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GENERALIZED KAC–PALJUTKIN ALGEBRAS

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Abstract In this note, we construct a family of non-commutative, non-cocommutative semisimple Hopf algebras $H_{n,m}$ of dimension $n^m m!$ over a field of characteristic zero containing a primitive nth root of unity, where $n, m \geq 2$ are integers. The well-known eight-dimensional Kac–Paljutkin algebra arises as the special case $H_{2,2}$, while the Hopf algebras previously constructed by Pansera correspond to the instances $H_{n,2}$. Each algebra $H_{n,m}$ is defined as an extension of the group algebra $\mathbb{K}\Sigma_m$ of the symmetric group by the m-fold tensor product $R = \mathbb{K}\mathbb{Z}_n^{\otimes m}$, where $\mathbb{Z}_n$ denotes the cyclic group of order n. This extension admits a realization as a crossed product: $H_{n,m} = \mathbb{K}\mathbb{Z}_n^{\otimes m} \#_\gamma \Sigma_m$. In the final section, we construct a family of irreducible m-dimensional representations of $H_{n,m}$ that are inner faithful as R-modules and exhibit a non-trivial inner-faithful action of a subalgebra of $H_{n,m}$ on a quantum polynomial algebra.

Keywords: Kac–Paljutkin; semisimple Hopf algebras

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1. Introduction

Let $\mathbb{K}$ be a field of characteristic zero. In [5] (see also [6]), Kac and Paljutkin introduced an eight-dimensional semisimple Hopf algebra H_8 given by generators x, y, z subject to the relations

$$zx = yz, \quad zy = xz, \quad xy = yx, \quad x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \tag{1}$$

such that x and y are group-like and

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + x \otimes 1 + 1 \otimes y - x \otimes y)(z \otimes z). \tag{2}$$

This paper is dedicated to my father.

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The antipode of H_8 is the identity map. The algebra H_8 can be seen as a quotient of a skew polynomial ring $R[z;\sigma]/I$, where the ring of coefficients is the group ring $R = \mathbb{K}[\mathbb{Z}_2 \times \mathbb{Z}_2]$, with an automorphism σ that interchanges the generators x and y of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and I is the ideal generated by the relation of z^2 in Equation (1). In [9, 10], Pansera introduced a $2n^2$-dimensional semisimple Hopf algebra H_{2n^2} as $R[z;\sigma]/I$, where $R = \mathbb{K}[\mathbb{Z}_n \times \mathbb{Z}_n]$, for a given integer $n \geq 2$. The comultiplication of z is given by

$$\Delta(z) = \frac{1}{n}\left(\sum_{i,j=0}^{n-1} q^{-ij} x^i \otimes y^j\right)(z \otimes z), \tag{3}$$

where q is a primitive nth root of unity in $\mathbb{K}$. The group algebra R is a Hopf subalgebra of H_{2n^2} and the antipode S of H_{2n^2} is extended from R by setting $S(z) = z$. Motivated by Etingof–Walton's Theorem, which says that if a semisimple Hopf algebra acts on a commutative domain then its action factors through a group algebra, Pansera showed in [9] that H_{2n^2} admits an action on a quantum plane A that does not factor through a group algebra, i.e. H_{2n^2} acts inner-faithfully on A.

The purpose of this note is to extend Pansera's construction and to provide further examples of non-trivial semisimple Hopf algebras acting inner-faithfully on non-commutative domains. The basic observation is that, in the construction of H_{2n^2}, the action of the automorphism σ of order 2 can be regarded as the action of the symmetric group Σ_2 on the tensor product $\mathbb{K}[\mathbb{Z}_n] \otimes \mathbb{K}[\mathbb{Z}_n] = \mathbb{K}[\mathbb{Z}_n \times \mathbb{Z}_n]$. As a generalization, we consider Σ_m acting on the m-fold tensor $R = B^{\otimes m}$, for some bialgebra B.[1] The standard generators $s_1, \ldots, s_{m-1}$ of Σ_m, i.e. the transpositions $s_i = (i, i+1)$, generate a free monoid $M = \langle \overline{s}_1, \ldots, \overline{s}_{m-1} \rangle$ that acts on R and allows to consider the skew monoid algebra $R\#M$, which in the case $m = 2$ corresponds to the skew polynomial ring $R[z;\sigma]$. The comultiplication of $\overline{s_i}$ can be defined as $\Delta(\overline{s}_i) = J_i(\overline{s}_i \otimes \overline{s}_i)$, for a suitable twist $J_i \in R \otimes R$. Under further assumptions on these twists J_i, we define a Hopf structure on the quotient $R\#M/I$, where I is the ideal generated by the usual relations of the symmetric group, $\overline{s}_i\overline{s}_{i+1}\overline{s}_i = \overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1}$ and $\overline{s}_i\overline{s}_j = \overline{s}_j\overline{s}_i$ for $|i-j| > 1$, but with $\overline{s}_i^2 = t_i := \mu_R(J_i)$, where μ_R is the multiplication of R. The obtained Hopf algebra H is an extension of $\mathbb{K}\Sigma_m$ by R and can be considered a crossed product $R\#_\gamma\Sigma_m$ for a suitable 2-cocycle $\gamma : \Sigma_m \times \Sigma_m \to R^\times$. For $B = \mathbb{K}[\mathbb{Z}_n]$ and $\mathbb{K}$ containing a primitive nth root of unity, we provide twists J_i that satisfy all our assumptions and yield a family of semisimple Hopf algebras $H_{n,m} = \mathbb{K}[\mathbb{Z}_n]^{\otimes m}\#_\gamma\Sigma_m$ of dimension $n^m m!$. The original Kac–Paljutkin Hopf algebra appears as $H_{2,2}$, while Pansera's algebras appear as $H_{n,2}$.

In the last section we examine actions of $H_{n,m}$ on a quantum polynomial algebra A in m generators $u_1, \ldots, u_m$. We will show that there exists an mn^m-dimensional semisimple Hopf subalgebra H of $H_{n,m}$ that acts inner-faithfully on A and whose ring of invariants A^H is the subring of cyclic polynomials in $u_1^n, \ldots, u_m^n$, in case n is even.

[1] We will denote the symmetric group by Σ_m instead of S_m in order to avoid confusion with the antipode S.

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2. Constructing Hopf algebras via twists

An invertible element $J \in B \otimes B$ of a bialgebra B is called a *twist for* B if

$$(\Delta_B \otimes \mathrm{id}_B)(J)(J \otimes 1_B) = (\mathrm{id}_B \otimes \Delta_B)(J)(1_B \otimes J) \tag{4}$$

and $(\epsilon \otimes \mathrm{id})(J) = 1 = (\mathrm{id} \otimes \epsilon)(J)$ hold. Writing symbolically $J = J^{(1)} \otimes J^{(2)}$, Equation (4) means

$$J_1^{(1)} j^{(1)} \otimes J_2^{(2)} j^{(2)} \otimes J^{(2)} = J^{(1)} \otimes J_1^{(2)} j^{(1)} \otimes J_2^{(2)} j^{(2)}, \tag{5}$$

where $j^{(1)} \otimes j^{(2)}$ is another copy of J and where $\Delta_B(J^{(i)}) = J_1^{(i)} \otimes J_2^{(i)}$ denotes the comultiplication of one of the legs of $J^{(i)}$ of J. Clearly $1_B \otimes 1_B$ is a twist for B. For $B = \mathbb{K}\mathbb{Z}_n$, with generator x, the element $J = \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x^i \otimes x^j$ is a twist for B. We will use twists to deform the comultiplication of a group-like element of a bialgebra as they appear naturally in our construction. Suppose $R \subseteq S$ is an extension of bialgebras with $x \in S \setminus R$ an element such that Rx is a free R-module. Suppose further that $J \in R \otimes R$ is an invertible element, such that $\Delta_S(x) = J(x \otimes x)$ and $\epsilon_S(x) = 1$. Then the coassociativity for $\Delta_S(x)$ and the freeness of Rx imply Equation (4), while the counity will imply $(\epsilon_R \otimes \mathrm{id}_R)(J) = (\mathrm{id}_R \otimes \epsilon_R)(J) = 1$. Hence J is a twist for R. Since our aim is to extend the bialgebra structure of some algebra $R = B^{\otimes m}$ to a skew monoid algebra $R\#M$, where M is generated by some elements $\overline{s}_1, \ldots, \overline{s}_k$, such that $\Delta(\overline{s}_i) = J_i(\overline{s}_i \otimes \overline{s}_i)$ holds for some $J_i \in R \otimes R$, we will need that J_i is a twist for R.

For two bialgebra-twist pairs (B, J) and (B', J'), the element $(\mathrm{id}_B \otimes \tau_{B,B'} \otimes \mathrm{id}_{B'})(J \otimes J')$ is a twist for $R = B \otimes B'$, where $\tau_{B,B'} : B \otimes B' \to B' \otimes B$ denotes the usual flip map. Hence, given a twist $J = J^{(1)} \otimes J^{(2)}$ for B, we could consider

$$J' = \left(J^{(1)} \otimes 1_B\right) \otimes \left(J^{(2)} \otimes 1_B\right) = (\mathrm{id}_B \otimes \tau_{B,B} \otimes \mathrm{id}_B)(J \otimes 1_{B^{\otimes 2}}),$$

which is a twist for $B \otimes B$. However, we would like to embed J differently into $(B \otimes B) \otimes (B \otimes B)$. For this, let $m \geq 2$ and denote by $e_i^m : B \to B^{\otimes m}$, the embedding of B into the ith tensorand of $B^{\otimes m}$. We would like that $(e_i^m \otimes e_j^m)(J) \in B^{\otimes m} \otimes B^{\otimes m}$ is a twist for $B^{\otimes m}$, for any $1 \leq i < j \leq m$. For $m = 2$, we need in particular that $(e_1^2 \otimes e_2^2)(J)$ is a twist for $B \otimes B$. For $J = J^{(1)} \otimes J^{(2)}$, this condition is equivalent to the condition that

$$J' = (e_1^2 \otimes e_2^2)(J) = \left(J^{(1)} \otimes 1_B\right) \otimes \left(1_B \otimes J^{(2)}\right)$$

satisfies Equation (4) for the ring $B \otimes B$, which boils down to

$$J_1^{(1)} j^{(1)} \otimes J_2^{(1)} \otimes j^{(2)} \otimes J^{(2)} = J^{(1)} \otimes j^{(1)} \otimes J_1^{(2)} \otimes J_2^{(2)} j^{(2)}, \tag{6}$$

where $j^{(1)} \otimes j^{(2)}$ is another copy of J. Conversely, Equation (6) is actually equivalent to $(e_1^2 \otimes e_2^2)(J)$ being a twist for $B \otimes B$, because the counit condition is automatically verified. Comparing (6) and (5) shows that the difference lies in the middle tensorands.

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If J is central in $B \otimes B$, then Equation (6) implies (5). It remains unclear whether the converse holds.

Lemma 1. *Let J be a twist for B. Suppose*

$$(e_1^2 \otimes e_2^2)(J) \text{ is a twist for } B \otimes B. \tag{7}$$

Then $(e_i^m \otimes e_j^m)(J)$ is a twist for $B^{\otimes m}$, for any $1 \leq i < j \leq m$.

Proof. Set $R = B^{\otimes m}$. To ease notation, we will drop the superscript m from e_i^m. Let $J' = (e_i \otimes e_j)(J) \otimes R \otimes R$. Note that $\Delta_R e_i = (e_i \otimes e_i)\Delta_B$ and $1_R = e_i(1_B)$. Then

$$\begin{aligned}(\Delta_R \otimes id_R)(J')(J' \otimes 1_R) &= (\Delta_R e_i \otimes e_j)(J)(e_i \otimes e_j \otimes e_j)(J \otimes 1_B)\\ &= (e_i \otimes e_i \otimes e_j)(\Delta_B \otimes \mathrm{id}_B)(J)(e_i \otimes e_j \otimes e_j)(J \otimes 1_B)\\ &= e_i(J_1^{(1)} j^{(1)}) \otimes e_i(J_2^{(1)}) e_j(j^{(2)}) \otimes e_j(J^{(2)})\\ &= (e_i \otimes e_{i,j} \otimes e_j)\left(J_1^{(1)} j^{(1)} \otimes J_2^{(1)} \otimes j^{(2)} \otimes J^{(2)}\right),\end{aligned}$$

where $j^{(1)} \otimes j^{(2)}$ is another copy of J and $e_{i,j} : B^{\otimes 2} \to R$ denotes the embedding of $B \otimes B$ into the ith and jth tensorands of R. Similarly one shows $(\mathrm{id}_R \otimes \Delta_R)(J')(1_R \otimes J') = (e_i \otimes e_{i,j} \otimes e_j)\left(J^{(1)} \otimes j^{(1)} \otimes J_1^{(2)} \otimes J_2^{(2)} j^{(2)}\right)$. Since J satisfies Equation (6), $J_1^{(1)} j^{(1)} \otimes J_2^{(1)} \otimes j^{(2)} \otimes J^{(2)} = j^{(1)} \otimes J_1^{(2)} \otimes J_2^{(2)} j^{(2)}$ and therefore $(\Delta_R \otimes id_R)(J')(J' \otimes 1_R) = (\mathrm{id}_R \otimes \Delta_R)(J')(1_R \otimes J')$. □

The main idea of this note is to use the action of the symmetric group Σ_m on the tensor product $R = B^{\otimes m}$ and to extend the bialgebra structure of R to that of a suitable quotient of a skew monoid algebra. The comultiplication of the newly added generators, however, shall be twisted by some twists of R. A pair (σ, J) of an algebra homomorphism $\sigma : R \to R$ and a twist J is called a twisted homomorphism (see [1]) if $\epsilon\sigma = \epsilon$ and $\Delta\sigma(h) = J(\sigma \otimes \sigma)\Delta(h)J^{-1}$, for $h \in R$. We will be mostly interested in the case where J commutes with the elements of the image of Δ, in which case (σ, J) is a twisted homomorphism if and only if σ is a coalgebra homomorphism. Pansera showed in [9] that if R is a bialgebra with twisted homomorphism (σ, J), then the bialgebra structure of R can be extended to the skew polynomial ring $R[x; \sigma]$, such that $\Delta(x) = J(x \otimes x)$ and $\epsilon(x) = 1$. We will extend this result to $R = B^{\otimes m}$ and a twist J for B that satisfies (7). Here the skew polynomial ring will be replaced by a skew monoid algebra.

Each permutation $s \in \Sigma_m$ acts on $R = B^{\otimes m}$ by $\sigma_s : R \to R$, with $\sigma_s(a_1 \otimes \cdots \otimes a_n) = a_{s(1)} \otimes \cdots \otimes a_{s(n)}$. Note that σ_s is a bialgebra homomorphism of R. Thus $\Delta\sigma_s = (\sigma_s \otimes \sigma_s)\Delta$ and $\epsilon\sigma_s = \epsilon$. Given a twist J for B, such that $(e_1^2 \otimes e_2^2)(J)$ is a twist for $B \otimes B$, we define for each transposition $s = (ij) \in \Sigma_m$, with $i < j$, the element $J_s = (e_i^m \otimes e_j^m)(J) \in R \otimes R$, which is a twist for R by Lemma 1.

Theorem 2. *Let B be a bialgebra with central twist J of B that satisfies (7). Let $m \geq 2$ and $X \subseteq \Sigma_m$ a set of transpositions. Then the free monoid M, generated by $\{\overline{s} : s \in X\}$, acts on $R = B^{\otimes m}$ and the bialgebra structure on R extends to a bialgebra structure on the skew monoid algebra $R\#M$, such that $\Delta(\overline{s}) = J_s(\overline{s} \otimes \overline{s})$ and $\epsilon(\overline{s}) = 1$,*

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for all $s \in X$. Moreover, $\pi : R\#M \to \mathbb{K}[M]$ with $\pi(a\#\overline{w}) = \epsilon(a)\overline{w}$ is a homomorphism of bialgebras.

Proof. Denote by $\mathrm{Aut}(R)$ the group of algebra automorphisms of R. Since M is the free monoid on X, the map $X \to \mathrm{Aut}(R)$ given by $s \mapsto \sigma_s$ extends to a map of monoids $M \to \mathrm{Aut}(R)$, which allows us to consider the skew monoid ring $H = R\#M$, whose basis as a left R-module is given by the words in $\{\overline{s} : s \in X\}$. We identify elements $a \in R$ with elements $a\#1_M$, where 1_M is the identity of M. In particular, for each $s \in X$ and $a \in R$, we have $\overline{s}a = \sigma_s(a)\overline{s}$. To define $\Delta_H : H \to H \otimes H$, we set

$$\Delta_H(a\overline{w}) := \Delta_R(a)J_{s_1}(\overline{s}_1 \otimes \overline{s}_1)\cdots J_{s_k}(\overline{s}_k \otimes \overline{s}_k), \tag{8}$$

for $a \in R$ and $w = s_1 \dots s_k$ a word in X, In particular, $\Delta_{H|_R} = \Delta_R$ and $\Delta_H(\overline{s}) = J_s(\overline{s}\otimes\overline{s})$, for $s \in X$. Since J is central in $B^{\otimes 2}$, J_s is also central in $R\otimes R$, and since σ_s is a bialgebra homomorphism we have

$$\Delta_H(\overline{s})\Delta_R(a) = J_s(\overline{s} \otimes \overline{s})\Delta_R(a) = J_s(\sigma_s \otimes \sigma_s)\Delta_R(a)(\overline{s} \otimes \overline{s}) = \Delta_R(\sigma_s(a))\Delta_H(\overline{s}). \tag{9}$$

Thus, Δ_H is a well-defined algebra homomorphism. The coassociativity follows from the fact that J_s is a twist for R, by Lemma 1:

$$\begin{aligned}(\Delta_H \otimes \mathrm{id}_H)\Delta_H(\overline{s}) &= (\Delta_R \otimes \mathrm{id}_R)\Delta_R(J_s)\,(\Delta_H(\overline{s}) \otimes \overline{s})\\ &= (\Delta_R \otimes \mathrm{id}_R)\Delta_R(J_s)(J_s \otimes 1_R)(\overline{s} \otimes \overline{s} \otimes \overline{s})\\ &= (\mathrm{id}_R \otimes \Delta_R)\Delta_R(J_s)(1_R \otimes J_s)(\overline{s} \otimes \overline{s}) \otimes \overline{s})\\ &= (\mathrm{id}_R \otimes \Delta_R)\Delta_R(J_s)\,(\overline{s} \otimes \Delta_H(\overline{s}))\\ &= (\mathrm{id}_H \otimes \Delta_H)\Delta_H(\overline{s})\end{aligned}$$

Furthermore, set $\epsilon_H(a\overline{w}) = \epsilon_R(a)$, for all $a \in R$ and $w \in M$. Since $\epsilon_R\sigma_s = \epsilon_R$, we conclude that ϵ_H is an algebra homomorphism. For any $a \in R$ and $s \in X$:

$$(\epsilon_H \otimes \mathrm{id}_H)\Delta_H(a\overline{s}) = (\epsilon_H \otimes \mathrm{id}_H)\,(\Delta_R(a)J_s(\overline{s} \otimes \overline{s})) = a(\epsilon_R \otimes \mathrm{id}_R)(J_s)\overline{s} = a\overline{s},$$

since $(\epsilon_R \otimes \mathrm{id}_R)(J_s) = (\epsilon_B \otimes id_B)(J) = 1$. Similarly, one checks $(id_H \otimes \epsilon_H)\Delta_H(a\overline{s}) = 1$. □

Remark 3. Note that $R\#\overline{1} = \{\gamma \in R\#M \mid (id \otimes \pi)\Delta(\gamma) = \gamma \otimes 1\}$. Hence $R\#M$ is a bialgebra extension of $\mathbb{K}[M]$ by R.

Remark 4. Let $s \in \Sigma_m$ be an element of order 2. Then s is a product of disjoint transpositions, say $s = s_1 \cdots s_k$. Since the elements $J_{s_1}, \dots, J_{s_k}$ commute pairwise, the product $J_s := J_{s_1} \cdots J_{s_k}$ is also a twist for R and one could also prove a version of Theorem 2 for subsets $X \subseteq \Sigma_m$ of permutations of order 2.

Denote by μ_R the multiplication of R.

Theorem 5. *Let B be a bialgebra with J a central twist for B that satisfies (7). Let $m \geq 2$, $X \subseteq \Sigma_m$ a set of transpositions, M the free monoid generated by $\{\overline{s} : s \in X\}$ acting on $R = B^{\otimes m}$.*

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(1) *Suppose J satisfies*

$$\Delta_{B\otimes B}(J) = (e_1^2 \otimes e_2^2)(J)(e_2^2 \otimes e_1^2)(J)(J \otimes J). \tag{10}$$

Then the ideal $I = \langle \overline{s}^2 - t_s \,:\, s \in X\rangle$ is a biideal of $R\#M$, where $t_s = \mu_R(J_s)$.

(2) *Suppose moreover that B is a Hopf algebra with antipode S, such that $(S\otimes S)(J) = J$ and $(S \otimes \mathrm{id})(J) = J^{-1} = (\mathrm{id} \otimes S)(J)$. Then $H = (R\#M)/I$ is a Hopf algebra with $S(\overline{s} + I) = \overline{s} + I$ in H, for $s \in X$.*

Proof. (1) Let $s = (ij) \in X$, with $i < j$. Note that $e_{ij}^m := \mu_R(e_i^m \otimes e_j^m) : B \otimes B \to R$ is an algebra homomorphism that satisfies

$$\Delta_R \circ e_{ij}^m = (e_{ij}^m \otimes e_{ij}^m) \circ \Delta_{B\otimes B}, \tag{11}$$

as well as $e_{ij}^m e_1^2 = e_i^m$ and $e_{ij}^m e_2^2 = e_j^m$. Set $J_s := (e_i^m \otimes e_j^m)(J) \in R^{\otimes 2}$ and $t_s = \mu_R(J_s) = e_{ij}^m(J) \in R$. Since J is central in $B \otimes B$, so is t_s in R. Then Equations (10) and (11) imply

$$\begin{aligned}
\Delta_R(t_s) = \Delta_R(e_{ij}^m(J)) &= e_{ij}^m \otimes e_{ij}^m\,(\Delta_{B\otimes B}(J)) \\
&= e_{ij}^m \otimes e_{ij}^m\left((e_1^2 \otimes e_2^2)(J)(e_2^2 \otimes e_1^2)(J)(J \otimes J)\right) \\
&= (e_i^m \otimes e_j^m)(J)(e_j^m \otimes e_i^m)(J)(e_{ij}^m(J) \otimes e_{ij}^m(J)) \\
&= J_s(\sigma_s \otimes \sigma_s)(J_s)(t_s \otimes t_s).
\end{aligned}$$

Hence, $\Delta(\overline{s}^2 - t_s) = J_s(\overline{s} \otimes \overline{s})J_s(\overline{s} \otimes \overline{s}) - \Delta(t_s) = J_s(\sigma_s \otimes \sigma_s)(J)\left(\overline{s}^2 \otimes \overline{s}^2 - t_s \otimes t_s\right)$ is an element of $I \otimes (R\#M) + (R\#M) \otimes I$. Since $\epsilon_H(t_s) = \epsilon_R(\mu_R(J))) = 1 = \epsilon_H(\overline{s}^2)$, we conclude that I is a coideal of $R\#M$.

(2) Set $H = (R\#M)/I$ and assume that B is a Hopf algebra with antipode S. Then R is a Hopf algebra with antipode $S^{\otimes m}$, which we will also denote by S. Since $(S\otimes S)(J) = J$ holds, we have $S(t_s) = S(e_{ij}^m(J)) = e_{ij}^m(S \otimes S)(J) = t_s$. Hence, $S(\overline{s}) = \overline{s}$ modulo I is well defined. We also set

$$S(a\overline{s}_1 \cdots \overline{s}_k + I) = \overline{s}_k \cdots \overline{s}_1 S(a) + I, \tag{12}$$

for all $a \in R$ and $s_1, \ldots, s_k \in X$. Since $S\sigma_s = \sigma_s S$, definition (12) leads to a well-defined algebra anti-homomorphism $S : H \to H$. Writing J again symbolically as $J = J^{(1)} \otimes J^{(2)}$, we calculate modulo I:

$$\begin{aligned}
\mu_H(\mathrm{id}_H \otimes S)\Delta_H(\overline{s}) &= \mu_H(\mathrm{id}_H \otimes S)J_s(\overline{s} \otimes \overline{s}) \\
&= e_i^m(J^{(1)})\overline{s}S(\overline{s})S(e_j^m(J^{(2)})) \\
&= e_i^m(J^{(1)})S(e_j^m(J^{(2)}))t_s \\
&= e_{ij}^m\left((id \otimes S)(J)\right) e_{ij}^m(J) = e_{ij}^m\left(J^{-1}J\right) = 1
\end{aligned}$$

Similarly $\mu_H(S \otimes \mathrm{id}_H)\Delta_H(\overline{s}) = 1$ holds modulo I, because of $(S \otimes \mathrm{id})(J) = J^{-1}$. Thus S defines an antipode for the bialgebra H. □

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Since our aim is to construct semisimple Hopf algebra quotients of $R\#M$, they must be necessarily finite dimensional over $\mathbb{K}$. Thus we will consider quotients of $R\#M$, by using the usual Coxeter presentation of Σ_m with generators $s_i = (i, i+1)$.

Theorem 6 (Extension by the symmetric group). *Let B be a Hopf algebra with antipode S and J a central twist for B that satisfies (7) and (10), such that $(S\otimes S)(J) = J$ and $(\mathrm{id}\otimes S)(J) = J^{-1} = (S\otimes \mathrm{id})(J)$. For any $m \geq 2$, let $X = \{s_1, \ldots, s_{m-1}\}$ be the set of transposition $s_i = (i, i+1)$ that generates Σ_m and let M be the free monoid generated by $\{\overline{s_i} : 1 \leq i < m\}$, which acts naturally on $R = B^{\otimes m}$. Consider the ideal I of $R\#M$ generated by*

$$\overline{s}_i^2 - t_{s_i}, \qquad \overline{s}_i\overline{s}_{i+1}\overline{s}_i - \overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1}, \qquad \overline{s}_i\overline{s}_j - \overline{s}_j\overline{s}_i, \qquad \forall i, j \text{ with } |i-j| > 1.$$

Then $H = (R\#M)/I$ is a Hopf algebra of dimension $\dim(B)^m m!$ with Hopf subalgebra $R = B^{\otimes m}$ and Hopf quotient $\mathbb{K}\Sigma_m$. Furthermore,

(1) *$\{\overline{w} : w \in \Sigma_m\}$ is an R-basis of H;*
(2) *there exists a 2-cocycle $\gamma : \Sigma_m \times \Sigma_m \to Z(R)^\times$ such that $H \simeq R\#_\gamma\Sigma_m$ with multiplication given by*

$$(a\#\overline{w})(b\#\overline{v}) = a\sigma_w(b)\gamma(w, v)\#\overline{wv}. \tag{13}$$

for all $a, b \in R$ and $w, v \in \Sigma_m$, where $\gamma(w, v)$ is a product of elements $\sigma_u(t_s)$, for some $u \in \Sigma_m$ and $s \in X$.
(3) *$\epsilon(\gamma(w, v)) = 1$, for all $w, v \in \Sigma_m$.*
(4) *if B is semisimple, then H is semisimple with integral $\int \sum_{w\in\Sigma_m} \overline{w}$, where $\int$ is the integral of R.*

Proof. From Theorem 5 we know already that $\langle \overline{s}_i^2 - t_{s_i} : 1 \leq i < m\rangle$ is a coideal of $R\#M$. Furthermore, for i, j such that $|i-j| > 1$, the transpositions s_i and s_j are disjoint. Hence J_{s_i} and $\overline{s}_j \otimes \overline{s}_j$, as well as J_{s_j} and $\overline{s}_i \otimes \overline{s}_i$ commute in $R\#M$. Thus

$$\Delta(\overline{s}_i\overline{s}_j - \overline{s}_j\overline{s}_i) = J_{s_i}J_{s_j}\,(\overline{s}_i\overline{s}_j \otimes \overline{s}_i\overline{s}_j - \overline{s}_j\overline{s}_i \otimes \overline{s}_j\overline{s}_i) \in (R\#M)\otimes I + I \otimes (R\#M).$$

For $i < m-1$, we have $\sigma_{s_i}e_i^m = e_{i+1}^m$ and $\sigma_{s_i}e_{i+1}^m = e_i^m$, while $\sigma_{s_i}e_j^m = e_j^m$ if $j \notin \{i, i+1\}$. Hence

$$(\sigma_{s_i} \otimes \sigma_{s_i})J_{s_{i+1}} = (e_i^m \otimes e_{i+2}^m)(J) = (\sigma_{s_{i+1}} \otimes \sigma_{s_{i+1}})J_{s_i} \tag{14}$$

In particular, $(\sigma_{s_i}\sigma_{s_{i+1}} \otimes \sigma_{s_i}\sigma_{s_{i+1}})(J_{s_i}) = J_{s_{i+1}}$ and $(\sigma_{s_{i+1}}\sigma_{s_i} \otimes \sigma_{s_{i+1}}\sigma_{s_i})(J_{s_{i+1}}) = J_{s_i}$. Thus,

$$J_{s_i}(\sigma_{s_i} \otimes \sigma_{s_i})(J_{s_{i+1}})(\sigma_{s_i}\sigma_{s_{i+1}} \otimes \sigma_{s_i}\sigma_{s_{i+1}})(J_{s_i}) = J_{s_i}(\sigma_{s_i} \otimes \sigma_{s_i})(J_{s_{i+1}})J_{s_{i+1}} \tag{15}$$

$$J_{s_{i+1}}(\sigma_{s_{i+1}} \otimes \sigma_{s_{i+1}})(J_{s_i})(\sigma_{s_{i+1}}\sigma_{s_i} \otimes \sigma_{s_{i+1}}\sigma_{s_i})(J_{s_{i+1}}) = J_{s_{i+1}}(\sigma_{s_i} \otimes \sigma_{s_i})(J_{s_{i+1}})J_{s_i} \tag{16}$$

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Since J is central, the values in (15) and (16) are equal; call it γ. Then

$$\Delta(\overline{s}_i\overline{s}_{i+1}\overline{s}_i) = J_{s_1}(\overline{s}_i \otimes \overline{s}_i)J_{s_{i+1}}(\overline{s}_{i+1} \otimes \overline{s}_{i+1})J_{s_1}(\overline{s}_i \otimes \overline{s}_i) = \gamma(\overline{s}_i\overline{s}_{i+1}\overline{s}_i \otimes \overline{s}_i\overline{s}_{i+1}\overline{s}_i)$$

and similarly $\Delta(\overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1}) = \gamma(\overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1} \otimes \overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1})$. Hence $\Delta(\overline{s}_i\overline{s}_{i+1}\overline{s}_i - \overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1})$ is an element of $I \otimes (R\#M) + (R\#M) \otimes I$. Defining $S(\overline{s}_i) = \overline{s}_i$, for all i, we obtain that $H = (R\#M)/I$ is a Hopf algebra. Note that $R\#M$ is graded by the length of words in M. Since elements of the form $\overline{s}_i\overline{s}_j - \overline{s}_j\overline{s}_i$ and $\overline{s}_i\overline{s}_{i+1}\overline{s}_i - \overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1}$ are homogeneous of degree 2 and 3 respectively, the only possible $a \in R \cap I$ must be of the form $a = \sum_{i=1}^{m-1} a_i(\overline{s}_i^2 - t_i)$. Since words in M form an R-basis of $R\#M$, we must have $a_i = 0$, for all i; hence $R \cap I = 0$ and R embeds into $H = (R\#M)/I$ as a Hopf subalgebra of H. Actually, R is a normal Hopf subalgebra and $HR^+ = R^+H$ is a Hopf ideal of H, where $R^+ = R \cap \mathrm{Ker}(\epsilon)$.

(1) Since $s_1, \ldots, s_{n-1}$ generate Σ_m, we choose for each $w \in \Sigma_m$ a representation $w = s_{i_1} \cdots s_{i_k}$, with k minimal. We will write $\overline{w} = \overline{s}_{i_1} \cdots \overline{s}_{i_k}$, for the corresponding element in $R\#M$. Since the s_i satisfy $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ and $s_is_j = s_js_i$, for $|i-j| > 1$, any word in $\overline{s}_1, \ldots, \overline{s}_{m-1}$ reduces to an element $\overline{w}$ modulo I. Thus $\{\overline{w} + I : w \in \Sigma_m\}$ generates H as a left R-module. On the other hand, an element of the form $\sum_{w\in\Sigma_m} a_w\overline{w}$, with $a_w \in R$ belongs to I if and only if $a_w = 0$, for any $w \in \Sigma_m$. Thus H is free over R with basis $\{\overline{w} : w \in \Sigma_m\}$ and $\dim(H) = \dim(R)^m m!$ follows.

(2) Let $w, v \in \Sigma_m$ then there exists a unique element $\gamma(w, v) \in R$ such that $\overline{w}\,\overline{v} = \gamma(w, v)\overline{wv}$. More precisely, let $w = s_{i_1} \cdots s_{i_k}$ and $v = s_{j_1} \cdots s_{j_l}$ be the chosen minimal presentations of w and v. If wv has length $k + l$, then set $\gamma(w, v) = 1$. If the length of wv is less than $k + l$, then some reduction occurred, which means that some relations of the type $s_i^2 = \mathrm{id}$ must have reduced the length of wv. In H, the reduction takes the form of replacing $\overline{s}_i^2$ by $t_{s_i} \in R^\times$. Depending on where in wv the replacement takes place, some automorphism of the form σ_u is applied to shift t_{s_i} to the 'left' of $\overline{wv}$. Hence $\overline{w}\,\overline{v} = \gamma(w, v)\overline{wv}$, where $\gamma(w, v)$ is a product of elements of the form $\sigma_u(t_{s_i})$, for some $u \in \Sigma_m$ and some i.

(3) Since J_{s_i} is central in $R \otimes R$, also $t_{s_i} = \mu_R(J_{s_i})$ is central in R. Furthermore, as $(\epsilon_B \otimes \mathrm{id}_B)(J) = 1$, also $\epsilon(t_{s_i}) = 1$ and therefore $\epsilon(\sigma_u(t_{s_i})) = 1$. Thus $\epsilon(\gamma(w, v)) = 1$, for any $w, v \in \Sigma_m$.

(4) If B is semisimple, then R is semisimple and the antipode of R is an involution. Since $S^2(\overline{w}) = \overline{w}$, for $w \in \Sigma_m$, the antipode of H is an involution and H is semisimple. Let $\int_B$ be the integral of B, then $\int = \int_B \otimes \cdots \otimes \int_B \in R = B^{\otimes m}$ is an integral for R. Note that $\int$ is invariant under the action of Σ_m. Hence, for $v \in \Sigma_m$, we have

$$\overline{v} \int \sum_{w\in\Sigma_m} \overline{w} = \sigma_v(\int) \sum_{w\in\Sigma_m} \gamma(v, w)\overline{vw} = \sum_{w\in\Sigma_m} \int \epsilon(\gamma(v, w))\overline{vw} = \int \sum_{w\in\Sigma_m} \overline{w}. \qquad (17)$$

□

3. Generalized Kac–Paljutkin algebras

The group algebra of a finite cyclic group provides a twist that satisfies all the requirements of the last Theorem. Let $n, m \geq 2$, q a primitive nth root of unity and $\mathbb{Z}_n = \langle x : x^n = 1\rangle$ the cyclic group of order n. For each $0 \leq k < n$ set

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$$e_k = \frac{1}{n}\sum_{i=0}^{n-1} q^{-ik}x^i.$$

Then $\{e_0, \dots, e_{n-1}\}$ is a complete set of orthogonal idempotents of $B = \mathbb{K}\mathbb{Z}_n$.

Lemma 7. *Let $B = \mathbb{K}\mathbb{Z}_n$. Then $J = \sum_{k=0}^{n-1} e_k \otimes x^k \in B \otimes B$ is a twist for B satisfying (7) and (10), such that $(S \otimes S)(J) = J$ and $(\mathrm{id} \otimes S)(J) = J^{-1} = (S \otimes \mathrm{id})(J)$ hold.*

Proof. It is well-known that J is a twist for B. We calculate $(S \otimes S)(J) = \frac{1}{n}\sum_{i,k=0}^{n-1} q^{-ik}x^{-i} \otimes x^{-k} = J$. Moreover,

$$(\mathrm{id} \otimes S)(J)J = \sum_{k,l=0}^{n-1} e_k e_l \otimes x^{l-k} = \sum_{k=0}^{n-1} e_k \otimes 1_B = 1_B \otimes 1_B, \tag{18}$$

shows $(\mathrm{id} \otimes S)(J) = J^{-1}$. Since $S^2 = \mathrm{id}$ and $(S \otimes S)(J^{-1}) = J^{-1}$, also $(S \otimes \mathrm{id})(J) = J^{-1}$ holds. Next we show that

$$J' = (e_1^2 \otimes e_2^2)(J) = \sum_{k=0}^{n-1} e_k \otimes 1_B \otimes 1_B \otimes x^k = \sum_{k=0}^{n-1} x^k \otimes 1_B \otimes 1_B \otimes e_k$$

is a twist for $B \otimes B$:

$$\begin{aligned}
&(\Delta_{B\otimes B} \otimes \mathrm{id}_{B\otimes B})(J')(J' \otimes 1_{B\otimes B}) \\
&\quad = \sum_{k,l} (\Delta_{B\otimes B}(x^k \otimes 1) \otimes 1 \otimes e_k)(e_l \otimes 1 \otimes 1 \otimes x^l \otimes 1 \otimes 1) \\
&\quad = \sum_{k,l} x^k e_l \otimes 1 \otimes x^k \otimes x^l \otimes 1 \otimes e_k \\
&\quad = \sum_{k,l} e_l \otimes 1 \otimes x^k \otimes x^l \otimes 1 \otimes x^l e_k \\
&\quad = \sum_{k,l} \left(e_l \otimes 1 \otimes \Delta_{B\otimes B}(1 \otimes x^l)\right)(1 \otimes 1 \otimes x^k \otimes 1 \otimes 1 \otimes e_k) \\
&\quad = (\mathrm{id}_{B\otimes B} \otimes \Delta_{B\otimes B}(J')(1_{B\otimes B} \otimes J')
\end{aligned}$$

where we use $x^k e_l = q^{kl} e_l$ and $q^{kl} e_k = x^l e_k$. Lastly, we will verify Equation (10). For this consider

$$\begin{aligned}
(e_1 \otimes e_2)(J)(e_2 \otimes e_1)(J)(J \otimes J) &= \sum_{k,l,s,t} (e_k \otimes x^l \otimes e_l \otimes x^k)(e_s \otimes x^s \otimes e_t \otimes x^t) \\
&= \sum_{k,l} e_k \otimes x^{k+l} \otimes e_l \otimes x^{k+l} \\
&= \sum_{k,j} e_k \otimes x^j \otimes e_{j-k} \otimes x^j = \Delta_{B\otimes B}(J),
\end{aligned}$$

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because $\Delta_B(e_j) = \sum_k e_k \otimes e_{j-k}$ holds. □

From Theorem 6 and Lemma 7, we deduce that there exists a non-trivial semisimple Hopf algebra $H_{n,m} = (\mathbb{K}\mathbb{Z}_n)^{\otimes m} \#_\gamma \Sigma_m$ of dimension $n^m m!$, for any integers $n, m \geq 2$ (provided $\mathbb{K}$ contains a primitive nth root of unity).

Theorem 8 (Generalized Kac–Paljutkin algebras). *For any $n, m \geq 2$, $\mathbb{K}$ a field of characteristic zero with primitive nth root of unity q, there exists a semisimple Hopf algebra $H_{n,m} = \mathbb{K}\mathbb{Z}_n^{\otimes m} \#_\gamma \Sigma_m$ of dimension $n^m m!$ that is an extension of $\mathbb{K}\Sigma_m$, by $\mathbb{K}\mathbb{Z}_n^{\otimes m}$, which is generated by elements $x_1, \ldots, x_m, z_1, \ldots, z_{m-1}$ such that*

$$x_i^n = 1, \qquad x_i x_j = x_i x_j, \qquad z_k x_i = x_{s_k(i)} z_k, \qquad \forall\, i, j, k,$$

where s_k is the transposition $s_k = (k, k+1)$. Furthermore, for all k, l with $|k - l| > 1$:

$$z_k z_l = z_l z_k, \qquad z_k z_{k+1} z_k = z_{k+1} z_k z_{k+1}, \qquad z_k^2 = \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x_k^i x_{k+1}^j. \tag{19}$$

The elements x_i are group-like, while

$$\Delta(z_k) = \left(\frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x_k^i \otimes x_{k+1}^j \right) (z_k \otimes z_k), \quad \epsilon(z_k) = 1, \quad S(z_k) = z_k. \tag{20}$$

As before, we write $t_k := \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x_k^i x_{k+1}^j$ and $J_k := \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x_k^i \otimes x_{k+1}^j$. Since $(\mathrm{id} \otimes S)(J_k) = J_k^{-1}$, we also have

$$t_k^{-1} = \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x_k^i x_{k+1}^{-j}.$$

Remark 9. Let $G := \mathbb{Z}_n^m$. Then $\mathbb{K}G \simeq \mathbb{K}\mathbb{Z}_n^{\otimes m}$. By construction, $\mathbb{K}G$ is a Hopf subalgebra of $H := H_{n,m}$ and $\mathbb{K}\Sigma_m$ is a Hopf quotient algebra of H. Since the ring of $\mathbb{K}\Sigma_m$-coninvariants of H is $\mathbb{K}G$ and since $\mathbb{K}G \simeq \mathbb{K}^G$ as Hopf algebras,[2] as G is finite Abelian, the Hopf algebra H is an extension of $\mathbb{K}\Sigma_m$ by $\mathbb{K}^G$ and hence equivalent to an element of $\mathrm{Opext}(\mathbb{K}\Sigma_m, \mathbb{K}^G)$, the set of equivalence classes of all extensions of $\mathbb{K}\Sigma_m$ by $\mathbb{K}^G$, associated with a matched pair (Σ_m, G). Here the matched pair is given by Σ_m acting on G by permutations of its components and G acting trivially on Σ. The set $\mathrm{Opext}(\mathbb{K}\Sigma_m, \mathbb{K}^G)$ forms a group with identity element being the group ring $\mathbb{K}[G \rtimes \Sigma_m]$ of the semidirect product of G and Σ_m. Since $H_{n,m}$ is not cocommutative, $H_{n,m}$ is equivalent to a non-trivial element of $\mathrm{Opext}(\mathbb{K}\Sigma_m, \mathbb{K}^G)$. By [8, Proposition 1.5], elements

[2] More precisely, for any $\alpha \in \{0, \ldots, n-1\}^m$ set $e_\alpha := \frac{1}{n^m} \prod_{k=1}^m \sum_{i=0}^{n-1} q^{-i\alpha_k} x_k^i \in \mathbb{K}[G]$. Then $\{e_\alpha : \alpha \in \{0, \ldots, n-1\}^m\}$ is a complete set of orthogonal idempotents of $\mathbb{K}[G]$. Equally, the projections $p_\alpha \in \mathbb{K}^G$ defined by $p_\alpha(x^\beta) = 1$ if and only if $\alpha = \beta$ and 0 otherwise, for any $\beta \in \{0, \ldots, n-1\}^m$, where $x^\beta := x_1^{\beta_1} \cdots x_m^{\beta_m}$, form a complete set of orthogonal idempotents of $\mathbb{K}^G$. Defining $\psi : \mathbb{K}[G] \to \mathbb{K}^G$ with $\psi(e_\alpha) = p_\alpha$ yields an algebra homomorphism, which is actually a Hopf algebra isomorphism.

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of $\mathrm{Opext}(\mathbb{K}\Sigma_m, \mathbb{K}^G)$ are equivalent to a bicrossed products $\mathbb{K}^G \#_\gamma^\tau \mathbb{K}\Sigma_m$, for cocycles $\gamma : \Sigma_m \times \Sigma_m \to \mathbb{K}^G$ and $\tau : G \times G \to \mathbb{K}^{\Sigma_m}$. From the multiplication of $H_{n,m}$, we see that τ is constant $1_{K^{\Sigma_m}}$, while γ depends on the chosen twist J.

For $m = 2$, the elements of $\mathrm{Opext}(\mathbb{K}\Sigma_2, \mathbb{K}^{\mathbb{Z}_n \times \mathbb{Z}_n})$ have been classified in [7, Theorem 2.1] and are in bijective correspondence with all nth roots of unity in $\mathbb{K}$. Thus, the Hopf algebras $H_{n,2}$ had already been discovered in [7], prior to [10]. In particular, for each root of unity q, there exists a twist J (as defined above) that determines the comultiplication of such an extension via $\Delta(z) = J(z \otimes z)$. The cocycle γ is determined by the element $t = \mu(J)$, where $\gamma(s, s) = t$ for the generator s of $\Sigma_2 = \mathbb{Z}_2$.

For $m \geq 3$, it would be interesting to classify $\mathrm{Opext}(\mathbb{K}\Sigma_m, \mathbb{K}^G)$ in general, whereas $H_{n,m}$ provides a non-trivial example of it. Again, for each n-th root of unity q, we use the same twist J of $B = \mathbb{K}\mathbb{Z}_n$ as above. For each Coxeter generator s_i of Σ_m, we deform the comultiplication of $z_i = \overline{s_i}$ by defining $\Delta(z_i) = J_i(z_i \otimes z_i)$, where $J_i = (e_i^m \otimes e_{i+1}^m)(J) \in R \otimes R$ for $R = B^{\otimes m} = \mathbb{K}G$ means that J is placed into the i-th and $(i+1)$-st components, respectively. As Example 11 illustrates, the cocycle γ is determined by $t = \mu(J)$, which is embedded suitably into R such that $\gamma(s_i, s_i) = t_i := \mu(J_i)$ and $\gamma(s_i, s_j) = 1$ for $i \neq j$. Hence, the non-triviality of $H_{n,m}$ as an element of $\mathrm{Opext}(\mathbb{K}\Sigma_m, \mathbb{K}^G)$ stems essentially from the non-triviality of the Kac–Paljutkin algebra $H_{n,2}$ and has just been placed into an m-fold tensor product of $\mathbb{K}\mathbb{Z}_n$.

Note that it is shown in [4] that $H_{n,m}$ is isomorphic, as an algebra, to the group ring $\mathbb{K}[\mathbb{Z}_n \wr \Sigma_m]$ of the wreath product of $\mathbb{Z}_n$ and Σ_m. Since the wreath product $\mathbb{Z}_n \wr \Sigma_m$ is the semidirect $\mathbb{Z}_n^m \rtimes \Sigma_m$, the Hopf algebra $H_{n,m}$ is isomorphic as an algebra to the smash product $\mathbb{K}^G \# \mathbb{K}\Sigma_m$ (without cocycle). However, due to the fact that $H_{n,m}$ is not cocommutative, this isomorphism is not an isomorphism of Hopf algebras.

Example 10 (Case $m = 2$). Write x for the generator of $B = \mathbb{K}\mathbb{Z}_n$ and write the elements of $R = B^{\otimes 2} = \mathbb{K}\mathbb{Z}_n \otimes \mathbb{K}\mathbb{Z}_n$ as linear combinations of elements $x^i y^j$, where x stands for $x \otimes 1$ and y for $1 \otimes x$. Since $m = 2$, we have only one transposition, which is $s = (12)$. Hence $\sigma_s(x) = y$, $\sigma_s(y) = x$ and the twist $J = \sum_{j=0}^{n-1} e_j \otimes x^j$ for B becomes the twist

$$J_s = \sum_{j=0}^{n-1} e_j \otimes y^j = \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x^i \otimes y^j$$

for R. Instead of $\overline{s}$ write z for the generator of the monoid $M = \langle \overline{s} \rangle$. Set $t := \mu(J_s) = \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x^i y^j$. The Hopf algebra $H_{n,2} = (R \# M)/\langle z^2 - t \rangle$ has therefore generators x, y, z subject to

$$xy = yx, \quad x^n = 1 = y^n, \quad zx = yz, \quad zy = xz, \quad z^2 = t.$$

While x and y are group-like,

$$\Delta(z) = \left(\frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x^i \otimes y^j \right) (z \otimes z).$$

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A basis for $H_{n,2}$ are the monomials $\{x^iy^jz^k : 0 \leq i,j < n, k \in \{0,1\}\}$. Hence $H_{n,2}$ is precisely the generalized Kac–Paljutkin algebra H_{2n^2} considered by Pansera. In particular, $H_{2,2}$ is the Kac–Paljutkin Hopf algebra itself. Note that $H_{n,2} = \mathbb{K}[\mathbb{Z}_n \times \mathbb{Z}_n]\#_\gamma \Sigma_2$, for $\gamma : \Sigma_2 \times \Sigma_2 \to R^\times$ given by

$$\gamma(\mathrm{id},\mathrm{id}) = \gamma(\mathrm{id},(12)) = \gamma((12),\mathrm{id}) = 1, \qquad \gamma((12),(12)) = t.$$

Example 11 (Case $m = 3$). Let q be a primitive third root of unity. Write x for the generator of $B = \mathbb{K}\mathbb{Z}_n$ and $R = B^{\otimes 3} = \mathbb{K}\mathbb{Z}_n \otimes \mathbb{K}\mathbb{Z}_n \otimes \mathbb{K}\mathbb{Z}_n$. Elements of R are linear combinations of monomials $x_1^i x_2^j x_3^k$, where x_1 stands for $x \otimes 1 \otimes 1$, x_2 stands for $1 \otimes x \otimes 1$ and x_3 stands for $1 \otimes 1 \otimes x$. The transpositions $s_1 = (12)$ and $s_2 = (23)$ are generators of Σ_3. Write $\sigma_i = \sigma_{s_i}$ for the corresponding automorphism of R. The twist $J = \sum_{j=0}^{n-1} e_j \otimes x^j$ for $\mathbb{K}\mathbb{Z}_n$ yields two twists

$$J_{s_1} = \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x_1^i \otimes x_2^j \qquad \text{and} \qquad J_{s_2} = \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x_2^i \otimes x_3^j$$

for R. Set $z_1 := \overline{s_1}$ and $z_2 := \overline{s_2}$ for the generator of the free monoid $M = \langle \overline{s_1}, \overline{s_2} \rangle$. The Hopf algebra

$$H_{n,3} = (R\#M)/\langle z_1^2 - t_1, z_2^2 - t_2, z_1z_2z_1 - z_2z_1z_2\rangle = (\mathbb{K}\mathbb{Z}_n)^{\otimes 3}\#_\gamma \Sigma_3 \tag{21}$$

has therefore generators x_1, x_2, x_3, z_1, z_2 subject to $x_ix_j = x_jx_i$ and $x_i^n = 1$, for all i, j and

$$z_1x_1 = x_2z_1, \quad z_1x_2 = x_1z_1, \quad z_1x_3 = x_3z_1$$

$$z_2x_1 = x_1z_2, \quad z_2x_2 = x_3z_2, \quad z_2x_3 = x_2z_2$$

$$z_1z_2z_1 = z_2z_1z_2, \qquad z_1^2 = t_1 := \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x_1^i x_2^j, \qquad z_2^2 = t_2 := \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x_2^i x_3^j.$$

While the x_i are group-like, we have

$$\Delta(z_1) = \left(\frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x_1^i \otimes x_2^j\right)(z_1 \otimes z_1) \qquad \Delta(z_2) = \left(\frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x_2^i \otimes x_3^j\right)(z_2 \otimes z_2).$$

$H_{n,3}$ has dimension $6n^3$ and basis $\left\{x_1^i\ x_2^j\ x_3^k\ \overline{w} : 0 \leq i,j,l < n,\ \overline{w} \in \{1, z_1, z_2, z_1z_2, z_2z_1, z_1z_2z_1\}\right\}$. The 2-cocycle γ is determined by the multiplication in $H_{n,3}$ and given by $\gamma(\mathrm{id}, w) = \gamma(w, \mathrm{id}) = 1$, for $w \in \Sigma_3$, and the following table. Note that $t_2 = \sigma_1\sigma_2(t_1) = t^{\sigma_2\sigma_1}$ and $\sigma_1(t_2) = \sigma_2(t_1) = \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x_1^i x_3^j = t^{\sigma_2}$. The values of the 2-cocycle γ can be calculated and are as follows:

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γ	s_1	s_2	s_1s_2	s_2s_1	$s_1s_2s_1$
s_1	t_1	1	t_1	1	t_1
s_2	1	t_2	1	t_2	t_2
s_1s_2	1	t_3	t_1	t_1t_3	t_1t_3
s_2s_1	t_3	1	t_2t_3	t_2	t_2t_3
$s_1s_2s_1$	t_2	t_1	t_2t_3	t_1t_3	$t_1t_2t_3$

Remark 12. We see that $H_{n,2}$ embeds into $H_{n,3}$ by sending x, y, z of $H_{n,2}$ to the corresponding elements x_1, x_2, z_1 in $H_{n,3}$. More generally, for $n, m \geq 2$, the algebra $R = \mathbb{K}\mathbb{Z}_n^{\otimes m}$ embeds into $R' = \mathbb{K}\mathbb{Z}_n^{\otimes m+1}$ by sending $a_1 \otimes \cdots \otimes a_m$ into $a_1 \otimes \cdots \otimes a_m \otimes 1_B$. Similarly, Σ_m can be considered a subgroup of Σ_{m+1}, where we identify the generators $s_1, \ldots, s_{m-1}$ with the first $m-1$ generators of Σ_{m+1}. Hence, if $x_1, \ldots, x_m$ denotes the basis of $R = \mathbb{K}\mathbb{Z}_n^{\otimes m+1}$ and $z_1, \ldots, z_{m-1}$ the remaining algebra generators of $H_{n,m}$ and if $x'_1, \ldots, x'_{m+1}$ denote the basis of R' and $z'_1, \ldots, z'_{m+1}$ the remaining generators of $H_{n,m+1}$, then mapping x_i to x'_i and z_i to z'_i yields an algebra embedding of $H_{n,m}$ into $H_{n,m+1}$, which is also an embedding of Hopf algebras.

Remark 13. Let $H = H_{n,m}$ and $w \in \Sigma_m$. We claim that the comultiplication of $\overline{w}$ in H looks like $\Delta(\overline{w}) = J(w)(\overline{w} \otimes \overline{w})$ for an invertible element $J(w) \in R \otimes R$ with $\epsilon(\mu_R(J(w))) = 1$. We will prove this by induction on the length of the chosen representation $w = s_{i_1} \ldots s_{i_k}$ in the generators $s_1, \ldots s_{m-1}$. For $k = 1$, we have $w = s_i$, for some i and by definition $\Delta(z_i) = J_i(z_i \otimes z_i)$ for $z_i = \overline{s_i}$ and $J_i = J_{s_i} = (e_i^m \otimes e_{i+1}^m)(J)$, which is invertible, since J is invertible. Moreover, $\mu_R(J_i) = e_{i,i+1}^m(J) = t_i$ and $\epsilon(t_i) = 1$. Now suppose that $w = s_i v$ has length greater than 1, where $v \in \Sigma_m$ and there exists $J(v) \in R \otimes R$ such that $\Delta(\overline{v}) = J(v)(\overline{v} \otimes \overline{v})$. Then

$$\Delta(\overline{w}) = J_i(z_i \otimes z_i)J(v)(\overline{v} \otimes \overline{v}) = J_i J(v)^{\sigma_i}(\overline{w} \otimes \overline{w}), \tag{22}$$

where we denote by $J(w)^{\sigma_i} = (\sigma_i \otimes \sigma_i)(J(w))$. Setting $J(w) = J(s_i v) = J_i J(v)^{\sigma_i}$ proves our claim, since also $\mu_R(J(w)) = t_i \sigma_i(\mu_R(J(v)))$ has counit 1.

Thus, given a subgroup N of Σ_m, we can consider the Hopf subalgebra of H generated by R and $\{\overline{w} : w \in N\}$, which we shall denote by $R\#_\gamma N$. Since for any $w \in N$, $\Delta(\overline{w}) = J(w)(\overline{w} \otimes \overline{w}) \in (R\#_\gamma N) \otimes (R\#_\gamma N)$ we conclude that the subalgebra $R\#_\gamma N$ is a semisimple Hopf subalgebra of H.

Example 14. Consider $H_{n,m} = R\#_\gamma \Sigma_m$ with generators as above and consider θ, the product of all $z'_i s$, i.e. $\theta = z_1 \cdots z_{m-1}$. Let H be the subalgebra of $H_{n,m}$ generated by R and θ. We claim that H is equal to $R\#_\gamma\langle s\rangle$, where $s = (12\cdots m)$ is the cycle of length m. Clearly, $\theta = \overline{s}$ and we claim that $\theta^k = \left(\prod_{i=1}^{k-1} \gamma(s^i, s)\right) \overline{s^k}$, for $2 \leq k \leq m$, since inductively, if $\theta^k = c_k \overline{s^k}$, for some $c_k \in R^\times$, then $\theta^{k+1} = c_k \overline{s^k}\,\overline{s} = c_k \gamma(s^k, s)\overline{s^{k+1}}$. Thus $H = R\#_\gamma\langle s\rangle$, as $\left(\prod_{i=1}^{k-1} \gamma(s^i, s)\right) \in R^\times$. Note that $\theta^m = \left(\prod_{i=1}^{m-1} \gamma(s^i, s)\right) =: t \in R^\times$. Hence $H \simeq R[\theta; \sigma]/\langle \theta^m - t\rangle$, where $\sigma = \sigma_s$. In particular, H is generated by $x_1, \ldots, x_m$ and θ subject to $\theta x_i = x_{s(i)}\theta$ and $\theta^m = t$. While $\Delta(\theta) = J(s)(\theta \otimes \theta)$, with $J(s)$ as

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in Remark 13. Note that $\theta^{m-1} = \tilde{t}\,\overline{s}^{m-1}$, for $\tilde{t} = t\gamma(s^{m-1}, s)^{-1}$. Since $s^{-1} = s^{m-1}$ we obtain

$$\theta^{m-1} = \tilde{t}\,\overline{s}^{-1} = \tilde{t}\,z_{m-1}\cdots z_1 = \tilde{t}S(\theta). \tag{23}$$

Hence $S(\theta) = \tilde{t}^{-1}\theta^{m-1} = t^{-1}\gamma(s^{m-1}, s)\theta^{m-1}$. The dimension of H is mn^m. For $m = 2$, we obtain $H' = H_{n,2}$.

4. Actions of the generalized Kac–Paljutkin algebra

As before, let $H = H_{n,m} = R\#_\gamma\Sigma_m$, with $R = B^{\otimes m}$ and $B = \mathbb{K}\mathbb{Z}_n$, for some $n, m \geq 2$.

The aim of this section is to define an action of H on a non-commutative algebra A such that the action does not factor through a group algebra – in other words, the action is inner-faithful. To achieve this, we first examine m-dimensional simple H-modules V (Lemma 15) in order to define a quantum polynomial algebra A generated by the basis $u_1, \ldots, u_m$ of V, such that V is an H-submodule of A. After introducing commutation relations between the generators u_i, we obtain a Noetherian domain with an inner-faithful H-action (Theorem 20).

Let q be the primitive nth root of unity used in the definition of the twist J of B. Since $\mathbb{K}\Sigma_m$ is a factor ring of H, any irreducible representation over Σ_m is a simple module over H. As above, let $x_1, \ldots, x_m$ and $z_1, \ldots, z_{m-1}$ be the generators of H. For each transposition $s_i = (i, i+1)$ we have $z_i = \overline{s}_i$ and set $\sigma_i := \sigma_{s_i}$.

Assume that there exists a square root p of q. Given $0 \leq a, b < n$ we define a left H-module structure on an m-dimensional vector space $V_{a,b}$ with basis $u_1, \ldots, u_m$ by letting x_i and z_k respectively act as $m \times m$-matrices:

$$X_i = \begin{bmatrix} q^b & & & & \\ & \ddots & & & \\ & & q^a & & \\ & & & \ddots & \\ & & & & q^b \end{bmatrix} \qquad Z_k = \begin{bmatrix} p^{b^2} & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & q^{ab} & 0 & & \\ & & & & \ddots & \\ & & & & & p^{b^2} \end{bmatrix} \tag{24}$$

where X_i has on its diagonal the value q^b, except on the ith row, where it has q^a. The matrix Z_k has on its diagonal the value p^{b^2} except for the kth and $k+1$st rows (resp. columns), where one has a 2×2 block of the form $\begin{bmatrix} 0 & 1 \\ q^{ab} & 0 \end{bmatrix}$. It is clear, that if $1 \leq k < l < m$ such that $|k - l| > 1$, then Z_k and Z_l commute, since their 2×2-block is in different rows and columns. Hence $z_k z_l \cdot u_j = z_l z_k \cdot u_j$ holds, for $|k - l| > 1$.

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For the braid relation $z_k z_{k+1} z_k = z_{k+1} z_k z_{k+1}$ it is enough to look at the 3×3-submatrix of the rows and columns of index $k, k+1$ and $k+2$ of the product $Z_k Z_{k+1} Z_k$, i.e.

$$\begin{bmatrix} 0 & 1 & 0 \\ q^{ab} & 0 & 0 \\ 0 & 0 & p^{b^2} \end{bmatrix} \begin{bmatrix} p^{b^2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & q^{ab} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ q^{ab} & 0 & 0 \\ 0 & 0 & p^{b^2} \end{bmatrix} = p^{b^2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & q^{ab} & 0 \\ q^{2ab} & 0 & 0 \end{bmatrix} \tag{25}$$

Similarly, for $Z_{k+1} Z_k Z_{k+1}$ one obtains:

$$\begin{bmatrix} p^{b^2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & q^{ab} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ q^{ab} & 0 & 0 \\ 0 & 0 & p^{b^2} \end{bmatrix} \begin{bmatrix} p^{b^2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & q^{ab} & 0 \end{bmatrix} = p^{b^2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & q^{ab} & 0 \\ q^{2ab} & 0 & 0 \end{bmatrix} \tag{26}$$

It is not difficult to check that $Z_k X_i = X_{s_k(i)} Z_k$ holds for all i and k. Moreover, using $p^2 = q$,

$$Z_k^2 = \begin{bmatrix} q^{b^2} & & & & & & \\ & \ddots & & & & & \\ & & q^{ab} & & & & \\ & & & q^{ab} & & & \\ & & & & \ddots & & \\ & & & & & q^{b^2} \end{bmatrix} = \frac{1}{n} \sum q^{-st} X_k^s X_{k+1}^t, \tag{27}$$

since the latter matrix is a diagonal matrix with $\frac{1}{n}\sum_{s,t} q^{-st+bs+bt} = q^{b^2}$ on its ith row, for $i \notin \{k, k+1\}$, and $\frac{1}{n}\sum_{s,t} q^{-st+as+bt} = q^{ab}$ on its kth and $k+1$st row. Thus, $V_{a,b}$ is a left H-module.

Lemma 15. *$V_{a,b}$ is a simple left $H_{n,m}$-module, if $a \neq b \pmod n$.*

Proof. Let $H = H_{n,m}$ and assume $a \neq b \pmod n$. Let $v = \sum_{i=1}^m \lambda_i u_i$ be a non-zero element of v with $\lambda_k \neq 0$. Then $\lambda_k^{-1}(x_k - q^b) \cdot v = (q^a - q^b) u_k \neq 0$. Thus, $u_k \in H \cdot v$. However, since Σ_m acts transitively on $\{1, \ldots, m\}$, any basis element u_j of $V_{a,b}$ can be obtained by multiplying u_k by a suitable product of $Z_i's$. Hence $V_{a,b}$ is simple. □

Lemma 16. *Let $m \geq 3$ and $a \neq b \pmod n$ and $a' \neq b' \pmod n$. Then $V_{a,b} \simeq V_{a',b'}$ as left $H_{n,m}$-module if and only if $(a,b) = (a',b')$ in $\mathbb{Z}_n^2$.*

Proof. Suppose $f : V_{a,b} \to V_{a',b'}$ is an isomorphism of $H_{n,m}$-modules and $(\lambda_{ij})_{1\leq i,j\leq m}$ is a scalar matrix such that $f(u_i) = \sum_{j=1}^m \lambda_{ij} u_j'$, where $\{u_1, \ldots, u_m\}$ respectively $\{u_1', \ldots, u_m'\}$ is a basis for $V_{a,b}$ respectively $V_{a',b'}$. We will show that (λ_{ij}) is a diagonal matrix. Suppose there exists $i \neq j$ with $\lambda_{ij} \neq 0$. Let $l \neq i, j$ (using $m \geq 3$) and consider the following equations:

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$$\sum_{r=1}^{m} \lambda_{ir} q^b u'_r = q^b f(u_i) = x_j \cdot f(u_i) = \lambda_{ij} q^{a'} u'_j + \sum_{r \neq j}^{m} \lambda_{ir} q^{b'} u'_r \tag{28}$$

$$\sum_{r=1}^{m} \lambda_{ir} q^b u'_r = q^b f(u_i) = x_l \cdot f(u_i) = \lambda_{il} q^{a'} u'_l + \sum_{r \neq l}^{m} \lambda_{ir} q^{b'} u'_r. \tag{29}$$

Comparing the coefficients of u'_j on both sides of each equation, we conclude $b = a' (\mathrm{mod} n)$ and $b = b' (\mathrm{mod} n)$, which contradicts the assumption $a' \neq b' (\mathrm{mod} n)$. Hence $f(u_i) = \lambda_{ii} u'_i$ and $\lambda_{ii} \neq 0$, for all i. Thus $\lambda_{ii} q^a u'_i = q^a f(u_i) = x_i \cdot f(u_i) = \lambda_{ii} q^{a'} u'_i$ and $\lambda_{ii} q^b u'_i = q^b f(u_i) = x_j \cdot f(u_i) = \lambda_{ii} q^{b'} u'_i$, for any $j \neq i$ shows $a = a'$ and $b = b'$ modulo n. The converse statement holds, since the map $f : V_{a,b} \to V_{a',b'}$ with $f(u_i) = u'_i$ is an isomorphism of $H_{n,m}$-modules, in case $(a, b) = (a', b')$ in $\mathbb{Z}_n^2$. □

Denote by $M_{m,a,b} \in M_m(\mathbb{Z})$ the $m \times m$-matrix, such that the diagonal entries are all equal to a and all other entries are equal to b. For example, for $m = 2$ and $m = 3$, we have the matrices

$$M_{2,a,b} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad M_{3,a,b} = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}. \tag{30}$$

The determinant of such matrices are $\det(M_{2,a,b}) = a^2 - b^2$ and $\det(M_{3,a,b}) = a^3 + 2b^3 - 3ab^2$. More generally, if e_i denotes the canonical basis of $\mathbb{K}^m$, then $v_1 = \sum_{i=1}^m e_i$ is an eigenvector of $M_{m,a,b}$ with eigenvalue $a + (m-1)b$ and the vectors $v_i := e_{i-1} - e_i$, for $2 \leq i \leq n$ are eigenvectors with eigenvalue $a - b$. Hence,

$$\det(M_{m,a,b}) = (a + (m-1)b)(a-b)^{m-1}. \tag{31}$$

An H-module V is inner-faithful if the only Hopf ideal that annihilates V is the zero ideal.

Lemma 17. *$V_{a,b}$ is an inner-faithful R-module if* $\gcd((a+(m-1)b)(a-b)^{m-1}, n) = 1$.

Proof. Let I be a Hopf ideal of R. Since R is a group algebra, there exists a subgroup $N \subseteq G = \mathbb{Z}_n^m$ such that $I = \mathbb{K}G(\mathbb{K}N)^+$ (see [2, Lemma 4] or [11, p. 329]). Suppose $I \cdot V_{a,b} = 0$. Then $\omega = 1 - x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in I$, for some $x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in N$. For any $1 \leq j \leq m$ we obtain an equation $\omega \cdot u_j = 0$, i.e. $1 = q^{\alpha_j a + \sum_{l \neq j} \alpha_l b}$. Thus, $\alpha_j a + \sum_{l \neq j} \alpha_l b = 0 (\mathrm{mod} n)$, or in matrix notation $\alpha M_{m,a,b} = 0 \ (\mathrm{mod}\ n)$. If $\gcd(\det(M_{m,a,b}), n) = 1$, then $M_{m,a,b}$ is invertible modulo n. Thus $\alpha_i = 0$, for all i, which means that N is trivial and so is I. □

Next we show that $H_{n,m}$ acts on a quantum polynomial algebra A in m variables, such that $V_{a,b}$ is its degree one space.

Theorem 18. *Let $a \neq b \, (\mathrm{mod}\, n)$, q a primitive nth root of unity in $\mathbb{K}$ and a square root p of q. Set $\lambda = q^{b^2 - ab}$ and $\mu = p^{b^2 - a^2}$ and let (r_{ij}) be the $m \times m$-matrix such*

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that $r_{ii} = 1$, $r_{ij} = \lambda^{j-i-1}\mu$ *for* $i < j$ *and* $r_{ij} = r_{ji}^{-1}$ *for* $j < i$. *Consider the quantum polynomial algebra in* m *variables* $u_1, \dots, u_m$ *subject to* $u_i u_j = r_{ij} u_j u_i$, *for all* i, j.

$$A_{a,b} = \frac{\mathbb{K}\langle u_1, \dots, u_m\rangle}{\langle u_i u_j - r_{ij} u_j u_i : i, j\rangle}. \tag{32}$$

Then $A_{a,b}$ *is a left* H*-module algebra for the semisimple Hopf algebra* $H = \mathbb{K}\mathbb{Z}_n^{\otimes m} \#_\gamma \Sigma_m$, *where* $V_{a,b}$ *determines the* H*-action on the generators* u_i, *i.e.*

$$x_i \cdot u_i = q^a u_i, \qquad x_j \cdot u_i = q^b u_i, \quad \text{for } i \neq j$$

$$z_i \cdot u_i = q^{ab} u_{i+1}, \qquad z_i \cdot u_{i+1} = u_i, \qquad z_i \cdot u_j = p^{b^2} u_j, \qquad \text{for } j \neq \{i, i+1\}.$$

Proof. We claim that A is a left H-module algebra. As before, $R = \mathbb{K}\mathbb{Z}_n^{\otimes m}$. Since the group-likes x_i act as scalars on u_j, we conclude that A is an R-module algebra. Recall that

$$\Delta(z_i) = J_i(z_i \otimes z_i) = \frac{1}{n} \sum_{s,t} q^{-st} x_i^s z_i \otimes x_{i+1}^t z_i. \tag{33}$$

Hence for $1 \leq i < m$:

$$z_i \cdot (u_i u_{i+1}) = \frac{1}{n} \sum_{s,t} q^{-st} \left(x_i^s z_i \cdot u_i\right) \left(x_{i+1}^t z_i \cdot u_{i+1}\right) = q^{ab} q^{b^2} u_{i+1} u_i$$

$$z_i \cdot (u_{i+1} u_i) = \frac{1}{n} \sum_{s,t} q^{-st} \left(x_i^s z_i \cdot u_{i+1}\right) \left(x_{i+1}^t z_i \cdot u_i\right) = q^{ab} q^{a^2} u_i u_{i+1} = q^{ab} q^{a^2} \mu u_{i+1} u_i,$$

where $r_{i,i+1} = \mu = p^{b^2 - a^2}$. Thus $z_i \cdot (u_i u_{i+1}) = q^{ab} q^{b^2} u_{i+1} u_i = q^{ab} \mu^2 q^{a^2} u_{i+1} u_i = z_i \cdot (r_{i,i+1} u_{i+1} u_i)$.

A similar calculation for $j > i + 1$ shows

$$z_i \cdot (u_i u_j) = p^{b^2} q^{ab+b^2} u_{i+1} u_j$$
$$z_i \cdot (u_j u_i) = p^{b^2} q^{2ab} u_j u_{i+1} = p^{b^2} q^{2ab} r_{j,i+1} u_{i+1} u_j$$

Since $r_{ij} r_{j,i+1} = r_{ij} r_{i+1,j}^{-1} = \lambda = q^{b^2 - ab}$ we obtain

$$z_i \cdot (u_i u_j) = p^{b^2} q^{ab+b^2} u_{i+1} u_j = p^{b^2} r_{ij} r_{j,i+1} q^{2ab} u_{i+1} u_j = z_i \cdot (r_{ij} u_j u_i). \tag{34}$$

Similar calculations show $z_i \cdot (u_l u_k) = r_{lk} z_i \cdot (u_k u_l)$, for all i, k, l. □

Proposition 19. *Let* $H_{n,m} = R \#_\gamma \Sigma_m$ *and* $A_{a,b}$ *as above, for some* $a \neq b \pmod n$. *If* $\gcd(\det(M_{m,a,b}), n) = 1$ *and* n *is even, then* $A_{a,b}^{H_{n,m}}$ *is contained in the commutative polynomial ring* $A_{a,b}^R = \mathbb{K}[u_1^n, \dots, u_m^n]$.

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Proof. Set $A = A_{a,b}$. Clearly, $A^H \subseteq A^R$. Let $u_1, \dots, u_m$ be the algebra generators of A. The monomials $u^\alpha := u_1^{\alpha_1} \cdots u_m^{\alpha_m}$, with $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, form a $\mathbb{K}$-basis for A. Let $w = \sum a_\alpha u^\alpha \in A^R$, with only finitely many non-zero $a_\alpha \in \mathbb{K}$. Denote by $\sup(w) = \{\alpha : a_\alpha \neq 0\}$ the support of w. For any $1 \leq i \leq m$ we have

$$w = x_i \cdot w = \sum a_\alpha q^{\alpha_i a + \left(\sum_{j \neq i} \alpha_j\right) b} u^\alpha.$$

Hence $\alpha_i a + \left(\sum_{j \neq i} \alpha_j\right) b = 0 \pmod n$, for any $\alpha \in \sup(w)$ and for any i, which means that $\alpha M_{m,a,b} = 0 \pmod n$. Since $\gcd(\det(M_{m,a,b}), n) = 1$, $M_{m,a,b}$ is invertible modulo n and $\alpha = (0, \dots, 0) \pmod n$, i.e. $n \mid \alpha_i$, for all i. Note that if n is even, then $r_{ij}^{n^2} = q^{n^2(b^2-ab)} p^{n^2(b^2-a^2)} = (-1)^{n(b^2-a^2)} = 1$. Hence $u_i^n u_j^n = u_j^n u_i^n$ and $A^R \subseteq \mathbb{K}[u_1^n, \dots, u_m^n]$. The converse inclusion is clear. □

The condition $\gcd(\det(M_{m,a,b}), n) = 1$ is obviously satisfied when $a = 1$ and $b = 0$. Then $\lambda = q^{b^2-ab} = 1$ and $\mu = p^{b^2-a^2} = p^{-1}$. Thus the matrix (r_{ij}) satisfies $r_{ji} = p$ for $i < j$.

Theorem 20. *Let $n, m \geq 2$, $H = R\#_\gamma \Sigma_m$ and $R = \mathbb{K}\mathbb{Z}_n^{\otimes m}$. Let q be the primitive nth root of unity used in the definition of the twist J and assume there exists a square root p of q. Then*

$$A = \frac{\mathbb{K}\langle u_1, \dots, u_m \rangle}{\langle u_j u_i - p u_i u_j : i < j \rangle}$$

is a left H-module algebra with action of the generators x_i and z_i of H given by

$$x_i u_i = q u_i, \quad x_i u_j = u_j \text{ for } j \neq i, \quad z_i u_i = u_{i+1}, \quad z_i u_{i+1} = u_i, \quad z_i u_j = u_j \text{ for } \jmath \neq i, i{+}1.$$

Moreover, A is an inner-faithful R-module and if n is even then $A^H \subseteq \mathbb{K}[u_1^n, \dots, u_m^n]$.

Proof. Combine Lemma 17, Theorem 18 and Proposition 19. □

In [3], Ferraro et al. showed that Pansera's Hopf algebras $H_{n,2}$ act inner-faithfully on an Artin-Schelter regular algebra A, such that its ring of invariants is again Artin-Schelter regular – an analogous property to the Chevalley–Shephard–Todd Theorem for group actions on polynomial algebras. The authors of [3] termed such Hopf algebras 'reflection Hopf algebra' for A.

Example 21. Like in Example 14, consider the Hopf subalgebra $H = R\#_\gamma \langle s \rangle$ of $H_{n,m}$ generated by R and $\theta = z_1 z_2 \cdots z_{m-1}$, where $s = (12 \cdots m)$. Then H has dimension mn^m and is generated as algebra by $x_1, \dots, x_m$ and θ. By Theorem 20, $H_{n,m}$ and hence H acts on

$$A = A_{1,0} = \frac{\mathbb{K}\langle u_1, \dots, u_m \rangle}{\langle u_j u_i - p u_i u_j : i < j \rangle},$$

where θ acts cyclically on the generators of A, i.e. $\theta \cdot u_j = u_{s(j)}$, for all j. We claim that A is an inner-faithful H-module algebra. Suppose I is a Hopf ideal of H with $I \cdot A = 0$

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and let $\omega = \sum_{i=0}^{m-1} a_i\theta^i \in I$, with $a_i \in R$. Then $\omega \cdot u_j = \sum_{i=0}^{m-1} a_i \cdot u_{s(j)} = 0$, for all $1 \leq j \leq m$. Since the spaces $\mathbb{K}u_j$ are stable under the action of R, we get $a_i \cdot u_j = 0$, for all i and j. Thus $a_i \in I \cap R$, for all i. As I is also a Hopf ideal of R, we conclude by Lemma 17, $I \cap R = 0$. Hence $\omega = 0$, which means that H acts inner faithfully on A.

If n is even, then by Theorem 20, $A^H \subseteq \mathbb{K}[u_1^n, \ldots, u_m^n]$. Since θ acts cyclically on the generators u_i, it also acts cyclically on the elements u_i^n. Hence A^H consists of the cyclic polynomials in u_i^n, i.e. $A^H = \mathbb{K}[u_1^n, \ldots, u_m^n]^{\langle s\rangle}$.

Note that the quantum polynomial algebra $A = A_{1,0}$ is an Artin-Schelter regular ring. In case $m = 2$, $H = H_{n,2}$ and $A^H = \mathbb{K}[u_1^n, u_2^n]^{\langle(12)\rangle} = \mathbb{K}[u_1^n + u_2^n, u_1^n u_2^n]$ is a polynomial algebra, hence Artin-Schelter regular and H is a reflection Hopf algebra for A. In case $m \geq 3$, the ring of invariants $A^H = \mathbb{K}[u_1^n, \ldots, u_m^n]^{\langle s\rangle}$ is not a polynomial ring and H is not a reflection Hopf algebra for A in the sense of [3].

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