PROCEEDINGS OF THE
Cambridge Philosophical Society

VOL. 69 JANUARY 1971 PART 1

Proc. Camb. Phil. Soc. (1971), **69**, 1
PCPS 69–1
Printed in Great Britain

Bounded and invariant elements in 2-firs

BY A. J. BOWTELL AND P. M. COHN
Westfield College and Bedford College, University of London

(*Received* 6 *January* 1970)

1. *Introduction.* In a principal ideal domain R, any two-sided ideal is of the form $Rc = cR$, i.e. it has an invariant element as generator, and the customary development of ideal theory in a principal ideal domain (cf. e.g. (10), ch. III) takes on a more transparent form when expressed in terms of invariant elements. Likewise, the one-sided bounded ideals may be studied in terms of their generators.

If instead of a principal ideal domain one considers more generally a 2-fir (= weak Bezout ring, cf. (5) and section 2 below), it is no longer possible to describe all ideals by means of invariant generators, but the structure theory of invariant elements can be taken over unchanged, and can be used to describe the ideals of a 2-fir that are principal as left and right ideals.

Our aim in this note is to present a study of invariant and bounded elements in 2-firs (sections 3–5). This then amounts to a study of principal left and right ideals, and the one-sided ideals they bound, in a 2-fir. On the one hand this throws some light on the structure of general 2-firs, while on the other hand, by specializing to the Noetherian case we get an alternative development of the ideal theory in principal ideal domains. The final section (section 6) shows how invariant elements can be used to obtain an embedding of 2-firs in matrix rings over skew fields, which appear to be new even in the case of principal ideal domains.

One might also be tempted to specialize to the case of firs, were it not for the fact that a fir has no non-unit invariant element except when it is a principal ideal domain (3). The same reference also provides us with examples of genuine (i.e. non-Ore) 2-firs with non-unit invariant elements (see section 3 below), thus assuring us that the generalization discussed here is not illusory. For a more detailed discussion of a particular case we refer to (4) and (1).

We are indebted to the referee whose careful reading led to the removal of some obscurities.

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

2. *Background.* For the reader's convenience we briefly summarize the facts needed about 2-firs. Throughout, all rings are associative, with 1, all homomorphisms preserve the 1 and subrings have the same 1.

By a *2-fir* (= weak Bezout ring) we understand a ring R in which every right ideal on at most two generators is free, of unique rank. Equivalently, a 2-fir may be defined as an integral domain in which any two principal right ideals with non-zero intersection have a principal sum. It can be shown that their intersection is also principal, and that the class of rings so defined is left-right symmetric (cf. (5, 6)).

Two elements a, a' of a ring R are said to be *right similar*, if $R/aR \cong R/a'R$. In an integral domain this holds precisely when $R/Ra \cong R/Ra'$ (cf. e.g. (6)), so we can omit the qualifier 'right' in this case. In the particular case of a 2-fir one has the following useful criterion for similarity (cf. (5)):

Two elements a, a' of a 2-fir R are similar if and only if there is a coprime relation

$$ab' = ba'. \tag{2·1}$$

Here 'coprime' means: a, b have no common left factor and a', b' no common right factor, apart from units. In fact, in a 2-fir R, any coprime relation (2·1) is *comaximal*, in the sense that $aR + bR = R$ and $Ra' + Rb' = R$. The existence of a comaximal relation (2·1) is also expressed by saying: a, b' (in that order) are *comaximally transposable*.

By an *atom* in a ring R one understands a non-unit which is not a product of non-units. If every element apart from units and zero can be expressed as a product of atoms, the ring R is said to be *atomic*. For atomic 2-firs a unique factorization theorem was proved in (5); we shall not need the precise form here, but only the result that the number of factors in each factorization of a given element c is the same. This number is denoted by $l(c)$ and called the *length* of c. We also note that similar elements in an atomic 2-fir have the same length.

3. *Invariant elements in 2-firs.* An element c of a ring R is said to be *invariant* if it is not a zero-divisor and

$$Rc = cR.$$

Clearly, left and right divisibility of a by an invariant element c mean the same thing, and may be denoted by $c|a$. In an integral domain, moreover, being a right factor of an invariant element is the same as being a left factor. For if $c = ab$, then $cb = b'c$, hence $cb = b'ab$, and by cancelling b we find that $cb = b'a$; the converse follows by symmetry. Note that every unit of R is trivially invariant.

We now take a closer look at invariant elements in 2-firs. The following example, taken from (3), shows that there exist 2-firs that are not Ore rings but have non-unit invariant elements. In the free associative algebra on x and y over the rationals, $\mathbf{Q}\langle x, y\rangle$, consider the subring of elements whose constant term has an odd denominator. This is easily seen to be a 2-fir (in fact a semifir, cf. (3)) but 2 is a central (hence invariant) non-unit. This ring is not atomic; in fact no examples of non-unit invariant elements in atomic 2-firs (other than principal ideal domains) are known, but the discussion in (3) makes the existence of such rings very plausible.

Let a, a' be similar elements in an atomic 2-fir R, and suppose that a' is invariant. Then $R/aR \cong R/a'R$ and $a'R$ is the annihilator of the module $R/a'R$. Hence $a'R \subseteq aR$,

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

i.e. $a' = au$, where u must be a unit, by unique factorization and the fact that a is similar to a'. Moreover, a is invariant, for $Ra = Ra'u^{-1} = a'Ru^{-1} = aR$. Thus we have proved

PROPOSITION 3·1. *In an atomic 2-fir, any element similar to an invariant element a is associated to a and hence is itself invariant.*

From the definition it follows that the right or left ideal generated by an invariant element is actually two-sided. The next result shows when the converse holds:

PROPOSITION 3·2. (i) *In any ring R a non-zero-divisor c is invariant if and only if the left and right ideals generated by c are both two-sided.*

(ii) *If R is an integral domain, then any non-zero ideal that is principal both as left and as right ideal has an invariant generator.*

(iii) *If R is a 2-fir, then the ideals with invariant generators form a sublattice of the lattice of all ideals.*

Proof. (i) is clear; for (ii) take $aR = Ra'$ and let $a = ua'$, $a' = av$, then $a' = ua'v$. Now $a'v \in Ra'$, say $a'v = wa'$; so $a' = uwa'$, i.e. $uw = 1$, u is a unit and $aR = Ra$.

(iii) Let $aR = Ra$, $bR = Rb$ be two non-zero ideals in a 2-fir R, then $aR \cap Rb \neq 0$, hence $aR + bR = dR = Ra + Rb = Rd'$ and $aR \cap bR = mR = Ra \cap Rb = Rm'$. By (ii), d and m are invariant and the proof is complete.

COROLLARY. *In a principal ideal domain every two-sided ideal has an invariant generator.*

Proposition 3·2 shows that in any 2-fir R the semigroup $\mathbf{I}(R)$ of invariant elements is lattice-ordered by divisibility. This means that any two invariant elements have a highest common factor, and this is the same whether calculated in $\mathbf{I}(R)$ or in R; similarly for the least common multiple. Since an atomic 2-fir satisfies the ascending chain condition (ACC) on principal right ideals, the semigroup $\mathbf{I}(R)$ in an atomic 2-fir R clearly also satisfies ACC on principal right ideals, and we can therefore apply Theorem 6·1 of (7). Let us define an *I-atom* in a ring R as a non-unit invariant element which cannot be written as a product of non-unit invariant elements. Then we obtain

THEOREM 3·3. *Let R be an atomic 2-fir. Then every non-unit invariant element c can be written as a product of I-atoms*

$$c = a_1 \dots a_r \tag{3·1}$$

and if

$$c = b_1 \dots b_s \tag{3·2}$$

is any other factorization of c into I-atoms, then $r = s$ and there exists a permutation $i \to i'$ of $1 \dots r$, such that $b_{i'}$ is associated to a_i. Moreover, every order of the factors in (3·1) *can be realized, in fact we can write c in the form $c = uc_1^{d_1} \dots c_t^{d_t}$, where the c_i are dissimilar I-atoms and u is a unit.*

In terms of ideals Theorem 3·3 states that in an atomic 2-fir the ideals with invariant generator form a free commutative semigroup under ideal multiplication. In the special case of principal ideal domains we thus recover a well-known result by using Proposition 3·2, Corollary (cf. (10), p. 38):

COROLLARY 1. *In a principal ideal domain the two-sided ideals form a free commutative semigroup under ideal multiplication.*

The corollary to Proposition 3·2 also gives the following criterion for simplicity:

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

COROLLARY 2. *A principal ideal domain is simple if and only if it has no non-unit invariant elements.*

The quotient of a 2-fir by an ideal with an invariant generator has a rather special form, which is described in

THEOREM 3·4. *Let R be a 2-fir and c a non-unit invariant element. Then R/cR is a left and right Bezout ring. If R is moreover atomic, R/cR is an Artinian principal ideal ring, which is a skew field if and only if c is an atom in R.*

Proof. The first assertion is that every finitely generated left or right ideal is principal. Now cR meets every non-zero right ideal non-trivially, hence every principal right ideal of R/cR comes from a principal right ideal of R containing cR, and the sum of any two (and hence any finite number of) such principal right ideals is again principal. If R is also atomic, every chain of principal (left or) right ideals is bounded by the length of c, hence R/cR is then left and right Artinian. Finally, an Artinian ring is a skew field if and only if it is an integral domain, and R/cR is an integral domain precisely if c is an atom. This proves the last assertion and the proof is complete.

Let R be an atomic 2-fir and c an invariant element, and consider the ring $Q = R/cR$. We saw that any right ideal of Q has the form aR/cR, where $c = ab$ for some $b \in R$. Let $cb = b'c$, then $cb = b'c = b'ab$, hence $c = ab = b'a$, Rb' annihilates aR (in Q). Let Rd be the precise annihilator of aR in Q, then

$$Rd \supseteq Rb', \tag{3·3}$$

and since $da \in cR$, $l(da) \geqslant l(c) = l(b'a)$, whence $l(d) \geqslant l(b')$. This implies that we have equality in (3·3), and so Rb' is the precise annihilator of aR in Q; by symmetry aR is the precise annihilator of Rb' in Q. This means that Q is a quasi-Frobenius ring (cf. e.g. (8)), so we have the

COROLLARY. *Let R be an atomic 2-fir and c a non-unit invariant element of R, then R/cR is a quasi-Frobenius ring.*

A well-known result of Asano (2) states that any Artinian principal ideal ring is a direct product of primary rings. For the rings in Theorem 3·4 this follows directly by expressing c as a product of powers of distinct I-atoms: $c = uq_1 \dots q_r$, where $q_i = p_i^{\alpha_i}$, $p_i R \not\cong p_j R$ for $i \neq j$, and u is a unit. The Krull–Schmidt theorem shows that

$$R/cR \cong \prod_1^r R/q_i R, \tag{3·4}$$

as R-modules, and by taking endomorphism rings, as isomorphism of rings, if we bear in mind that $\mathrm{Hom}_R(R/q_i R, R/q_j R) = 0$ for $i \neq j$ (because a homomorphism would lead to a factor of q_i being similar to a factor of q_j). Each factor $R/q_i R$ is primary, as endomorphism ring of the 'I-indecomposable' module $R/q_i R$. We shall study these factors in section 5, but before we can do so, we need to look at factors of invariant elements. They form the subject of section 4.

4. *Bounded elements in 2-firs.* Let R be a 2-fir. In general there is no reason to suppose that I-atoms will be atoms in R, but if R is also atomic, we can at least factorize any I-atom into atoms. We now turn to the study of such factorizations. We recall that

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

any left factor of an invariant element is also a right factor; such a factor is said to be *bounded*. A number of equivalent conditions are listed in

PROPOSITION 4·1. *In an integral domain R, the following conditions on an element a of R are equivalent*:

(α) *a is bounded, i.e. a divides an invariant element of R,*

(β) *R/aR is annihilated by an invariant element of R,*

(γ) *the right ideals dR, where d runs over all elements similar to right factors of a, contain an ideal with an invariant generator,*

$(\alpha)^* - (\gamma^*)$ *The left-right analogues of* $(\alpha) - (\gamma)$.

Proof. Let c be an invariant element of R, then R/aR is annihilated by cR if and only if $cR \subseteq aR$, i.e. $c = ab$ for some $b \in R$, whence $(\alpha) \Leftrightarrow (\beta)$. If $a = bd$, then R/aR has the submodule $bR/bdR \cong R/dR$; thus the right factors of a define submodules of R/aR and each dR is the annihilator of an element of R/aR. Thus $\operatorname{Ann}(R/aR) \subseteq \bigcap dR$, where d runs over all elements similar to right factors of a; hence $(\beta) \Rightarrow (\gamma)$. Conversely, let $c \in \bigcap dR$ be invariant, then in particular, $c = ab$ for some b, and c annihilates R/aR. Now the rest follows by the left-right symmetry of (α).

In the course of the proof we obtained the inequality

$$\operatorname{Ann}(R/aR) \subseteq \bigcap \{dR | d \text{ is similar to a right factor of } a\}. \tag{4·1}$$

In general we cannot assert that equality holds here. But in an atomic 2-fir, if both sides contain an invariant element c, then by DCC the intersection on the right can be taken finite and moreover, every element of R/aR is annihilated by some dR. For if $x \neq 0$, then $xR \cap aR \neq 0$ ($xc = cx' \neq 0$), hence $xd_1 = ax_1$, where d_1 is similar to a right factor of a. So we have equality in (4·1) when R is an atomic 2-fir and a is bounded. If a is bounded by c say, then the precise annihilator of R/aR corresponds to an ideal containing cR, necessarily principal, by Theorem 3·4. If we denote this ideal by a^*R, then the invariant element a^* is determined up to associates; it is called the *bound* of a, clearly it depends only on the similarity class of a and what has been said shows that it can also be defined by

$$a^*R = \bigcap \{dR | \, d \text{ is similar to a right factor of } a\}. \tag{4·1′}$$

This intersection can be taken to be a finite intersection (by the DCC in R/a^*R) and if we take it to be irredundant, we obtain a subdirect sum representation of R/a^*R, qua right R-module, by modules R/dR. This shows that every atomic factor of a^* is similar to a factor of a. Since $a|a^*$, this characterizes the atomic factors of a^* as the atoms similar to factors of a.

Let $a \in R$ have a bound a^*; if $a = bcd$, then c is again bounded, with bound dividing a^*. This follows by observing that R/cR is a factor (i.e. a quotient of a submodule) of R/aR. Likewise the product of any bounded elements is bounded.

The next result gives the connexion between bounded atoms and I-atoms. We recall that a *strictly cyclic* module over a 2-fir R is a module with a single generator and a single irredundant defining relation, i.e. a module of the form R/aR ($a \neq 0$). The category of strictly cyclic right R-modules and all homomorphisms is denoted by $\mathscr{C}_R$ or simply by $\mathscr{C}$. We note that R/aR is a simple object in $\mathscr{C}$, briefly: $\mathscr{C}$-simple, if and

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

only if a is an atom in R. The category $\mathscr{C}$ is known to admit kernels and cokernels (cf. (6), (7)), hence Schur's lemma holds, and we conclude that the eigenring of aR, i.e. $\mathrm{End}_R(R/aR)$, is a skew field whenever a is an atom.

PROPOSITION 4·2. *In an atomic 2-fir R, let p be an atom which is bounded. Then its bound p^* is an I-atom whose atomic factors are precisely the atoms similar to p. If K is the eigenring of pR and $n = l(p^*)$, then*

$$R/p^*R \cong K_n.$$

Conversely, every I-atom of R occurs in this way.

Proof. Let p have the bound $p^* = a^*b^*$, where a^*, b^* are non-unit invariant elements. Then p cannot be comaximal with both a^* and b^*, therefore it divides either a^* or b^* and so has a smaller bound, a contradiction. Thus p^* is an I-atom. By the remarks following Proposition 4·1, $p^*R = \bigcap p'R$, where p' runs over all elements similar to p, and in fact we can take a finite intersection. Thus R/p^*R is a submodule of $(R/pR)^N$ for some N, and since R/pR is $\mathscr{C}$-simple, every submodule has the form $(R/pR)^n$ for some $n \leqslant N$. Hence

$$R/p^*R \cong (R/pR)^n \tag{4·2}$$

as right R-modules. By comparing the lengths of composition series within $\mathscr{C}$, we see that $l(p^*) = n$, and comparing endomorphism rings in (4·2) we find

$$R/p^*R \cong K_n,$$

where K a skew field, is the eigenring of pR.

By (4·1′) an atom divides p^* if and only if it is similar to p, and to establish the final part, let p^* be any I-atom and p an atom dividing it. Then p^* is the bound of p and we can complete the proof by applying the first part.

Let us take a bounded element a in an atomic 2-fir R. By the Krull–Schmidt theorem (cf. (5)) we have a direct decomposition

$$aR = q_1R \cap \ldots \cap q_rR, \tag{4·3}$$

where each q_i is bounded indecomposable. We shall take up the study of such decomposition in the next sections, but for the moment ask: if a in (4.3) is any element of R, not necessarily bounded, can we separate out the bounded components on the right? This can be answered by the method used by Asano (2) in the case of principal ideal domains.

We begin with a lemma which is generally useful, and part of which was proved for principal ideal domains by Ore (12).

LEMMA 4·3. *Let a, b be non-units in a 2-fir R. If a, b, can be comaximally transposed then there exist $x, y \in R$ such that*

$$xa + by = 1; \tag{4·4}$$

if, moreover R is atomic, then conversely, (4·4) *implies that a, b can be comaximally transposed.*†

Proof. Assume that there is a comaximal relation

$$ab = b'a'. \tag{4·5}$$

† This last condition holds even without atomicity; we shall not give the proof here, as it will not be needed.

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

Then there exist $x', y \in R$ such that $x'a' + yb = 1$, hence

$$(1 - by)\, b = bx'a'.$$

But (4·5) is a least common left multiple of a' and b, hence there exists $x \in R$ such that $1 - by = xa$, i.e. (4·4). Conversely, assume (4·4), then

$$xab = b - byb, \tag{4·6}$$

and the left-hand side is non-zero, by (4·4), because a, b are non-units. Thus $xR \cap bR \neq 0$, and hence there is a comaximal relation

$$xb' = bx', \tag{4·7}$$

which represents the least common right multiple of x and b. A comparison with (4·6) shows that

$$ab = b'a', \tag{4·8}$$

$$1 - yb = x'a'. \tag{4·9}$$

By (4·9), a' and b are left comaximal, hence by (4·8), b is similar to a right factor of b', but by (4·7), b is similar to b' itself, therefore $l(b) = l(b')$ and it follows that (4·8) is left coprime, and hence right (as well as left) comaximal. Therefore (4·8) is a comaximal relation.

Let us digress for a moment to show how Lemma 4·3 can be used to characterize the Jacobson radical of a 2-fir.

THEOREM 4·4. *Let R be a 2-fir. Then $\mathbf{J}(R)$, the Jacobson radical of R, consists of those elements of R that cannot be comaximally transposed with any non-unit of R.*

Proof. Suppose that c cannot be comaximally transposed with any non-unit. For any $x \in R$ we have the comaximal relation

$$c(xc + 1) = (cx + 1)\, c,$$

hence $cx + 1$ is a unit for all $x \in R$, and this shows that $c \in \mathbf{J}(R)$. Conversely, assume that $c \in \mathbf{J}(R)$ and suppose that c, b can be comaximally transposed. Then by Lemma 4·3, $cx + yb = 1$, and by the choice of c, $yb = 1 - cx$ is a unit, whence b is a unit.

Returning to the problem of finding direct decompositions, let us define two elements a, b of a 2-fir R to be *totally coprime* if no factor of a is similar to a factor of b, i.e. if R/Ra and R/bR have no isomorphic factors other than zero. Let a, b be totally coprime and let a be bounded, with bound $a^* = a_1 a$ say, then a^* and b are left coprime, and hence right comaximal, i.e.

$$a^*u + bv = 1.$$

It follows that $u'a^* + bv = 1$ for some $u' \in R$, i.e. $u'a_1 a + bv = 1$. In an atomic 2-fir we can apply Lemma 4·3 to obtain

PROPOSITION 4·5. *A bounded element in an atomic 2-fir can be comaximally transposed with any element totally coprime to it.*

Let a be any element in R and take an atomic factorization

$$a = p_1 p_2 \cdots p_r. \tag{4·10}$$

Suppose that a bounded atom occurs in this factorization, say $p_{i_1}, \ldots, p_{i_k}$ are all bounded similar atoms, while the remaining atoms in (4·10) are not similar to p_{i_1}. By a repeated application of Proposition 4·5 we can write a as

$$a = p'_{i_1} \cdots p'_{i_k} p'_{j_1} \cdots p'_{j_h},$$

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

where the p_i' but not the p_j', are similar to p_{i_1}. Bracketing the first k factors together, and the last h, we have $a = bc$, where b, c are totally coprime and b, like p_{i_1}, is bounded. Applying Proposition 4·5 again, we have $a = c'b'$, where b' is similar to b and c' is similar to c. Therefore $a = bc = c'b'$ leads to a direct decomposition of a

$$aR = bR \cap c'R.$$

If we repeat the process with a replaced by c', we eventually reach a direct decomposition of a into products of pairwise similar bounded atoms and a term containing no bounded atomic factors. Let us call an element *totally unbounded* if no atom occurring as factor is bounded. Then we can state our result as

THEOREM 4·6. *Any element a of an atomic 2-fir R has a direct decomposition*

$$aR = q_1R \cap \ldots \cap q_kR \cap uR, \tag{4·11}$$

where each q_i is a product of similar bounded atoms, while atoms in different q's are dissimilar, and u is totally unbounded. Moreover, the q_i and u are unique up to right associates.

Only the uniqueness remains to be proved. But (4·11) corresponds to a direct decomposition

$$R/aR \cong R/q_1R \oplus \ldots \oplus R/q_kR \oplus R/uR.$$

Here the R/q_iR are uniquely determined as the homogeneous components corresponding to a given $\mathscr{C}$-simple bounded isomorphism type, while R/uR contains all $\mathscr{C}$-simple submodules of unbounded isomorphism type.

Note that neither u nor the q_i are in general indecomposable, in the sense defined in (5).

5. *Decomposition of bounded elements in 2-firs.* Let R be a 2-fir and $\mathbf{I} = \mathbf{I}(R)$ the semigroup of its invariant elements. Any $c \in \mathbf{I}$ is said to be *I-decomposable* if it has a factorization

$$c = ab \tag{5·1}$$

into non-unit invariant elements a, b which are left (and hence also right) coprime; otherwise c is *I-indecomposable.* This corresponds to the notion of decomposability defined in (5), (6), since (5·1), for invariant elements, a, b entails $ab = ba'$, $ab = b'a$. From Theorem 3·3 it is clear that the I-indecomposable elements in an atomic 2-fir are just the powers of I-atoms and their associates. From another point of view, an invariant element c in a 2-fir R is I-indecomposable if and only if the ring R/cR cannot be written as a direct product of two non-zero rings, i.e. if and only if it contains no central idempotent apart from 0 and 1. (This follows from Proposition 5·6 of (5).)

The relation between bounded indecomposable elements and I-indecomposable elements is given by

PROPOSITION 5·1. *Let R be an atomic 2-fir. Then*

(i) *an invariant element is I-indecomposable if and only if it is a power of an I-atom,*

(ii) *if $q \in R$ is bounded and indecomposable then its bound is I-indecomposable, hence q is a product of similar atoms.*

Proof. (i) follows by Theorem 3·3. To prove (ii), let q be bounded, with bound q^*. Since q^*R annihilates $M = R/qR$, we may regard M as Q-module, where $Q = R/q^*R$.

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

If q^* is I-decomposable, Q is a non-trivial direct product and hence contains a central idempotent $e \neq 0, 1$. But then $M = Me \oplus M(1-e)$ is a direct decomposition of M into strictly cyclic modules, which contradicts the indecomposability of q. Hence q^* is I-indecomposable and so is a power of an I-atom, p^* say. Any atomic factor p of p^* has p^* as its bound and so p^* is a product of factors similar to p. The same holds for q^* and hence for q.

Of course an *un*bounded indecomposable element need not be a product of similar atoms, e.g. take xy in the free algebra on x and y. Further, the converse of Proposition 5·1 (ii) is false, i.e. a decomposable bounded element need not have an I-decomposable bound; thus in Proposition 5·1 (ii), q^* itself may well be decomposable.

Let q^* be an I-indecomposable element in an atomic 2-fir R, say $q^* = p^{*e}, l(p^*) = h$. By Proposition 4·2, $R/p^*R \cong K_h$, where the skew field K is the eigenring of an atomic factor of p^*. Now $R/p^*R \cong S/\mathbf{J}(S)$, where $S = R/q^*R$ and $\mathbf{J}(S)$ is the Jacobson radical of S. Since S is Artinian (Theorem 3·4), the matrix units can be lifted from R/p^*R to R/q^*R (cf. (11), p. 56), whence $R/q^*R \cong L_h$ where $L/\mathbf{J}(L) \cong K$, i.e. L is an Artinian local ring, or in other words a completely primary ring.

Now take a complete direct decomposition of R/q^*R as right R-module. The summands are necessarily again strictly cyclic; thus we have

$$R/q^*R = R/q_1R \oplus \ldots \oplus R/q_kR. \tag{5·2}$$

But since $R/q^*R \cong L_h$ has a complete direct decomposition into h isomorphic right ideals, we see that $k = h$ and all the R/q_iR are isomorphic. Thus $R/q^*R \cong (R/qR)^h$, as right R-modules, for some indecomposable q. Since $l(q^*) = eh = hl(q)$, we see that $l(q) = e$. The result may be summed up as

THEOREM 5·2. *In an atomic 2-fir R, let q be bounded indecomposable, with bound q^*. Express q^* as the power of an I-atom, say $q^* = p^{*e}$, then $l(q) = e$, and if $l(p^*) = h$, then*

$$R/q^*R \cong (R/qR)^h, \tag{5·3}$$

as right R-modules, while as ring, R/q^*R is a total matrix ring over a completely primary ring:

$$R/q^*R \cong L_h, \quad \text{where} \quad L = \mathrm{End}_R(R/qR). \tag{5·4}$$

If q, q^* are as in Theorem 5·2, then q^*R is determined by the similarity class of q as the annihilator of R/qR, while R/qR is determined by q^*R as an indecomposable part of R/q^*R. Hence we have the

COROLLARY. *In an atomic 2-fir R, two bounded indecomposable elements have the same bound if and only if they are similar.*

There remains the question of deciding when a given product of similar bounded atoms is indecomposable. Let p^* be any I-atom; its length h is called the *capacity* of p. For any integer $e \geqslant 0$ we have by Theorem 5·2, on decomposing p^{*e},

$$R/p^{*e}R \cong (R/q_eR)^h$$

for some indecomposable element q_e of length e. Thus q_e is a product of e atomic factors which are all similar.

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

Conversely, if p is a bounded atom with bound p^*, and p_i is similar to p for $i = 1,\ldots,e$, then $q = p_1 \ldots p_e$ is bounded by p^{*e}, and if its exact bound is p^{*e}, then q is indecomposable. For if q could be decomposed it would have a smaller bound. This proves

PROPOSITION 5·3. *In an atomic 2-fir R, each bounded indecomposable element is a product of similar atoms. Let p be a bounded atom and p^* its bound. Then a product $q = p_1 \ldots p_e$ of atoms similar to p is indecomposable if and only if p^{*e} is the exact bound of q. Moreover, for any e, there is a bounded indecomposable element q_e of length e such that*

$$R/p^{*e}R \cong (R/q_eR)^h, \quad \text{where} \quad h = l(p^*).$$

COROLLARY 1. *In an atomic 2-fir, if a has the bound p^{*e}, where p^* is an I-atom, then a direct decomposition of a has at most $l(p^*)$ terms.*

If an element a has bound p^{*e} and b has bound p^{*f}, then ab is bounded by p^{*e+f}. Applying this remark to a product of similar bounded atoms, we obtain

COROLLARY 2. *Any factor of a bounded indecomposable element is again bounded indecomposable.*

Now a product of two atoms p_1p_2 is indecomposable if and only if they cannot be comaximally transposed. For any coprime equation $p_1p_2 = q_1q_2$ gives rise to a comaximal transposition. Hence any product $q = p_1 \ldots p_n$ of bounded atoms has no indecomposable factors apart from atoms if and only if no pair of adjacent factors can be comaximally transposed, i.e. R/qR has only one composition series. Such an element q is said to be *rigid*. By Proposition 5·3, two similar atoms p_1, p_2 with common bound p^* are comaximally transposable if and only if p_1p_2 does not divide p^*. Thus we have

COROLLARY 3. *Let R be an atomic 2-fir and $q = p_1 \ldots p_e$ a bounded element in R (where the p_i are atoms), then q is rigid if and only if the p_i have a common bound p^* and*

$$p_{i-1}p_i \nmid p^* \quad (i = 2, \ldots, e).$$

In particular, p^r is rigid if and only if $p^2 \nmid p^*$.

We also have the following connexion with one-sided indecomposable elements (c.f. (5)), noted in the case of principal ideal domains by Feller (9).

COROLLARY 4. *A bounded element in an atomic 2-fir is indecomposable if and only if it is right (or left) indecomposable.*

6. *Embeddings in matrix rings over skew fields.* Let R be an atomic 2-fir and c an I-atom of R. Then $\bigcap c^nR = 0$, by unique factorization, so we can form the completion $\hat{R}$, consisting of all formal power series $\Sigma c^n a_n$, where the a_n are taken from a fixed transversal of cR in R. Clearly R is embedded in $\hat{R}$, and

$$\hat{R}/c\hat{R} \cong R/cR \cong k_r, \tag{6·1}$$

by Proposition 4·2, where k is a skew field, namely the eigenring of an atom dividing c, and $r = l(c)$. By (6·1), $\hat{R}/c\hat{R}$ contains a matrix basis (e_{ij}) which can be lifted to one in $\hat{R}$ (cf. (11), p. 54). It follows that

$$\hat{R} \cong L_r, \tag{6·2}$$

where L, the centralizer of the lifted matrix basis in $\hat{R}$, is a ring which modulo its Jacobson radical is k, i.e. L is a local ring with residue class sfield k.

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

Let L be any local ring with maximal ideal $\mathfrak{m}$ and residue class sfield $k = L/\mathfrak{m}$, then $V = \mathfrak{m}/\mathfrak{m}^2$ is a right k-space, of dimension d say. Consider the total matrix ring L_r; its radical is $\mathfrak{m}_r$ and $\mathfrak{m}_r/\mathfrak{m}_r^2 = V_r$ may be regarded as a right k_r-module. As such it is free, of rank d, as is easily verified. We apply this remark to the above situation: in L_r the radical modulo its square is $cR/c^2R \cong ck_r$; this is free of rank 1, and it follows that the radical $\mathfrak{m}$ of L is principal modulo $\mathfrak{m}^2$, say $\mathfrak{m} = aL + \mathfrak{m}^2$. Now $\mathfrak{m}$ is finitely generated as right ideal, because it corresponds to rad $(\hat{R})$ which is finitely generated. Hence by Nakayama's lemma, $\mathfrak{m}$ is principal, i.e. $\mathfrak{m} = aL$. Since rad $(\hat{R})$ is not nilpotent, neither is $\mathfrak{m}$ and in fact L is complete with respect to the $\mathfrak{m}$-adictopology because $\hat{R}$ is complete. We shall show that L is a *discrete valuation ring* (i.e. L is an integral domain in which every left or right ideal is two-sided, of the form a^nL for some n). Let $\mathfrak{a}$ be a non-zero right ideal of L, then there is a largest n such that $\mathfrak{a} \subseteq a^nL$. Thus there exists $u \in L$ such that $a^nu \in \mathfrak{a}$ and $u \notin aL$. Therefore u is a unit and $\mathfrak{a} = a^nL$, This shows that every right ideal has the form a^nL, in particular, every non-zero element has the form a^nu for some n and some unit u. Moreover, $a^n \neq 0$ for all n, because $a^nL = \mathfrak{m}^n$. Thus a is not a left zero-divisor and L is an integral domain, by an easy argument. Now by symmetry, $\mathfrak{m} = aL = Lb$, so by Proposition 3·2, $aL = La$, and repeating the argument for a left ideal $\mathfrak{a}$, we have the result. Clearly L is a right Ore ring and so has a skew field of fractions.

If we form rings of fractions on both sides of (6·2), we obtain on the left the ring $\hat{R}(c^{-1})$ of all Laurent series in c: $\Sigma c^i a_i$, while on the right we get K_r, where K is the skew field of fractions of L:

$$\hat{R}(c^{-1}) \cong K_r. \tag{6·3}$$

As an example to illustrate the above situation consider $R = \mathbf{H}[x]$, the ring of polynomials on one indeterminate over the quaternions. This is a principal ideal domain, and x^2+1 is an I-atom of capacity 2, as the factorization $x^2+1 = (x+i)(x-i)$ shows. It is easily seen that $R/(x^2+1)R \cong \mathbf{C}_2$, where $\mathbf{C}$ is the field of complex numbers. We note in passing that if we form the localization of R at the set of elements congruent to 1 (mod $(x^2+1)R$), we obtain a ring S which modulo its Jacobson radical is simple Artinian, i.e. a matrix ring over a sfield, even though the matrix units cannot be lifted to S, since S itself is an integral domain.

Returning to the general situation of an atomic 2-fir R with an I-atom, we may enquire (in view of (6.3)) whether R itself is embeddable in a skew field. This does not seem to be easy to answer, but at least we can show that every non-zero element of R has an inverse in $\hat{R}(c^{-1})$. This follows from

LEMMA 6·1. *Let $f = a_0 + ca_1 + c^2a_2 + \ldots \in \hat{R}$ be such that a_0 has no left or right factor similar to a factor of c. Then f is a unit in $\hat{R}$.*

Proof. By hypothesis a_0 and c are left coprime, and since c is invariant, they are right comaximal, say $a_0x + cy = 1$. Hence $fx = a_0x + cg = 1 - cy + cg$, for some $g \in \hat{R}$. This is congruent to 1 (mod $c\hat{R}$) and so is a unit. Thus $fh = 1$ for some h, and similarly $h'f = 1$ for some h' and this shows f to be a unit.

We observe that if $a \in R$ is left and right coprime to c then it satisfies the conditions of the lemma and so has an inverse. Now any $a \in R$ can be written in the form

$$a = p_1p_2 \ldots p_s f q_1 q_2 \ldots q_t,$$

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press

where p_i, q_j are divisors of c and hence units in $\hat{R}(c^{-1})$, and f has no left or right factor dividing c. By the lemma, f is a unit in $\hat{R}(c^{-1})$ and hence so is a itself. The result may be summarized as

THEOREM 6·2. *Let R be an atomic 2-fir with a non-unit invariant element. Then there exists a skew field K and an integer r such that R is embeddable in K_r, in such a way that all non-zero elements of R become units in K_r.*

In particular, this provides an embedding of the non-zero elements of R in a group. Further, we observe that Theorem 6·2 would prove R to be embeddable in a skew field if the following conjecture could be proved:

Let K be a skew field and r an integer. Then any ring of $r \times r$ matrices over K whose non-zero elements are invertible in K_r can be embedded in a skew field.

REFERENCES

(1) AMITSUR, S. A. A generalization of a theorem on differential equations. *Bull. Amer. Math. Soc.* **54** (1948), 937–941.

(2) ASANO, K. *Nichtkommutative Hauptidealringe* (Paris, 1938).

(3) BERGMAN, G. M. and COHN, P. M. The centers of 2-firs and hereditary rings (to appear).

(4) CARCANAGUE, J. Anneaux de polynomes gauches (to appear.)

(5) COHN, P. M. Noncommutative unique factorization domains. *Trans. Amer. Math. Soc.* **109** (1963), 313–331.

(6) COHN, P. M. Free associative algebras. *Bull. London Math. Soc.* **1** (1969), 1–39.

(7) COHN, P. M. Factorization in general rings and strictly cyclic modules. *J. Reine Angew. Math.* **239/240** (1970) 185–200.

(8) CURTIS, C. W. and REINER, I. *Representation theory of finite groups and associative algebras* (New York–London, 1962).

(9) FELLER, E. H. Intersection irreducible ideals of a non-commutative principal ideal domain. *Canad. J. Math.* **12** (1960), 592–596.

(10) JACOBSON, N. *Theory of rings* (New York, 1943).

(11) JACOBSON, N. *Structure of rings* (Providence, 1956, 1964).

(12) ORE, O. Formale Theorie der linearen Differentialgleichungen. *J. Reine Angew. Math.* **167** (1932), 221–234, **168** (1932), 233–252.

https://doi.org/10.1017/S0305004100046375 Published online by Cambridge University Press