J. Appl. Prob. **56,** 701–722 (2019)
*doi:*10.1017/jpr.2019.42
© *Applied Probability Trust* 2019

ON FIRST EXIT TIMES AND THEIR MEANS FOR BROWNIAN BRIDGES

CHRISTEL GEISS,* ** AND
ANTTI LUOTO,* *** *University of Jyväskylä*
PAAVO SALMINEN,**** *Åbo Akademi University*

Abstract

For a Brownian bridge from 0 to y, we prove that the mean of the first exit time from the interval $(-h, h)$, $h > 0$, behaves as $O(h^2)$ when $h \downarrow 0$. Similar behaviour is also seen to hold for the three-dimensional Bessel bridge. For the Brownian bridge and three-dimensional Bessel bridge, this mean of the first exit time has a puzzling representation in terms of the Kolmogorov distribution. The result regarding the Brownian bridge is applied to provide a detailed proof of an estimate needed by Walsh to determine the convergence of the binomial tree scheme for European options.

Keywords: Bessel process; Black–Scholes model; Brownian motion; diffusion process; h-transform; last exit time; Poisson summation formula

2010 Mathematics Subject Classification: Primary 60J65; 60J60
Secondary 91G60

1. Introduction

In this paper we consider diffusion bridges and present integral representations for the mean of the first exit time from an interval. With the help of these representations, we deduce the limiting behaviour of the mean when the length of the interval around the starting value of the bridge decreases to 0. For a continuous process $(X_t)_{t\geq 0}$ with $X_0 = x$, the first exit time from the interval (a, b) is defined by

$$\mathcal{T}^X_{(a,b)} := \begin{cases} \inf\{t > 0 \colon X_t \notin (a, b)\}, \\ +\infty \quad \text{if the set in brackets is empty.} \end{cases}$$

We write simply $\mathcal{T}_{(a,b)}$ if there is no ambiguity about the underlying process. The main part of the paper focuses on Brownian and three-dimensional Bessel bridges. For the Brownian bridge $(B^{x,T,y}_t)_{0\leq t<T}$ from x to y of length T (a similar result holds for Bessel bridges), it is shown in Theorem 3.1 that

$$\lim_{h\downarrow 0} \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]}{h^2} = 1, \qquad y \neq 0, \tag{1.1}$$

where $\mathbb{E}_{x,T,y}$ stands for the expectation associated with the Brownian bridge.

Received 17 November 2017; revision received 29 March 2019.
* Postal address: Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35 (MaD), FI-40014, Finland.
** Email address: christel.geiss@jyu.fi
*** Email address: antti.k.luoto@jyu.fi
**** Postal address: Faculty of Science and Engineering, Åbo Akademi University, Domkyrkotorget 3, 20500 Åbo, Finland. Email address: paavo.salminen@abo.fi

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

Let $(W_t^x)_{t\geq 0}$ be the Brownian motion started at $x \in \mathbb{R}$. Recall that

$$\mathbb{E}_x[\mathcal{T}_{(a,b)}] = (b-x)(x-a), \qquad a < x < b \tag{1.2}$$

(this can be deduced, for example, from the Laplace transform of $\mathcal{T}_{(a,b)}$ given in [2, part II, section 1, 3.0.1]), where $\mathbb{E}_x$ denotes the expectation for the Brownian motion. Consequently, by (1.1),

$$\lim_{h\downarrow 0} \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]}{\mathbb{E}_0[\mathcal{T}_{(-h,h)}]} = 1, \qquad y \neq 0. \tag{1.3}$$

For three-dimensional Bessel bridges a similar asymptotic behaviour can be found. Our conjecture is that a relation of the form (1.3) holds for a large class of diffusion bridges. For a regular diffusion X with $X_0 = x_o$, Proposition 3.3 describes the limiting behaviour of the mean of $\mathcal{T}_{(x_o-h,x_o+h)}$. We expect the same limiting behaviour for a class of diffusion bridges, implying that (1.3) would be true; however, we have not been able to prove this so far.

The paper is organized as follows. In Section 2 we describe our setting and the construction of a bridge measure for a regular diffusion by the Doob h-transform. We also recall some properties of the Brownian bridge. Section 3 contains the main results. We start by deriving integral representations for the mean of $\mathcal{T}^X_{(a,b)}$ when X is, firstly, a general regular diffusion and, secondly, the corresponding diffusion bridge. We also calculate for a regular diffusion X, $X_0 = x$, the limiting behaviour of the mean of $\mathcal{T}^X_{(x-h,x+h)}$ as $h \to 0$. In Section 3.3 we focus on Brownian and three-dimensional Bessel bridges starting from $x > 0$ and find the limiting behaviour of the mean of $\mathcal{T}_{(x-h,x+h)}$ as $h \to 0$. The convergence in distribution of the random variables $\mathcal{T}^X_{(x-h,x+h)}/h^2$ is discussed in Theorem 3.2. For the means of the first exit times – considered in Section 3.3 – we derive in Section 3.4 puzzling representations in terms of the Kolmogorov distribution function. We also discuss other properties of the Kolmogorov distribution, in particular its connection with the last exit time distribution of Brownian motion. As an application, in Section 4, we use our Brownian bridge results to give a detailed proof of an estimate in [17] needed to derive the rate of convergence of an option price – calculated from the binomial tree scheme – to the Black–Scholes price. The estimate is also used in a forthcoming paper [12].

2. Preliminaries

We start with the description of the setting. Let $C[0,\infty)$ denote the space of continuous functions $\omega : [0,\infty) \mapsto \mathbb{R}$, and let $\mathcal{C}_t := \sigma\{\omega(s) : s \leq t\}$ denote the smallest σ-algebra making the coordinate mappings up to time t measurable. Furthermore, let $\mathcal{C}$ be the smallest σ-algebra containing all $\mathcal{C}_t$, $t \geq 0$. For an interval $I \subset \mathbb{R}$, let $(\mathbb{P}^X_x)_{x\in I}$ be a family of probability measures defined in the filtered canonical space $(C[0,\infty), \mathcal{C}, (\mathcal{C}_t)_{t\geq 0})$ such that, under $\mathbb{P}^X_x$, for a given $x \in I$ the coordinate process $X = (X_t)_{t\geq 0} := (\omega_t)_{t\geq 0}$ is a regular diffusion taking values in I and starting from x. Here, X is considered in the sense of Itô and McKean [8]; see also [2]. A crucial property of X is that there exists a (speed) measure m^X such that the transition probability has a continuous strictly positive density $(t, x, y) \mapsto q_t(x, y)$, $t > 0, x, y \in I$ with respect to m^X, that is,

$$\mathbb{P}^X_x(X_t \in \mathrm{d}y) = q_t(x, y)m^X(\mathrm{d}y) \tag{2.1}$$

(see [8, pp. 149 and 157]).

For $(X_t)_{t\geq 0}$, $X_0 = x$, as defined above, and $T > 0$, we can construct a new non-homogeneous strong Markov process by conditioning X to be at a given point $y \in I$ at time T. Although the conditioning is generally with respect to a zero set $\{X_T = y\}$, it can be realized using the Bayes

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

formula and the notion of regular conditional distributions. Another approach is to apply the theory of Doob h-transforms. To explain this briefly, consider X in space–time, i.e. the process $\bar{X} = ((X_t, t))_{t\geq 0}$. For $z \in I$ and $t < T$, introduce the function

$$h(z, t) := h(z, t; y, T) := q_{T-t}(z, y).$$

By the Chapman–Kolmogorov equation, for $x \in I$ and $s < t$ we have

$$\begin{aligned}\mathbb{E}^X_{(x,s)}[h(X_t, t)] &= \int_I q_{t-s}(x, z)h(z, t)\, m(\mathrm{d}z)\\ &= \int_I q_{t-s}(x, z)q_{T-t}(z, y)\, m(\mathrm{d}z)\\ &= q_{T-s}(x, y)\\ &= h(x, s),\end{aligned}$$

where $\mathbb{E}^X_{(x,s)}$ refers to the expectation associated with the space–time process $\bar{X}$ initiated from x at time s. Consequently, for $f \in \mathcal{B}_b(I)$ (= bounded measurable functions on I) and $s < t < T$, we may define a non-homogeneous Markov semigroup $P^{X,h}_{s,t}$, $0 < s < t < T$, via

$$P^{X,h}_{s,t} f(x) := \mathbb{E}^X_x\left[f(X_{t-s})\frac{h(X_{t-s}, t)}{h(x, s)}\right]. \tag{2.2}$$

The process governed by the probability measure induced by this semigroup is called the X-bridge to y of length T. The notations $X^{x,T,y}$, $\mathbb{P}^X_{x,T,y}$, and $\mathbb{E}^X_{x,T,y}$ are used for this process, its probability measure, and the expectation, respectively, when the initial state is $x \in I$. From (2.2) one may deduce the following absolute continuity relation for $A_t \in \mathcal{C}_t$, $t < T$:

$$\mathbb{P}^X_{x,T,y}(A_t) = \mathbb{E}^X_x\left[\frac{h(X_t, t)}{h(x, 0)}\,; A_t\right] = \mathbb{E}^X_x\left[\frac{q_{T-t}(X_t, y)}{q_T(x, y)}\,; A_t\right]. \tag{2.3}$$

We refer to Chung and Walsh [4] for a general discussion of h-transforms, Fitzsimmons, Pitman, and Yor [6] (in particular Proposition 1) for general Markovian bridges, and Salminen [14] for some properties of diffusion bridges.

Brownian bridges enjoy several equivalent characterizations in addition to the general h-transform approach. Indeed, a Brownian bridge can be viewed as (i) a Gaussian process, (ii) a deterministic time change with an additional drift of standard Brownian motion, or (iii) as the solution of an SDE; see e.g. [2, p. 66]. In Section 4 we apply (ii) and (iii) in particular. Indeed, (ii) states that

$$B^{x,T,y}_t \overset{d}{=} \begin{cases}\left(1 - \dfrac{t}{T}\right)W_{Tt/(T-t)} + x + \dfrac{(y-x)t}{T} & t < T,\\ y & t = T,\end{cases} \tag{2.4}$$

where $\overset{d}{=}$ means that the processes on the left- and right-hand sides are identical in law, while (iii) says that

$$B^{x,T,y}_t \overset{d}{=} \begin{cases}(T-t)\displaystyle\int_0^t \frac{\mathrm{d}W_s}{T-s} + x + \frac{(y-x)t}{T} & t < T,\\ y & t = T.\end{cases} \tag{2.5}$$

Here it is assumed that the canonical filtration $(\mathcal{C}_t)_{t\geq 0}$ is augmented with the null sets of $\mathcal{C}$ with respect to $\mathbb{P}_0$, in order to have the usual conditions satisfied.

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

3. Main results

3.1. Integral representations for the mean of the first exit time

Let $X = (X_t)_{t\geq 0}$ be a regular diffusion taking values on an interval I, and recall from (2.1) the notation $q_t(x, y)$ for its transition density with respect to the speed measure m^X. For $a < b,\ a, b \in I$, let $\widehat{X}$ denote X killed at $\mathcal{T}_{(a,b)}$. Then $\widehat{X}$ is a regular diffusion on (a, b). The speed measure of $\widehat{X}$ is m^X, and $\widehat{X}$ has a continuous strictly positive transition density $\widehat{q}_t(x, y)$ such that, for $x, y \in (a, b)$,

$$\begin{aligned}\mathbb{P}^{\widehat{X}}_x(\widehat{X}_t \in \mathrm{d}y) &= \widehat{q}_t(x, y)m^X(\mathrm{d}y)\\ &= \mathbb{P}^X_x(\mathcal{T}_{(a,b)} > t, X_t \in \mathrm{d}y)\\ &= \mathbb{P}^X_x\Big(a < \inf_{0\leq s\leq t} X_s,\ \sup_{0\leq s\leq t} X_s < b, X_t \in \mathrm{d}y\Big).\end{aligned} \tag{3.1}$$

This immediately yields the following result.

Proposition 3.1. *If* $\mathbb{P}^X_x(\mathcal{T}_{(a,b)} < \infty) = 1$*, i.e. X does not die inside* (a, b)*, we obtain*

$$\mathbb{E}^X_x[\mathcal{T}_{(a,b)}] = \int_0^\infty \mathrm{d}t \int_a^b m^X(\mathrm{d}z)\,\widehat{q}_t(x, z). \tag{3.2}$$

The next proposition will serve as an important tool for the calculations below.

Proposition 3.2. *For* $x \in (a, b) \subset I$ *and* $y \in I\backslash(a, b)$*, we have*

$$\mathbb{E}^X_{x,T,y}[\mathcal{T}_{(a,b)}] = \int_0^T \mathrm{d}t \int_a^b m^X(\mathrm{d}z)\,\widehat{q}_t(x, z)\frac{q_{T-t}(z, y)}{q_T(x, y)}. \tag{3.3}$$

Proof. Since $x \in (a, b)$ and $y \in I\backslash(a, b)$, we have $\mathbb{P}^X_{x,T,y}(\mathcal{T}_{(a,b)} < T) = 1$. Consider

$$\begin{aligned}\mathbb{E}^X_{x,T,y}[\mathcal{T}_{(a,b)}] &= \int_0^T \mathrm{d}t \int_a^b \mathbb{P}^X_{x,T,y}(\mathcal{T}_{(a,b)} > t, X^{x,T,y}_t \in \mathrm{d}z)\\ &= \int_0^T \mathrm{d}t \int_a^b \mathbb{P}^X_x(\mathcal{T}_{(a,b)} > t, X_t \in \mathrm{d}z)\frac{q_{T-t}(z, y)}{q_T(x, y)},\end{aligned} \tag{3.4}$$

where the Markov property and formula (2.2) are used. To get the claimed formula (3.3), it remains to apply (3.1). □

3.2. The limit behaviour for regular diffusions

One of our main issues concerns the limiting behaviour of the mean of $\mathcal{T}^X_{(x-h,x+h)}$ as $h \to 0$ for diffusion bridges. For regular diffusions we have the following fairly complete characterization, which has also been proved in [3].

Proposition 3.3. *Assume that the differential operator associated with the regular diffusion X is given by*

$$\mathcal{G}u(x) := \frac{1}{2}a^2(x)u''(x) + b(x)u'(x), \qquad x \in I,$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

where $x \mapsto a^2(x) > 0$ and $x \mapsto b(x)$ are continuous in I. Let $x_o \in \mathrm{Int}(I)$. Then

$$\lim_{h\downarrow 0} \frac{\mathbb{E}^X_{x_o}[\mathcal{T}_{(x_o-h,x_o+h)}]}{h^2} = a^{-2}(x_o).$$

Proof. Recall from [2, part I, chapter II, no. 7, p. 17] that the speed measure m^X and the scale function s^X can be taken to be such that

$$m^X(\mathrm{d}x) = m^X(x)\,\mathrm{d}x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} s^X(x) = \mathrm{e}^{-B(x)}, \tag{3.5}$$

where

$$m^X(x) = 2a^{-2}(x)\,\mathrm{e}^{B(x)}, \qquad B(x) = \int^x 2a^{-2}(y)b(y)\,\mathrm{d}y. \tag{3.6}$$

Now consider X initiated at x_o and killed when it leaves the interval $(x_o - h, x_o + h)$. We let $(\widehat{X}_t)_{t\geq 0}$ denote this diffusion:

$$\widehat{X}_t = \begin{cases} X_t & t < \mathcal{T}_{(x_o-h,x_o+h)}, \\ \partial & t \geq \mathcal{T}_{(x_o-h,x_o+h)}, \end{cases}$$

where ∂ is a cemetery point. It is well known (see [2, part I, chapter II, no. 11]) that the 0-resolvent kernel of $\widehat{X}$ is given by

$$\widehat{G}_0(x,y) := \int_0^\infty \widehat{q}_t(x,y)\,\mathrm{d}t = \begin{cases} \dfrac{(s(x)-s(x_o-h))(s(x_o+h)-s(y))}{s(x_o+h)-s(x_o-h)} & x < y, \\ \dfrac{(s(y)-s(x_o-h))(s(x_o+h)-s(x))}{s(x_o+h)-s(x_o-h)} & x > y, \end{cases}$$

where $s := s^X$ and $\widehat{q}_t(x,y)$ is the transition density with respect to the speed measure m^X. Consequently, relation (3.2) implies that

$$\begin{aligned} \mathbb{E}^X_{x_o}[\mathcal{T}_{(x_o-h,x_o+h)}] &= \int_0^\infty \left(\int_{x_o-h}^{x_o+h} \widehat{q}_t(x_o,y) m^X(\mathrm{d}y) \right) \mathrm{d}t \\ &= \int_{x_o-h}^{x_o+h} \widehat{G}_0(x_o,y) m^X(\mathrm{d}y) \\ &= \frac{s(x_o+h)-s(x_o)}{s(x_o+h)-s(x_o-h)} \int_{x_o-h}^{x_o} (s(y)-s(x_o-h)) m^X(y)\,\mathrm{d}y \\ &\quad + \frac{s(x_o)-s(x_o-h)}{s(x_o+h)-s(x_o-h)} \int_{x_o}^{x_o+h} (s(x_o+h)-s(y)) m^X(y)\,\mathrm{d}y. \end{aligned} \tag{3.7}$$

Since s is continuously differentiable and $s'(x) > 0$ for all x (see (3.5), (3.6)), we have

$$\lim_{h\downarrow 0} \frac{s(x_o+h)-s(x_o)}{s(x_o+h)-s(x_o-h)} = \lim_{h\downarrow 0} \frac{s(x_o)-s(x_o-h)}{s(x_o+h)-s(x_o-h)} = \frac{1}{2},$$

and l'Hôpital's rule yields

$$\begin{aligned} &\lim_{h\downarrow 0} \frac{1}{h^2} \int_{x_o-h}^{x_o} (s(y)-s(x_o-h)) m^X(y)\,\mathrm{d}y \\ &\quad = \lim_{h\downarrow 0} \frac{1}{2h} \int_{x_o-h}^{x_o} \left(-\frac{\mathrm{d}}{\mathrm{d}h} s(x_o-h) \right) m^X(y)\,\mathrm{d}y \end{aligned}$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

$$= \lim_{h\downarrow 0}\left(-\frac{\mathrm{d}}{\mathrm{d}h}s(x_o-h)\right)\frac{1}{2h}\int_{x_o-h}^{x_o} m^X(y)\,\mathrm{d}y$$
$$= \frac{1}{2}s'(x_o)m^X(x_o)$$
$$= a^{-2}(x_o),$$

where the last equality follows from relations (3.5) and (3.6). Similarly,

$$\lim_{h\downarrow 0}\frac{1}{h^2}\int_{x_o}^{x_o+h}(s(x_o+h)-s(y))m^X(y)\,\mathrm{d}y = a^{-2}(x_o).$$

Hence, by (3.7),

$$\lim_{h\downarrow 0}\frac{\mathbb{E}^X_{x_o}[\mathcal{T}_{(x_o-h,x_o+h)}]}{h^2} = a^{-2}(x_o).$$

□

3.3. The mean of the first exit time for the Brownian bridge and three-dimensional Bessel bridge

Let

$$p_t(x,y) := \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-(y-x)^2/(2t)}$$

denote the transition density of the standard Brownian motion and let

$$r_t^{(3)}(x,y) := \frac{y}{x}(p_t(x,y)-p_t(x,-y)), \qquad x,y>0 \tag{3.8}$$

be the transition density of the three-dimensional Bessel process. We also introduce the function

$$\Delta(z,h,t) := \frac{1}{\sqrt{2\pi t}}\sum_{m=-\infty}^{\infty}(\mathrm{e}^{-(z+4mh)^2/(2t)}-\mathrm{e}^{-(z+(2m+1)2h)^2/(2t)})$$
$$= \frac{1}{\sqrt{2\pi t}}\sum_{m=-\infty}^{\infty}(-1)^m\,\mathrm{e}^{-(z-2mh)^2/(2t)}$$
$$= p_t(0,z)\sum_{m=-\infty}^{\infty}(-1)^m\,\mathrm{e}^{-2mh(mh-z)/t}, \qquad t>0. \tag{3.9}$$

Then, by [2, part II, section 1, 1.15.8 (p. 180)] and (3.9),

$$\mathbb{P}_0\Big(\inf_{0\le s\le t}W_s>-h,\ \sup_{0\le s\le t}W_s<h,\ W_t\in\mathrm{d}z\Big) = \mathbb{P}_0(\mathcal{T}_{(-h,h)}>t,\ W_t\in\mathrm{d}z)$$
$$= \Delta(z,h,t)\,\mathrm{d}z. \tag{3.10}$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

In addition, according to [2, part II, section 5, 1.15.8], for $z \in (x-h, x+h)$ we find that

$$\begin{aligned}
&\mathbb{P}_x^{(3)}\Big(\sup_{0\le s\le t}|X_s - x| < h,\, X_t \in \mathrm{d}z\Big) \\
&\quad = \frac{z}{x\sqrt{2\pi t}} \sum_{m=-\infty}^{\infty} \big(\mathrm{e}^{-((z-x)+2h(2m))^2/(2t)} - \mathrm{e}^{-((z-x)+2h(2m+1))^2/(2t)}\big)\,\mathrm{d}z \\
&\quad = \frac{z}{x}\Delta(z-x, h, t)\,\mathrm{d}z, \qquad x > 0,
\end{aligned} \tag{3.11}$$

where $\mathbb{P}^{(3)}$ stands for the probability measure associated with the three-dimensional Bessel process. Finally, let us recall that by [2, part II, section 1, 3.0.6(a)]

$$\mathbb{P}_x(\mathcal{T}_a \in \mathrm{d}t,\, \mathcal{T}_a < \mathcal{T}_b) = \mathrm{ss}_t(b-x, b-a)\,\mathrm{d}t, \qquad a < x < b, \tag{3.12}$$

where the special function ss_t is given by (see [2, appendix 2, no. 13])

$$\mathrm{ss}_t(a, b) := \mathcal{L}_{t,\gamma}^{-1}\left(\frac{\sinh(a\sqrt{2\gamma})}{\sinh(b\sqrt{2\gamma})}\right) = \sum_{m=-\infty}^{\infty} \frac{b-a+2mb}{\sqrt{2\pi t^3}}\,\mathrm{e}^{-(b-a+2mb)^2/(2t)}, \quad a < b, \tag{3.13}$$

and $\mathcal{L}_{t,\gamma}^{-1}$ denotes the inverse Laplace transform.

Theorem 3.1. (i) *For the Brownian bridge with $|y| \ge h$,*

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = \int_0^T \int_{-h}^{h} \frac{p_{T-t}(z, y)}{p_T(0, y)}\Delta(z, h, t)\,\mathrm{d}z\,\mathrm{d}t \tag{3.14}$$

and

$$\lim_{h\downarrow 0} \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]}{h^2} = 1, \qquad y \neq 0. \tag{3.15}$$

(ii) *For the three-dimensional Bessel bridge with $x > h$ and $y \notin (x-h, x+h)$,*

$$\mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] = \int_0^T \int_{-h}^{h} \frac{z+x}{x}\Delta(z, h, t)\frac{r_{T-t}^{(3)}(z+x, y)}{r_T^{(3)}(x, y)}\,\mathrm{d}z\,\mathrm{d}t$$

and

$$\lim_{h\downarrow 0} \frac{\mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}]}{h^2} = 1. \tag{3.16}$$

(iii) *If $y \in (-h, h)$ (or $y \in (x-h, x+h)$ with $x > h$, $y > 0$), the mean of the first exit time is infinite for both bridges, that is,*

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = \mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] = \infty. \tag{3.17}$$

Proof. (i) According to (3.4), (3.1), and (3.10),

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = \int_0^T \int_{-h}^{h} \frac{p_{T-t}(z, y)}{p_T(0, y)}\Delta(z, h, t)\,\mathrm{d}z\,\mathrm{d}t.$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

To show (3.15), substitute $z = hu$ and $t = h^2 s$, and note that by the scaling property we have $\Delta(hu, h, h^2 s)h = \Delta(u, 1, s)$, so that

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = h^2 \int_0^{T/h^2} \int_{-1}^{1} \frac{p_{T-h^2 s}(hu, y)}{p_T(0, y)} \Delta(u, 1, s)\, \mathrm{d}u\, \mathrm{d}s.$$

To apply dominated convergence for $h \downarrow 0$, let y be fixed and assume $h < |y|/2$. Then, for any $u \in (-1, 1)$ we have $|y - hu| > |y| - h > |y|/2$. Moreover, note that for $y \neq 0$ there exists a constant $C > 0$ such that

$$\sup_{t>0} p_t(0, y) \leq \frac{C}{|y|}. \tag{3.18}$$

This implies that

$$\sup_{u\in(-1,1),\, s\in(0,T/h^2)} \frac{p_{T-h^2 s}(hu, y)}{p_T(0, y)} \leq \frac{2\, C}{|y| p_T(0, y)}.$$

Therefore, as $(u, s) \mapsto \Delta(u, 1, s)$ is integrable and $(u, s, h) \mapsto p_{T-h^2 s}(hu, y)/p_T(0, y)$ is bounded on $(-1, 1) \times (0, T/h^2) \times (0, |y|/2)$, dominated convergence yields

$$\begin{aligned} \lim_{h\downarrow 0} \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]}{h^2} &= \int_0^{\infty} \int_{-1}^{1} \Delta(u, 1, s)\, \mathrm{d}u\, \mathrm{d}s \\ &= \int_0^{\infty} \int_{-1}^{1} \mathbb{P}_0(\mathcal{T}_{(-1,1)} > s,\, W_s \in \mathrm{d}u)\, \mathrm{d}s \\ &= \mathbb{E}_0[\mathcal{T}_{(-1,1)}] \\ &= 1, \end{aligned} \tag{3.19}$$

where (1.2) is used for the last equality.

(ii) By (3.4), (3.1), and (3.11),

$$\mathbb{E}^{(3)}_{x,T,y}[\mathcal{T}_{(x-h,x+h)}] = \int_0^T \int_{x-h}^{x+h} \frac{z}{x} \Delta(z - x, h, t) \frac{r^{(3)}_{T-t}(z, y)}{r^{(3)}_T(x, y)}\, \mathrm{d}z\, \mathrm{d}t. \tag{3.20}$$

For the proof of (3.16) we substitute $t = h^2 s$ and $z = hu + x$ and recall that $\Delta(hu, h, h^2 s)h = \Delta(u, 1, s)$, so that

$$\mathbb{E}^{(3)}_{x,T,y}[\mathcal{T}_{(x-h,x+h)}] = h^2 \int_0^{T/h^2} \int_{-1}^{1} \frac{hu + x}{x} \Delta(u, 1, s) \frac{r^{(3)}_{T-h^2 s}(hu + x, y)}{r^{(3)}_T(x, y)}\, \mathrm{d}u\, \mathrm{d}s.$$

Using relation (3.8) yields

$$\left| \frac{hu + x}{x} \frac{r^{(3)}_{T-h^2 s}(hu + x, y)}{r^{(3)}_T(x, y)} \right| = \left| \frac{p_{T-h^2 s}(hu + x, y) - p_{T-h^2 s}(hu + x, -y)}{p_T(x, y) - p_T(x, -y)} \right|.$$

To see that this expression is bounded in $s \in (0, T/h^2)$ and $u \in (-1, 1)$, note that if $h < |y - x|/2$ with $x > h$ and $y > 0$, then

$$|x - y + hu| > |x - y| - h > |x - y|/2 \quad \text{and} \quad |x + y + hu| > x + y - h > |x - y|/2,$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

so that, by (3.18)

$$|p_{T-h^2 s}(hu+x, y) - p_{T-h^2 s}(hu+x, -y)| \le \frac{C}{|x-y+hu|} + \frac{C}{|x+y+hu|} < \frac{4C}{|x-y|}.$$

Hence, dominated convergence gives

$$\lim_{h\downarrow 0} \frac{\mathbb{E}^{(3)}_{x,T,y}[\mathcal{T}_{(x-h,x+h)}]}{h^2} = \int_0^\infty \int_{-1}^1 \Delta(u, 1, s)\, \mathrm{d}u\, \mathrm{d}s = 1,$$

where the last equality was shown in (3.19).

(iii) For a regular diffusion X, (3.17) follows from $\mathbb{P}^X_{x,T,y}(\mathcal{T}_{(x-h,x+h)} \ge T) > 0$. To see that this holds, consider

$$\mathbb{P}^X_{x,T,y}(\mathcal{T}_{(x-h,x+h)} \ge T) = \lim_{\varepsilon\downarrow 0} \frac{\mathbb{P}^X_x(\mathcal{T}_{(x-h,x+h)} \ge T,\, y \le X_T \le y+\varepsilon)}{\mathbb{P}^X_x(y \le X_T \le y+\varepsilon)} \tag{3.21}$$

and

$$\mathbb{P}^X_x(\mathcal{T}_{(x-h,x+h)} \ge T,\, y \le X_T \le y+\varepsilon) = \int_y^{y+\varepsilon} \widehat{q}_T(x, z)\, \mathrm{d}z,$$

where $\widehat{q}$ is the transition density of X killed at $\mathcal{T}_{(x-h,x+h)}$. By [8, p. 157], the transition density of any regular diffusion is strictly positive. In our case this means

$$\widehat{q}_T(x, z) > 0 \quad \text{for all } |z| < h.$$

Using l'Hôpital's rule in (3.21) yields

$$\mathbb{P}^X_{x,T,y}(\mathcal{T}_{(x-h,x+h)} \ge T) = \frac{\widehat{q}_T(x, y)}{q_T(x, y)} > 0.$$

□

We also have the following result on convergence in distribution.

Theorem 3.2. *For each $y \neq 0$ and $T > 0$,*

$$\lim_{h\downarrow 0} \mathbb{E}_{0,T,y}[\mathrm{e}^{-\gamma \mathcal{T}_{(-h,h)}/h^2}] = \mathbb{E}_0[\mathrm{e}^{-\gamma \mathcal{T}_{(-1,1)}}], \qquad \gamma > 0, \tag{3.22}$$

and, for each x, $y > 0$,

$$\lim_{h\downarrow 0} \mathbb{E}^{(3)}_{x,T,y}[\mathrm{e}^{-\gamma \mathcal{T}_{(x-h,x+h)}/h^2}] = \lim_{h\downarrow 0} \mathbb{E}_{x,T,y}[\mathrm{e}^{-\gamma \mathcal{T}_{(x-h,x+h)}/h^2}] = \mathbb{E}_0[\mathrm{e}^{-\gamma \mathcal{T}_{(-1,1)}}], \qquad \gamma > 0.$$

Proof. We will sketch the proof of (3.22) since the other result follows similarly. Let

$$\mathcal{T}^X_c := \inf\{t > 0 \colon X_t = c\}$$

denote the first hitting time of c by a continuous process X. Since

$$\mathcal{T}_{(a,b)} = \mathcal{T}_a \wedge \mathcal{T}_b = \mathcal{T}_a \mathbf{1}_{\{\mathcal{T}_a < \mathcal{T}_b\}} + \mathcal{T}_b \mathbf{1}_{\{\mathcal{T}_a < \mathcal{T}_b\}},$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

it suffices to consider $\mathbb{E}_{x,T,y}[\mathbf{1}_{\{\mathcal{T}_a<\mathcal{T}_b\}}\,\mathrm{e}^{-\gamma\mathcal{T}_a}]$. We recall that by (2.3)

$$Z_t(\omega) := \frac{\mathrm{d}\mathbb{P}_{x,T,y}}{\mathrm{d}\mathbb{P}_x}\bigg|_{\mathcal{C}_{t+}}(\omega) = \frac{p_{T-t}(\omega(t),y)}{p_T(x,y)}, \qquad t<T.$$

Since $(Z_t)_{t\in[0,T)}$ is a $(\mathbb{P}_x,(\mathcal{C}_t)_{t\in[0,T)})$-martingale and the random variable $\mathbf{1}_{\{\mathcal{T}_a\le t,\mathcal{T}_a<\mathcal{T}_b\}}\,\mathrm{e}^{-\gamma\mathcal{T}_a}$ (with $\gamma>0$) is $\mathcal{C}_{\mathcal{T}_a\wedge t}$ -measurable, we conclude for $a<x<b$ that

$$\begin{aligned}
\mathbb{E}_{x,T,y}[\mathbf{1}_{\{\mathcal{T}_a\le t,\mathcal{T}_a<\mathcal{T}_b\}}\,\mathrm{e}^{-\gamma\mathcal{T}_a}] &= \mathbb{E}_x[\mathbf{1}_{\{\mathcal{T}_a\le t,\mathcal{T}_a<\mathcal{T}_b\}}\,\mathrm{e}^{-\gamma\mathcal{T}_a}Z_t]\\
&= \mathbb{E}_x[\mathbb{E}_x[\mathbf{1}_{\{\mathcal{T}_a\le t,\mathcal{T}_a<\mathcal{T}_b\}}\,\mathrm{e}^{-\gamma\mathcal{T}_a}Z_t \mid \mathcal{F}_{\mathcal{T}_a\wedge t}]]\\
&= \mathbb{E}_x[\mathbf{1}_{\{\mathcal{T}_a\le t,\mathcal{T}_a<\mathcal{T}_b\}}\,\mathrm{e}^{-\gamma\mathcal{T}_a}Z_{\mathcal{T}_a\wedge t}]\\
&= \mathbb{E}_x\left[\mathbf{1}_{\{\mathcal{T}_a\le t,\mathcal{T}_a<\mathcal{T}_b\}}\,\mathrm{e}^{-\gamma\mathcal{T}_a}\frac{p_{T-\mathcal{T}_a\wedge t}(W_{\mathcal{T}_a\wedge t},y)}{p_T(x,y)}\right]\\
&= \int_0^t \mathrm{e}^{-\gamma r}\frac{p_{T-r}(a,y)}{p_T(x,y)}\mathbb{P}_x(\mathcal{T}_a\in\mathrm{d}r,\,\mathcal{T}_a<\mathcal{T}_b).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}_{0,T,y}[\mathrm{e}^{-\gamma\mathcal{T}_{(-h,h)}/h^2}]\\
&\quad= \int_0^T \mathrm{e}^{-\gamma r/h^2}\left[\frac{p_{T-r}(-h,y)}{p_T(0,y)}\mathbb{P}_0(\mathcal{T}_{-h}\in\mathrm{d}r,\,\mathcal{T}_{-h}<\mathcal{T}_h)+\frac{p_{T-r}(h,y)}{p_T(0,y)}\mathbb{P}_0(\mathcal{T}_h\in\mathrm{d}r,\,\mathcal{T}_h<\mathcal{T}_{-h})\right].
\end{aligned}$$

The result follows by the variable transform $t=r/h^2$, the scaling property of the Brownian motion and dominated convergence as $h\downarrow 0$. □

Finally we consider the behaviour of the expected hitting time in the case of a three-dimensional Bessel bridge starting at 0.

Theorem 3.3. *For the three-dimensional Bessel bridge starting at* 0 *with* $y>h$,

$$\mathbb{E}^{(3)}_{0,T,y}[\mathcal{T}_h] = \int_0^T\int_0^h \mathbb{P}^{(3)}_0\left(\sup_{0\le s\le t}X_s<h,\,X_t\in\mathrm{d}z\right)\frac{r^{(3)}_{T-t}(z,y)}{r^{(3)}_T(0,y)}\,\mathrm{d}t \tag{3.23}$$

and

$$\lim_{h\downarrow0}\frac{\mathbb{E}^{(3)}_{0,T,y}[\mathcal{T}_h]}{h^2} = \lim_{h\downarrow0}\frac{\mathbb{E}^{(3)}_0[\mathcal{T}_h]}{h^2} = \frac{1}{3}, \qquad y>0.$$

Proof. The representation (3.23) follows from (3.1) and (3.3), which can be extended to $x=0$ since 0 is an entrance boundary point. Indeed, by [2, part II, section 5, 1.1.8 (p. 435)], for $0<x\vee z<h$ we have

$$\mathbb{P}^{(3)}_x\left(\sup_{0\le s\le t}X_s<h,\,X_t\in\mathrm{d}z\right) = \frac{z}{x\sqrt{2\pi\,t}}\sum_{m=-\infty}^{\infty}(\mathrm{e}^{-(z-x+2mh)^2/(2t)}-\mathrm{e}^{-(z+x+2mh)^2/(2t)})\,\mathrm{d}z$$

which implies, as $x\downarrow0$,

$$\mathbb{P}^{(3)}_0\left(\sup_{0\le s\le t}X_s<h,\,X_t\in\mathrm{d}z\right) = 2z\mathrm{ss}_t(h-z,h)\,\mathrm{d}z$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

with ss_t as defined in (3.13). Then, relations (3.23) and (3.12) yield

$$\begin{aligned}\lim_{h\downarrow 0}\frac{\mathbb{E}^{(3)}_{0,T,y}[\mathcal{T}_h]}{h^2} &= 2\int_0^1 u\int_0^\infty \mathbb{P}_u(\mathcal{T}_0\in \mathrm{d}r,\,\mathcal{T}_0<\mathcal{T}_1)\,\mathrm{d}u\\ &= 2\int_0^1 u\mathbb{P}_u(\mathcal{T}_0<\mathcal{T}_1)\,\mathrm{d}u\\ &= 2\int_0^1 u(1-u)\,\mathrm{d}u\\ &= \frac{1}{3}.\end{aligned}$$

To check that

$$\lim_{h\downarrow 0}\frac{\mathbb{E}^{(3)}_{0}[\mathcal{T}_h]}{h^2} = \frac{1}{3},$$

we differentiate the Laplace transform of $\mathcal{T}_h$ given in [2, part II, section 5, 2.0.1 (p. 469)] as

$$\mathbb{E}^{(3)}_0[\mathrm{e}^{-\gamma\mathcal{T}_h}] = \frac{h\sqrt{2\gamma}}{\sinh(h\sqrt{2\gamma})}$$

with respect to γ, and let $\gamma\to 0$. □

Remark 3.1. We conjecture that for a d-dimensional Bessel bridge starting at $x>0$, the same limit behaviour

$$\lim_{h\downarrow 0}\frac{\mathbb{E}^{(d)}_{x,T,y}[\mathcal{T}_{(x-h,x+h)}]}{h^2} = 1$$

as in Theorem 3.1 holds true. For a d-dimensional Bessel process it can be proved as above in the three-dimensional case that

$$\lim_{h\downarrow 0}\frac{\mathbb{E}^{(d)}_{0}[\mathcal{T}_h]}{h^2} = \frac{1}{d}.$$

Hence we conjecture that for a d-dimensional Bessel bridge starting at 0 the limit should also be

$$\lim_{h\downarrow 0}\frac{\mathbb{E}^{(d)}_{0,T,y}[\mathcal{T}_h]}{h^2} = \frac{1}{d}.$$

3.4. The Kolmogorov distribution function and the mean of the first exit time

A classical result due to Doob [5] is that the distribution of the supremum of the absolute value of the standard Brownian bridge is given by

$$F(h) := \mathbb{P}_{0,1,0}\Big(\sup_{0\le s\le 1}|B^{0,1,0}_s|\le h\Big) = \sum_{m=-\infty}^{\infty}(-1)^m\,\mathrm{e}^{-2m^2h^2} = \sqrt{2\pi}\,\Delta(0,h,1),$$

where Δ is as defined in (3.9). The function $h\mapsto F(h)$, $h>0$, is called the Kolmogorov distribution function due to Kolmogorov's fundamental work [10] (see also Smirnov [16]) on empirical distributions. We refer also to [13] and [11, section 5.7]. The main result of this section – Theorem 3.4 – provides representations for the mean of the first exit time of

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

the Brownian bridge and of the three-dimensional Bessel bridge involving the Kolmogorov distribution function.

We now present some formulas related to the Kolmogorov distribution, which we will need later. The following Jacobi theta function identity, an instance of the Poisson summation formula, is stated in [1, equation (2.1)]):

$$\sum_{m=-\infty}^{\infty} \cos(2m\pi v)\, \mathrm{e}^{-m^2\pi^2 u} = \frac{1}{\sqrt{\pi u}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{-(m+v)^2/u}, \qquad u > 0,\ v \in \mathbb{R}.$$

Setting $u = 2x^2/\pi^2$ and $v = 1/2$ yields

$$F(x) = \sum_{m=-\infty}^{\infty} (-1)^m \mathrm{e}^{-2m^2x^2} = \frac{\sqrt{2\pi}}{x} \sum_{k=1}^{\infty} \exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}\right). \tag{3.24}$$

Note also that

$$\begin{aligned} F(h/\sqrt{t}) &= \mathbb{P}_{0,1,0}\left(\sup_{0\le s\le 1} |B_s^{0,1,0}| \le h/\sqrt{t}\right) \\ &= \mathbb{P}_{0,t,0}\left(\sup_{0\le s\le 1} |B_{ts}^{0,t,0}| \le h\right) \\ &= \mathbb{P}_{0,t,0}\left(\sup_{0\le s\le t} |B_s^{0,t,0}| \le h\right), \end{aligned}$$

where we have used

$$(\sqrt{t}B_s^{0,1,0})_{0\le s\le 1} \overset{d}{=} (B_{st}^{0,t,0})_{0\le s\le 1},$$

which can be seen by applying the scaling property of Brownian motion to the representation (2.4). Consequently,

$$F(h/\sqrt{t}) = \mathbb{P}_{0,t,0}(\mathcal{T}_{(-h,h)} > t) = \mathbb{P}_{0,t,0}(\mathcal{T}_{(-h,h)} = \infty).$$

In the following, we will denote the Laplace transform of a function f by

$$\mathcal{L}_{t,\gamma}(f(t)) := \int_0^{\infty} \mathrm{e}^{-\gamma t} f(t)\, \mathrm{d}t, \qquad \gamma > 0.$$

Note that in this notation the integration variable t is also indicated.

Lemma 3.1. *For $|y| \ge h$,*

$$\int_0^T \int_{-h}^{h} p_{T-t}(x, y)\Delta(x, h, t)\, \mathrm{d}x\, \mathrm{d}t = h \int_0^T p_{T-t}(0, y)\Delta(0, h, t)\, \mathrm{d}t. \tag{3.25}$$

Proof. We prove Lemma 3.1 by showing that the Laplace transforms of the two sides of (3.25) coincide. Using Fubini's theorem and the convolution formula, the Laplace transform of the left-hand side of (3.25) is

$$\mathcal{L}_{T,\gamma}\left(\int_0^T \int_{-h}^{h} p_{T-t}(x, y)\Delta(x, h, t)\, \mathrm{d}x\, \mathrm{d}t\right) = \int_{-h}^{h} \mathcal{L}_{t,\gamma}(p_t(x, y))\mathcal{L}_{t,\gamma}(\Delta(x, h, t))\, \mathrm{d}x. \tag{3.26}$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

To compute the second Laplace transform expression of the right-hand side of (3.26), we use the series representation (3.9). For $x \in (-h, h)$,

$$\left| \sum_{|m|\geq 2} (-1)^m \, \mathrm{e}^{-2mh(mh-x)/t} \right| \leq \sum_{|m|\geq 2} \mathrm{e}^{-2mh(mh-x)/t} \leq 2 \sum_{m\geq 2} (\mathrm{e}^{-2h^2/t})^m \leq \frac{2}{1-\mathrm{e}^{-2h^2/t}}.$$

Hence we may interchange the summation and the Laplace transform, and since

$$\mathcal{L}_{t,\gamma}(p_t(x, z)) = \frac{1}{\sqrt{2\gamma}} \, \mathrm{e}^{-\sqrt{2\gamma}|z-x|},$$

we obtain

$$\begin{aligned}
\mathcal{L}_{t,\gamma}(\Delta(x, h, t)) &= \mathcal{L}_{t,\gamma}\left(\sum_{m=-\infty}^{\infty} (-1)^m \, \mathrm{e}^{-2mh(mh-x)/t} \frac{\mathrm{e}^{-x^2/(2t)}}{\sqrt{2\pi t}} \right) \\
&= \sum_{m=-\infty}^{\infty} (-1)^m \mathcal{L}_{t,\gamma}(p_t(x, 2mh)) \\
&= \frac{1}{\sqrt{2\gamma}} \left(\mathrm{e}^{-\sqrt{2\gamma}|x|} + (\mathrm{e}^{\sqrt{2\gamma}x} + \mathrm{e}^{-\sqrt{2\gamma}x}) \sum_{m=1}^{\infty} (-1)^m \, \mathrm{e}^{-\sqrt{2\gamma}2mh} \right) \\
&= 1 \Big/ \left[\sqrt{2\gamma} \left(\mathrm{e}^{-\sqrt{2\gamma}|x|} - (\mathrm{e}^{\sqrt{2\gamma}x} + \mathrm{e}^{-\sqrt{2\gamma}x}) \frac{\mathrm{e}^{-\sqrt{2\gamma}2h}}{1+\mathrm{e}^{-\sqrt{2\gamma}2h}} \right) \right].
\end{aligned}$$

By a straightforward calculation, for the right-hand side of (3.26) we have

$$\begin{aligned}
\int_{-h}^{h} \mathcal{L}_{t,\gamma}(p_t(x, y)) \mathcal{L}_{t,\gamma}(\Delta(x, h, t)) \, \mathrm{d}x &= \frac{h}{2\gamma} \, \mathrm{e}^{-\sqrt{2\gamma}|y|} \frac{1-\mathrm{e}^{-\sqrt{2\gamma}\,2h}}{1+\mathrm{e}^{-\sqrt{2\gamma}\,2h}} \\
&= \frac{h}{2\gamma} \, \mathrm{e}^{-\sqrt{2\gamma}|y|} \tanh(h\sqrt{2\gamma}). \qquad (3.27)
\end{aligned}$$

We continue by calculating the Laplace transform of the right-hand side of (3.25). From (3.24),

$$\begin{aligned}
\mathcal{L}_{t,\gamma}(\Delta(0, h, t)) &= \frac{1}{h} \sum_{k=1}^{\infty} \int_0^{\infty} \exp\left(-\frac{(2k-1)^2\pi^2 t}{8h^2} \right) \exp(-\gamma t) \, \mathrm{d}t \\
&= h \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2\pi^2 + 8h^2\gamma} \\
&= \frac{\tanh(h\sqrt{2\gamma})}{\sqrt{2\gamma}}, \qquad (3.28)
\end{aligned}$$

where we have used

$$\tanh\left(\frac{\pi x}{2}\right) = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2}, \qquad x \in \mathbb{R}$$

(see [7, section 1.421]). By the convolution formula,

$$\begin{aligned}
\mathcal{L}_{T,\gamma}\left(h \int_0^T p_{T-t}(0, y) \Delta(0, h, t) \, \mathrm{d}t \right) &= h \mathcal{L}_{t,\gamma}(p_t(0, y)) \mathcal{L}_{t,\gamma}(\Delta(0, h, t)) \\
&= \frac{h}{\sqrt{2\gamma}} \, \mathrm{e}^{-\sqrt{2\gamma}|y|} \, \frac{\tanh(h\sqrt{2\gamma})}{\sqrt{2\gamma}}. \qquad (3.29)
\end{aligned}$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

Hence (3.27) and (3.29) coincide, and this finishes the proof of Lemma 3.1. □

Theorem 3.4. (i) *For the Brownian bridge with* $|y| \geq h$,

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = h \int_0^T \frac{p_{T-t}(0,y)}{p_T(0,y)} \Delta(0,h,t)\,\mathrm{d}t. \tag{3.30}$$

(ii) *For the three-dimensional Bessel bridge with positive* $y \notin (x-h, x+h)$ *and* $x > 0$,

$$\mathbb{E}^{(3)}_{x,T,y}[\mathcal{T}_{(x-h,x+h)}] = h \int_0^T \frac{r^{(3)}_{T-t}(x,y)}{r^{(3)}_T(x,y)} \Delta(0,h,t)\,\mathrm{d}t. \tag{3.31}$$

Proof. (i) We derive (3.30) from (3.3), (3.1) and Lemma 3.1.
(ii) Recall the relation (3.20). Since, by (3.8),

$$\frac{r^{(3)}_{T-t}(z,y)}{r^{(3)}_T(x,y)} = \frac{x}{z} \frac{p_{T-t}(z,y) - p_{T-t}(z,-y)}{p_T(x,y) - p_T(x,-y)}, \tag{3.32}$$

we have

$$\begin{aligned}
&(p_T(x,y) - p_T(x,-y))\mathbb{E}^{(3)}_{x,T,y}[\mathcal{T}_{(x-h,x+h)}] \\
&\quad = \int_0^T \int_{x-h}^{x+h} \Delta(z-x,h,t)(p_{T-t}(z,y) - p_{T-t}(z,-y))\,\mathrm{d}z\,\mathrm{d}t \\
&\quad = \int_0^T \int_{-h}^{h} \Delta(u,h,t)(p_{T-t}(u,y-x) - p_{T-t}(u,-(y+x)))\,\mathrm{d}u\,\mathrm{d}t \\
&\quad = h \int_0^T \Delta(0,h,t)(p_{T-t}(0,y-x) - p_{T-t}(0,y+x))\,\mathrm{d}t,
\end{aligned}$$

where the last equality holds by (3.25). Then (3.32) yields (3.31). □

Remark 3.2. (i) We have not been able to find a probabilistic explanation for the appearance of the Kolmogorov distribution function in the representations (3.30) and (3.31).

Note that the integrands with respect to t of the expressions in [Theorem 3.1, equation (3.14)] and in [Theorem 3.4, equation (3.30)] do not coincide, that is,

$$\int_{-h}^{h} p_{T-t}(z,y)\Delta(z,h,t)\,\mathrm{d}z \neq h p_{T-t}(0,y)\Delta(0,h,t).$$

To see this, we can compare the double Laplace transform of

$$h\,p_{T-t}(0,y)\Delta(0,h,t) \quad \text{and} \quad \int_{-h}^{h} p_{T-t}(z,y)\Delta(z,h,t)\,\mathrm{d}z.$$

A computation similar to that in the proof of Lemma 3.1 yields, for $\lambda > 0$,

$$\mathcal{L}_{T,\gamma}\left(h \int_0^T p_{T-t}(0,y)\Delta(0,h,t)\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t\right) = \frac{h}{\sqrt{2\gamma}}\,\mathrm{e}^{-\sqrt{2\gamma}|y|}\,\frac{\tanh\left(h\sqrt{2(\gamma+\lambda)}\right)}{\sqrt{2(\gamma+\lambda)}}.$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

On the other hand, after some calculations,

$$\mathcal{L}_{T,\gamma}\left(\int_0^T \int_{-h}^{h} p_{T-t}(z, y)\Delta(z, h, t)\,\mathrm{d}z\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t\right) = \frac{1}{\lambda\sqrt{2\gamma}}\,\mathrm{e}^{-\sqrt{2\gamma}|y|}\left[1 - \frac{\mathrm{e}^{-\sqrt{2\gamma}h} + \mathrm{e}^{\sqrt{2\gamma}h}}{\mathrm{e}^{-\sqrt{2(\gamma+\lambda)}h} + \mathrm{e}^{\sqrt{2(\gamma+\lambda)}h}}\right].$$

(ii) It is also possible to use (3.30) and (3.31) to show (3.15) and (3.16). For example, after the variable transform $t = h^2 r$, we find an integrable majorant and conclude by dominated convergence that

$$h\int_0^T \frac{p_{T-t}(0, y)}{p_T(0, y)}\Delta(0, h, t)\,\mathrm{d}t \to \int_0^\infty \Delta(0, 1, t)\,\mathrm{d}t, \qquad h \downarrow 0.$$

Then, by monotone convergence, thanks to relation (3.28) we obtain

$$\int_0^\infty \Delta(0, 1, t)\,\mathrm{d}t = \lim_{\gamma\to 0+}\int_0^\infty \mathrm{e}^{-\gamma t}\Delta(0, 1, t)\,\mathrm{d}t = \lim_{\gamma\to 0+}\frac{\tanh(\sqrt{2\gamma})}{\sqrt{2\gamma}} = 1.$$

We conclude this section by pointing out a connection between the Kolmogorov distribution function and the density of the last visit of 0 by a Brownian motion before $\mathcal{T}_{(-h,h)}$. This connection was noted by Knight in [9, Corollary 2.1]. Here we provide a different proof of this fact.

Proposition 3.4. *When $\widehat{W}$ is a Brownian motion killed at $\mathcal{T}_{(-h,h)}$, then $t \mapsto (1/h)\Delta(0, h, t)$ is the density of the last passage time $\lambda_0 := \sup\{t > 0 \colon \widehat{W}_t = 0\}$.*

Proof. It is well known that

$$\mathbb{P}_0^{\widehat{W}}(\lambda_0 \in \mathrm{d}t) = \frac{\widehat{q}_t(0, 0)}{\widehat{G}_0(0, 0)}\,\mathrm{d}t \tag{3.33}$$

(see [2, part I, chapter II, no. 20, p. 26]), where

$$\widehat{q}_t(x, y) = \frac{1}{\sqrt{2\pi t}}\sum_{m=-\infty}^{\infty}\left(\mathrm{e}^{-(x-y+4mh)^2/(2t)} - \mathrm{e}^{-(x+y+(2m+1)2h)^2/(2t)}\right)$$

is the transition density (with respect to the Lebesgue measure) of $\widehat{W}$ (see [2, part I, appendix I, no. 6, p. 126]), and

$$\widehat{G}_0(x, y) = \begin{cases} \dfrac{(x+h)(h-y)}{h} & -h \le x \le y \le h \\ \dfrac{(y+h)(h-x)}{h} & -h \le y \le x \le h \end{cases}$$

denotes the 0-resolvent kernel (see [2, part I, appendix I, no. 6, p. 126]). Note that $\widehat{q}_t(0, 0) = \Delta(0, h, t)$ and $\widehat{G}_0(0, 0) = h$, so from (3.33) we obtain

$$\mathbb{P}_0^{\widehat{W}}(\lambda_0 \in \mathrm{d}t)/\mathrm{d}t = \frac{\Delta(0, h, t)}{h}.$$

□

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

4. Application

In this section we apply the previous results and prove in detail an estimate needed in the convergence analysis, presented in Walsh [17], of a functional of the binomial tree model for a corresponding functional of the Black–Scholes model. This convergence analysis succeeds in identifying the leading constants C_1, C_2 appearing in the error expansion

$$\mathbb{E}[g(X_n^{(n)}) - g(X_T)] = C_1\sqrt{\frac{T}{n}} + C_2\frac{T}{n} + \mathrm{O}\left(\frac{1}{n^{3/2}}\right), \qquad n \text{ even} \tag{4.1}$$

(see [17, equation (14)]) in terms of expressions depending on the function g (which is assumed to be exponentially bounded and piecewise twice continuously differentiable). Here $X_t = \sigma W_t$ for $t \geq 0$ and $(X_k^{(n)})_{k=0}^n$ denotes a symmetric simple random walk with time step T/n and space step size $\sigma\sqrt{T/n}$. For simplicity, we will set $\sigma = 1$ and hence consider

$$\mathbb{E}[g(W_n^{(n)}) - g(W_T)].$$

The central idea of this error analysis is to build the random walk from a given Brownian motion $(W_t)_{t\geq 0}$ so that both processes are on the same probability space. Fix $T > 0$ and $n \in \mathbb{N}$. For $h := \sqrt{T/n}$, define $\tau_0 := 0$ and

$$\tau_k := \inf\{t > \tau_{k-1} : |W_t - W_{\tau_{k-1}}| = h\}, \qquad k \geq 1.$$

Then $(W_{\tau_k} - W_{\tau_{k-1}})_{k=1}^\infty$ is a sequence of i.i.d. random variables with

$$\mathbb{P}(W_{\tau_k} - W_{\tau_{k-1}} = h) = \mathbb{P}(W_{\tau_k} - W_{\tau_{k-1}} = -h) = \frac{1}{2}.$$

Let k_* be such that $\tau_{k_*} \leq T < \tau_{k_*+1}$. One central idea to get (4.1) is to split the error into a local part (exploiting the closeness of W_T and $W_{\tau_{k_*+1}}$) and a global part (using the binomial distribution), that is,

$$\mathbb{E}[(g(W_T) - g(W_{\tau_{k_*+1}}))\mathbf{1}_{\{k_* \text{ odd}\}}] + \mathbb{E}[(g(W_{\tau_{k_*+1}}) - g(W_{\tau_n})\mathbf{1}_{\{k_* \text{ odd}\}}],$$

respectively, while the splitting is done with $W_{\tau_{k_*+2}}$ if k_* is even.

For this, in [17, section 9], where L denotes our k_*, the conditional probability

$$q(x) := \mathbb{P}(k_* \text{ is even} \mid W_T = x), \qquad x \in \mathbb{R}$$

is introduced. We want to study and estimate q. The reason for considering only even n and even $k_* + 1$ or $k_* + 2$ is to avoid even/odd fluctuations known to appear in binomial tree schemes. Note that k_* is an even number if and only if $W_{\tau_{k_*}}$ is an even multiple of h.

The process $(W_t)_{0\leq t\leq T}$ given $W_0 = 0$, $W_T = x$ is identical in law to the bridge $(B_t^{0,T,x})_{0\leq t\leq T}$. See Figure 1.

Fix $k \in \mathbb{Z}$ and assume $x \in ((2k-1)h, (2k+1)h)$. For simplicity, set

$$\underline{h} := (2k-1)h, \qquad \overline{h} := (2k+1)h, \qquad h_e := 2kh \tag{4.2}$$

for the lower and upper value, and for the 'even' midpoint of the interval. Then, given $W_T = x$, we find that 'k_* is even' is the same as '$W_{\tau_{k_*}} = h_e$'. For the time-reversed bridge,

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

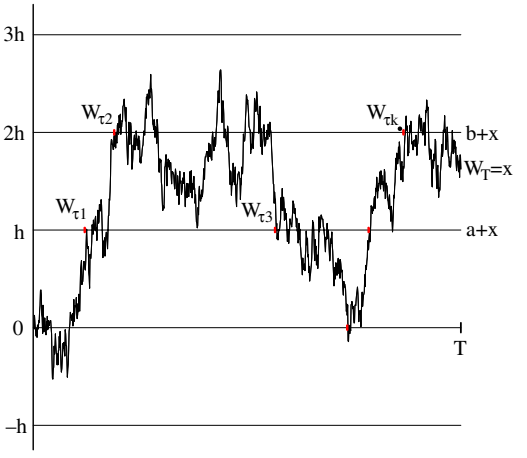


FIGURE 1: A Brownian motion satisfying $W_0 = 0$, $W_T = x$.

i.e. $(B_t^{x,T,0})_{0\le t\le T}$, we get

$$\begin{aligned} q(x) &= \mathbb{P}(k_* \text{ is even} \mid W_T = x) \\ &= \begin{cases} \mathbb{P}_{x,T,0}(\mathcal{T}_{h_e} < \mathcal{T}_{\underline{h}}) & \underline{h} < x < h_e, \\ \mathbb{P}_{x,T,0}(\mathcal{T}_{h_e} < \mathcal{T}_{\overline{h}}) & h_e < x < \overline{h}, \end{cases} \\ &= \begin{cases} \mathbb{P}_{0,T,-x}(\mathcal{T}_{h_e-x} < \mathcal{T}_{\underline{h}-x}) & \underline{h} < x < h_e, \\ \mathbb{P}_{0,T,-x}(\mathcal{T}_{h_e-x} < \mathcal{T}_{\overline{h}-x}) & h_e < x < \overline{h}, \end{cases} \end{aligned}$$

where $\mathcal{T}_y(\omega) := \inf\{t > 0 \colon \omega(t) = y\}$ for $\omega \in C[0, T]$. Note that, for any $a < 0 < b$ and $y \notin (a, b)$,

$$\mathbb{E}_{0,T,y}[B_{\mathcal{T}_{(a,b)}}] = a(1 - \mathbb{P}_{0,T,y}(\mathcal{T}_b < \mathcal{T}_a)) + b\,\mathbb{P}_{0,T,y}(\mathcal{T}_b < \mathcal{T}_a),$$

and, consequently,

$$\mathbb{P}_{0,T,y}(\mathcal{T}_b < \mathcal{T}_a) = \frac{-a}{b-a} + \frac{\mathbb{E}_{0,T,y}[B_{\mathcal{T}_{(a,b)}}]}{b-a}.$$

Hence

$$q(x) = \begin{cases} \dfrac{x-\underline{h}}{h} + \dfrac{\mathbb{E}_{0,T,-x}[B_{\mathcal{T}_{(\underline{h}-x,h_e-x)}}]}{h} & \underline{h} < x < h_e, \\ \dfrac{\overline{h}-x}{h} - \dfrac{\mathbb{E}_{0,T,-x}[B_{\mathcal{T}_{(h_e-x,\overline{h}-x)}}]}{h} & h_e < x < \overline{h}. \end{cases} \tag{4.3}$$

Arguing that a Brownian bridge and Brownian motion have similar exit behaviour for small h, Walsh [17, equation (20)] stated that

$$q(x) = \frac{\operatorname{dist}(x, \mathbb{N}_o^h)}{h} + \mathrm{O}(h), \tag{4.4}$$

where $x \in \mathbb{R}$, $\mathbb{N}_o^h = \{(2k+1)h \colon k \in \mathbb{Z}\}$ and $\operatorname{dist}(x, \mathbb{N}_o^h) := \inf\{|x - y| \colon y \in \mathbb{N}_o^h\}$. Clearly,

$$\operatorname{dist}(x, \mathbb{N}_o^h) \text{ is equal to } x - \underline{h} \text{ or } \overline{h} - x \text{ for some } k,$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

so for (4.4) to hold we should have

$$\left|q(x) - \frac{\operatorname{dist}(x, \mathbb{N}_o^h)}{h}\right| = \left|\frac{\mathbb{E}_{0,T,-x}[B_{\mathcal{T}(x)}]}{h}\right| = \mathrm{O}(h), \tag{4.5}$$

where (4.2) was used to rewrite $\mathcal{T}_{(\underline{h}-x,h_e-x)}$ and $\mathcal{T}_{(h_e-x,\overline{h}-x)}$ as

$$\mathcal{T}(x) := \mathcal{T}_{(kh-x,(k+1)h-x)} \text{ if } x \in (kh, (k+1)h). \tag{4.6}$$

We will show below that (4.5) is indeed true for a fixed x. However, we will see that this relation does not hold uniformly in x, and therefore it cannot be used to estimate the second term on the right-hand side of (4.11) below. We will estimate that second term in Corollary 4.1 below.

Lemma 4.1. *Suppose that $a < 0 < b$, $y \notin (a, b)$, $T > 0$ and $h > 0$. Then*

$$|\mathbb{E}_{0,T,y}[B_{\mathcal{T}_{(a,b)}}]| \le \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}]}{T}(2(|a| \vee b) + |y| + 3\sqrt{2T}). \tag{4.7}$$

Proof. We use the Markov inequality to deduce

$$|\mathbb{E}_{0,T,y}[B_{\mathcal{T}_{(a,b)}}\mathbf{1}_{\{\mathcal{T}_{(a,b)}>T/2\}}]| \le (|a| \vee b)\mathbb{P}_{0,T,y}(\mathcal{T}_{(a,b)} > T/2) \le \frac{2(|a| \vee b)}{T}\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}].$$

To estimate $|\mathbb{E}_{0,T,y}[B_{\mathcal{T}_{(a,b)}}\mathbf{1}_{\{\mathcal{T}_{(a,b)}\le T/2\}}]|$, we use the representation (2.5) for a Brownian bridge:

$$\tilde{B}_t^{0,T,y} := (T-t)\int_0^t \frac{\mathrm{d}W_s}{T-s} + \frac{t}{T}y, \quad t \in [0, T), \quad \text{and} \quad \tilde{B}_T^{0,T,y} := y.$$

Introducing

$$\tilde{\mathcal{T}} := \inf\{t \in [0, T] \colon \tilde{B}_t^{0,T,y} \notin (a, b)\},$$

we have

$$\mathbb{E}[\tilde{B}_{\tilde{\mathcal{T}}}^{0,T,y}\mathbf{1}_{\{\tilde{\mathcal{T}}\le T/2\}}] = \mathbb{E}_{0,T,y}[B_{\mathcal{T}_{(a,b)}}\mathbf{1}_{\{\mathcal{T}_{(a,b)}\le T/2\}}]$$

and

$$\begin{aligned}
&|\mathbb{E}[\tilde{B}_{\tilde{\mathcal{T}}}^{0,T,y}\mathbf{1}_{\{\tilde{\mathcal{T}}\le T/2\}}]| \\
&\quad\le \left|\mathbb{E}\left[(T-\tilde{\mathcal{T}})\int_0^{\tilde{\mathcal{T}}} \frac{\mathrm{d}W_s}{T-s}\mathbf{1}_{\{\tilde{\mathcal{T}}\le T/2\}}\right]\right| + \frac{|y|}{T}\mathbb{E}[\tilde{\mathcal{T}}] \\
&\quad\le T\left|\mathbb{E}\left[\int_0^{\tilde{\mathcal{T}}\wedge T/2} \frac{\mathrm{d}W_s}{T-s}\,\mathbf{1}_{\{\tilde{\mathcal{T}}\le T/2\}}\right]\right| + \mathbb{E}\left|(\tilde{\mathcal{T}} \wedge (T/2))\int_0^{\tilde{\mathcal{T}}\wedge T/2} \frac{\mathrm{d}W_s}{T-s}\right| + \frac{|y|}{T}\mathbb{E}[\tilde{\mathcal{T}}].
\end{aligned}$$

We note that, by the optional stopping theorem,

$$\mathbb{E}\left[\int_0^{\tilde{\mathcal{T}}\wedge T/2} \frac{\mathrm{d}W_s}{T-s}\right] = 0,$$

so that

$$\left|\mathbb{E}\left[\int_0^{\tilde{\mathcal{T}}\wedge T/2} \frac{\mathrm{d}W_s}{T-s}\,\mathbf{1}_{\{\tilde{\mathcal{T}}\le T/2\}}\right]\right| = \left|\mathbb{E}\left[\int_0^{\tilde{\mathcal{T}}\wedge T/2} \frac{\mathrm{d}W_s}{T-s}\,\mathbf{1}_{\{\tilde{\mathcal{T}}> T/2\}}\right]\right|.$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

Then, by the Cauchy–Schwarz and Markov inequalities,

$$\begin{aligned}\left|\mathbb{E}\left[\mathbf{1}_{\{\tilde{\mathcal{T}}>T/2\}}\int_0^{\tilde{\mathcal{T}}\wedge T/2}\frac{\mathrm{d}W_s}{T-s}\right]\right| &\le \left(\frac{2}{T}\mathbb{E}[\tilde{\mathcal{T}}]\right)^{1/2}\left(\mathbb{E}\left[\int_0^{\tilde{\mathcal{T}}\wedge T/2}\frac{\mathrm{d}s}{(T/2)^2}\right]\right)^{1/2}\\ &\le \left(\frac{2}{T}\mathbb{E}[\tilde{\mathcal{T}}]\right)^{1/2}\left(\left(\frac{2}{T}\right)^2\mathbb{E}\left(\tilde{\mathcal{T}}\wedge\frac{T}{2}\right)\right)^{1/2}\\ &\le (\sqrt{2/T})^3\,\mathbb{E}[\tilde{\mathcal{T}}].\end{aligned}$$

Again by the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathbb{E}\left|\left(\tilde{\mathcal{T}}\wedge\frac{T}{2}\right)\int_0^{\tilde{\mathcal{T}}\wedge T/2}\frac{\mathrm{d}W_s}{T-s}\right| &\le \left(\mathbb{E}\left(\tilde{\mathcal{T}}\wedge\frac{T}{2}\right)^2\right)^{1/2}\left(\mathbb{E}\left[\int_0^{\tilde{\mathcal{T}}\wedge T/2}\frac{\mathrm{d}s}{(T-s)^2}\right]\right)^{1/2}\\ &\le \left(\frac{T}{2}\mathbb{E}[\tilde{\mathcal{T}}]\right)^{1/2}\left(\left(\frac{2}{T}\right)^{1/2}\mathbb{E}\left(\tilde{T}\wedge\frac{T}{2}\right)\right)^{1/2}\\ &\le \sqrt{2/T}\,\mathbb{E}[\tilde{\mathcal{T}}].\end{aligned}$$

The above estimates and $\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] = \mathbb{E}[\tilde{\mathcal{T}}]$ yield (4.7). □

Now we derive estimates for $\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}]$.

Lemma 4.2. *Let $T > 0$ be fixed. Suppose that $a < 0 < b$ and $y \notin (a, b)$.*

(i) *We have*

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] \le \begin{cases}[b(2|a|+y)]\wedge T & \text{if } y \ge b,\\ [|a|(2b+|y|)]\wedge T & \text{if } y \le a.\end{cases} \tag{4.8}$$

(ii) *For $|y| \ge h$, define*

$$C(T,h,y) := \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]}{h^2}. \tag{4.9}$$

Then

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] \le C(b-a)^2 \quad \text{if } |y| \ge b-a,$$

where $C = C(T, b-a, y)$, and

$$\lim_{b-a\to 0,\ a<0<b} C(T,b-a,y) = 1.$$

Proof. (i) We first assume that $y \ge b$. Recall the representation (2.4) for a Brownian bridge:

$$\bar{B}_t^{0,T,y} := \left(1-\frac{t}{T}\right)W_{Tt/(T-t)} + y\frac{t}{T}, \quad t\in[0,T) \quad \text{and} \quad \bar{B}_T^{0,T,y} := y.$$

Further, set

$$\bar{\mathcal{T}} := \inf\{t\in[0,T]\colon \bar{B}_t^{0,T,y} \notin (a,b)\}.$$

Since by definition $\mathcal{T}_{(a,b)} \le T$ if $y \notin (a,b)$, we have $\mathcal{T}_{(a,b)} \overset{d}{=} \bar{\mathcal{T}}\wedge T$. Setting $u = Tt/(T-t)$, $t\in[0,T)$, $u \ge 0$, we get

$$\begin{aligned}\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] &= \mathbb{E}_0\left[\inf\left\{t\in[0,T]\colon W_{Tt/(T-t)}\notin\left(\frac{a-yt/T}{1-t/T},\frac{b-yt/T}{1-t/T}\right)\right\}\wedge T\right]\\ &= \mathbb{E}_0\left[\inf\left\{u\ge 0\colon W_u\notin\left(a+(a-y)\frac{u}{T},b+(b-y)\frac{u}{T}\right)\right\}\wedge T\right].\end{aligned}$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

Since $y \geq b$, for any $u \in [0, T]$ we have

$$\left(a + (a-y)\frac{u}{T}, b + (b-y)\frac{u}{T}\right) \subseteq (2a - y, b)$$

and then, by (1.2),

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] \leq \mathbb{E}_0[\inf\{u \geq 0 \colon W_u \notin (2a-y, b)\} \wedge T] = \mathbb{E}_0[\mathcal{T}_{(2a-y,b)}] \wedge T \leq [b(2|a| + y)] \wedge T.$$

The case $y \leq a$ follows similarly.

(ii) For $|y| \geq b - a$ we have $\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] \leq \mathbb{E}_{0,T,y}[\mathcal{T}_{(-(b-a),b-a)}]$. Using (4.9) with $h = b - a$ gives

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = h^2\, C(T, h, y) \quad \text{for } |y| \geq h,$$

and from Theorem 3.1 (i) we find that $C(T, h, y)$ converges to 1 as $h \to 0$. □

Hence, for $x \in (kh, (k+1)h)$ with $k \in \{-1, 0\}$, we get by Lemma 4.2(i)

$$\mathbb{E}_{0,T,-x}[\mathcal{T}_{(kh-x,(k+1)h-x)}] \leq ch^2.$$

For $x \in (kh, (k+1)h)$ with $k \notin \{-1, 0\}$, we have $x \notin (-h, h)$ and $(kh - x, (k+1)h - x) \subseteq (-h, h)$, so Lemma 4.2(ii) applies, and we get

$$\mathbb{E}_{0,T,-x}[\mathcal{T}_{(kh-x,(k+1)h-x)}] \leq C(T, h, -x)h^2.$$

Then Lemma 4.1 implies that, for any fixed $x \in \mathbb{R}$,

$$\mathbb{E}_{0,T,-x}[B_{\mathcal{T}(x)}] = \mathrm{O}(h^2).$$

However, this equality does not hold uniformly in x, with the consequence that we cannot use (4.4) in integrals like (4.11) below. For example, for the sequence $x_k := (k + 0.5)h$,

$$\mathbb{E}_{0,T,-x_k}[B_{\mathcal{T}(x_k)}] = \mathbb{E}_{0,T,-x_k}[B_{\mathcal{T}_{(-h/2,h/2)}}] \to -h/2, \qquad k \to \infty, \tag{4.10}$$

which contradicts the claim that $\mathbb{E}_{0,T,-x}[B_{\mathcal{T}(x)}] = \mathrm{O}(h^2)$ holds uniformly in x. The limit in (4.10) can be easily seen from the representation

$$(B_t^{0,T,-x_k})_{0 \leq t \leq T} \overset{d}{=} \left(B_t^{0,T,0} - \frac{x_k t}{T}\right)_{0 \leq t \leq T}.$$

For any path $t \mapsto B_t^{0,T,0}(\omega)$, we can find a sufficiently large x_k, such that the transformed path $t \mapsto B_t^{0,T,0}(\omega) - x_k t/T$ exits $(-h/2, h/2)$ first at $-h/2$.

Nevertheless, we can prove the estimates needed in [17], where q was used inside an integral over the real line. Here we only discuss [17, equation (38)], because the calculations for the other cases where the function q appears are similar. For $\sigma = 1$ and denoting $\mathbb{N}_e^h := \{2kh \colon k \in \mathbb{Z}\}$, the last term in [17, equation (38)] can be written as

$$\begin{aligned}
&\int_{-\infty}^{\infty} (2h^2 - \mathrm{dist}^2(x, \mathbb{N}_e^h))q(x)p_T(0, x)\,\mathrm{d}x \\
&\quad = \int_{-\infty}^{\infty} (2h^2 - \mathrm{dist}^2(x, \mathbb{N}_e^h))\mathrm{dist}(x, \mathbb{N}_o^h)h^{-1}p_T(0, x)\,\mathrm{d}x \\
&\qquad + \int_{-\infty}^{\infty} (2h^2 - \mathrm{dist}^2(x, \mathbb{N}_e^h))(q(x) - \mathrm{dist}(x, \mathbb{N}_o^h)h^{-1})p_T(0, x)\,\mathrm{d}x.
\end{aligned} \tag{4.11}$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

The calculation for the first integral on the right-hand side is carried out in [17]. To justify (4.1), it remains to show that the other integral behaves like $O(h^3)$. Since $(2h^2 - \operatorname{dist}^2(x, \mathbb{N}_e^h)) \le 2h^2$ and

$$\left| q(x) - \frac{\operatorname{dist}(x, \mathbb{N}_o^h)}{h} \right| = \frac{|\mathbb{E}_{0,T,-x}[B_{\mathcal{T}(x)}]|}{h}$$

by (4.3) and (4.6), we get the desired estimate from the next corollary.

Corollary 4.1. *For $T > 0$ and $h = \sqrt{T/n}$, there exists a $C = C(T) > 0$ such that*

$$\int_{-\infty}^{\infty} |\mathbb{E}_{0,T,-x}[B_{\mathcal{T}(x)}]| p_T(0, x)\, \mathrm{d}x \le Ch^2,$$

where $\mathcal{T}(x)$ is given in (4.6).

Proof. Since $B^{0,T,-x}_{\mathcal{T}(x)} \overset{d}{=} -B^{0,T,x}_{\mathcal{T}(-x)}$, it suffices to estimate the integral over $[0, \infty)$. By (4.7),

$$|\mathbb{E}_{0,T,-x}[B_{\mathcal{T}(x)}]| \le \frac{\mathbb{E}_{0,T,-x}[\mathcal{T}(x)]}{T}(2h + |x| + 3\sqrt{2T}).$$

If $x \in (0, h)$, then $\mathcal{T}(x) = \mathcal{T}_{(-x,h-x)}$, and estimate (4.8) gives

$$\mathbb{E}_{0,T,-x}[\mathcal{T}(x)] \le 2x\left(h - x + \frac{x}{2}\right) \le h^2.$$

For $x \ge h$,

$$\mathbb{E}_{0,T,-x}[\mathcal{T}(x)] \le C(T, h, -x)h^2$$

by Lemma 4.2. From the above estimates we get

$$\int_0^h |\mathbb{E}_{0,T,-x}[B_{\mathcal{T}(x)}]|\, p_T(0, x)\, \mathrm{d}x \le h^2 \int_0^h \frac{2h + x + 3\sqrt{2T}}{T} p_T(0, x)\, \mathrm{d}x \le C(T)h^2, \tag{4.12}$$

and

$$\begin{aligned} \int_h^{\infty} |\mathbb{E}_{0,T,-x}[B_{\mathcal{T}(x)}]| p_T(0, x)\, \mathrm{d}x &\le h^2 \int_h^{\infty} C(T, h, -x)\frac{2h + x + 3\sqrt{2T}}{T} p_T(0, x)\, \mathrm{d}x \\ &\le C(T)h^2 \int_h^{\infty} (1 + x)C(T, h, -x)p_T(0, x)\, \mathrm{d}x \\ &\le C(T)h^2, \end{aligned} \tag{4.13}$$

where the constant $C(T)$ varies from line to line. The last inequality in (4.13) can be seen as follows. We use the representation (3.30) for (4.9) (it is also possible to use (3.14) instead of (3.30)) and substitute $t = h^2u$, so that

$$\begin{aligned} C(T, h, -x) &= \mathbb{E}_{0,T,-x}[\mathcal{T}_{(-h,h)}]h^{-2} \\ &= h^{-1} \int_0^T \frac{p_{T-t}(0, x)}{p_T(0, x)} \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}}\, \mathrm{d}t \\ &= \int_0^{T/h^2} \frac{p_{T-h^2u}(0, x)}{p_T(0, x)} \frac{F(1/\sqrt{u})}{\sqrt{2\pi u}}\, \mathrm{d}u. \end{aligned}$$

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press

By Fubini's theorem,

$$\int_0^\infty \frac{F(1/\sqrt{u})}{\sqrt{2\pi u}} \int_h^\infty (1+x) p_{T-h^2 u}(0,x) \mathbf{1}_{[0,T/h^2)}(u)\,\mathrm{d}x\,\mathrm{d}u \le C(T),$$

where for the last line we used the fact that

$$u \mapsto \frac{F(1/\sqrt{u})}{\sqrt{2\pi u}} \mathbf{1}_{(0,\infty)}(u)$$

is a density. The claim then follows by (4.12) and (4.13). □

Acknowledgements

We thank the reviewers for careful reading and comments which led to improvements of the presentation. Christel Geiss would like to thank the Erwin Schrödinger Institute, Vienna, where part of this work was written, for hospitality and support.

References

[1] P. Biane, J. Pitman and M. Yor (2001). Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bull. Amer. Math. Soc.* (N.S.) **38**, 435–465.
[2] A. N. Borodin and P. Salminen (2015). *Handbook of Brownian Motion: Facts and Formulae*, 2nd edn (Probability and Its Applications). Birkhäuser, Basel.
[3] S. Christensen and K. Lindensjö (2018). On time-inconsistent stopping problems and mixed strategy stopping times. Available at arXiv:1804.07018.
[4] K. L. Chung and J. B. Walsh (2005). *Markov Processes, Brownian Motion, and Time Symmetry*, 2nd edn (Grundlehren der Mathematischen Wissenschaften **249**). Springer, New York.
[5] J. L. Doob (1949). Heuristic approach to the Kolmogorov–Smirnov theorems. *Ann. Math. Stat.* **20**, 393–403.
[6] P. Fitzsimmons, J. Pitman and M. Yor (1993). Markovian bridges: construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes* (Progress in Probability **32**), pp. 101–134. Springer.
[7] I. S. Gradshteyn and I. M. Ryzhik (2014). *Table of Integrals, Series, and Products*, 8th edn. Elsevier/Academic Press, Amsterdam.
[8] K. Itô and H. P. McKean (1974). *Diffusion Processes and their Sample Paths* (Grundlehren der Mathematischen Wissenschaften **125**). Springer, Berlin and New York.
[9] F. B. Knight (1969). Brownian local times and taboo processes. *Trans. Amer. Math. Soc.* **143**, 173–185.
[10] A. N. Kolmogorov (1933). Sulla determinazione empirica delle leggi di probabilità. *Giorn. Ist. Ital. Attuari* **4**, 1–11.
[11] D. Kroese, T. Taimre and Z. Botev (2011). *Handbook of Monte Carlo Methods* (Wiley Series in Probability and Statistics). Wiley, New York.
[12] A. Luoto (2017). Time-dependent weak rate of convergence for functions of generalized bounded variation. Available at arXiv:1609.05768v3.
[13] J. Pitman and M. Yor (1999). The law of the maximum of a Bessel bridge. *Electron. J. Prob.* **4**, 1–15.
[14] P. Salminen (1997). On last exit decomposition of linear diffusions. *Studia Sci. Math. Hungar.* **33**, 251–262.
[15] P. Salminen and M. Yor (2011). On hitting times of affine boundaries by reflecting Brownian motion and Bessel processes. *Period. Math. Hungar.* **62**, 75–101.
[16] N. V. Smirnov (1939). On the estimation of the discrepancy between empirical curves of distribution for two independent samples. *Bul. Math. de l'Univ. de Moscou* **2**, 3–14 (in Russian).
[17] J. B. Walsh (2003). The rate of convergence of the binomial tree scheme. *Finance Stoch.* **7**, 337–361.

https://doi.org/10.1017/jpr.2019.42 Published online by Cambridge University Press