The trisectrix and Langley's problem

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Introduction

If a circle rolls without slipping around an equal fixed circle, then a point carried by the rolling circle traces out a *limaçon* of Pascal. (This is Etienne Pascal, father of Blaise. The word *limaçon* is derived from the Latin *limax*, a snail.) If the fixed and rolling circles have radius 1, and the point P carried by the rolling wheel is distant a from its centre, then for $a > 1$ the limaçon has an inner and an outer loop, joining up at a node. For $a = 1$ it has a cusp, and is then a *cardioid*, so-called because it is heart-shaped. See Figure 1, where we have plotted the cases $a = \frac{3}{4}$, $a = 1$ and $a = \frac{3}{2}$.

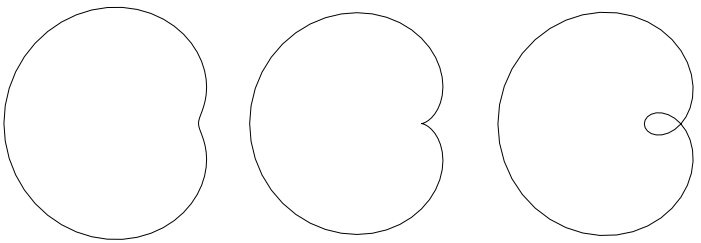


FIGURE 1

In this Article we shall obtain a few standard properties of the limaçon, and then concentrate on the special case when $a = 2$, the *trisectrix*. This leads to a generalisation of *Langley's problem* [1].

The limaçon

In the complex plane, let the fixed circle be $|z| = 1$, a general point of which is given by the *turn* $t = e^{i\theta}$. If the centre of the rolling circle is initially at $z = 2$, and the point P is initially at $2 - a$, then when the circle has rolled so that its point of contact is at t, it has turned through an angle 2θ and P is now at $z = 2t - at^2$. This is the equation of the limaçon, parametrised by the turn t. See Figure 2. For $a > 1$, the curve meets the real axis four times, at $2 - a$ (when $\theta = 0$), at $-2 - a$ (when $\theta = \pi$), and twice at a (when $\theta = \pm \arccos \frac{1}{a}$).

If we move the origin to a, then we obtain the polar equation of the limaçon as $r = 2 - 2a\cos\theta$. See Figure 2, again. Halving the scale, and multiplying by r, we obtain $r^2 = r - ax$, or $(x^2 + y^2 + ax)^2 = x^2 + y^2$, a quartic curve, so that every line meets the limaçon in four points (properly counted: real, imaginary, or coincident). Note that, in the case $a < 1$, this curve has an isolated point at the origin, an *acnode*—from the same root as *acne*: the curve has a pimple! Note also that the inverse of the limaçon,

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obtained by mapping the point (r, θ) to (r^{-1}, θ), has polar equation $r = (1 - a\cos\theta)^{-1}$, which is a conic of eccentricity a, with the origin at a focus. Thus the limaçon with two loops is the inverse of a hyperbola with respect to a focus, the two loops corresponding to the two branches of the hyperbola. The limaçon without two loops is the inverse of an ellipse, and the cardioid is the inverse of a parabola, both with respect to a focus.

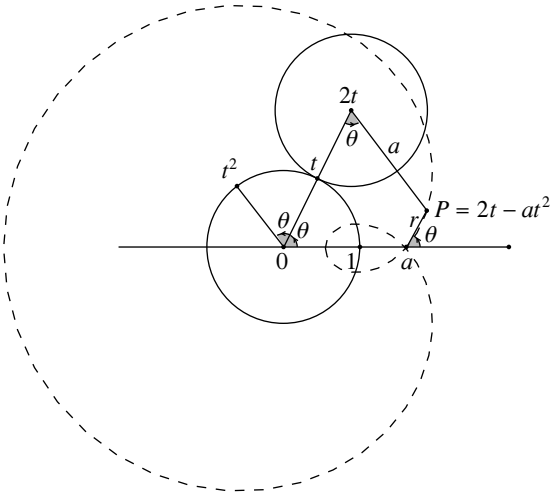


FIGURE 2

The trisectrix

The special case when $a = 2$ is the *trisectrix*, $z = 2t - 2t^2$, so-called because it can be used to trisect angles, as we shall show. Of course, this is not a straight edge and compasses construction, which is in general impossible, essentially because in terms of algebra straight edge and compasses can only be used to solve linear and quadratic equations, whereas trisection involves solving a cubic equation $4x^3 - 3x = k$, where $k = \cos 3\theta$ and $x = \cos\theta$.

Halving the scale, take the trisectrix as $\Gamma : z = t - t^2$. Let $\omega = e^{2\pi i/3}$, so that ω and ω^2 are the complex cube roots of 1, and $1 + \omega + \omega^2 = 0$. Then the node of Γ is at $z = 1$, when $t = -\omega$ or $-\omega^2$.

For a general turn t, consider the points P, P' given by the turns t and $-t$, that is, $p = t - t^2$ and $p' = -t - t^2$, using lower-case for the corresponding complex numbers. Let U be the unit point 1, so $u = 1$. Then

$$\frac{p-u}{p-p'} = \frac{t - t^2 - 1}{2t} = \frac{1 - t - t^{-1}}{2} = \frac{1 - t - \bar{t}}{2},$$

which is real, so that U, P and P' are collinear. See Figure 3, where we have taken P on the inner loop of Γ. Next, $|p - p'| = |2t| = 2$, and, if M is the mid-point of PP', then $|MP| = |MP'| = 1$, and also $m = \frac{1}{2}(p + p') = -t^2$, so $|MO| = |m| = 1$. Thus ΔUOM and ΔOMP are both isosceles.

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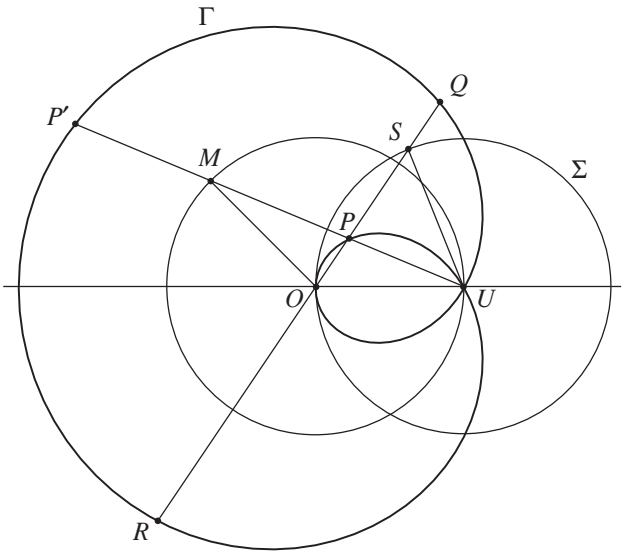


FIGURE 3

Let $\angle PUO = \angle MUO = \alpha$. Then $\angle OMU = \alpha$, and $\angle UOM = \pi - 2\alpha$. Also, $\angle POM = \angle MPO = \frac{1}{2}(\pi - \alpha)$, so that $\angle UOP = \frac{1}{2}(\pi - 3\alpha)$. Let OP meet the circle Σ (centre U, through O) again at S. See Figure 3, again. Then ΔSUO is isosceles, so that $\angle OSU = \frac{1}{2}(\pi - 3\alpha)$, and $\angle SUO = 3\alpha$. Therefore the line UP trisects $\angle SUO$.

So, to trisect a given angle, proceed as follows. Mark S on Σ so that $\angle SUO$ is the given angle β, and let P be where the line SO crosses the inner loop of Γ. Then $\angle PUO$ is $\frac{1}{3}\beta$.

Now, the line US is obtained by rotating the line OU about U through β; but it can also be obtained by rotating OU about U through $\beta + n\pi$, where $n \in \mathbb{Z}$. So the lines obtained by rotating OU about U through $\frac{1}{3}(\beta + n\pi)$ have equal claim of being a trisector of $\angle SUO$. These lines are UP and two lines at an angle $\pm\frac{1}{3}\pi$ to UP, and we now locate them.

Let P, Q, R be the points of Γ with parameters t, ωt, $\omega^2 t$ respectively, that is, $p = t - t^2$, $q = \omega t - \omega^2 t^2$ and $r = \omega^2 t - \omega t^2$. Then

$$\frac{q}{p} = \frac{\omega t - \omega^2 t^2}{t - t^2} = \frac{\omega - \bar{\omega} t}{1 - t} = \frac{(\omega - \bar{\omega} t)(1 - \bar{t})}{(1 - t)(1 - \bar{t})} = \frac{\omega + \bar{\omega} - \bar{\omega} t - \omega \bar{t}}{(1 - t)(1 - \bar{t})}$$

which is real, so O, P, Q are collinear. Similarly, so are O, P and R, so in fact O, P, Q and R are collinear, and Q, R are the points in which OP meets the outer loop of Γ. See Figure 3, yet again.

Now let H, K be defined by $h = -\omega^2 t$, $k = h^2 = \omega t^2$, so H, K lie on the circle $|z| = 1$. Then

$$p - h = t - t^2 + \omega^2 t = -t(t + \omega)$$
$$q - h = \omega t - \omega^2 t^2 + \omega^2 t = -\omega^2 t(t + \omega)$$

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$$k - h = \omega t^2 + \omega^2 t = \omega t\,(t + \omega)$$
$$u - h = 1 + \omega^2 t = \omega^2\,(t + \omega)$$

from which we deduce that $|PH| = |QH| = |KH| = |UH| = |t + \omega|$, so that U, K, P and Q lie on a circle, centre H. Further,

$$p - k = t - t^2 - \omega t^2 = \omega^2 t\,(t + \omega)$$

and

$$q - k = \omega t - \omega^2 t^2 - \omega t^2 = t\,(t + \omega),$$

so $|PK| = |QK| = |t + \omega|$, and therefore ΔPHK and ΔQKH are equilateral. See Figure 4. It follows that the line OPQ bisects $\angle HOK$, and of course the line OH bisects $\angle UOK$ (because $k = h^2$), so that U is on the opposite side of PQ from K. Therefore $\angle QUP = \pi - \angle PKQ = \frac{1}{3}\pi$. By a similar argument, $\angle PUR = \frac{1}{3}\pi$ also, so that UP, UQ and UR are the three trisectors referred to, above.

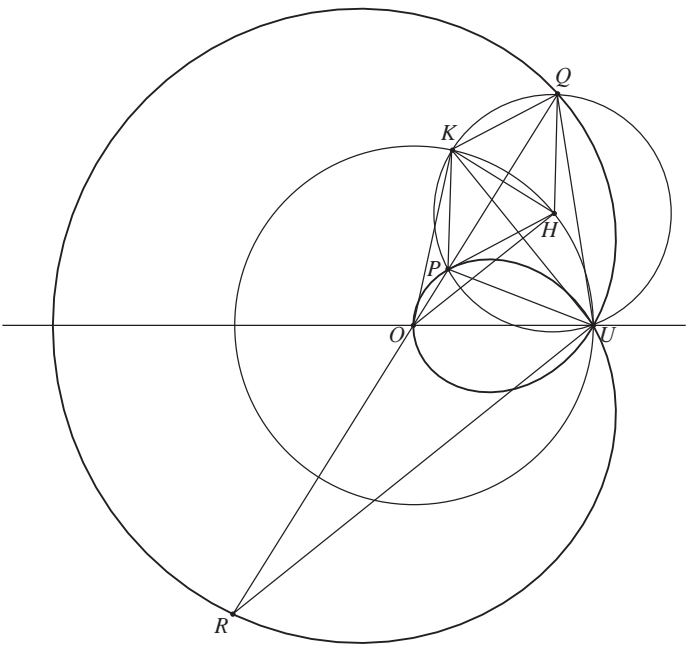


FIGURE 4

Note, for use in the next section, that $\angle KQP = \angle KUP = \frac{1}{2}\angle KHP = \frac{1}{6}\pi$, and that $\angle QUK = \frac{1}{6}\pi$ as well. Note also that, since $\angle QOK = \frac{1}{2}\angle HOK$ and $\angle HOK = \frac{1}{2}\angle UOK$, we have $\angle QOK = \frac{1}{3}\angle UOQ$.

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Langley's problem

As above, if $\angle PUO = \alpha$, then $\angle UOQ = \angle UOP = \frac{1}{2}(\pi - 3\alpha)$. Put $\varphi = \frac{1}{2}\alpha$ and change to degrees: then $\angle PUO = 2\varphi$ and $\angle UOQ = 90° - 3\varphi$, where $0° < \varphi < 30°$. Also, $\angle QOK = \frac{1}{3}\angle UOQ = 30° - \varphi$, $\angle KUO = 30° + 2\varphi$ and $\angle QUK = 30°$. Let the lines OK, UQ meet at L. Then, given ΔOUL with K, Q on OL, UL respectively, and given the above angles, we already know that $\angle KQO = 30°$ (see Figure 5).

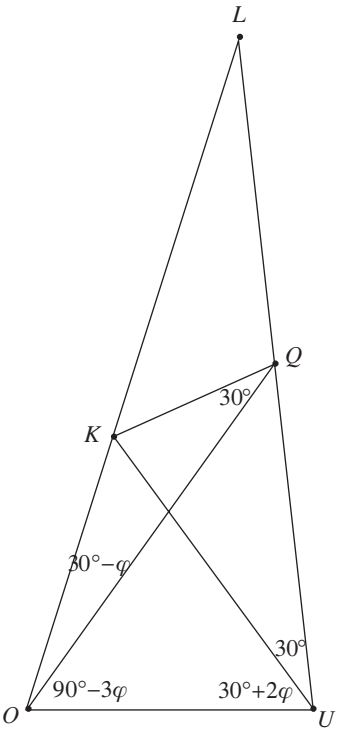


FIGURE 5

In the case where $\varphi = 10°$, ΔOUL is an $80°-80°-20°$ isosceles triangle, and $\angle QUK = 30°$, $\angle KUO = 50°$, $\angle UOQ = 60°$ and $\angle QOK = 20°$. The determination of $\angle KQO$ in this case appeared for the first time at [1], after which it became known, at least in the pages of the *Gazette*, as *Langley's problem*, or *Langley's adventitious angles*. It is a deceptively difficult problem. See also [2], especially the footnote on p. 174. The problem, and its generalisation, also features in some handwritten notes of my late friend and colleague John Tyrrell as *Mahatma's puzzle*. Where he got that name I do not know, though Tony Barnard has pointed out that the problem appears as *Mahatma's triangle* in [3].

There is an incomplete trigonometric proof of Langley's problem at [4], where, after two applications of the sine rule, the equation

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$2 \sin p \sin 50° = \sin(20° + p)$ is obtained, where $p = \angle KQO$. The note then states that this is clearly satisfied by $p = 30°$, and leaves it at that, ignoring the possibility of other solutions. To finish the proof properly, put $p = q + 30°$, which on expanding reduces the equation to $\sin q\,(2 \cos 30° \sin 50° - \cos 50°) = 0$. The bracketed expression is non-zero, so $\sin q = 0$, and q is acute, so $q = 0°$ and $p = 30°$.

We finish by outlining two of the standard purely geometric proofs of Langley's problem, adapted to the more general problem of Figure 5.

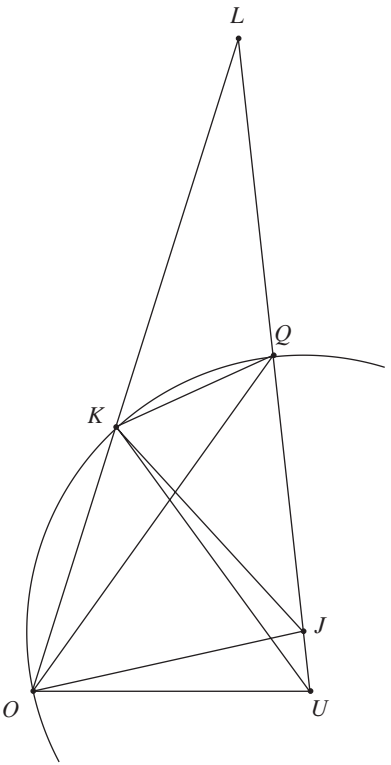


FIGURE 6

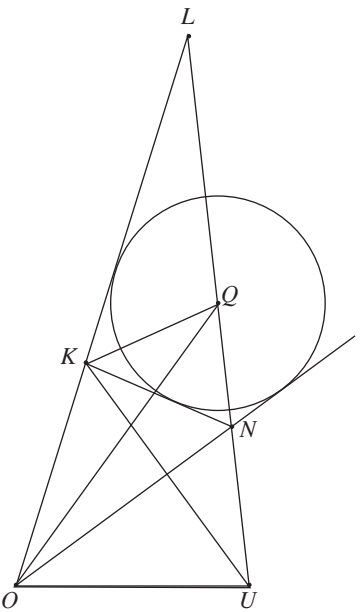


FIGURE 7

For the first proof, put J on UL so that $\angle JOK = 60°$. See Figure 6. Then it is easily seen that ΔUOK, ΔUOJ and ΔQJO are all isosceles, so that $|OK| = |OU| = |OJ| = |JQ|$. Therefore ΔJOK is equilateral, and there is a circle centre J through O, K and Q, whence $\angle KQO = \frac{1}{2}\angle KJO = 30°$.

For the second proof, put N on UL so that $ON \perp UK$. See Figure 7. Then OQ bisects $\angle NOK$ and, the angles at N being 60°, NQ is the external bisector of $\angle KNO$. Therefore Q is an excentre of ΔONK, so that KQ is the external bisector of $\angle NKO$, and an easy angle-chase shows that $\angle KQO = 30°$, once more.

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4. Robert J. Clarke, The return of the cotangent rule, *Math. Gaz.* **88** (March 2004) pp. 116-118.

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Quotations for *Nemo* (continued from page 20)

3. He dived in and out of those crooked lanes, which were more like cracks than thoroughfares; and by the time that he had completed about twenty alternate angles and described an unthinkable polygon, he paused to listen for any sound of pursuit.

4. Then came forward Maister Rombus, and with many speciall graces made this learned oration.

5. Its stables and outhouses were built in the shape of a rough octangle surrounding the farmhouse itself, which was built in the shape of a rough triangle. The left point of the triangle abutted on the farthest point of the octangle, which was formed by the cowsheds, which lay parallel with the big barn.

6. The day being the sixth of April, the Durbeyfield waggon met many other waggons with families on the summit of the load, which was built on a wellnigh unvarying principle, as peculiar, probably, to the rural labourer as the hexagon to the bee.

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