Proceedings of the Edinburgh Mathematical Society (2026) **69**, 1168–1182
doi:10.1017/S0013091526101400

C^1 SOLUTIONS OF A GENERAL COHOMOLOGICAL EQUATION ON THE PLANE

LIN LI[1], JANUSZ MATKOWSKI[2] AND ZHIHENG YU[3]

[1] *College of Data Science, Jiaxing University, Jiaxing, Zhejiang, P. R. China*
[2] *Institute of Mathematics, University of Zielona Góra, Szafrana 4a, Zielona Góra, Poland*
[3] *School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan, P. R. China* (yuzhiheng9@163.com)

(Received 20 August 2025; first published online 22 May 2026)

Abstract Cohomological equation is of special interest because it concerns the study of time change for flows, topological stability and topological conjugacy in dynamical systems, which is also an auxiliary equation to study the problem of linearization. In this paper, we consider a general form of cohomological equation for planar contractions. By using the ideas of invariant manifold and estimations in [W. Zhang and W. Zhang, C^1 linearization for planar contractions, *J. Funct. Anal.* **260** (2011), 2043–2063.], we present new criteria on eigenvalues of the linear parts for the existence of C^1 solutions in the Poincaré domain. Our results are a generalization of C^1 linearization for contractions.

Keywords: Cohomological equation; contractions; Hölder continuous; invariant curve

2020 Mathematics Subject Classification: 37C05; 37D05; 39B22

1. Introduction

In this paper, we consider the general cohomological equation

$$\psi(x) = M(x)\varphi(F(x)) - \varphi(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

proposed in [3] to solve the normal form problem, where $\psi : \mathbb{R}^n \to \mathbb{R}^m$, $M : \mathbb{R}^n \to \mathcal{M}^{m\times m}$ ($\mathcal{M}^{m\times m}$ is denoted by the space of $m \times m$ real valued matrices), $F : \mathbb{R}^n \to \mathbb{R}^n$ are given and $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is unknown.

Cohomological equation and its generalized forms play an important role in many areas of mathematics, such as group representations [13, 19], ergodic theory [1, 10], dynamical systems [11, 15, 18], singularity theory [2] etc. Specifically, if M is a constant matrix and $\psi \equiv 0$, equation (1.1) becomes

$$M\varphi \circ F = \varphi,$$

© The Author(s), 2026. Published by Cambridge University Press on Behalf of The Edinburgh Mathematical Society.

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

which is the fundamental and important problem, called linearization [3, 21]. For a general non-constant matrix M, if $\psi \equiv 0$, equation (1.1) is concerned with the problem of normal form, which is also a basic and important tool in bifurcation theory [5, 8]. As a kind of auxiliary equation, cohomological equation (1.1) arises in various important problems in dynamical systems, for example, the study of the existence of invariant measures, conjugacy problems, reparametrization of flows, rigidity and stability questions in dynamical systems etc.[4, 6, 7, 11, 12].

The general cohomology equation can also induce a cocycle. Let G denote either $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{R}$, $\rho : G \to GL(k, \mathbb{R})$ a homeomorphism, that is a linear representation of G, and $T : G \times X \to X$ a dynamical system with phase space X and time G. A one-cocycle twisted by ρ is a map $\eta : G \times X \to \mathbb{R}^k$ such that

$$\eta(g_2 + g_1, x) = \rho(g_1)\eta(g_2, T(g_1)x) + \eta(g_1, x), \tag{1.2}$$

where $k \in \mathbb{N}$, $g_1, g_2 \in G$ and $T(g)x := T(g, x)$ (see [11, Definition 2.9.1]). If ρ is the identity representation then such an η is called an untwisted cocycle. A cocycle can also be defined via a function $\varphi : X \to \mathbb{R}^k$ as

$$\eta(g, x) := \rho(g)\varphi(T(g)x) - \varphi(x), \tag{1.3}$$

which is called coboundary. Two cocycles are cohomologous if their difference is a coboundary. In the discrete-time case, every cocycle is determined by the function $\psi(x) = \eta(1, x)$. Let $F(\cdot) = T(1)$ and $M = \rho(1)$. Equation (1.3) becomes the vector form of (1.2)

$$\psi(x) = M\varphi(F(x)) - \varphi(x), \quad x \in X.$$

The following fact is a special solvability criterion for equation (1.1) in the case when $M = \mathrm{id}$. Equation (1.1) is solvable if and only if for every integer $p \geq 1$ and every periodic point x of the period p the sum $\sum_{k=0}^{p-1} \psi(F^k(x))$ vanishes [11, 18]. As a consequence, if the mapping F has no periodic points, then with any ψ equation (1.1) is solvable [11, 18]. Concretely, with $\psi > 0$ (in particular for the Abel equation) equation (1.1) is solvable if and only if the mapping F has no periodic points. Concerning the case of hyperbolic ρ, i.e. $\rho(1) = M$ is hyperbolic, the most celebrated results on this kind are the Livšic-type theorems for hyperbolic diffeomorphisms and flows [9, 11, 14–17], which reads that for any bounded $\psi = \eta(1, .)$ equation (1.3) has a unique bounded solution φ. If ψ is continuous then the solution of (1.3) is also continuous [11, Theorem 2.9.2]). In addition, as shown in [11], it is worth mentioning that even if ψ, F and ρ are smooth, the solution φ cannot be expected to be very regular. If $\eta(n, x)$ is bounded uniformly in n and x, it is proved that $\varphi(x) = \sup_{n\in\mathbb{N}} - \sum_{i=0}^{n} \psi(F^i(x))$ is a solution of (1.3) [11], also see [18]. For the partially hyperbolic case, the first attempt was addressed by Katok and Kononenko [12], who introduced a new method to solve (1.3) when F is partially hyperbolic. By introducing the notion of periodic cycles functional, they [12] proved that the subspace of cocycles cohomologous to a constant is the common zero set of the periodic cycles functionals under some transitive condition. With an additional hypothesis of accessibility on F, Wilkinson [20] obtained the existence and higher regularity of solutions to equation (1.3).

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

In 1996, by adding the smooth assumption on ψ and F, Banyaga et al. [3] considered the local smooth solutions to equation (1.1). The main result for a contraction F is given as follows.

Theorem 1 [3, Theorem 5.1]. *Assume that in (1.1) we have (i) F, ψ, M are C^r ($r \geq 2$) and $F(O) = O$, where O is the origin; (ii) $D^i\psi(O) = O$ for $i \leq k < r$; (iii) $\|DF(O)\| \leq \tilde{\alpha} < 1$; (iv) $\|M(O)\| \leq \tilde{\beta}$; (v) $\tilde{\alpha}^{k+1}\tilde{\beta} < 1$. Then there exists a C^r solution φ of equation (1.1) defined in a sufficiently small neighborhood of zero.*

In this paper, we continue to consider cohomological equation (1.1) for planar contractions near the origin. Since $M(O)$ plays an important role in the study of equation (1.1), we first consider the case that the mapping M is a real constant matrix. By using the idea of invariant manifold and the estimations of linearization, given in [21], we present new criteria on the existence of C^1 solutions to equation (1.1), that is, conditions on the eigenvalues of M ($M(O)$ in Theorem 1) and $DF(O)$. Furthermore, as mentioned in Remark 1, the linearization result [21, Theorem 1] is a special case of our Theorem 4. In the case when M is a matrix mapping, we also give sufficient conditions for the existence of C^1 solutions to equation (1.1) in the Poincaré domain, which requires lower smoothness on the given mappings F, γ than Theorem 1.

Throughout this paper, let $F : U \to \mathbb{R}^2$ be a diffeomorphism such that

$$F(O) = O, \quad DF(O) = S := \operatorname{diag}(\lambda_1, \lambda_2), \tag{1.4}$$

where $U \subseteq \mathbb{R}^2$ is a neighborhood of O and $\lambda_1, \lambda_2 \in \mathbb{C}$ satisfy

$$0 < |\lambda_1| \leq |\lambda_2| < 1. \tag{1.5}$$

For $x := (x_1, x_2) \in U$ we denote the norm $\|.\|$ as $\|x\| := \max\{|x_1|, |x_2|\}$.

2. M is a constant matrix

Before presenting the main results in this section, we give some necessary conditions on ψ if equation (1.1) has a C^1 diffeomorphic solution φ with $\varphi(O) = O$.

Lemma 1. *Let $F : U \to \mathbb{R}^2$ be a C^1 mapping such that $F(O) = O$ and M is a constant matrix. Assume that equation (1.1) has a C^1 diffeomorphic solution φ in U with $\varphi(O) = O$ and $D\varphi(O) = \mathrm{id}$. Then ψ is also C^1 with $\psi(O) = O$ and $D\psi(O) = M \cdot DF(O) - \mathrm{id}$, where* id *denotes the identity mapping.*

Proof. It is easy to prove that ψ is C^1 in U and $\psi(O) = O$ because φ, F are C^1 and $\varphi(O) = F(O) = O$. Differentiating both sides of equation (1.1), we obtain

$$D\psi(O) = M \cdot D\varphi(F(O))DF(O) - D\varphi(O) = M \cdot DF(O) - \mathrm{id}.$$

□

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

Under the conditions of Lemma 1 and putting $\varphi = \mathrm{id} + \Phi$, equation (1.1) becomes

$$M\Phi(F(x)) - \Phi(x) = \gamma(x), \quad x \in U, \tag{2.1}$$

where

$$\gamma := \psi - MF + \mathrm{id}, \tag{2.2}$$

Clearly, $D\gamma(O) = D\Phi(O) = O$.

As mentioned in our introduction, for the special case of $\psi \equiv 0$, equation (1.1) becomes the linearization problem $M\varphi \circ F - \varphi = 0$, which can be seen as the homogeneous form of (1.1). Actually, using Lemma 1, we have $M = \{DF(O)\}^{-1}$. In this case, the mapping γ in (2.1) becomes $\gamma = -MF + \mathrm{id}$.

When M is a constant matrix, we can easily check that equation (1.1) has the formal solution

$$\varphi(x) = \sum_{n=0}^{\infty}(M^{n+1}F^{n+1}(x) - M^nF^n(x)) + x - \sum_{n=0}^{\infty} M^n\psi(F^n(x)), \quad x \in U. \tag{2.3}$$

Substituting $\psi = \gamma + MF - \mathrm{id}$ into (2.3), we get

$$\begin{aligned}\varphi(x) = & \sum_{n=0}^{\infty}(M^{n+1}F^{n+1}(x) - M^nF^n(x)) + x - \sum_{n=0}^{\infty} M^n\gamma(F^n(x)) \\ & - \left(\sum_{n=0}^{\infty} M^{n+1}F^{n+1}(x) - \sum_{n=0}^{\infty} M^nF^n(x)\right) \\ = & -\sum_{n=0}^{\infty} M^n\gamma(F^n(x)) + x,\end{aligned}$$

which means that $-\sum_{n=0}^{\infty} M^n\gamma(F^n(x)) = \varphi(x) - x = \Phi(x)$ is a solution of equation (2.1) and it is easy to check the fact. So we only need to consider the existence of C^1 solution to the equivalent equation (2.1), that is to prove the convergence of the series $-\sum_{n=0}^{\infty} M^n\gamma(F^n(x))$.

Based on the relationship between the eigenvalues of matrix M and λ_i, $i = 1, 2$ (see condition (iv) in Theorem 1), we may assume that $M \in \mathcal{M}^{2\times 2}$ is of the diagonal form, i.e.

$$M := \mathrm{diag}(M_1, M_2), \tag{2.4}$$

where $M_1, M_2 \in \mathbb{R}$ such that $|M_1| \geq |M_2|$. Due to the different techniques in the proofs, we will show our results in the following two cases: $|M_1\lambda_2| < 1$ and $|M_1\lambda_2| \geq 1$.

The following Theorems 2 and 3 are our main results in this section.

Theorem 2. *Let $F, \psi : U \to \mathbb{R}^2$ be C^1 mappings fixing O and $D\psi(O) = M \cdot DF(O) -$* id. *Assume that two eigenvalues λ_1, λ_2 of $DF(O)$ satisfy (1.5), and M_1, M_2 in (2.4)*

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

satisfy $|M_1\lambda_2| < 1$. *Then there exist a neighborhood* $\Omega \subset U$ *of* O *and a* C^1 *diffeomorphism* $\varphi : \Omega \to \mathbb{R}^2$ *such that equation* (1.1) *holds.*

Theorem 3. *Let* $F, \psi : U \to \mathbb{R}^2$ *be* $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ *mappings fixing* O *and* $D\psi(O) = M \cdot DF(O) - \mathrm{id}$. *Assume that two eigenvalues* λ_1, λ_2 *of* $DF(O)$ *satisfy* (1.5), *and* M_1, M_2 *in* (2.4) *satisfy* $|M_1\lambda_2| \geq 1$. *If* $\alpha > \alpha_3 := -\log|M_1|/\log|\lambda_2| - 1$, *then there exist a neighborhood* $\Omega \subset U$ *of* O *and a* C^1 *diffeomorphism* $\varphi : \Omega \to \mathbb{R}^2$ *such that equation* (1.1) *holds.*

In order to prove Theorems 2 and 3, we need the following useful lemma, which helps us to find a suitable transformation for F.

Lemma 2 (21, Lemma 1). *Suppose that* $F : U \to \mathbb{R}^2$ *is a* $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ *contraction and that the two eigenvalues* λ_1 *and* λ_2 *of* $DF(O)$ *satisfy* (1.5). *Then there exists a closed disk* $V \subset U$ *centered at* O *such that* F *has a* $C^{1,\alpha}$ *invariant curve*

$$\Gamma := \{(x_1, x_2) \in V : x_1 = g(x_2)\},$$

where $g : V \cap \mathbb{R} \to \mathbb{R}$ *is* $C^{1,\alpha}$ *and* $g(0) = g'(0) = 0$, *if the constant* α *satisfies that* $0 < \alpha < \alpha_1$ *where*

$$\alpha_1 := \log|\lambda_1|/\log|\lambda_2| - 1.$$

The existence of an invariant curve guaranteed in Lemma 2 allows us to define a $C^{1,\alpha}$ transformation $\Theta : V \to \mathbb{R}^2$ by

$$\Theta(x) := (x_1 - g(x_2), x_2), \quad x \in V. \tag{2.5}$$

Clearly, $\Theta(O) = O, D\Theta(O) = \mathrm{id}$ and its inverse mapping Θ^{-1} is also $C^{1,\alpha}$ since g is $C^{1,\alpha}$, that is, there exists a constant $L_1 > 0$ such that

$$\|(D\Theta^{-1})(x) - (D\Theta^{-1})(y)\| \leq L_1\|x - y\|^\alpha$$

for all x, y in a neighborhood of the origin. So we can consider the mapping

$$\tilde{F} := \Theta \circ F \circ \Theta^{-1}$$

instead of F, which satisfies $\tilde{F}(O) = O, D\tilde{F}(O) = S$ and has a straight line x_2-axis as its invariant curve, i.e. $\tilde{F}_1(0, x_2) = x_2$, $x_2 \in V \cap \mathbb{R}$, where $\tilde{F}_1 := \pi_1\tilde{F}$ and π_1 denotes the projection onto the x_1-axis.

Furthermore, for each $n \in \mathbb{N}$ and a sufficiently small closed disk $\Omega \subset V$ centered at O, let

$$D\tilde{F}^n(x) := \begin{bmatrix} a_n(x) & b_n(x) \\ c_n(x) & d_n(x) \end{bmatrix}, \tag{2.6}$$

where $a_n, b_n, c_n, d_n : \Omega \to \mathbb{R}$ are functions. The following result provides useful estimations of a_n, b_n, c_n, d_n, which were given in [21].

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

Lemma 3 (21, Lemma 3). *The mapping $\tilde{F}$ is $C^{1,\alpha}$ $(0 < \alpha < \alpha_1)$ in U and*

$$|a_n(x)| \le K_1|\lambda_1|^n, \qquad |b_n(x)| \le K_1|\lambda_1|^n,$$
$$|c_n(x)| \le K_1|\lambda_2|^n, \qquad |d_n(x)| \le K_1|\lambda_2|^n$$

for all $n \in \mathbb{N}$ and all $x \in \Omega$, where $K_1 > 0$ is a constant independent of n and x.

Let $(\tilde{F}^n)_1 := \pi_1\tilde{F}^n$ and $\psi_1 := \pi_1\psi$. From Lemma 3 and its proof in [21], we have

$$\|\tilde{F}^n(x)\| \le K_1|\lambda_2|^n, \quad \|D\tilde{F}^n(x)\| \le K_1|\lambda_2|^n, \quad |(\tilde{F}^n)_1(x)| \le K_2|\lambda_1|^n \tag{2.7}$$

for all $x \in \Omega$, where $K_2 > 0$ is a constant independent of n and x.

Case 1, when $|M_1\lambda_2| < 1$.

For this case, we consider equation (2.1) directly and get that (1.1) has a C^1 solution under the assumption that the known mappings F, ψ are C^1.

Proof of Theorem 2. As proved in [21, p. 9], there exist a closed disk $\Omega \subset U$ centered at O and a constant $K_3 > 0$ independent of n and x such that

$$\|F^n(x)\| \le K_3|\lambda_2|^n, \quad \|DF^n(x)\| \le K_3|\lambda_2|^n, \quad x \in \Omega. \tag{2.8}$$

Since F, ψ are C^1 and $\gamma = \psi - MF + \mathrm{id}$, we conclude that γ is also C^1 with $\gamma(O) = O$ and there exists a constant $L_2 > 0$ such that

$$\|D\gamma(F^n(x))\| \le L_2, \quad x \in \Omega, \ n \in \mathbb{N}. \tag{2.9}$$

In what follows, we only need to prove the uniform convergence of the series $-\sum_{n=0}^{\infty} M^n D\gamma(F^n(x))DF^n(x)$ near O. For this purpose, let us note that

$$\|M^n D\gamma(F^n(x))DF^n(x)\| \le |M_1|^n \cdot L_2K_3|\lambda_2|^n, \quad x \in \Omega,$$

which implies the uniform convergence of the series $-\sum_{n=0}^{\infty} M^n D\gamma(F^n(x))DF^n(x)$ in Ω, because $|M_1\lambda_2| < 1$ in the considered case. It follows that the series $-\sum_{n=0}^{\infty} M^n\gamma(F^n(x))$ also converges uniformly near O by the fact that

$$\|M^n\gamma(F^n(x))\| = \|M^n\gamma(F^n(x)) - M^n\gamma(F^n(O))\| \le \|M^n D\gamma(F^n(\tau_1 x))DF^n(\tau_1 x)\| \cdot \|x\|,$$

where $\tau_1 \in (0, 1)$ is a number depending on x. Moreover, we have

$$D\varphi(O) = -\sum_{n=0}^{\infty} M^n D\gamma(F^n(O))DF^n(O) + \mathrm{id} = \mathrm{id},$$

implying that φ is a C^1 diffeomorphism near O. □

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

Case 2, when $|M_1\lambda_2| \geq 1$.
For this case, we introduce another number

$$\alpha_3 = -\log|M_1|/\log|\lambda_2| - 1,$$

as given in Theorem 3. Clearly, $\alpha_3 \geq 0$.

Proof of Theorem 3. Since F, ψ are $C^{1,\alpha}$ and $\gamma = \psi - MF + \mathrm{id}$, we obtain that $\gamma : U \to \mathbb{R}^2$ is also $C^{1,\alpha}$, that is, there exists a constant $L_3 > 0$ such that

$$\|D\gamma(x) - D\gamma(y)\| \leq L_3\|x - y\|^\alpha, \quad x, y \in U.$$

By the above inequality and (2.8), we have

$$\|D\gamma(F^n(x))\| = \|D\gamma(F^n(x)) - D\gamma(F^n(O))\| \leq L_3K_3^\alpha|\lambda_2|^{n\alpha}, \tag{2.10}$$

and then

$$\|M^nD\gamma(F^n(x))DF^n(x)\| \cdot \|x\| \leq |M_1|^n \cdot L_3K_3^\alpha|\lambda_2|^{n\alpha}K_3|\lambda_2|^n, \ x \in \Omega.$$

For all $\alpha > \alpha_3$, we have $|M_1||\lambda_2|^{\alpha+1} < 1$ and thus $-\sum_{n=0}^{\infty} M^nD\gamma(F^n(x))DF^n(x)$ is uniformly convergent in Ω. As in the previous proof of Theorem 2, we conclude that equation (1.1) has a C^1 diffeomorphic solution φ. □

Theorems 2-3 show that equation (1.1) has a C^1 solution in the cases where $|M_1\lambda_2| < 1$ or $|M_1\lambda_2| \geq 1$ and $|M_1||\lambda_2|^{\alpha+1} < 1$. In order to estimate the smoothness on F, we continue to consider the case of $|M_1\lambda_2| \geq 1$ on the assumption that $|M_1\lambda_1| < 1$, which enable us to define a new number

$$\alpha_2 := \max\left\{-\frac{\log|\lambda_2| + \log|M_1|}{\log|\lambda_1|}, -\frac{\log|\lambda_2| + \log|M_2|}{\log|\lambda_2|}\right\}.$$

One can check that $0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_1$. Clearly, equation (1.1) has a C^1 diffeomorphic solution when $\alpha > \alpha_3$. In what follows, it suffices to discuss equation (1.1) for all $\alpha \in (\alpha_2, \alpha_3]$.

Theorem 4. *Let $F, \psi : U \to \mathbb{R}^2$ be $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ mappings fixing O and $D\psi(O) = M \cdot DF(O) - \mathrm{id}$. Assume that two eigenvalues λ_1, λ_2 of $DF(O)$ satisfy (1.5), and M_1, M_2 in (2.4) satisfy $|M_1\lambda_2| \geq 1$ and $|M_1\lambda_1| \leq 1$. If $\alpha > \alpha_2$ and ψ_1 vanishes on an invariant curve Γ of F, then there exist a neighborhood $\Omega \subset U$ of O and a C^1 diffeomorphism $\varphi : \Omega \to \mathbb{R}^2$ such that equation (1.1) holds.*

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

Proof. For $\alpha > \alpha_3$, the desired conclusion has been obtained in Theorem 3, so we only need to consider the case of $\alpha_2 < \alpha \le \alpha_3$. Recall that $\tilde{F} = \Theta \circ F \circ \Theta^{-1}$ and put

$$\Phi := \tilde{\Phi} \circ \Theta + \Theta - \mathrm{id}, \tag{2.11}$$

where Θ is defined in (2.5). Now equation (2.1) takes the form

$$M\tilde{\Phi} \circ \Theta \circ F + M\Theta \circ F - MF - \tilde{\Phi} \circ \Theta - \Theta + \mathrm{id} = \gamma. \tag{2.12}$$

By compositing both sides of (2.12) with Θ^{-1} on the right, it becomes

$$M\tilde{\Phi} \circ \tilde{F} - \tilde{\Phi} = \gamma \circ \Theta^{-1} - M\tilde{F} + MF \circ \Theta^{-1} + \mathrm{id} - \Theta^{-1}.$$

Thus, equation (2.1) can be rewritten as

$$M\tilde{\Phi}(\tilde{F}(x)) - \tilde{\Phi}(x) = \eta(x), \quad x \in U \tag{2.13}$$

where, by (2.2), $\eta := \psi \circ \Theta^{-1} - M\tilde{F} + \mathrm{id}$.

In order to make use of the properties of $\tilde{F}$, we may consider equation (2.13) instead of (1.1) and similar to the above proof, it suffices to prove the convergence of $\sum_{n=0}^{\infty} M^n D\eta(\tilde{F}^n(x)) D\tilde{F}^n(x)$.

Note that $\tilde{F}, \psi$ and Θ^{-1} are $C^{1,\alpha}$, by Lemma 1 we claim that η is also $C^{1,\alpha}$, satisfying $\eta(O) = D\eta(O) = O$. Actually, we have

$$\begin{aligned}
&\|D\eta(x) - D\eta(y)\| \\
&= \|D\psi(\Theta^{-1}(x))D\Theta^{-1}(x) - D\psi(\Theta^{-1}(y))D\Theta^{-1}(y) - M(D\tilde{F}(x) - D\tilde{F}(y))\| \\
&\le \|D\psi(\Theta^{-1}(x))D\Theta^{-1}(x) - D\psi(\Theta^{-1}(x))D\Theta^{-1}(y)\| \\
&+ \|D\psi(\Theta^{-1}(x))D\Theta^{-1}(y) - D\psi(\Theta^{-1}(y))D\Theta^{-1}(y)\| + |M_1|\|D\tilde{F}(x) - D\tilde{F}(y)\| \\
&\le \|D\psi(\Theta^{-1}(x))\| \cdot \|D\Theta^{-1}(x) - D\Theta^{-1}(y)\| \\
&+ \|D\psi(\Theta^{-1}(x)) - D\psi(\Theta^{-1}(y))\| \cdot \|D\Theta^{-1}(y)\| + |M_1|\|D\tilde{F}(x) - D\tilde{F}(y)\| \\
&= L_4\|x - y\|^{\alpha}
\end{aligned}$$

for all x, y in a neighborhood of the origin, where L_4 is a positive constant independent of x and y. Since $\tilde{F}_1(0, x_2) = 0$ and ψ_1 vanishes on the invariant curve Γ defined in Lemma 2, that is, $\psi_1(g(x_2), x_2) = 0$ for all $x_2 \in U \cap \mathbb{R}$, then $\eta(x) = \psi(\Theta^{-1}(x)) - M\tilde{F}(x) + \mathrm{id} = \psi(x_1 + g(x_2), x_2) - M\tilde{F}(x_1, x_2) + (x_1, x_2), x \in U$, which leads to

$$\eta_1(0, x_2) = 0 \quad \text{and} \quad \frac{\partial \eta_1(0, x_2)}{\partial x_2} = 0, \quad x_2 \in U \cap \mathbb{R}. \tag{2.14}$$

Let

$$D\eta(\tilde{F}^n(x)) := \begin{bmatrix} \tilde{a}_n(x) & \tilde{b}_n(x) \\ \tilde{c}_n(x) & \tilde{d}_n(x) \end{bmatrix}, \tag{2.15}$$

where $\tilde{a}_n, \tilde{b}_n, \tilde{c}_n, \tilde{d}_n : U \to \mathbb{R}$ are functions.

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

Concerning (2.15), we have

$$|\tilde{b}_n(x)| = \left|\frac{\partial\eta_1(\tilde{F}^n(x))}{\partial x_2}\right| = \left|\frac{\partial\eta_1(\tilde{F}^n(x))}{\partial x_2} - \frac{\partial\eta_1(0,(\tilde{F}^n)_2(x))}{\partial x_2}\right|$$
$$\leq L_4|(\tilde{F}^n)_1(x)|^\alpha \leq L_4K_2^\alpha|\lambda_1|^{n\alpha} \tag{2.16}$$

due to (2.14) and the last inequality of (2.7).

On the other hand, making use of the first inequality of (2.7), we get

$$\|D\eta(\tilde{F}^n(x))\| = \|D\eta(\tilde{F}^n(x)) - D\eta(\tilde{F}^n(O))\| \leq L_4\|\tilde{F}^n(x)\|^\alpha \leq L_4K_1^\alpha|\lambda_2|^{n\alpha},$$

which implies that

$$|\tilde{a}_n(x)|, |\tilde{c}_n(x)|, |\tilde{d}_n(x)| \leq L_4K_1^\alpha|\lambda_2|^{n\alpha}, \quad x \in \Omega. \tag{2.17}$$

Therefore, we obtain

$$\begin{aligned}
&\|M^nD\eta(\tilde{F}^n(x))D\tilde{F}^n(x)\| \\
&= \left\|\begin{bmatrix} M_1^n & 0 \\ 0 & M_2^n \end{bmatrix} \cdot \begin{bmatrix} \tilde{a}_n(x) & \tilde{b}_n(x) \\ \tilde{c}_n(x) & \tilde{d}_n(x) \end{bmatrix} \cdot \begin{bmatrix} a_n(x) & b_n(x) \\ c_n(x) & d_n(x) \end{bmatrix}\right\| \\
&\leq \left\|\begin{bmatrix} M_1^n\tilde{a}_n(x)a_n(x) + M_1^n\tilde{b}_n(x)c_n(x) & M_1^n\tilde{a}_n(x)b_n(x) + M_1^n\tilde{b}_n(x)d_n(x) \\ M_2^n\tilde{c}_n(x)a_n(x) + M_2^n\tilde{d}_n(x)c_n(x) & M_2^n\tilde{c}_n(x)b_n(x) + M_2^n\tilde{d}_n(x)d_n(x) \end{bmatrix}\right\|
\end{aligned}$$

for all $x \in \Omega$. According to the inequalities of Lemma 3 and formulas (2.16) and (2.17), we get

$$\max\{|M_1^n\tilde{a}_n(x)a_n(x)|, |M_1^n\tilde{a}_n(x)b_n(x)|\} \leq |M_1|^nL_3K_1^\alpha|\lambda_2|^{n\alpha}K_1|\lambda_1|^n, \tag{2.18}$$

$$\max\{|M_1^n\tilde{b}_n(x)c_n(x)|, |M_1^n\tilde{b}_n(x)d_n(x)|\} \leq |M_1|^nL_3K_2^\alpha|\lambda_1|^{n\alpha}K_1|\lambda_2|^n, \tag{2.19}$$

$$\max\{|M_2^n\tilde{c}_n(x)a_n(x)|, |M_2^n\tilde{c}_n(x)b_n(x)|\} \leq |M_2|^nL_3K_1^\alpha|\lambda_2|^{n\alpha}K_1|\lambda_1|^n, \tag{2.20}$$

$$\max\{|M_2^n\tilde{d}_n(x)c_n(x)|, |M_2^n\tilde{d}_n(x)d_n(x)|\} \leq |M_2|^nL_3K_1^\alpha|\lambda_2|^{n\alpha}K_1|\lambda_2|^n. \tag{2.21}$$

The assumption of $\alpha > \alpha_2$ confirms that $|\lambda_i|^\alpha|\lambda_2||M_i| < 1$ for $i = 1, 2$ and thus the series $\sum_{n=0}^\infty M^nD\eta(\tilde{F}^n(x))D\tilde{F}^n(x)$ and $\sum_{n=0}^\infty M^n\eta(\tilde{F}^n(x))$ are both uniformly convergent in Ω. Therefore, the mapping $\tilde{\Phi}(x) := -\sum_{n=0}^\infty M^n\eta(\tilde{F}^n(x))$ is a C^1 solution of equation (2.13) with $\tilde{\Phi}(O) = D\tilde{\Phi}(O) = O$, and then $\varphi = (\tilde{\Phi} + \mathrm{id}) \circ \Theta$ is a C^1 diffeomorphism solution to equation (1.1). □

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

Remark 1. In our introduction, we mentioned that when $\psi \equiv 0$, that is, $\gamma = -MF + \mathrm{id}$, equations (1.1) and (2.1) become the problem of linearization. Since $M \cdot DF(O) = \mathrm{id}$, it means that $|\lambda_1 M_1| = 1$. Consequently, we have $\alpha_2 < \alpha_3 = \alpha_1$, where

$$\alpha_2 = 1 - \log|\lambda_2| / \log|\lambda_1|,$$

which is exactly the number α_0 given in [21, Theorem 1]. Furthermore, the mapping η in (2.13) becomes $\eta = -M\tilde{F} + \mathrm{id}$, implying that $\eta_1(0, x_2) = 0$ by the straightening property of $\tilde{F}$. Therefore, our Theorem 4 is a generalization of the linearization result [21, Theorem 1] in the Poincaré domain.

Remark 2. Theorem 4 gives a sufficient condition for the existence of C^1 solutions to equation (1.1) in the case $|\lambda_1 M_1| \le 1$. When $|\lambda_1 M_1| > 1$, we have two subcases, that is $0 < \alpha_2 < \alpha_1 < \alpha_3$ and $0 < \alpha_1 \le \alpha_2 < \alpha_3$. By the proofs in Theorems 3-4, equation (1.1) can be solved similarly for all $\alpha_2 < \alpha < \alpha_1$ (resp. $\alpha_1 < \alpha < \alpha_2$) and $\alpha > \alpha_3$. Since α_2 is the sharp estimate to α, as proved in [21], the problem for $\alpha_1 \le \alpha \le \alpha_3$ is still unsolved.

3. M is a matrix mapping

According to the proof of Lemma 1, if φ is a C^1 diffeomorphic solution of equation (1.1) satisfying $\varphi(O) = O$, then ψ is also C^1 with $\psi(O) = O$ and $D\psi(O) = M(O)DF(O) - \mathrm{id}$. Consequently, by putting $\varphi = \mathrm{id} + \Phi$ and $D\varphi(O) = \mathrm{id}$, equation (1.1) becomes

$$M(x)\Phi(F(x)) - \Phi(x) = \gamma(x), \quad x \in U, \tag{3.1}$$

where

$$\gamma(x) := \psi(x) - M(x)F(x) + x$$

and $D\gamma(O) = D\Phi(O) = O$.

Theorem 5. *Let $F, \psi : U \to \mathbb{R}^2$ be $C^{1,\alpha}$ $(0 < \alpha \le 1)$ mappings fixing O, $D\psi(O) = M(O)DF(O) - \mathrm{id}$ and $M : U \to \mathcal{M}^{2\times 2}$ be a $C^{1,\alpha}$ matrix mapping. Assume that two eigenvalues λ_1, λ_2 of $DF(O)$ satisfy (1.5) and $M(O) = \mathrm{diag}(M_1, M_2)$ satisfying $|M_1| \ge |M_2|$. If*

(i) $|M_1\lambda_2| < 1$ *and* $\alpha \in (0, 1]$

or

(ii) $|M_1\lambda_2| \ge 1$ *and* $\alpha > \alpha_3$,

then there exist a neighborhood $\Omega \subset U$ of O and a C^1 diffeomorphism $\varphi : \Omega \to \mathbb{R}^2$ such that equation (1.1) holds.

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

Proof. It is easy to verify that the function

$$\Phi(x) = -\sum_{n=0}^{\infty}\left(\left[\prod_{j=0}^{n-1} M(F^j(x))\right]\cdot\gamma(F^n(x))\right)$$

is a formal solution of equation (3.1).

Since the inequality

$$\|F(x)\| \le (\|DF(O)\| + \|DF(\tau_2 x) - DF(O)\|)\cdot\|x\| \le (|\lambda_2| + L\|x\|^\alpha)\cdot\|x\|$$

is true for all $x \in U$ and some $\tau_2 \in (0,1)$, we can prove inductively that there exists a closed disk $V \subset U$ centered at O such that

$$\|F^j(x)\| \le (|\lambda_2| + \delta)^j, \quad j \in \mathbb{N},\ x \in V, \tag{3.2}$$

where $\delta := (1 - |\lambda_2|)/2$ (see the proof in [21, p. 9]). Note that the matrix mapping M is $C^{1,\alpha}$, there exists a real constant $K_4 > 0$ such that

$$\|M(x)\| = \|M(x) - M(O) + M(O)\| \le \|M(O)\| + \|DM(\tau_3 x)\|\cdot\|x\| \le |M_1| + K_4\|x\|$$

for $x \in U$ and some $\tau_3 \in (0,1)$, whence

$$\left\|\prod_{j=0}^{n-1} M(F^j(x))\right\| \le K_5|M_1|^n, \tag{3.3}$$

where

$$K_5 := \prod_{j=0}^{\infty}\left\{1 + \frac{K_4}{|M_1|}\left(\frac{1+|\lambda_2|}{2}\right)^j\right\} < \infty.$$

Since F, ψ are $C^{1,\alpha}$ and $\gamma(x) = \psi(x) - M(x)F(x) + x$, we get that $\gamma : U \to \mathbb{R}^2$ is also $C^{1,\alpha}$, that is, there exists a constant $L_5 > 0$ such that

$$\|D\gamma(x) - D\gamma(y)\| \le L_5\|x - y\|^\alpha, \quad x, y \in U.$$

By the above inequality and (2.8), we have

$$\|D\gamma(F^n(x))\| \le L_5\|F^n(x)\|^\alpha \le L_5 K_3^\alpha|\lambda_2|^{n\alpha}, \tag{3.4}$$

and thus

$$\|\gamma(F^n(x))\| \le \|D\gamma(F^n(\tau_4 x))DF^n(\tau_4 x)\|\cdot\|x\| \le L_5 K_3^\alpha|\lambda_2|^{n\alpha}K_3|\lambda_2|^n,\ x \in \Omega$$

for some $\tau_4 \in (0,1)$.

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

To prove the uniform convergence of the function series

$$\sum_{n=0}^{\infty} D\left(\left[\prod_{j=0}^{n-1} M(F^j(x))\right]\cdot\gamma(F^n(x))\right), \tag{3.5}$$

note that by (2.8) and (3.3) we have

$$\begin{aligned}
&\left\|D\left(\left[\prod_{j=0}^{n-1} M(F^j(x))\right]\cdot\gamma(F^n(x))\right)\right\|\\
&=\left\|\sum_{k=0}^{n-1}\left[\prod_{i=0}^{k-1} M(F^i(x))\right] DM(F^k(x))DF^k(x)\cdot\prod_{i=k+1}^{n-1} M(F^i(x))\cdot\gamma(F^n(x))\right.\\
&\quad\left.+\prod_{k=0}^{n-1} M(F^k(x))\cdot D\gamma(F^n(x))DF^n(x)\right\|\\
&\le K_5|M_1|^n\cdot\left(\sum_{k=0}^{n-1} K_3|\lambda_2|^k+1\right)\|D\gamma(F^n(x))DF^n(x))\|\\
&\le K_5K_6L_5K_3^{1+\alpha}|\lambda_2|^{n\alpha}|M_1\lambda_2|^n,
\end{aligned}$$

where

$$K_6:=K_3\sum_{k=0}^{\infty}|\lambda_2|^k+1<\infty.$$

The assumptions in condition (i) (resp. (ii)) guarantee $|M_1||\lambda_2|^{\alpha+1}<1$, and then the series (3.5) is uniformly convergent in Ω. Arguing similarly as in the proof of Theorem 2, we conclude that equation (1.1) has a C^1 diffeomorphic solution φ. □

Theorem 6. *Let $F,\psi: U\to\mathbb{R}^2$ be $C^{1,\alpha}$ $(0<\alpha\le 1)$ mappings fixing O, $D\psi(O)=M(O)DF(O)-\mathrm{id}$ and $M: U\to\mathcal{M}^{2\times 2}$ be a $C^{1,\alpha}$ matrix mapping. Assume that two eigenvalues λ_1,λ_2 of $DF(O)$ satisfy (1.5) and $M(O)=\mathrm{diag}(M_1,M_2)$ satisfying $|M_1|\ge|M_2|$, $|M_1\lambda_2|\ge 1$, $|M_1\lambda_1|\le 1$. If*

$$\alpha>\alpha_4:=\max\{\alpha_2,-\log q/\log|\lambda_2|-1\},\ q:=\|DM(O)\|$$

and ψ_1 vanishes on an invariant curve Γ of F, then there exist a neighborhood $\Omega\subset U$ of O and a C^1 diffeomorphism $\varphi:\Omega\to\mathbb{R}^2$ such that equation (1.1) holds.

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

Proof. Firstly, the formal solution of equation (3.1) can be decomposed into the following two parts

$$\begin{aligned}\Phi(x) &= -\sum_{n=0}^{\infty}\left[\prod_{j=0}^{n-1} M(F^j(x))\right]\cdot\gamma(F^n(x))\\ &= -\sum_{n=0}^{\infty} M^n(O)\gamma(F^n(x)) - \sum_{n=0}^{\infty}\left[\prod_{j=0}^{n-1} m(F^j(x))\right]\cdot\gamma(F^n(x))\\ &= \Phi_1(x)+\Phi_2(x),\end{aligned}$$

where $m(x) := M(x) - M(O)$ and $\Phi_1(x) := -\sum_{n=0}^{\infty} M^n(O)\gamma(F^n(x))$, $\Phi_2(x) := -\sum_{n=0}^{\infty}\left[\prod_{j=0}^{n-1} m(F^j(x))\right]\cdot\gamma(F^n(x))$ for $x\in U$. According to the proof of Theorem 4, we confirm the convergence of function Φ_1 because $\alpha>\alpha_2$. In what follows, it suffices to prove the convergence of Φ_2.

Clearly, the mapping m is $C^{1,\alpha}$ by its definition. So for all $x,y\in U$, there exists a constant $L_6>0$ such that

$$\|Dm(x)-Dm(y)\|\le L_6\|x-y\|^{\alpha}.$$

Consequently, we have

$$\begin{aligned}\|m(x)\| &= \|m(x)-m(O)\|\le\|m'(\tau_5 x)\|\cdot\|x\|\\ &= \|m'(O)+m'(\tau_5 x)-m'(O)\|\cdot\|x\|\le(\|m'(O)\|+L_4\|x\|^{\alpha})\end{aligned}\tag{3.6}$$

for all $x\in U$ and some $\tau_5\in(0,1)$. Furthermore, by (2.8) and the definition of q we obtain

$$\|m(F^j(x))\|\le(\|m'(O)\|+L_4\|F^j(x)\|^{\alpha})\le(q+L_4K_3^{\alpha}|\lambda_2|^{\alpha j}),\quad x\in U.$$

Therefore, we get

$$\left\|\prod_{j=0}^{n-1} m(F^j(x))\right\|\le K_7 q^n,\tag{3.7}$$

where

$$K_7 := \prod_{j=0}^{\infty}\left\{1+\frac{L_4K_3^{\alpha}}{q}|\lambda_2|^{\alpha j}\right\}<\infty.$$

In order to prove the convergence of the function series

$$\sum_{n=0}^{\infty}\left[\prod_{j=0}^{n-1} m(F^j(x))\right]\cdot\gamma(F^n(x)),\tag{3.8}$$

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

note that, by (2.8) and (3.7), for all $x \in U$ we have

$$\begin{aligned}
&\left\| D\left(\left[\prod_{j=0}^{n-1} m(F^j(x)) \right] \cdot \gamma(F^n(x)) \right) \right\| \\
&= \left\| \sum_{k=0}^{n-1} \left[\prod_{i=0}^{k-1} m(F^i(x)) \right] Dm(F^k(x)) DF^k(x) \cdot \prod_{i=k+1}^{n-1} m(F^i(x)) \cdot \gamma(F^n(x)) \right. \\
&\quad \left. + \prod_{k=0}^{n-1} m(F^k(x)) \cdot D\gamma(F^n(x)) DF^n(x) \right\| \\
&\le K_7 q^n \cdot \left(\sum_{k=0}^{n-1} K_3 |\lambda_2|^k + 1 \right) \|D\gamma(F^n(x)) DF^n(x)\| \\
&\le K_7 K_8 L_2 K_3^{1+\alpha} q^n |\lambda_2|^{(1+\alpha)n},
\end{aligned}$$

where

$$K_8 := K_3 \sum_{k=0}^{n-1} |\lambda_2|^k + 1 < \infty.$$

Applying the condition $\alpha > -\log|q|/\log|\lambda_2| - 1$ in this theorem, we get $0 < q \le |\lambda_2|^{-(\alpha+1)}$ and thus $q|\lambda_2|^{\alpha+1} < 1$. So the series (3.8) is uniformly convergent in U, which admits the convergence of Φ_2. Moreover, one checks

$$\Phi_2(O) = -\sum_{n=0}^{\infty} \left[\prod_{j=0}^{n-1} m(F^j(O)) \right] \cdot \gamma(F^n(O)) = O \quad \text{and} \quad D\Phi_2(O) = O.$$

Therefore, equation (1.1) has a C^1 solution φ near the origin O with $\varphi(O) = O$ and $D\varphi(O) = \mathrm{id}$. □

Remark 3. Compared with Theorem 1 (also see [3, Theorem 5.1]), our Theorem 6 requires lower smoothness on the given mappings F and γ, where F, γ are $C^r, r \ge 2$ in Theorem 1 and here we only need $C^{1,\alpha}$ for $\alpha > \alpha_2$.

Acknowledgments. The authors are very grateful to the editor and reviewer for their careful checking and helpful suggestions.

Funding statement. This work was supported by the National Science Foundation of China (#12571180), Sichuan Provincial Natural Science Foundation of China (#2023NSFSC0064) and Jiaxing Public Welfare Research Program of China (2025CGW024,2024AD30076).

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press

References

1. D. V. Anosov, On an additive functional homological equation connected with an ergodic rotation on the circle, *Math. USSR-Izv.* **7** (1973), 1257–1271.
2. V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps*, Vol. I, (Monographs in Math., 82, Birkhäuser Verlag, 1985).
3. A. Banyaga, R. de la Llave and C. E. Wayne, Cohomology equations near hyperbolic points and geometric versions of Sternberg linearization theorem, *J. Geom. Anal.* **6** (1996), 613–649.
4. G. Belitskii and V. Tkachenko, *One-dimensional Functional Equations*, Operator Theory: Advances and Applications, vol. 144, (Birkhäuser Verlag, Basel, 2003).
5. S. N. Chow, C. Li and D. Wang , *Normal Forms and Bifurcation of Planar Vector Fields*, (Cambridge University Press, Cambridge, 1994).
6. D. Damjanović and A. Katok, Periodic cycle functionals and cocycle rigidity for certain partially hyperbolic actions, *Discrete Cont. Dyn-A* **13** (2005), 985–1005.
7. D. Damjanović and A. Katok. Local rigidity of partially hyperbolic actions I. KAM method and $\mathbb{Z}^k$ actions on the torus, *Ann. Math.* **172** (2010), 1805–1858.
8. J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, *Appl. Math. Sci.*, vol. 42, (Springer, 1983).
9. V. Guillemin and D. Kazhdan, On the cohomology of certain dynamical systems, *Topology* **19** (1980), 291–299.
10. A. Katok, Cocycles, cohomology and combinatorial constructions in ergodic theory, Smooth ergodic theory and its applications, *Proc. Sympos. Pure Math.*, vol. 69, AMS, 2001, In collaboration with E. A., Robinson, pp. 107–173.
11. A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, (Cambridge University Press, 1995).
12. A. Katok and A. Kononenko, Cocycles' stability for partially hyperbolic systems, *Math. Res. Lett.* **3** (1996), 191–210.
13. A. A. Kirillov, Dynamical systems, factors and group representations, *Russian Math. Surveys* **22** (1967), 63–75.
14. A. N. Livšic, Homology properties of U systems, *Math. Notes* **10** (1971), 758–763.
15. A. N. Livšic, Cohomology of dynamical systems, *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 1296–1320.
16. R. de la Llave, Analytic regularity of solutions of Livsic's cohomology equation and some applications to analytic conjugacy of hyperbolic dynamical systems, *Ergodic Theory Dynam. Systems* **17** (1997) 649–662.
17. R. de la Llave, J. M. Marco and R. Moriyon, Canonical perturbation theory of Anosov systems and regularity results for Livsic cohomology equation, *Ann. Math.* **123** (1986), 537–612.
18. Yu. I. Lyubich, The cohomological equations in nonsmooth categories, *Banach Center Publications* **112** (2017), 221–272.
19. C. C. Moore and K. Schmidt, Coboundaries and homomorphisms for nonsingular actions and a problem of H. Helson, *Proc. London Math. Soc.* **40** (1980), 2043–2063.
20. A. Wilkinson, The cohomological equation for partially hyperbolic diffeomorphisms, Asserisque **358** (2013), 75–165.
21. W. Zhang and W. Zhang, C^1 linearization for planar contractions, *J. Funct. Anal.* **260** (2011), 2043–2063.

https://doi.org/10.1017/S0013091526101400 Published online by Cambridge University Press