^2\beta_t = \emptyset$, where β_t is one of the two roots associated with the reflection t. Note that $\partial^2\beta_t = \partial^2(-\beta_t)$. A set T of three roots is called a *combinatorial triangle* (or simply a *triangle*) if the following hold.

(CT1) The set $\{r_\alpha \mid \alpha \in T\}$ is a reflection triangle.
(CT2) For each $\alpha \in T$, there exists $\sigma \in \partial^2\beta \cap \partial^2\gamma$ with $\sigma \subseteq \alpha$, where $\{\beta, \gamma\} = T \backslash \{\alpha\}$.

REMARK 2.11. Let R be a reflection triangle. Then, there exist three roots $\beta_1, \beta_2, \beta_3 \in \Phi$ such that $R = \{r_{\beta_1}, r_{\beta_2}, r_{\beta_3}\}$. Let $\{i, j, k\} = \{1, 2, 3\}$. As $o(r_{\beta_i} r_{\beta_j}) < \infty$, there exists $\sigma_k \in \partial^2\beta_i \cap \partial^2\beta_j$ by Lemma 2.8. Since R is a reflection triangle, we have $\sigma_k \notin \partial^2\beta_k$ and Lemma 2.4 yields $\sigma_k \subseteq \beta_k$ or $\sigma_k \subseteq -\beta_k$. Let $\varepsilon_k \in \{+, -\}$ with $\sigma_k \subseteq \varepsilon_k\beta_k$ and define $\alpha_k := \varepsilon_k\beta_k$. Then, $\{\alpha_1, \alpha_2, \alpha_3\}$ is a triangle, which induces the reflection triangle R.

LEMMA 2.12. *Suppose that (W, S) is 2-spherical and let T be a triangle.*

(a) *If the Coxeter diagram is the complete graph, then $(-\alpha, \beta) = \emptyset$ for all $\alpha \neq \beta \in T$.*
(b) *If (W, S) is cyclic hyperbolic, then T contains a unique chamber, that is, $|\bigcap_{\alpha \in T} \alpha| = 1$.*

PROOF. Assertion (a) is [Bis22, Proposition 2.3] and assertion (b) is a consequence of the classification in [Fel98] (see Figure 8 in § 5.1 in loc. cit.). □

LEMMA 2.13. *Suppose that (W, S) is 2-spherical and that the Coxeter diagram is the complete graph. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a triangle and let $\beta \in (\alpha_1, \alpha_2)$.*

(a) *We have $o(r_\beta r_{\alpha_3}) = \infty$ and $-\alpha_3 \subseteq \beta$.*
(b) *If (W, S) is cyclic hyperbolic, then $(-\alpha_3, \beta) = \emptyset$.*

PROOF. Let $i \in \{1, 2\}$ and assume that $o(r_{\alpha_3} r_\beta) < \infty$. Then, $\{r_{\alpha_i}, r_\beta, r_{\alpha_3}\}$ is a reflection triangle by Lemmas 2.8(b) and 2.9(a). We apply Remark 2.11 to determine the triangle that induces $\{r_{\alpha_1}, r_\beta, r_{\alpha_3}\}$. We note that $\alpha_2 \in (-\alpha_1, \beta)$ and $\alpha_1 \in (-\alpha_2, \beta)$ hold by Lemma 2.10.

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There exist $R \in \partial^2\alpha_1 \cap \partial^2\alpha_2 = \partial^2\alpha_1 \cap \partial^2\beta$ with $R \subseteq \alpha_3$, and $R' \in \partial^2\alpha_1 \cap \partial^2\alpha_3$ with $R' \subseteq \alpha_2$. Thus, $\emptyset \neq R' \cap \alpha_1 \subseteq \alpha_1 \cap \alpha_2 \subseteq \beta$ and Lemma 2.4 yields $R \subseteq \beta$. This implies that there exists $\varepsilon \in \{+,-\}$ such that $\{\varepsilon\alpha_1, \beta, \alpha_3\}$ is a triangle. If $\varepsilon = +$, then $(-\alpha_1, \beta) = \emptyset$ by Lemma 2.12. However, this is a contradiction to the fact that $\alpha_2 \in (-\alpha_1, \beta)$. Thus, we have $\varepsilon = -$. We show that $\{\beta, \alpha_2, \alpha_3\}$ is a triangle.

There exists $R_3 \in \partial^2\alpha_1 \cap \partial^2\alpha_2 = \partial^2\beta \cap \partial^2\alpha_2$ with $R_3 \subseteq \alpha_3$. There exists $R_1 \in \partial^2\alpha_2 \cap \partial^2\alpha_3$ with $R_1 \subseteq \alpha_1$. In particular, $\emptyset \neq R_1 \cap \alpha_2 \subseteq \alpha_1 \cap \alpha_2 \subseteq \beta$. Now, as $\{-\alpha_1, \beta, \alpha_3\}$ is a triangle, there exists $T \in \partial^2\beta \cap \partial^2\alpha_3$ with $T \subseteq (-\alpha_1)$. Thus, $\emptyset \neq \beta \cap T \subseteq \beta \cap (-\alpha_1) \subseteq \alpha_2$. We infer that $\{\beta, \alpha_2, \alpha_3\}$ is a triangle and hence $(-\alpha_2, \beta) = \emptyset$ by Lemma 2.12. However, this is again a contradiction. We conclude $o(r_{\alpha_3} r_\beta) = \infty$.

Note that $\emptyset \neq R_3 \cap (-\beta) \subseteq \alpha_3$ and $\emptyset \neq R_1 \cap \alpha_2 \cap (-\alpha_3) \subseteq \alpha_1 \cap \alpha_2 \subseteq \beta$. In particular, $\{-\alpha_3, \beta\}$ is a prenilpotent pair and by Lemma 2.8, it is nested. We have $\emptyset \neq R_3 \cap \beta \subseteq \alpha_3$. However, this implies $\beta \not\subseteq (-\alpha_3)$. We deduce $(-\alpha_3) \subseteq \beta$. This finishes the proof of assertion (a).

Let $\{x, y\} \in \partial\alpha_3$ such that $\bigcap_{i=1}^3 \alpha_i = \{y\}$ (see Lemma 2.12) and let $R \in \partial^2\alpha_1 \cap \partial^2\alpha_2$. By Lemma 2.7(b), R is unique and hence $y \in R$. Let $d \in R$ be opposite to y in R and let $(c_0 = x, c_1 = y, \ldots, c_n = d)$ be a minimal gallery. Then, $c_i \in R$ for each $1 \leq i \leq n$. Let $(\beta_1, \ldots, \beta_n)$ be the sequence of roots crossed by $(c_0, \ldots, c_n)$. Then, $\beta_1 = -\alpha_3$ and $o(r_{\beta_i} r_\beta) < \infty$ for each $2 \leq i \leq n$ by Lemma 2.8. Assume $(-\alpha_3, \beta) \neq \emptyset$. Then, [AB08, Lemma 3.69] implies that for each $\gamma \in (-\alpha_3, \beta)$, there exists $2 \leq i \leq n-1$ with $\gamma = \beta_i$. As $\gamma \subsetneq \beta$, this is a contradiction and hence, $(-\alpha_3, \beta) = \emptyset$. □

LEMMA 2.14. *Assume that (W, S) is 2-spherical and that the Coxeter diagram is the complete graph. Suppose $w \in W$ and $s \neq t \in S$ with $\ell(ws) = \ell(w) + 1 = \ell(wt)$ and suppose $w' \in \langle s, t\rangle$ with $\ell(w') \geq 2$. Then, we have $\ell(ww'r) = \ell(w) + \ell(w') + 1$ for each $r \in S \backslash \{s, t\}$.*

If, moreover, $m_{pq} \neq 3$ for all $p, q \in S$, then we have $\ell(ww'rf) = \ell(w) + \ell(w') + 2$ for each $r \in S\backslash\{s,t\}$ and $f \in S\backslash\{r\}$.

PROOF. The first part is [Bis25a, Corollary 3.3] and the second part is a consequence of the first. □

LEMMA 2.15. *Assume that (W, S) is 2-spherical and that the Coxeter diagram is the complete graph. Let $\alpha \in \Phi_+$ be a root and let $P, Q \in \partial\alpha$. Let $P_0 = P, \ldots, P_n = Q$ and $R_1, \ldots, R_n$ be as in Lemma 2.5. If $\mathrm{proj}_{R_i} 1_W = \mathrm{proj}_{P_{i-1}} 1_W$ for some $1 \leq i \leq n$, then $\mathrm{proj}_{R_n} 1_W = \mathrm{proj}_{P_{n-1}} 1_W$.*

PROOF. We show the hypothesis by induction on $n - i$. If $n - i = 0$, then there is nothing to show. Thus, we suppose $n - i > 0$. Let J_i be the type of R_i and let $w := \mathrm{proj}_{R_i} 1_W = \mathrm{proj}_{P_{i-1}} 1_W \in P_{i-1}$. As $P_{i-1} \neq P_i$ are contained and opposite in R_i by Lemma 2.5, there exists $w' \in P_i$ such that w and w' are opposite in R_i, that is, $w' = wr_{J_i}$. For $t \in S$ with $J_i \cap J_{i+1} = \{t\}$, we have $R_i \cap R_{i+1} = P_i = \mathcal{P}_t(w') = \mathcal{P}_t(wr_{J_i})$. As $w = \mathrm{proj}_{R_i} 1_W$, we deduce $\ell(\mathrm{proj}_{P_i} 1_W) = \ell(w(r_{J_i} t)) \geq \ell(w) + 2$. Using Lemma 2.14, we

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infer $\ell((\mathrm{proj}_{P_i} 1_W)s) = \ell(\mathrm{proj}_{P_i} 1_W) + 1$ for $s \in J_{i+1} \backslash J_i$ and hence $\mathrm{proj}_{R_{i+1}} 1_W = \mathrm{proj}_{P_i} 1_W$. Using induction, the claim follows. □

LEMMA 2.16. *Assume that (W, S) is 2-spherical and that the Coxeter diagram is the complete graph. Let $w \in W$ and $r, s, t \in S$ be pairwise distinct such that $\ell(ws) = \ell(w) + 1 = \ell(wt)$. If $\ell(wsr) = \ell(w)$, then $\ell(wsrt) = \ell(w) + 1$.*

PROOF. This is a consequence of Lemma 2.14. □

LEMMA 2.17 [Bis25a, Lemma 3.4]. *Assume that (W, S) is 2-spherical and that $m_{st} \geq 4$ for all $s \neq t \in S$. Suppose $w \in W$ and $s \neq t \in S$ with $\ell(ws) = \ell(w) + 1 = \ell(wt)$. Then, we have $\ell(w) + 2 \in \{\ell(wsr), \ell(wtr)\}$ for all $r \in S \backslash \{s, t\}$.*

LEMMA 2.18. *Suppose that $m_{st} = 4$ for all $s \neq t \in S$. Let $r, s, t \in S$ be pairwise distinct and let $p \in S \backslash \{s, t\}$. Let $H = (d_0, \ldots, d_4)$ be a minimal gallery of type (r, s, t, p) and let $\alpha \in \Phi$ with $\{d_0, d_1\} \in \partial\alpha$ and $d_0 \in \alpha$. Let β be a root containing d_0 and $\{\{d_2, d_3\}, \{d_3, d_4\}\} \cap \partial\beta \neq \emptyset$. Then, we have $\alpha \subsetneq \beta$.*

PROOF. We use the canonical linear representation of (W, S) (see [AB08, Ch. 2.5]). Let $V := \mathbb{R}^S$ be the vector space over $\mathbb{R}$ with standard basis $(e_s)_{s \in S}$ and let $(\cdot, \cdot)$ be the symmetric bilinear form on V given by

$$(e_s, e_t) := -\cos\left(\frac{\pi}{m_{st}}\right) = \begin{cases} 1 & \text{if } s = t, \\ -\frac{\sqrt{2}}{2} & \text{otherwise.} \end{cases}$$

Then, W acts on V via $\sigma : W \to \mathrm{GL}(V), s \mapsto (\sigma_s : V \to V, x \mapsto x - 2(x, e_s)e_s)$ and $(\cdot, \cdot)$ is invariant under this action. Let α and β be as in the statement. Without loss of generality, we can assume $\alpha = \alpha_r$ and $\beta \in \{rs\alpha_t, rst\alpha_p\}$. At first, we consider the case $\beta = rs\alpha_t$. We compute

$$(e_r, \sigma(rs)(e_t)) = (e_r, \sigma_r(\sigma_s(e_t))) = (\sigma_r(e_r), \sigma_s(e_t)) = (-e_r, e_t + \sqrt{2}e_s) = \frac{\sqrt{2}}{2} + 1 > 1.$$

Now, we assume $\beta = rst\alpha_p$ and we compute

$$\begin{aligned} (e_r, \sigma(rst)(e_p)) &= (e_r, \sigma_r(\sigma_s(\sigma_t(e_p)))) \\ &= (\sigma_s(-e_r), \sigma_t(e_p)) \\ &= (-e_r - 2(-e_r, e_s)e_s, e_p - 2(e_p, e_t)e_t) \\ &= -(e_r, e_p) + 2(e_p, e_t)(e_r, e_t) + 2(e_r, e_s)(e_s, e_p) - 4(e_r, e_s)(e_p, e_t)(e_s, e_t) \\ &= -(e_r, e_p) + 1 + 1 + \sqrt{2} > 1. \end{aligned}$$

Using [AB08, Lemma 2.77], we obtain that $o(r_\alpha r_\beta) = \infty$. As $\{\alpha, \beta\}$ is a prenilpotent pair, Lemma 2.8 yields that $\{\alpha, \beta\}$ is a pair of nested roots and hence $\alpha \subsetneq \beta$. □

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2.5. Roots in Coxeter systems.

CONVENTION 2.19. For the rest of this section, we assume that (W, S) is 2-spherical and $m_{st} \geq 4$ for all $s \neq t \in S$.

Let R be a residue and let $\alpha \in \Phi_+$. Then, we call α a *simple root of* R if there exists $P \in \partial\alpha$ such that $P \subseteq R$ and $\mathrm{proj}_R 1_W = \mathrm{proj}_P 1_W$. In this case, R is also stabilized by r_α and hence $R \in \partial^2\alpha$. For a positive root $\alpha \in \Phi_+$, we define

$$k_\alpha := \min\{k \in \mathbb{N} \mid \text{there exists } (c_0, \dots, c_k) \in \mathrm{Min} : c_k \notin \alpha\}.$$

We remark that $k_\alpha = 1$ if and only if α is a simple root.

REMARK 2.20. Let $\alpha \in \Phi_+$ be a positive root with $k_\alpha > 1$. Let $(c_0, \dots, c_k) \in \mathrm{Min}$ be a minimal gallery with $k = k_\alpha$ and $c_k \notin \alpha$. Then, $\{c_{k-1}, c_k\} \in \partial\alpha$ and α is not a simple root of the rank-2 residue containing c_{k-2}, c_{k-1}, c_k. In particular, there exists $R \in \partial^2\alpha$ such that α is not a simple root of R.

Let $\alpha \in \Phi_+$ be a root with $k_\alpha > 1$ and let $R \in \partial^2\alpha$ be a residue such that α is not a simple root of R. Let $P \neq P' \in \partial\alpha$ be the two parallel panels contained in R. Then, $\ell(1_W, \mathrm{proj}_P 1_W) \neq \ell(1_W, \mathrm{proj}_{P'} 1_W)$ and we can assume that $\ell(1_W, \mathrm{proj}_P 1_W) < \ell(1_W, \mathrm{proj}_{P'} 1_W)$. Let $G = (c_0, \dots, c_k) \in \mathrm{Min}$ be of type $(s_1, \dots, s_k)$ such that $c_i = \mathrm{proj}_R 1_W$ for some $0 \leq i \leq k$, $c_i, \dots, c_k \in R$, $c_{k-1} = \mathrm{proj}_P 1_W$ and $c_k \in P \backslash \{c_{k-1}\}$. For $P \neq Q := \{x, y\} \in \partial\alpha$ with $x \in \alpha$ and $y \notin \alpha$, we let $P_0 = P, \dots, P_n = Q$ and $R_1, \dots, R_n$ be as in Lemma 2.5.

LEMMA 2.21. *We have* $\mathrm{proj}_{R_n} 1_W = \mathrm{proj}_{P_{n-1}} 1_W$ *if at least one of the following holds:*

(a) $R_1 \neq R$ *and* $\ell(s_1 \cdots s_{k-1} r) = k$, *where* $\{r, s_k\}$ *is the type of* R_1*;*
(b) $n > 1$.

PROOF. Suppose $R_1 \neq R$ and $\ell(s_1 \cdots s_{k-1} r) = k$. Then, $\mathrm{proj}_{R_1} c_0 = \mathrm{proj}_{P_0} c_0$ and the claim follows from Lemma 2.15. Now, we suppose $n > 1$. Assume that $R_1 = R$. Then, Lemma 2.14 implies $\mathrm{proj}_{R_2} 1_W = \mathrm{proj}_{P_1} 1_W$ and the claim follows from Lemma 2.15. Now, we suppose $R_1 \neq R$. If $\ell(s_1 \cdots s_{k-1} r) = k$, the claim follows from assertion (a). Thus, we can assume that $\ell(s_1 \cdots s_{k-1} r) = k - 2$. Then, Lemma 2.14 yields $k - 1 = i + 1$, where $c_i = \mathrm{proj}_R c_0$. Define $d := \mathrm{proj}_{R_1} c_0$ and replace G by a minimal gallery $(d_0 = c_0, \dots, d, c_{k-1}, c_k)$. Now, we are in the case $R_1 = R$ and the claim follows. □

LEMMA 2.22. *We have* $k = k_\alpha$ *and the panel* $P_\alpha := P$ *is the unique panel in* $\partial\alpha$ *with the property* $\ell(1_W, \mathrm{proj}_{P_\alpha} 1_W) = k_\alpha - 1$.

PROOF. We have $\ell(1_W, \mathrm{proj}_P 1_W) = k - 1$. Thus, it suffices to show $\ell(1_W, \mathrm{proj}_Q 1_W) > k - 1$. For $n = 1$, we obtain $\ell(1_W, \mathrm{proj}_Q 1_W) \geq k$. Now, we assume $n > 1$. Then, we have $\mathrm{proj}_{R_n} 1_W = \mathrm{proj}_{P_{n-1}} 1_W$ by Lemma 2.21. Since $Q \subseteq R_n$, we deduce $\ell(1_W, \mathrm{proj}_Q 1_W) \geq \ell(1_W, \mathrm{proj}_{R_n} 1_W) = \ell(1_W, \mathrm{proj}_{P_{n-1}} 1_W)$ and the claim follows by induction. □

LEMMA 2.23. *We define* $R_{\alpha,Q}$ *to be the residue* R_1 *if* $R \neq R_1$ *and* $\ell(s_1 \cdots s_{k-1} r) = k - 2$. *In all other cases, we define* $R_{\alpha,Q} := R$. *Then, there exists a minimal gallery* $H = (d_0 = c_0, \dots, d_m = \mathrm{proj}_Q c_0, y)$ *with the following properties.*

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- *There exists* $0 \le i \le m$ *such that* $d_i = \mathrm{proj}_{R_{\alpha,Q}} 1_W$.
- *For each* $i+1 \le j \le m$, *there exists* $L_j \in \partial^2\alpha$ *with* $\{d_{j-1}, d_j\} \subseteq L_j$. *In particular, we have* $d_j \in C(\partial^2\alpha)$.

PROOF. We define

$$d := \begin{cases} \mathrm{proj}_{P_0} c_0 & \text{if } R \neq R_1 \text{ and } \ell(s_1 \cdots s_{k-1} r) = k, \\ \mathrm{proj}_{P_1} c_0 & \text{otherwise.} \end{cases}$$

We first show that $\ell(c_0, \mathrm{proj}_Q c_0) = \ell(c_0, \mathrm{proj}_{R_{\alpha,Q}} c_0) + \ell(\mathrm{proj}_{R_{\alpha,Q}} c_0, d) + \ell(d, \mathrm{proj}_Q c_0)$. By definition, we have $R_{\alpha,Q} = R_{\alpha,P_i}$ for all $1 \le i \le n$. We prove the hypothesis by induction on n. Suppose first $n = 1$ and that one of the following holds:

- $R = R_1$;
- $R \neq R_1$ and $\ell(s_1 \cdots s_{k-1} r) = k - 2$.

Then, $Q = P_1 \subseteq R_{\alpha,Q}$, $d = \mathrm{proj}_Q c_0$ and the claim follows. We prove the case $R \neq R_1$ and $\ell(s_1 \cdots s_{k-1} r) = k$ together with the case $n > 1$ simultaneously. Lemma 2.21 provides in both cases $\mathrm{proj}_{R_n} c_0 = \mathrm{proj}_{P_{n-1}} c_0$. If $n > 1$, we have $R_{\alpha,Q} = R_{\alpha,P_{n-1}}$; if $n = 1$, we have $P_{n-1} = P_0 \subseteq R_{\alpha,Q}$ and $d = \mathrm{proj}_{P_{n-1}} c_0$. This is used in the third equation below. We compute the following:

$$\begin{aligned} \ell(c_0, \mathrm{proj}_Q c_0) &= \ell(c_0, \mathrm{proj}_{R_n} c_0) + \ell(\mathrm{proj}_{R_n} c_0, \mathrm{proj}_Q c_0) \\ &= \ell(c_0, \mathrm{proj}_{P_{n-1}} c_0) + \ell(\mathrm{proj}_{P_{n-1}} c_0, \mathrm{proj}_Q c_0) \\ &= \ell(c_0, \mathrm{proj}_{R_{\alpha,Q}} c_0) + \ell(\mathrm{proj}_{R_{\alpha,Q}} c_0, d) + \ell(d, \mathrm{proj}_{P_{n-1}} c_0) \\ &\quad + \ell(\mathrm{proj}_{P_{n-1}} c_0, \mathrm{proj}_Q c_0) \\ &\ge \ell(c_0, \mathrm{proj}_{R_{\alpha,Q}} c_0) + \ell(\mathrm{proj}_{R_{\alpha,Q}} c_0, d) + \ell(d, \mathrm{proj}_Q c_0) \\ &\ge \ell(c_0, \mathrm{proj}_Q c_0). \end{aligned}$$

Thus, concatenating a minimal gallery from c_0 to $\mathrm{proj}_{R_{\alpha,Q}} c_0$, a minimal gallery from $\mathrm{proj}_{R_{\alpha,Q}} c_0$ to d and a minimal gallery from d to $\mathrm{proj}_Q c_0$ yields a minimal gallery from c_0 to $\mathrm{proj}_Q c_0$. Using Lemma 2.6, there exists a minimal gallery from d to $\mathrm{proj}_Q c_0$ such that every chamber of this gallery is contained in $C(\partial^2\alpha)$ and for two adjacent chambers, there exists a residue in $\partial^2\alpha$ containing both. Since $R_{\alpha,Q} \in \{R, R_1\} \subseteq \partial^2\alpha$ and, as $R_{\alpha,Q}$ is convex, each chamber of a minimal gallery from $\mathrm{proj}_{R_{\alpha,Q}} c_0$ to d is contained in $R_{\alpha,Q}$; the claim follows. □

REMARK 2.24. In the lemma, we use the following notation. For a minimal gallery $G = (c_0, \ldots, c_k)$, $k \ge 1$, we denote the unique root containing c_{k-1} but not c_k by α_G.

LEMMA 2.25. *Suppose that* (W, S) *is of type* $(4, 4, 4)$ *and suppose* $S = \{s_{k-1}, s_k, r\}$. *Let* $\beta \in \Phi_+ \backslash \{\alpha_s \mid s \in S\}$ *be a root with* $k_\beta \le k$, $o(r_\alpha r_\beta) < \infty$ *and* $R \notin \partial^2\beta$. *Moreover, we assume that* $\ell(s_1 \cdots s_{k-1} r) = k$. *Then, one of the following holds:*

(a) $\beta = \alpha_F$, *where* $F \in \mathrm{Min}$ *is the minimal gallery of type* $(s_1, \ldots, s_{k-1}, r)$;
(b) $\beta = \alpha_F$, *where* $F \in \mathrm{Min}$ *is the minimal gallery of type* $(s_1, \ldots, s_{k-2}, s_k, s_{k-1}, r)$, *and we have* $\ell(s_1 \cdots s_{k-2} s_k r) = k - 2$.

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PROOF. Recall that $\alpha = \alpha_G$. As $R \in \partial^2\alpha$, we have $\alpha \neq \pm\beta$. By Lemma 2.8(a), there exists $C \in \partial^2\alpha \cap \partial^2\beta$. Then, there exists a panel $Q' \in \partial\alpha$ that is contained in C. We let $\mathrm{proj}_{Q'}c_0 \neq y \in Q'$. Let P_i, R_i be as before (with $P_n = Q'$), let $G' := (c_0, \dots, c_{k-1})$ and let $G'' := (c_0, \dots, c_k, c_{k+1})$ be the minimal gallery of type $(s_1, \dots, s_k, s_{k-1})$. Let E be a minimal gallery from c_0 to y as in Lemma 2.23. We can extend this minimal gallery (if necessary) to a minimal gallery from c_0 to $e \in C$, where $\ell(e) = \ell(\mathrm{proj}_C c_0) + 4$. Let $Q'' \in \partial\beta$ be a panel contained in C and let $\mathrm{proj}_{Q''}c_0 \neq y' \in Q''$. Take a minimal gallery $H = (d_0 = c_0, \dots, d_{m-2} = \mathrm{proj}_{R_{\beta,Q''}}c_0, \dots, d_q := \mathrm{proj}_{Q''}d_0, d_{q+1} := y')$ as in Lemma 2.23. Then, $m = k_\beta \leq k$. As before, we can extend H (if necessary) to a minimal gallery from d_0 to e. Note that $R \neq C$ by assumption, and since $R \in \partial^2\alpha_{G'} \cap \partial^2\alpha_{G''}, R \notin \partial^2\beta$, we have $\alpha_{G'} \neq \pm\beta \neq \alpha_{G''}$.

(i) Assume that $R = R_1$. Since $R \in \partial^2\alpha_{G''} \cap \partial^2\alpha, C \in \partial^2\alpha$ and $\alpha_{G''} \neq \pm\beta$, Lemma 2.7(b) implies $C \notin \partial^2\alpha_{G''}$ and hence the gallery H has to cross the wall $\partial\alpha_{G''}$. Assume that $(d_0, \dots, d_{m-2})$ crosses the wall $\partial\alpha_{G''}$. Let $1 \leq j \leq m-2$ be such that $\{d_{j-1}, d_j\} \in \partial\alpha_{G''}$. Then, $k = k_{\alpha_{G''}} \leq j \leq m-2 \leq k-2$, which is a contradiction. Thus, the gallery $(d_0, \dots, d_{m-2})$ does not cross the wall $\partial\alpha_{G''}$ and hence $(d_{m-1}, \dots, d_{q+1})$ has to cross the wall $\partial\alpha_{G''}$. Let $m \leq j \leq q+1$ be such that $\{d_{j-1}, d_j\} \in \partial\alpha_{G''}$. By Lemma 2.23, there exists $L \in \partial^2\beta$ such that $\{d_{j-1}, d_j\} \subseteq L$. Then, $L \in \partial^2\beta \cap \partial^2\alpha_{G''}$ and hence $o(r_{\alpha_{G''}}r_\beta) < \infty$. As $\partial^2\alpha \cap \partial^2\alpha_{G''} = \{R\} \neq \{C\} = \partial^2\alpha \cap \partial^2\beta$ (see Lemma 2.7(b)), we have $\partial^2\alpha \cap \partial^2\beta \cap \partial^2\alpha_{G''} = \emptyset$ and hence $\{r_\alpha, r_{\alpha_{G''}}, r_\beta\}$ is a reflection triangle. As $\mathrm{proj}_R c_0 \in R \cap \beta \neq \emptyset$ and $R \notin \partial^2\beta$, we deduce $R \subseteq \beta$. As $e \in C \cap (-\alpha_{G''}) \neq \emptyset$ and $C \notin \partial^2\alpha_{G''}$, we deduce $C \subseteq (-\alpha_{G''})$. As $L \in \partial^2\alpha_{G''} \cap \partial^2\beta, \{d_{j-1}, d_j\} \subseteq L \cap \alpha$ and $L \notin \partial^2\alpha$, we deduce $L \subseteq \alpha$. Thus, $T := \{\alpha, -\alpha_{G''}, \beta\}$ is a triangle. For $d \in W$ with $\delta(c_{k-2}, d) = s_k s_{k-1}$, we have $d \in \bigcap_{\gamma\in T}\gamma$ and Lemma 2.12(b) implies $\bigcap_{\gamma\in T}\gamma = \{d\}$. If $\ell(s_1 \cdots s_{k-2}s_k r) = k$, then $k_\beta = k+1$. Thus, $\ell(s_1 \cdots s_{k-2}s_k r) = k-2$ and part (b) follows.

(ii) Assume that $R \neq R_1$. Since $R \in \partial^2\alpha_{G'} \cap \partial^2\alpha$ and $R \neq C \in \partial^2\alpha$, Lemma 2.7(b) implies $C \notin \partial^2\alpha_{G'}$ and hence H has to cross the wall $\partial\alpha_{G'}$. Suppose that $(d_0, \dots, d_{m-2})$ does not cross the wall $\partial^2\alpha_{G'}$. Replacing $\alpha_{G''}$ by $\alpha_{G'}$ in part (i), we obtain that $T := \{\alpha, -\alpha_{G'}, \beta\}$ is a triangle. Using Lemma 2.12(b), we have $\bigcap_{\gamma\in T}\gamma = \{c_{k-1}\}$ and hence assertion (a) follows. Now, we suppose that $(d_0, \dots, d_{m-2})$ crosses the wall $\partial\alpha_{G'}$ and let $1 \leq j \leq m-2$ be such that $P' := \{d_{j-1}, d_j\} \in \partial\alpha_{G'}$. Note that $1 \leq m-2 \leq k-2$ and hence $k \geq 3$. Let Z be the $\{s_{k-1}, r\}$-residue containing c_{k-2}. Then, $\alpha_{G'}$ is not a simple root of Z and hence $k_{\alpha_{G'}} \in \{k-2, k-1\}$. This implies $k-2 \leq k_{\alpha_{G'}} \leq j \leq m-2 \leq k-2$. Lemma 2.22 implies $P' = P_{\alpha_{G'}}$ and hence P' is contained in Z. Moreover, we have $j = m-2$ and $R_{\beta,Q''} = R_{\{r,s_k\}}(d_j)$. Both nonsimple roots of $R_{\beta,Q''}$ contain $-\alpha$ by Lemma 2.18. As one of them is equal to β, we have a contradiction. □

3. Commutator blueprints

CONVENTION 3.1. From now on, we assume $m_{st} \in \{2, 3, 4, \infty\}$ for all $s \neq t \in S$.

We let $\mathcal{P}$ be the set of prenilpotent pairs of positive roots. For $w \in W$, we define $\Phi(w) := \{\alpha \in \Phi_+ \mid w \notin \alpha\}$. Let $G = (c_0, \dots, c_k) \in \mathrm{Min}$ and let $(\alpha_1, \dots, \alpha_k)$ be the

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sequence of roots crossed by G. We define $\Phi(G) := \{\alpha_i \mid 1 \le i \le k\}$. Using the indices, we obtain an ordering $\le_G$ on $\Phi(G)$ and, in particular, on $[\alpha,\beta] = [\beta,\alpha] \subseteq \Phi(G)$ for all $\alpha,\beta \in \Phi(G)$. Note that $\Phi(G) = \Phi(w)$ holds for every $G \in \mathrm{Min}(w)$. We abbreviate $\mathcal{I} := \{(G,\alpha,\beta) \in \mathrm{Min} \times \Phi_+ \times \Phi_+ \mid \alpha,\beta \in \Phi(G), \alpha \le_G \beta\}$.

Given a family $(M^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$, where $M^G_{\alpha,\beta} \subseteq (\alpha,\beta)$ is ordered via $\le_G$, for $w \in W$, we define the group U_w via the following presentation:

$$U_w := \left\langle \{u_\alpha \mid \alpha \in \Phi(w)\} \;\middle|\; \begin{cases} \text{for all } \alpha \in \Phi(w) : u_\alpha^2 = 1, \\ \text{for all } (G,\alpha,\beta) \in \mathcal{I}, G \in \mathrm{Min}(w) : [u_\alpha, u_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma \end{cases} \right\rangle.$$

Here, the product $\prod_{\gamma\in M^G_{\alpha,\beta}} u_\gamma$ is understood to be ordered via the ordering $\le_G$, that is, if $(G,\alpha,\beta) \in \mathcal{I}$ with $G \in \mathrm{Min}(w)$ and $M^G_{\alpha,\beta} = \{\gamma_1 \le_G \ldots \le_G \gamma_k\} \subseteq (\alpha,\beta) \subseteq \Phi(G)$, then $\prod_{\gamma\in M^G_{\alpha,\beta}} u_\gamma = u_{\gamma_1} \cdots u_{\gamma_k}$. Note that there could be $G,H \in \mathrm{Min}(w), \alpha,\beta \in \Phi(w)$ with $\alpha \le_G \beta$ and $\beta \le_H \alpha$. In this case, we have two commutation relations, namely

$$[u_\alpha, u_\beta] = \prod_{\gamma\in M^G_{\alpha,\beta}} u_\gamma \quad \text{and} \quad [u_\beta, u_\alpha] = \prod_{\gamma\in M^H_{\beta,\alpha}} u_\gamma.$$

From now on, we implicitly assume that each product $\prod_{\gamma\in M^G_{\alpha,\beta}} u_\gamma$ is ordered via the ordering $\le_G$.

DEFINITION 3.2. A *commutator blueprint of type* (W,S) is a family $\mathcal{M} = (M^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$ of subsets $M^G_{\alpha,\beta} \subseteq (\alpha,\beta)$ ordered via $\le_G$ satisfying the following axioms.

(CB1) Let $G = (c_0,\ldots,c_k) \in \mathrm{Min}$ and let $H = (c_0,\ldots,c_m)$ for some $1 \le m \le k$. Then, $M^H_{\alpha,\beta} = M^G_{\alpha,\beta}$ holds for all $\alpha,\beta \in \Phi(H)$ with $\alpha \le_H \beta$.

(CB2) Suppose $s \ne t \in S$ with $m_{st} < \infty$. For $(G,\alpha,\beta) \in \mathcal{I}$ with $G \in \mathrm{Min}(r_{\{s,t\}})$, we have

$$M^G_{\alpha,\beta} = \begin{cases} (\alpha,\beta) & \text{if } \{\alpha,\beta\} = \{\alpha_s,\alpha_t\}, \\ \emptyset & \text{if } \{\alpha,\beta\} \ne \{\alpha_s,\alpha_t\}. \end{cases}$$

(CB3) For each $w \in W$, we have $|U_w| = 2^{\ell(w)}$, where U_w is defined as above.

A commutator blueprint $\mathcal{M} = (M^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$ is called *Weyl-invariant* if for all $w \in W$, $s \in S$, $G \in \mathrm{Min}_s(w)$ and $\alpha,\beta \in \Phi(G)\backslash\{\alpha_s\}$ with $\alpha \le_G \beta$, we have $M^{sG}_{s\alpha,s\beta} = sM^G_{\alpha,\beta} := \{s\gamma \mid \gamma \in M^G_{\alpha,\beta}\}$.

LEMMA 3.3 [Bis25b, Lemma 3.6]. *Let $\mathcal{M} = (M^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$ be a commutator blueprint of type (W,S). Let $w \in W, G = (c_0,\ldots,c_k) \in \mathrm{Min}(w)$ and let $(\alpha_1,\ldots,\alpha_k)$ be the sequence of roots crossed by G. Then, $\Phi(w) = \{\alpha_1,\ldots,\alpha_k\}$ and the group U_w has the following presentation:*

$$U_G := \left\langle u_{\alpha_1},\ldots,u_{\alpha_k} \;\middle|\; \text{for all } 1 \le i \le j \le k : u_{\alpha_i}^2 = 1, [u_{\alpha_i}, u_{\alpha_j}] = \prod_{\gamma\in M^G_{\alpha_i,\alpha_j}} u_\gamma \right\rangle.$$

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3.1. Pre-commutator blueprints.

DEFINITION 3.4. A *pre-commutator blueprint of type* (W,S) is a family $\mathcal{M} = (M^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$ of subsets $M^G_{\alpha,\beta} \subseteq (\alpha,\beta)$ ordered via $\leq_G$ satisfying Axioms (CB1) and (CB2) from Definition 3.2, and additionally the following axiom.

(PCB) For all $w \in W$ and $G \in \mathrm{Min}(w)$, the canonical homomorphism $U_G \to U_w$ is an isomorphism, where U_G is defined as in Lemma 3.3.

LEMMA 3.5. *Let* $\mathcal{M} = (M^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$ *be a pre-commutator blueprint of type* (W,S). *Then, the following are equivalent.*

(i) $\mathcal{M}$ *is a commutator blueprint.*

(ii) *Let* $G = (c_0,\dots,c_{k+1}) \in \mathrm{Min}$ *and let* $(\alpha_1,\dots,\alpha_{k+1})$ *be the sequence of roots crossed by* G. *For* $H := (c_0,\dots,c_k)$ *and for all* $1 \leq i \leq j \leq k$, *the following hold in the group* U_H:

(C1) $(\prod_{\gamma\in M^G_{\alpha_i,\alpha_{k+1}}} u_\gamma)\cdot(\prod_{\gamma\in M^G_{\alpha_i,\alpha_{k+1}}}(u_\gamma \prod_{\omega\in M^G_{\gamma,\alpha_{k+1}}} u_\omega)) = 1$;

(C2) $(u_{\alpha_i}\prod_{\gamma\in M^G_{\alpha_i,\alpha_{k+1}}} u_\gamma)^2 = 1$;

(C3) $[u_{\alpha_i}\prod_{\gamma\in M^G_{\alpha_i,\alpha_{k+1}}} u_\gamma, u_{\alpha_j}\prod_{\gamma\in M^G_{\alpha_j,\alpha_{k+1}}} u_\gamma] = \prod_{\gamma\in M^G_{\alpha_i,\alpha_j}}(u_\gamma \prod_{\omega\in M^G_{\gamma,\alpha_{k+1}}} u_\omega)$.

PROOF. One can easily check that the conditions in part (ii) must be satisfied if $\mathcal{M}$ is a commutator blueprint. Thus, we can assume part (ii). It suffices to show $|U_w| = 2^{\ell(w)}$. We show this by induction. For $\ell(w) \leq 1$, there is nothing to show. Thus, we can assume $\ell(w) > 1$. Let $G = (c_0,\dots,c_{k+1}) \in \mathrm{Min}(w)$ and let $(\alpha_1,\dots,\alpha_{k+1})$ be the sequence of roots crossed by G. We define $H := (c_0,\dots,c_k)$. Using induction, we have $|U_H| = 2^k$. We show that $U_G \cong U_H \rtimes_\varphi \mathbb{Z}_2$, where $\varphi \in \mathrm{Aut}(U_H)$ acts on U_H as $u_{\alpha_{k+1}}$. We consider the mapping

$$u_{\alpha_{k+1}} : \{u_\alpha \mid c_k \notin \alpha\} \to U_H, u_\alpha \mapsto u_\alpha \prod_{\gamma\in M^G_{\alpha,\alpha_{k+1}}} u_\gamma.$$

By Axioms (C2) and (C3), $u_{\alpha_{k+1}}$ extends to an endomorphism on U_H. Condition (C1) guarantees $u^2_{\alpha_{k+1}} = \mathrm{id}$ and hence $u_{\alpha_{k+1}} \in \mathrm{Aut}(U_H)$. Thus, we can define the semi-direct product $U_H \rtimes_{u_{\alpha_{k+1}}} \mathbb{Z}_2$. As $U_w \cong U_G \cong U_H \rtimes_{u_{\alpha_{k+1}}} \mathbb{Z}_2$, the claim follows. □

We see in Theorem 3.8 that there are weaker conditions than those in Lemma 3.5(ii), which imply that a pre-commutator blueprint is a commutator blueprint. In this case, the groups U_w are nilpotent of class at most 2. For that, we need the following two preparatory lemmas.

LEMMA 3.6. *Let* G *be a group and let* $X \subseteq G$ *be a symmetric generating set (that is,* $X = X^{-1}$*) such that* $[x,[y,z]] = 1$ *holds for all* $x,y,z \in X$. *Then,* G *is nilpotent of class at most* 2.

PROOF. Suppose $x,y,z \in G$ and let $x_1,\dots,x_k,y_1,\dots,y_l,z_1,\dots,z_m \in X$ be such that $x = x_1\cdots x_k, y = y_1\cdots y_l, z = z_1\cdots z_m$. We show $[x,[y,z]] = 1$. We first assume $l = 1 = m$. Induction on k yields $[x,[y,z]] = [xx_k^{-1},[y,z]]^{x_k}[x_k,[y,z]] = 1$.

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Now, assume $l = 1$. Induction on m implies $[x,[y,z]] = [x,[y,z_m][y,zz_m^{-1}]^{z_m}] = [x^{(z_m^{-1})},[y,zz_m^{-1}]]^{z_m}[x,[y,z_m]]^{[y,zz_m^{-1}]^{z_m}} = 1$. Now, induction on l yields

$$[x,[y,z]] = [x,[yy_l^{-1},z]^{y_l}[y_l,z]] = [x,[y_l,z]][x^{(y_l^{-1})},[yy_l^{-1},z]]^{y_l[y_l,z]} = 1. \qquad \square$$

LEMMA 3.7. *Let G be a nilpotent group of class at most 2 that is generated by a set X of involutions (that is, by elements $g \in G$ with $g^2 = 1$). Then, $[g,h]^2 = 1$ holds for all $g,h \in G$.*

PROOF. Let $f_1,\dots,f_r,h_1,\dots,h_m \in X$ such that $g = f_1\cdots f_r, h = h_1\cdots h_m$. We show the claim by induction on $r+m$. If $r+m \in \{0,1\}$, the claim follows directly. Thus, we assume $r+m \geq 2$. Again, if $0 \in \{r,m\}$, the claim follows directly. Thus, we can assume $r,m \geq 1$. Using the nilpotency class, we obtain

$$\begin{aligned}[g,h]^2 &= ([g,h_m][g,hh_m^{-1}]^{h_m})^2\\ &= ([gf_r^{-1},h_m]^{f_r}[f_r,h_m][g,hh_m^{-1}])^2\\ &= [gf_r^{-1},h_m]^2[f_r,h_m]^2[g,hh_m^{-1}]^2.\end{aligned}$$

Using the nilpotency class and the fact that $[f_r,h_m]^2 = [f_r,[f_r,h_m]]$ holds, the claim follows by induction. $\square$

THEOREM 3.8. *Let $\mathcal{M} = (M^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$ be a pre-commutator blueprint of type (W,S). Then, the following are equivalent.*

(i) *$\mathcal{M}$ is a commutator blueprint of type (W,S) and the groups U_w are nilpotent of class at most 2.*

(ii) *Let $G = (c_0,\dots,c_{k+1}) \in \mathrm{Min}$ and let $(\alpha_1,\dots,\alpha_{k+1})$ be the sequence of roots crossed by G. For $H := (c_0,\dots,c_k)$ and for all $1 \leq i < j \leq k+1$, we have the following:*

(2-n1) $\prod_{\gamma\in M^G_{\alpha_i,\alpha_{k+1}}} u_\gamma \in Z(U_H)$;

(2-n2) $(\prod_{\gamma\in M^G_{\alpha_i,\alpha_{k+1}}} u_\gamma)^2 = 1$ *holds in* U_H;

(2-n3) $\prod_{\gamma\in M^G_{\alpha_i,\alpha_j}}(u_\gamma \prod_{\delta\in M^G_{\gamma,\alpha_{k+1}}} u_\delta) = \prod_{\gamma\in M^G_{\alpha_i,\alpha_j}} u_\gamma$ *holds in* U_H.

PROOF. Suppose that $\mathcal{M}$ is a commutator blueprint of type (W,S) and that the groups U_w are nilpotent of class at most 2. Then, $\mathcal{M}$ is a pre-commutator blueprint of type (W,S) by Lemma 3.3 and it is not hard to see that $\mathcal{M}$ satisfies items (2-n1), (2-n2) and (2-n3) (see Lemma 3.7).

Now, we suppose that $\mathcal{M}$ satisfies the conditions in part (ii). We apply Lemma 3.5. Let $G = (c_0,\dots,c_{k+1}) \in \mathrm{Min}(w)$ be a minimal gallery, let $(\alpha_1,\dots,\alpha_{k+1})$ be the sequence of roots crossed by G and let $H := (c_0,\dots,c_k)$. Note that condition (C1) holds by items (2-n3) (applied with $j = k+1$) and (2-n2). Moreover, condition (C2) follows from items (2-n1) and (2-n2). Using items (2-n3) and (2-n1), one can show that condition (C3) also holds. Now, Lemma 3.5 implies that $\mathcal{M}$ is a commutator blueprint. It is left to show that the group $U_w \cong U_G$ is nilpotent of class at most 2.

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We prove this by induction on $\ell(w)$. The claim is obvious for $\ell(w) \leq 1$. Thus, we can assume $\ell(w) > 1$. Using item (2-n3), we compute for all $1 \leq i < j \leq k+1$:

$$[[u_{\alpha_i}, u_{\alpha_j}], u_{\alpha_{k+1}}] = [u_{\alpha_i}, u_{\alpha_j}]^{-1} \prod_{\gamma \in M^G_{\alpha_i,\alpha_j}} u_{\alpha_{k+1}}^{-1} u_\gamma u_{\alpha_{k+1}}$$

$$= \Big(\prod_{\gamma \in M^G_{\alpha_i,\alpha_j}} u_\gamma \Big)^{-1} \cdot \prod_{\gamma \in M^G_{\alpha_i,\alpha_j}} \Big(u_\gamma \prod_{\delta \in M^G_{\gamma,\alpha_{k+1}}} u_\delta \Big) = 1.$$

Now, the claim follows from Lemma 3.6. □

4. Examples of commutator blueprints

4.1. Examples of universal type.

CONVENTION 4.1. In this subsection, we assume that (W, S) is of universal type (that is, $m_{st} = \infty$ for all $s \neq t \in S$) and has rank at least 2.

REMARK 4.2. Note that for $w \in W$, we have $|\mathrm{Min}(w)| = 1$. This implies that each family $\mathcal{M} = (M^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$ of subsets $M^G_{\alpha,\beta} \subseteq (\alpha,\beta)$ ordered via $\leq_G$ satisfies Axiom (PCB). Moreover, we note that $(G,\alpha,\beta) \in \mathcal{I}$ implies $\alpha \subseteq \beta$.

DEFINITION 4.3. Let $\alpha \in \Phi$ be a root. As $|\partial^2\alpha| = 0$, we deduce $|\partial\alpha| = 1$ (see Lemma 2.5) and we call $c^{-1}d \in S$ the *type* of α, where $\{c,d\} \in \partial\alpha$.

DEFINITION 4.4.

(a) Let $s \neq t \in S$, $k \in \mathbb{N}$, $J \subseteq \{1,\dots,k\}$ and let $(G,\alpha,\beta) \in \mathcal{I}$. Assume that there exists a minimal gallery $H = (c_0,\dots,c_{2k+1})$ of type $(s,t,\dots,s,t,s)$ between α and β (that is α is the first and β is the last root crossed by H) such that s appears $k+1$ times and t appears k times in the type of H. Let $(\alpha_1 = \alpha,\dots,\alpha_{2k+1} = \beta)$ be the sequence of roots crossed by H. Then, we define

$$M(k,J,s,t)^G_{\alpha,\beta} := \{\alpha_{2j} \mid j \in J\}.$$

If there does not exist such a gallery between α and β, we define $M(k,J,s,t)^G_{\alpha,\beta} := \emptyset$.

(b) Let $s \neq t \in S$, let $K \subseteq \mathbb{N}$ be nonempty and let $\mathcal{J} = (J_k)_{k\in K}$ be a family of subsets $J_k \subseteq \{1,\dots,k\}$. For $(G,\alpha,\beta) \in \mathcal{I}$, we define

$$M(K,\mathcal{J},s,t)^G_{\alpha,\beta} := \bigcup_{k\in K} M(k,J_k,s,t)^G_{\alpha,\beta}.$$

Moreover, we define $\mathcal{M}(K,\mathcal{J},s,t) := (M(K,\mathcal{J},s,t)^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$.

REMARK 4.5. We note that for given $(G,\alpha,\beta) \in \mathcal{I}$ with $M(K,\mathcal{J},s,t)^G_{\alpha,\beta} \neq \emptyset$, there exists a unique $k \in K$ such that $M(K,\mathcal{J},s,t)^G_{\alpha,\beta} = M(k,J_k,s,t)^G_{\alpha,\beta}$.

THEOREM 4.6. *Let $s \neq t \in S$, let $K \subseteq \mathbb{N}$ be nonempty and let $\mathcal{J} = (J_k)_{k\in K}$ be a family of subsets $J_k \subseteq \{1,\dots,k\}$. Then, $\mathcal{M}(K,\mathcal{J},s,t)$ is a Weyl-invariant commutator blueprint and the groups U_w are nilpotent of class at most 2.*

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PROOF. We abbreviate $\mathcal{M} := \mathcal{M}(K, \mathcal{J}, s, t)$ and $M^G_{\alpha,\beta} := M(K, \mathcal{J}, s, t)^G_{\alpha,\beta}$ for all $(G, \alpha, \beta) \in \mathcal{I}$. By definition, $\mathcal{M}$ satisfies Axioms (CB1) and (CB2), and by Remark 4.2, it also satisfies Axiom (PCB). Hence, $\mathcal{M}$ is a pre-commutator blueprint. We apply Theorem 3.8. Thus, we suppose $(G, \alpha, \beta) \in \mathcal{I}$ with $M^G_{\alpha,\beta} \neq \emptyset$. Then, α, β are roots of type s and every $\gamma \in M^G_{\alpha,\beta}$ is a root of type t. Now, it is straight forward to verify that $\mathcal{M}$ satisfies the conditions in Theorem 3.8(ii). We infer that $\mathcal{M}$ is a commutator blueprint and the groups U_w are nilpotent of class at most 2. Moreover, $\mathcal{M}$ is Weyl-invariant, as $M^G_{\alpha,\beta}$ does only depend on the existence of a suitable gallery H and not on G. □

DEFINITION 4.7. Let $s_0, s_1 \in S$ be distinct. Let $(G, \alpha, \beta) \in \mathcal{I}$ and assume that there exists a minimal gallery $H = (c_0, \dots, c_k)$ of type $(s, t, s, t, \dots)$ between α and β. Let $(\alpha_1 = \alpha, \dots, \alpha_k = \beta)$ be the sequence of roots crossed by H. Then, we define

$$\begin{aligned}
k = 4, s = s_0, t = s_1 : &\quad M(s_0, s_1)^G_{\alpha_1,\alpha_4} := \{\alpha_2, \alpha_3\},\\
k = 5, s = s_0, t = s_1 : &\quad M(s_0, s_1)^G_{\alpha_1,\alpha_5} := \{\alpha_2, \alpha_3\},\\
k = 5, s = s_1, t = s_0 : &\quad M(s_0, s_1)^G_{\alpha_1,\alpha_5} := \{\alpha_3, \alpha_4\},\\
k = 6, s = s_1, t = s_0 : &\quad M(s_0, s_1)^G_{\alpha_1,\alpha_6} := \{\alpha_3, \alpha_4\}.
\end{aligned}$$

Otherwise, we define $M(s_0, s_1)^G_{\alpha,\beta} := \emptyset$. We define $\mathcal{M}(s_0, s_1) := (M(s_0, s_1)^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$.

THEOREM 4.8. *Let $s_0, s_1 \in S$ be distinct. Then, $\mathcal{M}(s_0, s_1)$ is a Weyl-invariant commutator blueprint of type (W, S). Moreover, the groups U_w are nilpotent of class at most 2.*

PROOF. We abbreviate $M^G_{\alpha,\beta} := M(s_0, s_1)^G_{\alpha,\beta}$ for all $(G, \alpha, \beta) \in \mathcal{I}$. Similarly as in Theorem 4.6, $\mathcal{M}(s_0, s_1)$ is a pre-commutator blueprint and we apply Theorem 3.8. Let $G = (c_0, \dots, c_{k+1}) \in \mathrm{Min}$ be a minimal gallery and let $(\alpha_1, \dots, \alpha_{k+1})$ be the sequence of roots crossed by G. We first show the following.

Claim. If α_i has type s_1 and α_{i+1} has type s_0 for some $1 \leq i \leq k$, then $u_{\alpha_i} u_{\alpha_{i+1}} \in Z(U_G)$.

Let $1 \leq j \leq k+1$. We have to show that u_j commutes with $u_{\alpha_i} u_{\alpha_{i+1}}$. If $j \in \{i, i+1\}$, the claim follows. We next suppose $j < i$. However, then it follows by construction that $M^G_{\alpha_j,\alpha_i} = M^G_{\alpha_j,\alpha_{i+1}}$. This follows for $j > i+1$ similarly.

Now, the conditions in Theorem 3.8(ii) follow essentially from the claim. Hence, $\mathcal{M}(s_0, s_1)$ is a commutator blueprint and the groups U_w are nilpotent of class at most 2. As $M^G_{\alpha,\beta}$ only depends on the existence of a suitable minimal gallery between α and β, it is also Weyl-invariant. □

In the rest of this subsection, we introduce a commutator blueprint, where the nilpotency class of the groups U_w becomes arbitrarily large and even infinite. To do that, we mimic the commutation relations from the upper triangular matrices. We first define (i, j, k, l)-galleries between two roots α and β for $i, j, k, l \in \mathbb{N}$ with $1 \leq i < j$, $1 \leq k < l$ and $j < k$. Any (i, j, k, l)-gallery has a certain type $(c_j, c_{j+1}, \dots, c_{k-1}, c_k)$ and the c_p indicates in which column we are. Roughly speaking, the first root crossed by

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such a gallery (which is α) corresponds to the (i,j)-entry of a matrix and the last root crossed by such a gallery (which is β) corresponds to the $(k+1,l)$-entry or, if $k = l-1$, to the $(1,k+1)$-entry. Every (i,j,k,l)-gallery crosses roots corresponding to the the following entries of a matrix:

$$(i,j),(i+1,j),\dots,(j-1,j),(1,j+1),\dots,(k,l).$$

As we mimic the commutation relations from the upper triangular matrices, the sets $M_{\mathrm{nil}}(n)^G_{\alpha,\beta}$ correspond to the (i,k)-entry, if $j = k$.

DEFINITION 4.9. Let (W,S) be of rank at least 3 and let $r,s,t \in S$ be pairwise distinct.

(a) For $i \in \mathbb{N}$, we let $k_i := tsts\cdots$ with $\ell(k_i) = i$, for example, $k_3 = tst$.

(b) Suppose $i_1,j_1,i_2,j_2 \in \mathbb{N}$ with $1 \le i_1 < j_1$, $1 \le i_2 < j_2$ and $j_1 < j_2$. A gallery is called a (i_1,j_1,i_2,j_2)-*gallery* if it has type

$$(c_{j_1},\dots,c_{j_2}),$$

where c_i is defined as follows:

$$\begin{aligned} r_{i,j} &= (r,k_i,r,k_j,r),\\ c_{j_1} &= (s,r_{i_1,j_1},s,r_{i_1+1,j_1},\dots,s,r_{j_1-1,j_1}),\\ c_i &= (s,r_{1,i},s,r_{2,i},\dots,s,r_{i-1,i}) \quad \text{for } j_1 < i < j_2,\\ c_{j_2} &= (s,r_{1,j_2},s,r_{2,j_2},\dots,s,r_{i_2-1,j_2},s). \end{aligned}$$

(c) Let $n \in \mathbb{N} \cup \{\infty\}$ and we define $z \le \infty$ for all $z \in \mathbb{N}$. Let $(G,\alpha,\beta) \in \mathcal{I}$ and assume that there exists an (i_1,j_1,i_2,j_2)-gallery between α and β. We let $(\alpha_{i_1,j_1},\alpha_{i_1+1,j_1},\dots,\alpha_{j_1-1,j_1},\alpha_{1,j_1+1},\dots,\alpha_{i_2,j_2})$ be the sequence of roots crossed by this gallery omitting the $r_{i,j}$ parts from above. Then, $\alpha = \alpha_{i_1,j_1}$ and $\beta = \alpha_{i_2,j_2}$. We define

$$M_{\mathrm{nil}}(n)^G_{\alpha,\beta} := \begin{cases} \{\alpha_{i_1,j_2}\} & \text{if } j_1 = i_2 \text{ and } j_2 \le n,\\ \emptyset & \text{otherwise.} \end{cases}$$

If there is no (i_1,j_1,i_2,j_2)-gallery between α and β for all $i_1,j_1,i_2,j_2 \in \mathbb{N}$, then we define $M_{\mathrm{nil}}(n)^G_{\alpha,\beta} := \emptyset$. Moreover, we define $\mathcal{M}_{\mathrm{nil}}(n) := (M_{\mathrm{nil}}(n)^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$.

EXAMPLE 4.10. A $(2,3,3,5)$-gallery G has type (c_3,c_4,c_5), where

$$\begin{aligned} c_3 &= (s,r_{2,3}), & r_{2,3} &= (r,t,s,r,t,s,t,r),\\ c_4 &= (s,r_{1,4},s,r_{2,4},s,r_{3,4}), & r_{1,4} &= (r,t,r,t,s,t,s,r),\\ & & r_{2,4} &= (r,t,s,r,t,s,t,s,r),\\ & & r_{3,4} &= (r,t,s,t,r,t,s,t,s,r),\\ c_5 &= (s,r_{1,5},s,r_{2,5},s), & r_{1,5} &= (r,t,r,t,s,t,s,t,r),\\ & & r_{2,5} &= (r,t,s,r,t,s,t,s,t,r). \end{aligned}$$

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If $(\alpha_1, \dots, \alpha_{61})$ is the sequence of roots crossed by G, then we have the following:

$$\alpha_{2,3} = \alpha_1, \quad \alpha_{1,4} = \alpha_{10}, \quad \alpha_{2,4} = \alpha_{19}, \quad \alpha_{3,4} = \alpha_{29}, \quad \alpha_{1,5} = \alpha_{40}, \quad \alpha_{2,5} = \alpha_{50}, \quad \alpha_{3,5} = \alpha_{61}.$$

If $\alpha_1, \alpha_{61} \in \Phi_+$, then we have

$$M_{\mathrm{nil}}(n)^G_{\alpha_1,\alpha_{61}} = M_{\mathrm{nil}}(n)^G_{\alpha_{2,3},\alpha_{3,5}} = \begin{cases} \{\alpha_{2,5} = \alpha_{50}\} & \text{if } 5 \le n, \\ \emptyset & \text{otherwise.} \end{cases}$$

REMARK 4.11.

(a) Let $\alpha, \beta, \gamma \in \Phi$ be three roots. Assume that $G = (d_0, \dots, d_k)$ is an (i_1, j_1, i_2, j_2)-gallery between α and β, and that $(e_0, \dots, e_l)$ is an (i_2, j_2, i_3, j_3)-gallery between β and γ. Then, $(d_0, \dots, d_{k-1} = e_0, \dots, e_l)$ is an (i_1, j_1, i_3, j_3)-gallery between α and γ.

(b) Let $\alpha, \beta \in \Phi$ be two roots with $M_{\mathrm{nil}}(n)^G_{\alpha,\beta} = \{\gamma\}$. Then, there exists an (i, j, j, k)-gallery between α and β for some $i, j, k \in \mathbb{N}$ with $i < j < k \le n$. Note that by definition, we have $M_{\mathrm{nil}}(n)^G_{\alpha,\gamma} = \emptyset = M_{\mathrm{nil}}(n)^G_{\gamma,\beta}$.

LEMMA 4.12. *Let $n \in \mathbb{N} \cup \{\infty\}$. Let $(G, \alpha, \beta) \in \mathcal{I}$ and suppose that there does not exist an (i, j, i', j')-gallery between α and β for all $i, j, i', j' \in \mathbb{N}$. Let $\delta \in \Phi(G)$ with $\alpha \le_G \delta \le_G \beta$. Then, the following hold.*

(a) *If $\gamma \in M_{\mathrm{nil}}(n)^G_{\alpha,\delta}$, then $M_{\mathrm{nil}}(n)^G_{\gamma,\beta} = \emptyset$.*

(b) *If $\gamma \in M_{\mathrm{nil}}(n)^G_{\delta,\beta}$, then $M_{\mathrm{nil}}(n)^G_{\alpha,\gamma} = \emptyset$.*

PROOF. We abbreviate $M^H_{\sigma,\rho} := M_{\mathrm{nil}}(n)^H_{\sigma,\rho}$ for all $(H, \sigma, \rho) \in \mathcal{I}$. We first show assertion (a). We can assume that $M^G_{\alpha,\delta} = \{\gamma\}$. By definition, there exists an (i, j, j, k)-gallery between α and δ for some $i, j, k \in \mathbb{N}$ with $i < j < k \le n$, and this gallery has type $(c_j, c_{j+1}, \dots, c_{k-1}, c_k)$. Note that

$$c_k = (s, r_{1,k}, s, r_{2,k}, \dots, s, r_{i-1,k}, \mathbf{s}, r_{i,k}, \dots, s, r_{j-1,k}, s)$$

and γ is the root of type $\mathbf{s}$ crossed by c_k directly before $r_{i,k}$.

Assume that $M^G_{\gamma,\beta} \neq \emptyset$. Then, there would exist an (i', j', j', k')-gallery between γ and β for some $i', j', k' \in \mathbb{N}$ with $i' < j' < k' \le n$. Because the type of G after crossing γ is $r_{i,k}$, we know that $i' = i$ and $j' = k$. However, then we have an (i, j, j', k')-gallery between α and β by Remark 4.11, which is a contradiction. This finishes the proof of the claim.

We now show assertion (b). Again, we can assume $M^G_{\delta,\beta} = \{\gamma\}$. By definition, there exists an (i, j, j, k)-gallery between δ and β for some $i, j, k \in \mathbb{N}$ with $i < j < k \le n$, and this gallery has type $(c_j, \dots, c_k)$. Note that

$$c_{k-1} = (\dots, s, r_{k-2,k-1}) \quad \text{and} \quad c_k = (s, r_{1,k}, \dots, s, r_{j-1,k}, s).$$

Assume that $M^G_{\alpha,\gamma} \neq \emptyset$. Then, there would exist an (i', j', j', k')-gallery between α and γ for some $i', j', k' \in \mathbb{N}$ with $i' < j' < k' \le n$. We distinguish the following cases.

(i) $i = 1$. Then, γ is the first root crossed by c_k and the type of G just before crossing γ is $r_{k-2,k-1}$. This implies that $j' = 1$ and $k' = k$, which is a contradiction.

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(ii) $i > 1$. In this case, the type of G just before crossing γ is $r_{i-1,k}$. This implies $j' = i$ and $k' = k$. However, then we have an (i', j', j, k)-gallery between α and β, which is a contradiction. □

THEOREM 4.13. *For each $n \in \mathbb{N} \cup \{\infty\}$, $\mathcal{M}_{\mathrm{nil}}(n)$ is a Weyl-invariant commutator blueprint. Moreover, the following hold.*

(i) *If $n \in \mathbb{N}_{\geq 3}$, then every group U_w is nilpotent of class at most $n-1$, but not all are nilpotent of class at most $n-2$.*
(ii) *If $n = \infty$, then for every $m \in \mathbb{N}$, there exists $w_m \in W$ such that U_{w_m} is not nilpotent of class at most m.*

PROOF. We abbreviate $\mathcal{M} := \mathcal{M}_{\mathrm{nil}}(n)$ and $M^G_{\alpha,\beta} := M_{\mathrm{nil}}(n)^G_{\alpha,\beta}$ for all $(G, \alpha, \beta) \in \mathcal{I}$. By definition, $\mathcal{M}$ satisfies Axioms (CB1) and (CB2), and by Remark 4.2 it also satisfies Axiom (PCB). Hence, $\mathcal{M}$ is a pre-commutator blueprint. We apply Lemma 3.5. Suppose $G = (d_1, \ldots, d_p) \in \mathrm{Min}$, let $\beta \in \Phi(G)$ be the last root that is crossed by G and let $\alpha \in \Phi(G)\backslash\{\beta\}$. If $M^G_{\alpha,\beta} = \emptyset$, then Axioms (C1) and (C2) are satisfied. Otherwise, there exists an (i, j, j, k)-gallery between α and β for some $i, j, k \in \mathbb{N}$ with $1 \leq i < j < k \leq n$ and we have $M^G_{\alpha,\beta} = \{\gamma\}$. Then, $M^G_{\alpha,\gamma} = \emptyset = M^G_{\gamma,\beta}$ holds by definition (see Remark 4.11). Thus, Axioms (C1) and (C2) are satisfied and it remains to check Axiom (C3).

Let $\alpha, \delta \in \Phi(G)\backslash\{\beta\}$ with $\alpha \leq_G \beta$. If there is an (i_1, j_1, i_2, j_2)-gallery between α and β for some $i_1, j_1, i_2, j_2 \in \mathbb{N}$, then Axiom (C3) follows essentially from the fact that Axiom (C3) holds for upper triangular matrices. Thus, we can assume that there does not exist any (i_1, j_1, i_2, j_2)-gallery between α and β for all $i_1, j_1, i_2, j_2 \in \mathbb{N}$. In particular, $M^G_{\alpha,\beta} = \emptyset$ and we have to check that the following is a relation in $U_{(d_0,\ldots,d_{p-1})}$:

$$\Big[u_\alpha, u_\delta \prod_{\gamma \in M^G_{\delta,\beta}} u_\gamma\Big] = \prod_{\gamma \in M^G_{\alpha,\delta}} \Big(u_\gamma \prod_{\omega \in M^G_{\gamma,\beta}} u_\omega\Big).$$

Suppose $M^G_{\alpha,\delta} \neq \emptyset$ and let $\gamma \in M^G_{\alpha,\delta}$. Then, Lemma 4.12 implies $M^G_{\gamma,\beta} = \emptyset$ and the right-hand side of the previous equation is equal to $\prod_{\gamma \in M^G_{\alpha,\delta}} u_\gamma$ in both cases $M^G_{\alpha,\delta} = \emptyset$ and $M^G_{\alpha,\delta} \neq \emptyset$.

If $M^G_{\delta,\beta} = \emptyset$, then Axiom (C3) holds and we are done. Thus, we can assume $M^G_{\delta,\beta} = \{\gamma\}$. Using Remark 4.11, we deduce $M^G_{\delta,\gamma} = \emptyset$. Moreover, Lemma 4.12 yields $M^G_{\alpha,\gamma} = \emptyset$. This implies that the following is a relation in $U_{(d_0,\ldots,d_{p-1})}$ and hence Axiom (C3) holds:

$$\Big[u_\alpha, u_\delta \prod_{\omega \in M^G_{\delta,\beta}} u_\omega\Big] = [u_\alpha, u_\delta u_\gamma] = [u_\alpha, u_\gamma][u_\alpha, u_\delta]^{u_\gamma} = [u_\alpha, u_\delta] = \prod_{\omega \in M^G_{\alpha,\delta}} u_\omega.$$

Now, Lemma 3.5 yields that $\mathcal{M}$ is a commutator blueprint. The Weyl-invariance follows from the fact that $M^G_{\alpha,\beta}$ does not depend on G, but only on the existence of a suitable minimal gallery between α and β.

We show parts (i) and (ii). Let $n \in \mathbb{N}_{\geq 3}$ and $G \in \mathrm{Min}(w)$ be a $(1, 2, n-1, n)$-gallery. Then, $U_w \cong U_G$ is not nilpotent of class at most $n-2$. This proves part (ii) and the

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second part of part (i). Thus, it is left to show that for every $w \in W$, the group U_w is nilpotent of class at most $n-1$.

Let $w \in W$. We prove the claim by induction on $\ell(w)$. If $\ell(w) \leq 1$, the claim holds. Thus, we can assume $\ell(w) > 1$. Let $G = (c_0, \dots, c_p) \in \mathrm{Min}(w)$, let $(\alpha_1, \dots, \alpha_p)$ be the sequence of roots crossed by G and let $H := (c_0, \dots, c_{p-1})$. Using induction, the group U_H is nilpotent of class at most $n-1$. If $M^G_{\alpha_i,\alpha_p} = \emptyset$ for all $1 \leq i \leq p-1$, then $U_w \cong U_H \times \mathbb{Z}_2$ and U_w is again nilpotent of class at most $n-1$. Thus, we can assume $M^G_{\alpha_i,\alpha_p} \neq \emptyset$ for some $1 \leq i \leq p-1$ and hence, there exists a (j,k,k,l)-gallery between α_i and α_p for some $j,k,l \in \mathbb{N}$ with $1 \leq j < k < l \leq n$. Then, the type of G ends with

$$(\dots, s, r_{1,l}, \dots, s, r_{k-1,l}, s).$$

Let $1 \leq i' \leq i$ be minimal with the property that there exists an (i_1,j_1,i_2,j_2)-gallery between $\alpha_{i'}$ and α_p for some $i_1,j_1,i_2,j_2 \in \mathbb{N}$ with $1 \leq i_1 < j_1, 1 \leq i_2 < j_2$ and $j_1 < j_2$. Note that $M^G_{\alpha_{i'},\alpha_p} = \emptyset$ is possible. Comparing the type of G and the type of the (i_1,j_1,i_2,j_2)-gallery, we deduce $i_2 = k$ and $j_2 = l \leq n$.

Let $\alpha \in \{\alpha_1, \dots, \alpha_{i'-1}\}$ and $\beta \in \{\alpha_{i'+1}, \dots, \alpha_p\}$, and assume $M^G_{\alpha,\beta} \neq \emptyset$. Then, there would exist a (j',k',k',l')-gallery between α and β for some $j',k',l' \in \mathbb{N}$ with $j' < k' < l' \leq n$. By definition, β must be of type s and the root crossed directly before β by G must be of type r. Note that β is a root on the (i_1,j_1,i_2,j_2)-gallery between $\alpha_{i'}$ and α_p. By considering the type of this gallery, it follows that β is a root of the form $\alpha_{x,y}$ (see Definition 4.9). However, then, we would also have an (i'_1,j'_1,i'_2,j'_2)-gallery between α and α_p by Remark 4.11, which is a contradiction of the minimality of i'. Thus, we deduce $M^G_{\alpha,\beta} = \emptyset$. However, we note that it is possible that $M^G_{\alpha,\alpha_{i'}} \neq \emptyset \neq M^G_{\alpha_{i'},\beta}$.

If $i' = 1$, then the claim holds. Thus, we can assume $1 < i'$. Using induction, the group $U_{(c_0,\dots,c_{i'})}$ is nilpotent of class at most $n-1$ and the subgroup of U_G generated by $\{u_{\alpha_{i'}}, \dots, u_{\alpha_p}\}$ is also nilpotent of class at most $n-1$. Moreover, we have $u_{\alpha_{i'}} \notin [U_G, U_G]$ because of the type of G and the fact that i' is minimal (see also Remark 4.11). This implies

$$[U_G, U_G] \leq \langle u_{\alpha_i} \mid 1 \leq i \leq i'-1 \rangle \times \langle u_{\alpha_i} \mid i'+1 \leq i \leq p \rangle.$$

It follows that the nilpotency class of U_G is equal to the maximum of the nilpotency classes of the groups $U_{(c_0,\dots,c_{i'})}$ and the subgroup of U_w generated by $\{u_{\alpha_{i'}}, \dots, \alpha_p\}$, which is at most $n-1$. This finishes the proof of the claim. □

4.2. Examples of type (4, 4, 4).

CONVENTION 4.14. For the rest of this paper, we assume that (W,S) is of rank 3 and that $m_{st} = 4$ for all $s \neq t \in S$.

PROPOSITION 4.15. *Let $\mathcal{M} = (M^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$ be a pre-commutator blueprint of type $(4,4,4)$ such that for all $(G,\alpha,\beta) \in \mathcal{I}$ with $o(r_\alpha r_\beta) < \infty$, we have*

$$M^G_{\alpha,\beta} = \begin{cases} (\alpha,\beta) & \text{if } |(\alpha,\beta)| = 2, \\ \emptyset & \text{if } |(\alpha,\beta)| < 2. \end{cases}$$

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Let $G = (c_0, \dots, c_k) \in \mathrm{Min}$ and let $(\alpha_1, \dots, \alpha_k)$ be the sequence of roots crossed by G. We let $\beta := \alpha_k$ and assume that the following hold for all $\alpha \in \Phi(G)\backslash\{\beta\}$.

(a) *Suppose that $o(r_\alpha r_\beta) < \infty$, $M^G_{\alpha,\beta} = \{\gamma, \delta\}$ and let $\varepsilon \in \Phi(G)$.*

  (i) *If $\varepsilon \subsetneq \gamma$ and $\varepsilon \subsetneq \delta$, then $M^G_{\varepsilon,\gamma} = M^G_{\varepsilon,\delta}$ holds.*
  (ii) *If $\gamma \subsetneq \varepsilon$ and $\delta \subsetneq \varepsilon$, then $M^G_{\gamma,\varepsilon} = M^G_{\delta,\varepsilon}$ holds.*

(b) *Suppose that $o(r_\alpha r_\beta) = \infty$. Then, the following hold:*

  (i) $\prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma \in Z(U_{(c_0,\dots,c_{k-1})})$.
  (ii) $(\prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma)^2 = 1$ *holds in* $U_{(c_0,\dots,c_{k-1})}$.

If item (2-n3) of Theorem 3.8(ii) holds, then $\mathcal{M}$ is a commutator blueprint and the groups U_w are nilpotent of class at most 2.

PROOF. Note that by assumption and Theorem 3.8, it suffices to show that $\mathcal{M}$ satisfies items (2-n1) and (2-n2). We abbreviate $u_i := u_{\alpha_i}$ for all $1 \le i \le k$.

Let $1 \le i \le k-1$. We first see that item (2-n2) holds in the case $o(r_{\alpha_i} r_\beta) < \infty$ by assumption and in the case $o(r_{\alpha_i} r_\beta) = \infty$ by condition (b)(ii). Thus, it suffices to show that item (2-n1) holds.

Let $1 \le i \le k-1$. We have to show $[u_i, [u_\alpha, u_\beta]] = 1$. If $o(r_\alpha r_\beta) = \infty$, then $[u_\alpha, u_\beta]$ commutes with u_i by condition (b)(i) and the claim follows. Thus, we assume $o(r_\alpha r_\beta) < \infty$. Moreover, we can assume that $M^G_{\alpha,\beta} \neq \emptyset$ and hence $|(\alpha, \beta)| = 2$. We suppose $M^G_{\alpha,\beta} = \{\gamma, \delta\}$ with $\gamma \le_G \delta$.

If $R \in \partial^2\alpha_i \cap \partial^2\alpha \cap \partial^2\beta$, then the claim follows from the assumptions. Thus, we can assume that $\partial^2\alpha_i \cap \partial^2\alpha \cap \partial^2\beta = \emptyset$. Suppose that $o(r_{\alpha_i} r_\gamma) = \infty = o(r_{\alpha_i} r_\delta)$. As $\{\alpha_i, \gamma\}, \{\alpha_i, \delta\} \in \mathcal{P}$, we deduce from Lemma 2.8(a) that both are nested. As $o(r_\gamma r_\delta) < \infty$, we deduce $\alpha_i \subseteq \gamma, \delta$ or $\gamma, \delta \subseteq \alpha_i$. If $\alpha_i \subseteq \gamma, \delta$, we have $M^G_{\alpha_i,\gamma} = M^G_{\alpha_i,\delta}$ by condition (a)(i) and we infer

$$\begin{aligned} u_i^{-1}\Big(\prod_{\varepsilon \in M^G_{\alpha,\beta}} u_\varepsilon\Big)u_i &= \prod_{\varepsilon \in M^G_{\alpha,\beta}} \Big(\prod_{\omega \in M^G_{\alpha_i,\varepsilon}} u_\omega\Big)u_\varepsilon \\ &= \Big(\prod_{\omega \in M^G_{\alpha_i,\gamma}} u_\omega\Big)u_\gamma\Big(\prod_{\omega \in M^G_{\alpha_i,\delta}} u_\omega\Big)u_\delta \\ &\overset{(b)(i)}{=} u_\gamma\Big(\prod_{\omega \in M^G_{\alpha_i,\gamma}} u_\omega\Big)^2 u_\delta \\ &\overset{(b)(ii)}{=} u_\gamma u_\delta = \prod_{\varepsilon \in M^G_{\alpha,\beta}} u_\varepsilon. \end{aligned}$$

The case $\gamma, \delta \subseteq \alpha_i$ follows similarly. Now, we suppose that there exists $\rho \in (\alpha, \beta)$ with $o(r_{\alpha_i} r_\rho) < \infty$. Assume that $o(r_\alpha r_{\alpha_i}) = \infty = o(r_\beta r_{\alpha_i})$. Then, we would have $\alpha_i \subseteq \beta$ (as $\beta = \alpha_k$) and, moreover, either $\alpha \subseteq \alpha_i$ or $\alpha_i \subseteq \alpha$. As $o(r_\alpha r_\beta) < \infty$, we deduce $\alpha_i \subseteq \alpha$, but then we would have $\alpha_i \subseteq \alpha \cap \beta \subseteq \rho$, which is a contradiction. We conclude

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$o(r_\alpha r_{\alpha_i}) < \infty$ or $o(r_\beta r_{\alpha_i}) < \infty$. Let $\varepsilon \in \{\alpha, \beta\}$ be a root such that $o(r_\varepsilon r_{\alpha_i}) < \infty$. Then, Lemma 2.8(b) yields

$$o(r_\varepsilon r_\rho) < \infty \quad \text{and} \quad \partial^2\alpha \cap \partial^2\beta \cap \partial^2\rho \neq \emptyset.$$

Let $\varepsilon' \in \{\alpha, \beta\} \backslash \{\varepsilon\}$. We deduce from Lemma 2.9 that $\partial^2\rho \cap \partial^2\varepsilon = \partial^2\alpha \cap \partial^2\beta = \partial^2\rho \cap \partial^2\varepsilon'$. As $\partial^2\alpha_i \cap \partial^2\alpha \cap \partial^2\beta = \emptyset$, we infer that $\{r_{\alpha_i}, r_\varepsilon, r_\rho\}$ is a reflection triangle. Using Remark 2.11, there exist $\beta_i \in \{\alpha_i, -\alpha_i\}, \beta_\varepsilon \in \{\varepsilon, -\varepsilon\}$ and $\beta_\rho \in \{\rho, -\rho\}$ such that $\{\beta_i, \beta_\varepsilon, \beta_\rho\}$ is a triangle. Note that $(-\beta_\varepsilon, \beta_\rho) = \emptyset$ by Lemma 2.12. Let $\sigma \in (\alpha, \beta) \backslash \{\rho\}$. Using Lemma 2.9, there exists $\beta_{\varepsilon'} \in \{\varepsilon', -\varepsilon'\}$ and $\beta_\sigma \in \{\sigma, -\sigma\}$ with $\beta_{\varepsilon'}, \beta_\sigma \in (\beta_\varepsilon, \beta_\rho)$. By Lemma 2.13, we have $o(r_{\alpha_i} r_{\varepsilon'}) = \infty$. Note that $\{\alpha_i, \varepsilon'\} \in \mathcal{P}$ and hence $\{\alpha_i, \varepsilon'\}$ is nested by Lemma 2.8. Recall that $\partial^2\varepsilon \cap \partial^2\varepsilon' \cap \partial^2\rho = \partial^2\alpha \cap \partial^2\beta \cap \partial^2\rho \neq \emptyset$. We distinguish the following two cases.

(a) $\alpha_i \subseteq \varepsilon'$. For $R \in \partial^2\varepsilon \cap \partial^2\rho = \partial^2\varepsilon' \cap \partial^2\rho = \partial^2\alpha \cap \partial^2\beta$ (see Lemma 2.9), we deduce $\emptyset \neq R \cap (-\varepsilon') \subseteq (-\alpha_i)$ and, as $R \notin \partial^2\alpha_i$, we have $R \subseteq (-\alpha_i)$. This yields $\beta_i = -\alpha_i$. For $R \in \partial^2\alpha_i \cap \partial^2\varepsilon$, we have $\emptyset \neq \alpha_i \cap R \subseteq \varepsilon'$. As $\partial^2\alpha_i \cap \partial^2\varepsilon \cap \partial^2\varepsilon' = \partial^2\alpha_i \cap \partial^2\alpha \cap \partial^2\beta = \emptyset$, we deduce $R \notin \partial^2\varepsilon'$ and hence $R \subseteq \varepsilon'$. In particular, we have $\emptyset \neq \varepsilon \cap R \subseteq \varepsilon \cap \varepsilon' = \alpha \cap \beta \subseteq \rho$. As $R \notin \partial^2\rho$ ($\{r_{\alpha_i}, r_\varepsilon, r_\rho\}$ is a reflection triangle), we infer $R \subseteq \rho$ and hence $\beta_\rho = \rho$. Lemma 2.12 implies $(\alpha_i, \rho) = (-\beta_i, \beta_\rho) = \emptyset$. Recall that $\beta_\sigma \in (\beta_\varepsilon, \beta_\rho)$. Using Lemma 2.13, we deduce $o(r_{\alpha_i} r_\sigma) = \infty, \alpha_i = -\beta_i \subseteq \beta_\sigma$ and $(\alpha_i, \beta_\sigma) = \emptyset$. As $\alpha_i \ni 1_W \notin (-\sigma)$, we deduce $\beta_\sigma = \sigma$. Since $\{\gamma, \delta\} = (\alpha, \beta) = \{\rho, \sigma\}$, we have $(\alpha_i, \gamma) = (\alpha_i, \delta) = \emptyset$.

(b) $\varepsilon' \subseteq \alpha_i$. For $R \in \partial^2\varepsilon \cap \partial^2\rho = \partial^2\varepsilon' \cap \partial^2\rho = \partial^2\alpha \cap \partial^2\beta$, we deduce $\emptyset \neq R \cap \varepsilon' \subseteq \alpha_i$ and, as $R \notin \partial^2\alpha_i$, we have $R \subseteq \alpha_i$. This yields $\beta_i = \alpha_i$. For $R \in \partial^2\alpha_i \cap \partial^2\varepsilon$, we have $\emptyset \neq R \cap (-\alpha_i) \subseteq (-\varepsilon')$. As $\partial^2\alpha_i \cap \partial^2\varepsilon \cap \partial^2\varepsilon' = \partial^2\alpha_i \cap \partial^2\alpha \cap \partial^2\beta = \emptyset$, we deduce $R \notin \partial^2\varepsilon'$ and hence, $R \subseteq (-\varepsilon')$. In particular, we have $\emptyset \neq (-\varepsilon) \cap R \subseteq (-\varepsilon) \cap (-\varepsilon') = (-\alpha) \cap (-\beta) \subseteq (-\rho)$. As $R \notin \partial^2\rho$ ($\{r_{\alpha_i}, r_\varepsilon, r_\rho\}$ is a reflection triangle), we infer $R \subseteq (-\rho)$ and hence, $\beta_\rho = -\rho$. Lemma 2.12 implies $(\alpha_i, \rho) = \emptyset$. Recall that $\beta_\sigma \in (\beta_\varepsilon, \beta_\rho)$. Using Lemma 2.13, we deduce $o(r_{\alpha_i} r_\sigma) = \infty, -\alpha_i = -\beta_i \subseteq \beta_\sigma$ and $(-\alpha_i, \beta_\sigma) = \emptyset$. As $\{\alpha_i, \sigma\} \in \mathcal{P}$, Lemma 2.8 implies that $\{\alpha_i, \sigma\}$ is nested and hence, $\{-\alpha_i, -\sigma\}$ is nested as well. As $-\alpha_i \subseteq \beta_\sigma$, we deduce $\beta_\sigma = -\sigma$ and hence, $(\alpha_i, \sigma) = -(-\alpha_i, -\sigma) = \emptyset$. Since $\{\gamma, \delta\} = (\alpha, \beta) = \{\rho, \sigma\}$, we have $(\alpha_i, \gamma) = (\alpha_i, \delta) = \emptyset$.

In both cases, we compute the following, where $N^G_{\alpha_i,\varepsilon} \in \{M^G_{\alpha_i,\varepsilon}, M^G_{\varepsilon,\alpha_i}\}$ depends on which expression is defined:

$$\prod_{\varepsilon \in M^G_{\alpha,\beta}} \Big(\prod_{\omega \in N^G_{\alpha_i,\varepsilon}} u_\omega \Big) u_\varepsilon = \prod_{\varepsilon \in M^G_{\alpha,\beta}} u_\varepsilon. \qquad \square$$

DEFINITION 4.16. Let $H = (c_0, \dots, c_k)$ be a gallery in $\Sigma(W, S)$. Then, H is called *of type* $(n, r) \in \mathbb{N}_{\geq 1} \times S$ if $S = \{r, s, t\}$ and the gallery H is of type $(u, r, r_{\{s,t\}}, \dots, r, r_{\{s,t\}}, r, v)$ for some $u, v \in \{1_W, s, t\}$, where $r_{\{s,t\}}$ appears n times in the type of H. We note that $(1_W, c_0^{-1}c_1, \dots, c_0^{-1}c_k)$ is a minimal gallery by Lemma 2.14 and [AB08, Lemma 2.15], and so is H.

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LEMMA 4.17. *Let $\alpha, \beta \in \Phi_+$ be two roots and let $(n, r) \in \mathbb{N}_{\geq 2} \times S$. Suppose that there exists a minimal gallery $H = (c_0, \dots, c_k)$ of type (n, r) between α and β. Then, the following hold.*

(a) *We can extend $(c_6, \dots, c_k)$ to a minimal gallery contained in* Min.
(b) *Let $R \in \partial^2\alpha$ be a residue such that α is a nonsimple root of R. If $\{c_0, c_1\} \not\subseteq R$, then there exists a simple root of R, say $\gamma \in \Phi_+$, such that $-\gamma \subseteq \beta$.*

PROOF. In the proof, we use the following notation. For a minimal gallery $G = (c_0, \dots, c_k)$, $k \geq 1$, we denote the unique root containing c_{k-1} but not c_k by α_G.

Let $S = \{r, s, t\}$ and recall that H is of type $(u, r, r_{\{s,t\}}, \dots, r, r_{\{s,t\}}, r, v)$, where $u, v \in \{1_W, s, t\}$ and $r_{\{s,t\}}$ appears n times. We first show assertion (a). Suppose $u = 1_W$. Then, $\ell(c_0 r) = \ell(c_0) + 1$. If $\ell(c_0 rs) = \ell(c_0 r) + 1 = \ell(c_0 rt)$, we can extend H to a gallery contained in Min by Lemma 2.14 and induction. If $\ell(c_0 rs) = \ell(c_0)$, it follows from Lemma 2.16 that $\ell(c_0 rst) = \ell(c_0) + 1$. Hence, we can extend $(c_5, \dots, c_k)$ to a gallery contained in Min. The same holds if $\ell(c_0 rt) = \ell(c_0)$. Now, we suppose $u = s$ (the case $u = t$ is analogous). Again, we note that $\ell(c_0 s) = \ell(c_0) + 1$. If $\ell(c_0 sr) = \ell(c_0)$, Lemma 2.16 yields $\ell(c_0 srt) = \ell(c_0) + 1$. Thus, we can extend $(c_6, \dots, c_k)$ to a gallery contained in Min. Suppose that $\ell(c_0 sr) = \ell(c_0) + 2$. Note that Lemma 2.14 implies that $\ell(c_0 srt) = \ell(c_0) + 3$. If $\ell(c_0 srs) > \ell(c_0 sr)$, then we can extend H to a gallery contained in Min. Otherwise, Lemma 2.16 again implies $\ell(c_0 srst) = \ell(c_0) + 2$ and we can extend $(c_6, \dots, c_k)$ to a gallery contained in Min. In any case, we can extend $(c_6, \dots, c_k)$ to a gallery $\Gamma \in$ Min. This proves assertion (a).

To prove assertion (b), we suppose $\{c_0, c_1\} \not\subseteq R$. As $P_\alpha \subseteq R$ by Lemma 2.22, we have $P_\alpha \neq \{c_0, c_1\}$. Let $P_0 = P_\alpha, \dots, P_n = \{c_0, c_1\}$ and $R_1, \dots, R_n$ be as in Lemma 2.5. For every $1 \leq i \leq n$, we define $w_i := \mathrm{proj}_{R_i} 1_W$, we let $\{x, y\}$ be the type of R_n, we let $\{x\}$ be the type of $\{c_0, c_1\}$ and we let $S = \{x, y, z\}$. We distinguish the following two cases.

(i) $\mathrm{proj}_{R_n} 1_W = \mathrm{proj}_{P_{n-1}} 1_W$. Depending on H, we have $\alpha_K \subseteq \beta$ by Lemma 2.18 for $K = (w_n, \dots, n)$, where the type of K is one of $(x, y, z, x), (x, y, x, z, y)$ and (x, y, x, y, z).
(ii) $\mathrm{proj}_{R_n} 1_W \neq \mathrm{proj}_{P_{n-1}} 1_W$. Depending on H, we have $\alpha_K \subseteq \beta$ by Lemma 2.18 for $K = (w_n, \dots, w)$, where the type of K is one of $(x, y, x, y, z), (x, y, x, z)$ and (y, x, y, z, x).

We note that for $L = (w_n, w_n x, w_n xy)$, we have $\alpha_L \subseteq \alpha_K$ by Lemma 2.18, where K is as in part (i) or (ii). We distinguish the following cases.

(a) $R = R_1$. Then, we have $n \geq 2$ (as $P_n \not\subseteq R$) and $\mathrm{proj}_{R_n} 1_W = \mathrm{proj}_{P_{n-1}} 1_W$ by Lemma 2.21. Let $\gamma \in \Phi_+$ be the simple root of R that does not contain P_α. We first suppose $n = 2$. Using Lemma 2.18, we deduce that $-\gamma$ is contained in all three roots α_K mentioned in case (i). Note also that $-\gamma$ is contained in one nonsimple root of R_2. Now, we assume $n \geq 3$. Using induction, $-\gamma$ is contained in a nonsimple root of R_{n-1}. As such a root is contained in both nonsimple roots of R_n by Lemma 2.18, it follows that $(-\gamma) \subseteq \alpha_L$, where $L = (w_n, w_n x, w_n xy)$ is as before.

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(b) $R \neq R_1$. Let $\gamma \in \Phi_+$ be the simple root of R containing P_α. We prove by induction on n that $(-\gamma) \subseteq \alpha_L$ holds, where L is as before. We first suppose $n = 1$. If $\mathrm{proj}_{R_1} 1_W \neq \mathrm{proj}_{P_0} 1_W$, then the claim follows from Lemma 2.18. Thus, we can assume that $\mathrm{proj}_{R_1} 1_W = \mathrm{proj}_{P_0} 1_W$. As before, the claim follows from Lemma 2.18. Now, we suppose $n > 1$. Using induction, $-\gamma$ is contained in a nonsimple root of R_{n-1}. As such a root is contained in both nonsimple roots of R_n by Lemma 2.18, the claim follows. □

LEMMA 4.18. *Let $\alpha, \beta \in \Phi$ be two roots. Let $G = (c_0, \ldots, c_k)$ be a gallery of type (n, r), let $H = (d_0, \ldots, d_l)$ be a gallery of type (n', r'), and suppose that G and H are minimal galleries between the same roots α and β. Then, we have $n = n'$ and $r = r'$.*

PROOF. Let $S = \{r, s, t\} = \{r', s', t'\}$. Recall that G is of type $(u, r, r_{\{s,t\}}, \ldots, r, v)$ with $u, v \in \{1_W, s, t\}$ and H is of type $(u', r', r_{\{s',t'\}}, \ldots, r', v')$ with $u', v' \in \{1_W, s', t'\}$. We first show that the last two chambers of the galleries G and H coincide, and that $r = r'$ and $v = v'$ hold. Using Lemma 4.17(a), we can extend $(c_6, \ldots, c_k)$ and $(d_6, \ldots, d_l)$ to minimal galleries contained in Min. We distinguish the following cases.

(a) $v = 1_W$. Then, there are exactly two different residues $R_1, R_2 \in \partial^2\beta$ such that β is not a simple root of R_1 and of R_2, and we have $\{c_{k-1}, c_k\} = R_1 \cap R_2$. This implies $v' = 1_W$, as otherwise, there would only be one such residue. However, then $\{c_{k-1}, c_k\} = P_\beta = \{d_{l-1}, d_l\}$ and hence $r = r'$.

(b) $v = s$ (the case $v = t$ is analogous). Then, by Lemmas 2.14 and 2.17, there is only one residue $R \in \partial^2\beta$ such that β is not a simple root of R. Note that c_{k-1}, c_k are contained in R and we have $P_\beta \neq \{c_{k-1}, c_k\}$. By the above, we have $v' \neq 1_W$ and we deduce $P_\beta \neq \{d_{l-1}, d_l\} \subseteq R$ similarly. This implies $v' = s$ and, in particular, $d_{l-1} = c_{k-1}$ and $d_l = c_k$. Note that there is only one element in S that decreases the length of $d_{l-1} = c_{k-1}$. As r, r' both decrease the length, we conclude $r = r'$.

Assume $n \neq n'$. Without loss of generality, we can assume $n < n'$. We consider the minimal galleries $H^{-1} = (d_l, d_{l-1}, \ldots, d_1, d_0)$ and $G^{-1} = (c_k, c_{k-1}, \ldots, c_1, c_0)$. Then, G^{-1} is a subgallery of H^{-1} because of the types. However, then H crosses the wall $\partial\alpha$ at least twice. This is a contradiction of the fact that H is minimal. We conclude $n = n'$ and the claim follows. □

In the next definition, we define subsets $M(n, r, L)^G_{\alpha,\beta} \subseteq (\alpha, \beta)$ for all $(G, \alpha, \beta) \in \mathcal{I}$, where $n \in \mathbb{N}$ with $n \geq 3$, $r \in S$ and $L \subseteq \{2, \ldots, n-1\}$. To have an intuition in mind, we describe these symbols here: n and r mean that there exists a minimal gallery of type (n, r) between α and β; the subset L roughly indicates which elements of (α, β) are contained in the set $M(n, r, L)^G_{\alpha,\beta}$.

DEFINITION 4.19.

(a) Let $r \in S$, let $n \in \mathbb{N}$ with $n \geq 3$, let $L \subseteq \{2, \ldots, n-1\}$ and let $(G, \alpha, \beta) \in \mathcal{I}$. If $o(r_\alpha r_\beta) < \infty$, then we define

$$M(n, r, L)^G_{\alpha,\beta} := \begin{cases} (\alpha, \beta) & \text{if } |(\alpha, \beta)| = 2; \\ \emptyset & \text{otherwise.} \end{cases}$$

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Now, we consider the case $o(r_\alpha r_\beta) = \infty$. If there exists no minimal gallery of type (n, r) between α and β, we define $M(n, r, L)^G_{\alpha,\beta} := \emptyset$. Now, suppose that there exists a minimal gallery $H = (c_0, \dots, c_k)$ of type (n, r) between α and β. Let $(\alpha_1 = \alpha, \dots, \alpha_k = \beta)$ be the sequence of roots crossed by H. For each $1 \le i \le n$, we define $\omega_i := \alpha_{k_i+2}$ and $\omega_i' := \alpha_{k_i+3}$, where $k_i = \ell(ur(r_{\{s,t\}}r)^{i-1})$. We define

$$M(n, r, L)^G_{\alpha,\beta} := \{\omega_i, \omega_i' \mid i \in L\}.$$

(b) Let $K \subseteq \mathbb{N}_{\ge 3}$ be nonempty, let $\mathcal{J} = (J_k)_{k\in K}$ be a family of nonempty subsets $J_k \subseteq S$ and let $\mathcal{L} = (L_k^j)_{k\in K, j\in J_k}$ be a family of subsets $L_k^j \subseteq \{2, \dots, k-1\}$. For $(G, \alpha, \beta) \in \mathcal{I}$, we define

$$M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta} := \bigcup_{k\in K, j\in J_k} M(k, j, L_k^j)^G_{\alpha,\beta}.$$

Moreover, we let $\mathcal{M}(K, \mathcal{J}, \mathcal{L}) := (M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta})_{(G,\alpha,\beta)\in\mathcal{I}}$.

CONVENTION 4.20. For the rest of this section, we let $K \subseteq \mathbb{N}_{\ge 3}$ be nonempty, $\mathcal{J} = (J_k)_{k\in K}$ be a family of nonempty subsets $J_k \subseteq S$ and $\mathcal{L} = (L_k^j)_{k\in K, j\in J_k}$ be a family of subsets $L_k^j \subseteq \{2, \dots, k-1\}$.

REMARK 4.21.

(a) Let H be a gallery of type (n, r) for some $n \in K$ and $r \in J_n$. We note that for $i > 1$, the roots ω_i, ω_i' are the nonsimple roots of $R_{\{s,t\}}(c_{k_i})$. Suppose that H is between α and β. Using Lemma 2.18, we deduce $\alpha \subsetneq \omega_1, \omega_1' \subsetneq \cdots \subsetneq \omega_n, \omega_n' \subsetneq \beta$ and hence, $M(n, r, L)^G_{\alpha,\beta} \subseteq (\alpha, \beta)$. We should remark that if $\alpha, \beta \in \Phi_+$, then not all of the roots crossed by H are necessarily positive roots. However, the roots ω_i, ω_i' are. Consider, for example, the case $c_0 = trt$ and H is of type $(r, r_{\{s,t\}}, r, \dots, r_{\{s,t\}}, r)$.

(b) We note that Lemma 4.18 implies that if $M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta} \ne \emptyset$, then there exist $k \in K$ and $j \in J_k$ with $M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta} = M(k, j, L_k^j)^G_{\alpha,\beta}$.

LEMMA 4.22. *$\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ is a pre-commutator blueprint of type $(4, 4, 4)$.*

PROOF. We abbreviate $M^G_{\alpha,\beta} := M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta}$ for all $(G, \alpha, \beta) \in \mathcal{I}$. We first note that the sets $M^G_{\alpha,\beta}$ do not depend on G: in the case $o(r_\alpha r_\beta) < \infty$, the set $M^G_{\alpha,\beta}$ only depends on $|(\alpha, \beta)|$; in the case $o(r_\alpha r_\beta) = \infty$, the set $M^G_{\alpha,\beta}$ only depends on the existence of a suitable minimal gallery between α and β. However, the order $\le_G$ on the set $M^G_{\alpha,\beta}$ depends on G. Note that $\omega_i, \omega_i' \subsetneq \omega_{i+1}, \omega_{i+1}'$ and hence $\omega_i, \omega_i' \le_G \omega_{i+1}, \omega_{i+1}'$, but the order on $\{\omega_i, \omega_i'\}$ depends on G.

Clearly, Axioms (CB1) and (CB2) hold. To show that Axiom (PCB) holds, we let $w \in W$ and $G \in \mathrm{Min}(w)$. Then, we have a homomorphism $U_G \to U_w$ and it suffices to show that we have a homomorphism $U_w \to U_G$ extending $u_\alpha \mapsto u_\alpha$. Let $F \in \mathrm{Min}(w)$ and let $(F, \alpha, \beta) \in \mathcal{I}$. We first assume $o(r_\alpha r_\beta) < \infty$. If $\alpha \le_G \beta$, then $M^F_{\alpha,\beta} = M^G_{\alpha,\beta}$ by definition and we are done. Thus, we can assume $\beta \le_G \alpha$. If $|(\alpha, \beta)| < 2$, then

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$M^F_{\alpha,\beta} = \emptyset = M^G_{\beta,\alpha}$ and we are done. Thus, we assume $(\alpha,\beta) = \{\gamma,\delta\}$ and $\gamma \leq_F \delta$. Then, $\delta \leq_G \gamma$ and we have the following relation in U_G:

$$[u_\alpha, u_\beta] = [u_\beta, u_\alpha]^{-1} = (u_\delta u_\gamma)^{-1} = u_\gamma u_\delta.$$

Thus, we can consider the case $o(r_\alpha r_\beta) = \infty$. Then, we have $\alpha \leq_G \beta$. If there is no gallery H of type (n,r) between α and β for all $n \in K$ and $r \in J_n$, then $M^F_{\alpha,\beta} = \emptyset = M^G_{\alpha,\beta}$. Suppose that there exists a gallery H of type (n,r) between α and β for some $n \in K$ and $r \in J_n$. Then, $M^F_{\alpha,\beta} = \{\omega_i, \omega_i' \mid i \in L^r_n\} = M^G_{\alpha,\beta}$ as sets. Note that $\omega_i, \omega_i' \leq_G \omega_{i+1}, \omega_{i+1}' \geq_F \omega_i, \omega_i'$. As $M^G_{\omega_i,\omega_i'} = \emptyset$, we deduce that $[u_\alpha, u_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma = \prod_{\gamma \in M^F_{\alpha,\beta}} u_\gamma$ is a relation in U_G. Thus, we obtain a homomorphism $U_w \to U_G$ and $\mathcal{M}(K,\mathcal{J},\mathcal{L})$ is a pre-commutator blueprint. □

LEMMA 4.23. *Let $(G,\alpha,\beta) \in \mathcal{I}$ with $o(r_\alpha r_\beta) < \infty$ and suppose that $M(K,\mathcal{J},\mathcal{L})^G_{\alpha,\beta} = \{\gamma,\delta\}$. Then, the following hold for all $\varepsilon \in \Phi(G)$.*

(i) *If $\varepsilon \subsetneq \gamma$ and $\varepsilon \subsetneq \delta$, then we have $M(K,\mathcal{J},\mathcal{L})^G_{\varepsilon,\gamma} = M(K,\mathcal{J},\mathcal{L})^G_{\varepsilon,\delta}$.*
(ii) *If $\gamma \subsetneq \varepsilon$ and $\delta \subsetneq \varepsilon$, then we have $M(K,\mathcal{J},\mathcal{L})^G_{\gamma,\varepsilon} = M(K,\mathcal{J},\mathcal{L})^G_{\delta,\varepsilon}$.*

PROOF. We abbreviate $M^G_{\alpha,\beta} := M(K,\mathcal{J},\mathcal{L})^G_{\alpha,\beta}$ for all $(G,\alpha,\beta) \in \mathcal{I}$. Let R be the unique residue of rank 2 contained in $\partial^2\alpha \cap \partial^2\beta$ (see Lemma 2.7(b)) and let $(\alpha,\beta) = \{\gamma,\delta\}$. Suppose first $\varepsilon \subsetneq \gamma$ and $\varepsilon \subsetneq \delta$. If $M^G_{\varepsilon,\gamma} = \emptyset = M^G_{\varepsilon,\delta}$, we are done. Thus, we can assume $M^G_{\varepsilon,\gamma} \neq \emptyset$. Then, there exists a minimal gallery $H = (d_0,\dots,d_k)$ of type (n,r) between the roots ε and γ for some $n \in K$ and $r \in J_n$. In particular, the type of H is given by $(u, r, r_{\{s,t\}}, r, \dots, r_{\{s,t\}}, r, v)$, where $u, v \in \{1_W, s, t\}$ and $r_{\{s,t\}}$ appears n times. Using Lemma 4.17(a), we can extend $(d_6,\dots,d_k)$ to a gallery $\Gamma \in \mathrm{Min}$. Using Lemma 2.14 and induction, γ is a nonsimple root of the residue of rank 2 containing d_{k-2}, d_{k-1} and d_k. We distinguish the following cases.

(i) $v = 1$. Then, we have $P_\gamma = \{d_{k-1}, d_k\}$ (see Lemma 2.22) and $P_\gamma \subseteq R$. There exists $x \in \{s,t\}$ such that the minimal gallery H extended by an x-adjacent chamber is still of type (n,r) and between the roots ε and δ. We deduce $M^G_{\varepsilon,\gamma} = M^G_{\varepsilon,\delta}$.
(ii) $v \neq 1$. Using Lemmas 2.17 and 2.14, we deduce that R is the only residue such that γ is a nonsimple root of R. Then, the minimal gallery $K := (d_0,\dots,d_{k-1})$ is still of type (n,r) and between the roots ε and δ. We deduce $M^G_{\varepsilon,\gamma} = M^G_{\varepsilon,\delta}$.

Now, we assume $\gamma \subsetneq \varepsilon$ and $\delta \subsetneq \varepsilon$. If $M^G_{\gamma,\varepsilon} = \emptyset = M^G_{\delta,\varepsilon}$, we are done. Thus, we can assume $M^G_{\gamma,\varepsilon} \neq \emptyset$. Then, there exists a minimal gallery $H = (d_0,\dots,d_k)$ of type (n,r) between γ and ε for some $n \in K$ and $r \in J_n$. In particular, the type of H is given by $(u, r, r_{\{s,t\}}, r, \dots, r_{\{s,t\}}, r, v)$, where $u, v \in \{1_W, s, t\}$ and $r_{\{s,t\}}$ appears n times. Clearly, γ is a nonsimple root of R. Note that $\alpha, \beta, \varepsilon \in \Phi(G)$ and hence $\{\alpha,\varepsilon\}, \{\beta,\varepsilon\} \in \mathcal{P}$. Then, Lemma 4.17(b) yields $\{d_0, d_1\} \subseteq R$. We distinguish the following two cases.

(i) $u = 1_W$. Assume that $\ell(\mathrm{proj}_R 1_W, d_0) = 1$. Then, Lemma 2.18 would imply that one of $-\alpha, -\beta$ is contained in one nonsimple root of the $\{s,t\}$-residue containing

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d_1 (that is, ω_1 or ω_1') and hence, one of $-\alpha, -\beta$ is contained in ε. As this is a contradiction, we deduce $\ell(\mathrm{proj}_R 1_W, d_0) = 2$. Let d be the chamber in R adjacent to both $\mathrm{proj}_R 1_W$ and d_0. Then, the gallery $(d, d_0, \dots, d_k)$ is of type (n, r) and between the roots δ and ε. We deduce $M^G_{\gamma,\varepsilon} = M^G_{\delta,\varepsilon}$.

(ii) $u \neq 1_W$. Assume $\ell(\mathrm{proj}_R 1_W, d_0) = 2$. In both cases ($d_2 \in R$ and $d_2 \notin R$), Lemma 2.18 would imply that one of $-\alpha, -\beta$ is contained in ω_2, ω_2' and hence in ε, which is a contradiction. Thus, $\ell(\mathrm{proj}_R 1_W, d_0) = 1$. Again, if $d_2 \notin R$, then Lemma 2.18 would imply that one of $-\alpha, -\beta$ is contained in ω_1, ω_1', which is a contradiction. Thus, we can assume $d_2 \in R$. As $(d_1, \dots, d_k)$ is still a gallery of type (n, r) between the roots δ and ε, we deduce $M^G_{\gamma,\varepsilon} = M^G_{\delta,\varepsilon}$ and the claim follows. □

LEMMA 4.24. *$\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ is a Weyl-invariant commutator blueprint and the groups U_w are nilpotent of class at most 2.*

PROOF. We abbreviate $M^G_{\alpha,\beta} := M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta}$ for all $(G, \alpha, \beta) \in \mathcal{I}$. We apply Proposition 4.15. Suppose $(G, \alpha, \beta) \in \mathcal{I}$ with $o(r_\alpha r_\beta) < \infty$. Then, we have

$$M^G_{\alpha,\beta} = \begin{cases} (\alpha, \beta) & \text{if } |(\alpha, \beta)| = 2, \\ \emptyset & \text{if } |(\alpha, \beta)| < 2, \end{cases}$$

by definition. Let $G = (c_0, \dots, c_k) \in \mathrm{Min}$, let $(\alpha_1, \dots, \alpha_k)$ be the sequence of roots crossed by G, let $1 \leq j \leq k-1$ and let $\alpha := \alpha_j, \beta := \alpha_k$. Then, condition (a) of Proposition 4.15 follows from Lemma 4.23. Now, we show that condition (b) holds. For that, we suppose $o(r_\alpha r_\beta) = \infty$. If $M^G_{\alpha,\beta} = \emptyset$, we are done. Thus, we can assume $M^G_{\alpha,\beta} \neq \emptyset$. Then, there exists a minimal gallery of type (n, r) between α and β for some $n \in K$ and $r \in J_n$. In particular, we have $M^G_{\alpha,\beta} = \{\omega_i, \omega_i' \mid i \in L^r_n\}$. Note that $\{\omega_i, \omega_i'\} = M^G_{\gamma_i,\gamma_i'}$ for some $\gamma_i \leq_G \gamma_i' \in \Phi(G)$ with $\alpha \subseteq \gamma_i \leq_G \gamma_i' \subseteq \beta$, as $L^r_n \subseteq \{2, \dots, n-1\}$. We remark that we really need $1, n \notin L^r_n$.

We first show that $u_{\omega_i} u_{\omega_i'} \in Z(U_{(c_0,\dots,c_{k-1})})$ for all $i \in L^r_n$. This implies that $\prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma = \prod_{i \in L^r_n} u_{\omega_i} u_{\omega_i'}$ is contained in $Z(U_{(c_0,\dots,c_{k-1})})$ and, moreover, we deduce

$$\Big(\prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma \Big)^2 = \Big(\prod_{i \in L^r_n} u_{\omega_i} u_{\omega_i'} \Big)^2 = \prod_{i \in L^r_n} (u_{\omega_i} u_{\omega_i'})^2 = 1.$$

Note that the order $\leq_G$ on $\{\omega_i, \omega_i'\}$ depends on G. Let $\varepsilon \in \Phi(G) \backslash \{\beta\}$. Then, we have to show that $u_{\omega_i} u_{\omega_i'}$ and u_ε commute in $U_{(c_0,\dots,c_{k-1})}$. However, this group is nilpotent of class at most 2 by induction and $u_{\omega_i} u_{\omega_i'} \in [U_{(c_0,\dots,c_{k-1})}, U_{(c_0,\dots,c_{k-1})}] \leq Z(U_{(c_0,\dots,c_{k-1})})$.

Now, it is left to show that item (2-n3) of Theorem 3.8ii is satisfied. Let $1 \leq i < j \leq k$. Note that by construction of the sets $M^G_{\alpha,\beta}$, it suffices to show the claim for the case $o(r_{\alpha_i} r_{\alpha_j}) < \infty$ and $M^G_{\alpha_i,\alpha_j} \neq \emptyset$. Suppose that $M^G_{\alpha_i,\alpha_j} = \{\gamma, \delta\}$ with $\gamma \leq_G \delta$. We distinguish the following cases.

(a) $o(r_\gamma r_{\alpha_k}) < \infty$. Assume that $\alpha_i \subseteq \alpha_k$ and $\alpha_j \subseteq \alpha_k$. Then, $(-\alpha_k) \subseteq (-\alpha_i) \cap (-\alpha_j) \subseteq (-\gamma)$ would yield a contradiction. We deduce $\alpha_i \nsubseteq \alpha_k$ or $\alpha_j \nsubseteq \alpha_k$. Suppose

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$\varepsilon \in \{\alpha_i, \alpha_j\}$ with $o(r_\varepsilon r_{\alpha_k}) < \infty$. If $\varepsilon = \alpha_j = \alpha_k$, then the claim follows. Thus, we can assume $\alpha_j \neq \alpha_k$. It follows that $\partial^2\varepsilon \cap \partial^2\gamma \cap \partial^2\alpha_k = \partial^2\alpha_i \cap \partial^2\alpha_j \cap \partial^2\alpha_k = \emptyset$ from Lemma 2.9 and hence $\{r_\varepsilon, r_\gamma, r_{\alpha_k}\}$ is a reflection triangle. By Remark 2.11, there exist $\beta_\varepsilon \in \{\varepsilon, -\varepsilon\}, \beta_\gamma \in \{\gamma, -\gamma\}$ and $\beta_k \in \{\alpha_k, -\alpha_k\}$ such that $\{\beta_\varepsilon, \beta_\gamma, \beta_k\}$ is a triangle.

Let $\varepsilon' \in \{\alpha_i, \alpha_j\}\backslash\{\varepsilon\}$. Then, we know by Lemmas 2.9 and 2.13(a) that $o(r_\delta r_{\alpha_k}) = \infty = o(r_{\varepsilon'} r_{\alpha_k})$. Lemma 2.8(a) together with the fact $\{\delta, \alpha_k\}, \{\varepsilon', \alpha_k\} \in \mathcal{P}$ imply that both pairs are nested. As $\delta \leq_G \alpha_k$ and $\varepsilon' \leq_G \alpha_k$, we deduce $\delta \subseteq \alpha_k$ and $\varepsilon' \subseteq \alpha_k$. Let $R \in \partial^2\varepsilon \cap \partial^2\gamma = \partial^2\alpha_i \cap \partial^2\alpha_j \cap \partial^2\delta$. Then, $\emptyset \neq R \cap \delta \subseteq \alpha_k$ and (as $R \notin \partial^2\alpha_k$) we deduce $R \subseteq \alpha_k$. This implies $\beta_k = \alpha_k$.

Note that there exist $\beta_\delta \in \{\delta, -\delta\}$ and $\beta_{\varepsilon'} \in \{\varepsilon', -\varepsilon'\}$ with $\beta_\delta, \beta_{\varepsilon'} \in (\beta_\gamma, \beta_\varepsilon)$ by Lemmas 2.9 and 2.12. Using Lemma 2.13(a), we obtain $(-\alpha_k) \subseteq \beta_\delta$ and $(-\alpha_k) \subseteq \beta_{\varepsilon'}$, and hence $(-\delta) = \beta_\delta, (-\varepsilon') = \beta_{\varepsilon'} \in (\beta_\gamma, \beta_\varepsilon)$. By Lemma 2.10, we have $(-\varepsilon') \in (\varepsilon, -\gamma)$ and hence $\beta_\gamma = -\gamma, \beta_\varepsilon = \varepsilon$. Lemma 2.13(b) implies $(\delta, \alpha_k) = -(-\delta, -\alpha_k) = \emptyset$. Moreover, Lemma 2.12 yields $(\gamma, \alpha_k) = (-\beta_\gamma, \beta_k) = \emptyset$ and hence $M^G_{\gamma,\alpha_k} = \emptyset = M^G_{\delta,\alpha_k}$. We conclude

$$\begin{aligned}\prod_{\sigma \in M^G_{\alpha_i,\alpha_j}} \left(u_\sigma \prod_{\rho \in M^G_{\sigma,\alpha_k}} u_\rho\right) &= \left(u_\gamma \prod_{\rho \in M^G_{\gamma,\alpha_k}} u_\rho\right)\left(u_\delta \prod_{\rho \in M^G_{\delta,\alpha_k}} u_\rho\right)\\ &= u_\gamma u_\delta\\ &= \prod_{\sigma \in M^G_{\alpha_i,\alpha_j}} u_\sigma.\end{aligned}$$

(b) $o(r_\gamma r_{\alpha_k}) = \infty = o(r_\delta r_{\alpha_k})$. It follows from Lemma 4.23 that $M^G_{\gamma,\alpha_k} = M^G_{\delta,\alpha_k}$. We have already shown that condition (b) of Proposition 4.15 holds. Thus, we know that $\prod_{\varepsilon \in M^G_{\gamma,\alpha_k}} u_\varepsilon \in Z(U_{(c_0,\ldots,c_{k-1})})$ as well as $(\prod_{\varepsilon \in M^G_{\gamma,\alpha_k}} u_\varepsilon)^2 = 1$ in the group $U_{(c_0,\ldots,c_{k-1})}$. We deduce the following:

$$\begin{aligned}\left(u_\gamma \prod_{\rho \in M^G_{\gamma,\alpha_k}} u_\rho\right)\left(u_\delta \prod_{\rho \in M^G_{\delta,\alpha_k}} u_\rho\right) &= \left(u_\gamma \prod_{\rho \in M^G_{\gamma,\alpha_k}} u_\rho\right)\left(u_\delta \prod_{\rho \in M^G_{\gamma,\alpha_k}} u_\rho\right)\\ &= u_\gamma u_\delta \left(\prod_{\rho \in M^G_{\gamma,\alpha_k}} u_\rho\right)^2\\ &= u_\gamma u_\delta.\end{aligned}$$

It is left to show that $\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ is Weyl-invariant. Let $w \in W, s \in S, G \in \mathrm{Min}_s(w)$ and $(G, \alpha, \beta) \in \mathcal{I}$ with $\alpha \neq \alpha_s \neq \beta$. If $o(r_\alpha r_\beta) < \infty$, then $o(r_{s\alpha} r_{s\beta}) < \infty$ and, as $(s\alpha, s\beta) = \{s\gamma \mid \gamma \in (\alpha, \beta)\}$, we infer $M^{sG}_{s\alpha,s\beta} = sM^G_{\alpha,\beta}$. Thus, we can assume $o(r_\alpha r_\beta) = \infty$. Suppose that there exists a gallery $H = (c_0, \ldots, c_k)$ of type (n, r) between α and β for some $n \in K, r \in J_n$. Then, $(sc_0, \ldots, sc_k)$ is a gallery of type (n, r) between the roots $s\alpha, s\beta$. This implies that a gallery of type (n, r) exists between the roots α and β if and only if a gallery of type (n, r) exists between the roots $s\alpha$ and $s\beta$. This finishes the proof of the claim. □

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SEBASTIAN BISCHOF, UCLouvain, IRMP, Chemin du Cyclotron 2,
1348 Louvain-la-Neuve, Belgium
e-mail: sebastian.bischof@math.uni-paderborn.de

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