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On the Sylow Theorem for Skew Braces

Andrea Caranti, Ilaria Del Corso, Massimiliano Di Matteo, Maria Ferrara, and Marco Trombetti

Abstract. We discuss the (first) Sylow theorem for certain classes of finite skew braces, proving it to hold true when the skew brace is two-sided, bi-skew, right nilpotent, λ-homomorphic or supersoluble. We also show it to hold true for soluble skew braces that are left nilpotent and address a number of more specialized settings, proving general Hall-type theorems.

1 Introduction

A *skew (left) brace* is a triple $(B, +, \circ)$ where $(B, +)$ and $(B, \circ)$ are (not necessarily abelian) groups that satisfy the *skew (left) distributivity*:

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

for all $a, b, c \in B$. We refer to $(B, +)$ as the *additive group* and to $(B, \circ)$ as the *multiplicative group* of the skew brace B. This algebraic structure was introduced in [11] as a generalization of the (left) braces defined by Rump in [14], which correspond to the special case of skew braces whose additive group is abelian. Skew braces were developed to tackle the challenge of identifying all set-theoretical, non-degenerate solutions to the Yang–Baxter equation, a fundamental consistency condition that plays a key role in quantum statistical mechanics, underpins the theory of quantum groups, and bridges various disciplines, including Hopf algebras, knot theory, braid theory, and Garside theory.

It is now well established that skew braces sometimes behave like rings (especially when the additive group is abelian) and in other instances like groups, particularly when the additive group is non-abelian. This dichotomy is clearly illustrated in [3], where numerous counterexamples to classical group-theoretic results are exhibited; for example, the product of two "nilpotent" ideals need not be "nilpotent."

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Probably, from a group-theorist's perspective, the most noticeable gap in the current theory is the lack, to date, of an analog of the Sylow theorems. This is by no means because such results would be irrelevant in this setting: on the contrary, the existence of well-behaved substructures often provides deep insight in many contexts related to the study of solutions of the Yang–Baxter equation itself (see, e.g., [2, 4, 18]). The reasons are essentially twofold: there is no a priori reason for such theorems to hold in the setting of skew braces, and, from a computational perspective, identifying counterexamples appears to be difficult (even by using GAP [16]). As proof of the difficulty of this task, one may consider that only very recently it has been possible to prove that the only finite skew braces with no proper nonzero sub-skew braces are the ones of prime order (see [4]). Thus, in a finite simple skew brace, we know by this result that there always exists a sub-skew brace of prime order, but we do not know (for example) if, given a prime dividing the order of the skew brace, there is a sub-skew brace of that order; in other words, we do not know whether an analog of the Cauchy theorem holds. Despite these discouraging premises, we chose to investigate the analogs of the Sylow theorems for skew braces. Our results show that, in many natural and significant settings, a Sylow-type theorem (and consequently a Cauchy-type theorem) does indeed hold.

Question 1 *Let p^n be the maximum power of the prime p dividing the order of the finite (soluble) skew brace $(B, +, \circ)$.*

Does B contain a sub-skew brace of order p^n?

Actually, our results for (certain classes of) finite soluble skew braces go well beyond proving the existence of Sylow substructures. In fact, they provide analogs of what is known as the Hall theorem for finite soluble groups, which deals with the existence of the so-called *Hall subgroups* (analogs of Sylow subgroups for more than one prime). We believe that the following somewhat stronger question might have a positive answer for soluble skew braces.

Question 2 *Let π be any set of primes, and $(B, +, \circ)$ be a finite soluble skew brace.*

Does B contain a sub-skew brace whose order is the maximum π-number dividing the order of B?

(The reason why we often need a strong solubility assumption in our arguments, even for the existence of Sylow subgroups, is explained in Remark 9.)

The unexplained terminology can be found in Section 2. For the reader's convenience, we summarize our main results as follows:

- Lemma 7: This is a key lemma that often allows us to conclude the existence of Sylow and Hall sub-skew braces in favorable contexts. As a consequence of this result, we can almost immediately deduce Sylow and Hall-type theorems for λ-homomorphic skew braces, right nilpotent skew braces and bi-skew braces (see Corollary 8).
- Theorem 11: A Hall-type theorem for left nilpotent skew braces.
- Theorem 12: A Hall-type theorem for a skew brace B such that both $(B, \circ)$ and $(B * B, +)$ have unique (i.e., normal) Hall subgroups. As a consequence, we obtain a

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Cauchy-type theorem when the multiplicative group has a normal Sylow subgroup (see Corollary 13).
- Theorem 14: A Hall-type theorem for two-sided skew braces, and so, in particular, for skew braces whose multiplicative group is abelian.
- Theorem 19: A Hall-type theorem for a supersoluble skew brace.
- Section 4: We prove that a well-known family of finite simple skew braces has Hall sub-skew braces.

2 Preliminaries

A *skew left brace* (resp. *skew right brace*) is a triple $(B, +, \circ)$ where $(B, +)$ and $(B, \circ)$ are (not necessarily abelian) groups (respectively called the *additive* and *multiplicative* groups) such that the *skew left distributivity* (resp. *skew right distributivity*) holds:

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

(resp., $(b + c) \circ a = b \circ a - a + c \circ a$) for all $a, b, c \in B$. Since skew left braces and skew right braces are algebraically equivalent, it is sufficient to restrict our attention to skew *left* braces. For simplicity, we will refer to skew left braces simply as skew braces, specifying "left" only when necessary. If $(B, +, \circ)$ satisfies both the skew left and right distributivity, then we say that B is a *two-sided* skew brace. If both $(B, +, \circ)$ and $(B, \circ, +)$ are skew braces, then B is said to be a *bi-skew brace*.

Now, let $(B, +, \circ)$ be a skew brace. If $\mathfrak{X}$ is any class of groups, then B is said to be of $\mathfrak{X}$ *type* if $(B, +) \in \mathfrak{X}$; thus, for example, B is of *abelian type* if $(B, +)$ is an abelian group. In this latter case, we also say that B is a *brace* (without the word "skew") because this is the name for the original structures constructed by Rump in [14].

The map

$$\lambda : a \in (B, \circ) \mapsto \lambda_a \in \mathrm{Aut}(B, +),$$

defined by $\lambda_a(b) = -a + a \circ b$, is a (well-defined) group homomorphism; for every $a, b \in B$, the following identities are verified: $a + b = a \circ \lambda_a^{-1}(b)$ and $-a = \lambda_a(a^{-1})$, where a^{-1} denotes the inverse of a in the group $(B, \circ)$. If λ is a homomorphism also with respect to the additive group $(B, +)$, then B is said to be λ-*homomorphic* (see [5]). A third operation, called the *star operation*, measures the interaction between the additive and the multiplicative groups of B: for each $a, b \in B$,

$$a * b := \lambda_a(b) - b = -a + a \circ b - b.$$

The following identities turn out to be useful on many occasions:

$$x * (y + z) = x * y + y + x * z - y \quad \text{and} \quad (x \circ y) * z = x * (y * z) + y * z + x * z$$

for all $x, y, z \in B$. Clearly, $a * b = 0$ for every $a, b \in B$ if and only if $(B, +) = (B, \circ)$; in this case, we say that B is *trivial* and we can consider B just as a group. In fact, given a group $(G, \cdot)$, we can always realize the trivial skew brace $(G, \cdot, \cdot)$ and the *almost trivial* skew brace $(G, \cdot, \cdot^{\mathrm{op}})$, where $g \cdot^{\mathrm{op}} h = h \cdot g$. The λ-action allows us to consider the natural semidirect product $(B, +) \rtimes_\lambda (B, \circ)$; a straightforward computation shows

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that the operation $a * b$ corresponds to a commutator of the form $[(0, a), (b, 0)] = (a * b, 0)$ for all $a, b \in B$.

Remark 3 Since every skew brace $(B, +, \circ)$ carries three group structures, we need to distinguish between the usual group-theoretic notations depending on which group operation is involved. This is done by adding a small symbol $+$, $\circ$, or λ as a subscript, and by using the adjectives "additive" and "multiplicative" appropriately. Thus, for example, the additive commutator $a + b - a - b$ is denoted by $[a, b]_+$, while if $a \in (B, +)$ and $b \in (B, \circ)$, we write $[a, b]_\lambda = a - \lambda_b(a)$.

It is presumably a well-known fact that bi-skew braces can be characterized as those skew braces for which the λ_a's, $a \in B$, are skew brace homomorphisms.

Lemma 4 *Let B be a skew brace. The following are equivalent:*

(1) *B is a bi-skew brace.*
(2) *λ_a is a skew brace automorphism of B for every $a \in B$.*

Proof For all $a, b, c \in B$

$$\lambda_c(a \circ b) = \lambda_c(a + \lambda_a(b)) = \lambda_c(a) + \lambda_c\lambda_a(b),$$

and

$$\lambda_c(a) \circ \lambda_c(b) = \lambda_c(a) + \lambda_{\lambda_c(a)}\lambda_c(b).$$

Thus, the two expressions coincide precisely when for all $a, c \in B$, one has

$$\lambda_{\lambda_c(a)} = \lambda_c\lambda_a\lambda_c^{-1},$$

which is, once one translates the notation, condition 4 of [9, Theorem 3.1].

The fact that (1) implies (2) also follows from the fact that if λ is the lambda function of the bi-skew brace $(B, +, \circ)$, then the lambda function of $(B, \circ, +)$ is given by

$$c \in (B, +) \mapsto \lambda_c^{-1} = \lambda_{c^{-1}} \in \mathrm{Aut}(B, \circ)$$

(see [9, Subsection 3.1]). ∎

Now, we briefly review the most important types of substructures that one may encounter in a skew brace $(B, +, \circ)$.

- *Sub-skew brace*: A subgroup of both $(B, +)$ and $(B, \circ)$. For example, it turns out that $\mathrm{Ker}(\lambda)$ is always a sub-skew brace. As usual in groups and rings, $C \leq B$ means that C is a sub-skew brace of B.
- (*Strong*) *left ideal*: A (normal) subgroup of $(B, +)$ that is also λ-invariant. Note that a left ideal is always a sub-skew brace, and that every characteristic subgroup of an additive normal subgroup of B is always a strong left ideal.
- *Ideal*: A strong left ideal of B that is also a multiplicative normal subgroup of B. If I is an ideal, then B/I inherits the structure of a skew brace with the induced operations on the additive cosets with respect to I. As usual in groups and rings, $I \trianglelefteq B$ means that I is an ideal of B. The most relevant ideals in a skew brace B (apart from the obvious ones B and $\{0\}$) are probably the following ones:

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- The *socle* $\operatorname{Soc}(B)$ of B: The intersection of $\operatorname{Ker}(\lambda)$ and the center $Z(B,+)$ of $(B,+)$.
- The *center* $Z(B)$ of B: The intersection of $\operatorname{Soc}(B)$ and the center $Z(B,\circ)$ of $(B,\circ)$.
- $B * B = \langle a * b : a, b \in B\rangle_+$. More generally, if I is any ideal of B and J is any left ideal of B, then $I * B = \langle i * b : b \in B,\ i \in I\rangle_+ \trianglelefteq B$, while $B * J = \langle b * j : b \in B,\ j \in J\rangle_+$ is a left ideal of B.

We say that B is *simple* if it does not have any nonzero proper ideal.

The previous ideals allow us to define natural solubility and nilpotency concepts for a skew brace $(B,+,\circ)$ as follows (see [3, 4, 12] for more information). The skew brace B is said to be *soluble* if it has a finite chain of ideals

$$\{0\} = I_0 \leq I_1 \leq \cdots \leq I_n = B$$

such that I_{j+1}/I_j is a trivial skew brace of abelian type for every $0 \leq j < n$; in this case, the smallest such nonnegative integer n is the *soluble length* of B.

Now, we recursively define the *upper central series* and the *upper socle series* of B by setting: $Z_0(B) = \{0\}$, $Z_1(B) = Z(B)$, and $Z_{i+1}(B)/Z_i(B) = Z(B/Z_i(B))$ for every $i \geq 0$: $\operatorname{Soc}_0(B)$, $\operatorname{Soc}_1(B) = \operatorname{Soc}(B)$, and $\operatorname{Soc}_{i+1}(B)/\operatorname{Soc}_i(B) = \operatorname{Soc}(B/\operatorname{Soc}_i(B))$. Then B is said to be *centrally nilpotent* (resp. *socle nilpotent*) if $B = Z_n(B)$ (resp. $B = \operatorname{Soc}_n(B)$) for some nonnegative integer n; in this case, the smallest such n is the *central nilpotency class* (resp. *socle nilpotency class*) of B. Clearly, every central nilpotent skew brace is socle nilpotent, and every socle nilpotent skew brace is soluble. Also, we observe that when B is a trivial skew brace, the solubility and nilpotency concepts here introduced coincide with the usual ones for groups. For this reason, a soluble skew brace of length 1 (resp. of length at most 2) is also said to be *abelian* (resp. *metabelian*).

Two other nilpotency concepts for a skew brace may be defined as follows. In this case, one tries to emulate nilpotency in the context of rings and not groups. We say that B is *right nilpotent* if there exists a nonnegative integer n such that

$$(\ldots(\underbrace{(B * B) * B \ldots) * B}_{n \text{ times}} = \{0\};$$

in this case, the smallest such n is the *right nilpotency class* of B. Note also that the sets appearing in the left-hand side of the above equality are always ideals of B. Similarly, B is *left nilpotent* if

$$\underbrace{B * (\cdots * (B * (B * B)))\ldots)}_{n \text{ times}} = \{0\}$$

for some nonnegative integer n; in this case, the smallest such n is the *left nilpotency class* of B. Note that the sets appearing in the left-hand side of the above equality are always left ideals of B, but they are not ideals in general. Clearly, every central nilpotent skew brace is both right and left nilpotent, while every socle nilpotent skew brace is right nilpotent. There are many other natural relations between these nilpotency concepts and the nilpotency of the additive/multiplicative groups of the skew brace, but since we do not directly deal with these we refer the interested reader to [3] and its bibliography.

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Somewhat between nilpotency and solubility, there is the concept of supersoluble skew brace (see [2]). The finite skew brace $(B, +, \circ)$ is said to be *supersoluble* if it has a finite chain of ideals whose factors have prime order. It turns out, for example, that every finite skew brace of square-free order is supersoluble (see [2, Corollary 3.9]) and that a finite supersoluble skew brace B of order $p_1^{r_1} \dots p_n^{r_n}$, where $p_n < \dots < p_1$ are primes, has a series of ideals

$$\{0\} = I_0 \leq I_1 \leq \dots \leq I_n = B,$$

where $|I_{i+1}/I_i| = p_{i+1}^{r_{i+1}}$ for $0 \leq i < n$ (see [2, Theorem 3.11]). Clearly, every centrally nilpotent skew brace is supersoluble, and every supersoluble skew brace is soluble.

Now, we turn to the main objects of this article. Let $(B, +, \circ)$ be a finite skew brace, and let p be a prime. We say that the sub-skew brace S_p is a *Sylow p-sub-skew brace* (or also a *p-Sylow* for short) of B if $|S_p|$ is the largest power of p dividing the order of B (this is equivalent to requiring that S_p is either an additive or a multiplicative Sylow p-subgroup of B). In a similar fashion, if π is a set of primes, we say that the sub-skew brace R_π is a *Hall π-sub-skew brace* (or also a *π-Hall* for short) of B if its order is the largest π-number dividing $|B|$. Again, this is equivalent to requiring that R_π is either an additive or a multiplicative Hall p-subgroup. Obviously, if $\pi = \{p\}$, then a π-Hall is just a p-Sylow. Note also that, for a trivial skew brace, these concepts coincide with the well-known group-theoretic analogs, and that in the almost trivial case, every additive/multiplicative Hall π-subgroup is a Hall π-sub-skew brace. The following result is easily proved and will be used several times without further notice.

Lemma 5 *Let $(B, +, \circ)$ be a finite skew brace, and let S be a Hall π-sub-skew brace for some set π of primes. If I is any ideal of B, then $S + I/I$ and $S \cap I$ are Hall π-sub-skew braces of B/I and I, respectively.*

Obviously, a finite skew brace will said to be a *π-skew brace* if its order is a π-number, or equivalently, if it coincides with its unique Hall π-sub-skew brace.

Clearly, every finite skew brace of nilpotent type has (unique) Hall π-sub-skew braces for every choice of the set π of primes because the additive Hall π-subgroups are additive characteristic subgroups and so sub-skew braces (see [1] for some additional piece of information on the structure of the Hall π-sub-skew braces in this case). More generally, a finite skew brace has a (unique) Hall π-sub-skew brace provided that its additive group has a normal Hall π-subgroup (because it is a characteristic additive subgroup and so a left ideal). Thus, for example, if p is the largest prime dividing the order of a finite supersoluble skew brace B, then B has (a unique) Sylow p-sub-skew brace.

We also remark that if $(B, +, \circ)$ is a finite skew brace of order a power of a prime p, then it is easy to see that B has a sub-skew brace of order p (see also the main theorem of [4]). Therefore, if we can prove the existence of a (nonzero) Sylow p-sub-skew brace in a certain environment, then we can automatically also prove the Cauchy theorem for the prime p.

One of the tools we will employ to prove our main result is the opposite skew brace. Recall that for any skew brace $(B, +, \circ)$, it is possible to define another skew brace which goes under the name of *opposite* skew brace of B and is denoted by B^{opp}.

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This skew brace is obtained by replacing the additive group $(B,+)$ with its *opposite* group $(B,+^{\mathrm{opp}})$, which is defined by: $a +^{\mathrm{opp}} b = b + a$ for every $a, b \in B$. The concept of opposite skew brace has been introduced in [13] (but see also [8]), and since then, it has proved to be very useful in the study of skew braces and their applications. From our point of view, the most relevant feature of the opposite skew brace is that it preserves sub-skew braces and (strong left) ideals. The following statement is certainly well known, but we were unable to find it in the literature. For completeness, we include a short proof.

Lemma 6 *Let $(B,+,\circ)$ be a skew brace, and let C be a sub-skew brace (resp., strong left ideal, or ideal) of B. Then C is a sub-skew brace (resp., strong left ideal, or ideal) of B^{opp}.*

Proof The fact that C is a sub-skew brace of B^{opp} is obvious because the (normal) subgroups of $(B,+)$ are also (normal) subgroups of $(B,+^{\mathrm{opp}})$. Suppose C is a (strong left) ideal of B. We only need to prove that C is λ-invariant. Thus, let $a \in B$ and $b \in C$. We have that

$$\lambda_a(b)^{\mathrm{opp}} = -a +^{\mathrm{opp}} a \circ b = a \circ b - a = a - a + a \circ b - a = a + \lambda_a(b) - a$$

belongs to C because C is a strong left ideal of B. ∎

3 Sylow and Hall theorems

We start this section with a key lemma that often allows us to conclude the existence of Sylow and Hall sub-skew braces under some restrictions on $\mathrm{Ker}(\lambda)$.

Lemma 7 *Let p be a prime (resp. π be a set of primes), and let $(B,+,\circ)$ be a finite skew brace (resp., a finite skew brace with $(B,+)$ and $(B,\circ)$ soluble) such that $(B,\circ)/\mathrm{Ker}(\lambda)$ or $(B,\circ)/\mathrm{Ker}(\lambda^{\mathrm{opp}})$ is a p-group (resp. a π-group). If P is any left ideal of B of p-power order (resp. of order a π-number), then B has a Sylow p-sub-skew brace P_2 (resp. a Hall π-sub-skew brace) containing P. (Actually, P_2 turns out to be a left ideal of B or of B^{opp}.)*

Proof We only prove the case for a prime p, the proof for the case of a set of primes π being totally analogous (the only difference is the use of the theorems of Hall instead of those of Sylow). Since the sub-skew braces of B and its opposite are the same by Lemma 6, it is enough to prove the case when $(B,\circ)/\mathrm{Ker}(\lambda)$ is a p-group. Let $G = (B,+) \rtimes_\lambda (B,\circ)$. Now, since P is a p-sub-skew brace of B, $S = (P,+) \rtimes_\lambda (P,\circ)$ is a p-subgroup of G. By the hypotheses, S can be extended to a p-subgroup of the form $S_1 = (P,+) \rtimes_\lambda (P_1,\circ)$, where $(P_1,\circ)$ is a Sylow p-subgroup of $(B,\circ)$ containing $(P,\circ)$, and subsequently to a Sylow p-subgroup of G, which is necessarily of the form $S_2 = (P_2,+) \rtimes_\lambda (P_1,\circ)$, where $(P_2,+)$ contains $(P,+)$. Consequently, $\lambda_a(P_2) = P_2$ for every $a \in P_1$. But $(B,\circ) = P_1 \circ \mathrm{Ker}(\lambda)$, and so P_2 is λ-invariant. Therefore, P_2 is the required Sylow p-sub-skew brace. ∎

Corollary 8 *Let p be a prime (resp. π be a set of primes), and let $(B,+,\circ)$ be a finite skew brace (resp., a finite skew brace with $(B,+)$ and $(B,\circ)$ soluble). Suppose that one of the following conditions holds:*

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(1) *B is a bi-skew brace.*
(2) *B is right nilpotent.*
(3) *B is λ-homomorphic.*

If S is any left ideal of B whose order is a p-number (resp. a π-number), then B has a Sylow p-sub-skew brace (resp. a Hall π-sub-skew brace) containing S.

Proof (1) In this case, λ is an antihomomorphism of $(B, +)$ (see [9, Theorem 3.1(3)]), so that the kernel $K = \mathrm{Ker}(\lambda)$ is an ideal of B. If $K = \{0\}$, then B is an almost trivial skew brace (see [9, Theorem 3.1(7)]), for which the results hold true trivially. Let B be a minimal counterexample, so that $K \neq \{0\}$. By the minimality of B, we have that B/K is a π-skew brace, and thus the results follow from Lemma 7.

(2) We use induction on the right nilpotency class n of B. Clearly, if $n \leq 1$, then the statement is trivial. Suppose $n > 1$, and let I be an ideal of B such that $I * B = \{0\}$ and B/I is right nilpotent of right nilpotency class $n - 1$. By induction, B/I has a Hall π-sub-skew brace, so, without loss of generality, we may assume that $|B/I|$ is a π-number. Since $I \subseteq \mathrm{Ker}(\lambda)$, we may apply Lemma 7 to conclude the proof.

(3) This follows from [15, Theorem 3.13] and (2). ∎

Left nilpotent skew braces cannot be dealt with in the same way as right nilpotent ones, the reason being that the sub structures that define them are usually only left ideals and not ideals.

Remark 9 The proof of Theorem 11 will also make it clear why (a strong) solubility assumption for skew braces is necessary for our arguments to work. In fact, we need a minimal ideal I of the skew brace B to be abelian, a fact that is only possible if B is soluble in the sense of [4]. In these circumstances, for example, every characteristic subgroup of *the group* I is an ideal of B, since it is a characteristic subgroup of both groups, so I must also be characteristically simple and consequently a q-group for some prime q. These are standard arguments that we will often employ.

Lemma 10 *Let $(B, +, \circ)$ be a finite, left nilpotent skew brace. Let $(B, +) = (I, +) + (J, +)$, where I and J are strong left ideals of B of coprime orders, and $I \trianglelefteq B$. Then $J \trianglelefteq B$.*

Proof We only need to prove that J is multiplicatively normal. Let B be a minimal counterexample to the statement. Since B is left nilpotent, $I + (J * J)$ is a proper ideal of B, so $J * J$ is an ideal of $I + (J * J)$, and hence also an ideal of B. By minimality of B, $J * J = \{0\}$. Now, since B is left nilpotent, if $B = B^1 \supseteq B^2 \supseteq \cdots \supseteq B^m = \{0\}$ denotes the left series of B, then the subgroups $I_r = I \cap B^r$ give a series of $(I, +)$. Moreover, for every $j \in J$ and every $x \in I_r$, one has $j * x \in I \cap B^{r+1} = I_{r+1}$, because I is an ideal and $J * B^r \subseteq B^{r+1}$. Thus, λ_j acts trivially on each factor I_r/I_{r+1}; in this sense, $(J, \circ)$ stabilizes this series of $(I, +)$.

Consider now the image of $(J, \circ)$ in $\mathrm{Aut}(I, +)$. It is contained in the group of automorphisms stabilizing the above series, and this group has order divisible only by primes dividing $|I|$. On the other hand, this image is a quotient of $(J, \circ)$, hence its order divides $|J|$. Since $|I|$ and $|J|$ are coprime, the image is trivial. Therefore, $\lambda_j(i) = i$ for

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all $j \in J$ and $i \in I$, and so $J * I = \{0\}$. Thus, $J * B = \{0\}$. Hence, $J \leq \mathrm{Ker}(\lambda)$ and hence $[J, I]_\circ = \{0\}$, which means that J is multiplicatively normal, a contradiction. ∎

Theorem 11 *Let π be a set of primes, and let $(B, +, \circ)$ be a finite soluble skew brace which is left nilpotent. Then the Hall π-sub-skew braces of B coincide with the Hall π-subgroups of $(B, \circ)$.*

Proof Let B be a minimal counterexample to the statement, and let I be a minimal ideal of B contained in $B * B$. By minimality of B, we have that $|B/I|$ is a π-number. Moreover, since B is soluble, I is an abelian q-group for some prime $q \notin \pi$. Let P be any Hall π-subgroup of $(B, \circ)$, and extend it to a Hall π-subgroup $P_1 \rtimes_\lambda P$ of $G = (B, +) \rtimes_\lambda (B, \circ)$. By hypothesis, B is left nilpotent. If $B = B^1 \supseteq B^2 \supseteq \cdots \supseteq B^m = \{0\}$ is the left series of B, then the subgroups $I_r = I \cap B^r$ form a finite series of $(I, +)$. Moreover, for every $p \in P$ and every $x \in I_r$, one has $p * x \in I \cap B^{r+1} = I_{r+1}$, because I is an ideal and $P * B^r \subseteq B^{r+1}$. Hence, λ_p acts trivially on each factor I_r/I_{r+1}. Thus, the image of P in $\mathrm{Aut}(I, +)$ is contained in the group of automorphisms acting trivially on this series. Since $(I, +)$ is an abelian q-group, this latter group is a q-group, and in particular, it is nilpotent. On the other hand, the image of P is a quotient of the π-group P. Since $q \notin \pi$, this image is trivial. Therefore, $P * I = \{0\}$.

Let $a \in I$, $b \in P_1$, and $c \in P$. Then $b + a - b = a_1$ for some $a_1 \in I$. Since $I \leq B * B$, $P * I = \{0\}$, so conjugating the previous equality by c in G yields $\lambda_c(b) + a - \lambda_c(b) = a_1$, so $-b + \lambda_c(b) = [-b, c]_\lambda$ additively centralizes a. The arbitrariness of a, b, and c shows that $P * P_1$ additively centralizes I.

It follows that $(B * B, +)$ is the direct sum of $(I, +)$ and $(P * P_1, +)$. By Lemma 10, $P * P_1$ is an ideal of $B * B$, and so also an ideal of B. Thus, the minimality of B yields that $P * P_1 = \{0\}$. But then $P * B = P * (P_1 + I) = \{0\}$, so $P * P = \{0\}$, and hence P is a Hall π-sub-skew brace of B. ∎

We remarked in Section 2 that the Hall-type theorems trivially hold when the additive group has a unique (and so normal) Hall subgroup. Our next result shows that such a hypothesis on the additive group can be weakened if we require a similar one on the multiplicative group.

Theorem 12 *Let π be a set of primes and $(B, +, \circ)$ a finite skew brace. If both $(B, \circ)$ and $(B * B, +)$ have normal Hall π-subgroups P and P_1, respectively, then P is a Hall π-sub-skew brace of B.*

Proof Suppose the statement is false, and let B be a minimal counterexample to the statement; in particular, $B * B \neq \{0\}$ otherwise B would be a group and the statement would trivially holds. Let I be a minimal ideal of B contained in $B * B$. Since B is soluble, I is a trivial skew brace of abelian type, and so its characteristic subgroups are ideals of B. This implies that $(I, +)$ is an elementary abelian q-group for some prime q. By minimality of B, we have that $|B/I|$ is a π-number and (obviously) that $q \notin \pi$. By hypothesis, $(B * B, +)$ contains a characteristic Hall π-subgroup P_1. But then P_1 is a sub-skew brace (and in particular a subgroup of $(B, \circ)$), which, by order considerations, coincides with the only Hall π-subgroup of $(B * B, \circ)$. Thus, P_1 is an

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ideal of B. Now, $B * B$ is the direct product (as a skew brace) of I and P_1, so if we factor P_1 out, then we may assume that $I = B * B$.

Let P be the unique Hall π-subgroup of $(B, \circ)$. In the natural semi direct product $G = (B, +) \rtimes_\lambda (B, \circ)$, extend P to a Hall π-subgroup R of G. Since $(B, +)$ is normal in G, there is a Hall π-subgroup P_2 of $(B, +)$ such that $R = (P_2, +) \rtimes_\lambda (P, \circ)$ (use Schur Zassenhaus). Thus, if $a \in P_2$ and $x \in P$, then $x * a \in P_2$; but B/I is a trivial skew brace, so $x * a \in I$, and hence $x * a \in P_2 \cap I = \{0\}$. Therefore, $P * P_2 = \{0\}$.

If $b \in (B, \circ)$, then the action by conjugation in G on R gives

$$^b R = \lambda_b(P_2) \rtimes_\lambda P,$$

which shows that $P * \lambda_b(P_2) = \{0\}$. Let U be the subgroup of $(B, +)$ generated by the $\lambda_b(P_2)$'s with $b \in B$. Then $P * U = \{0\}$. By its very definition, U (and consequently $U \cap I$) is λ-invariant, so U is a sub-skew brace. Since P_2 is an additive subgroup of U, the order of U is divisible by $|P_2| = |P|$, and this means that the multiplicative Hall π-subgroup of U must coincide with P (the latter being the unique Hall π-subgroup of $(B, \circ)$). Consequently, $P * P = \{0\}$ and so P is a Hall π-sub-skew brace of B. ∎

Corollary 13 *Let p be a prime dividing the order of the finite soluble skew brace $(B, +, \circ)$. If P is a multiplicative normal Sylow p-subgroup of B, then B has a sub-skew brace of order p.*

Proof Suppose the statement is false, and let B be a minimal counterexample to the statement. Let I be a minimal ideal of B. Since B is soluble, I is a trivial skew brace of abelian type. Hence, every characteristic subgroup of $(I, +)$ is an ideal of B. By the minimality of I, the group $(I, +)$ has no proper nonzero characteristic subgroups, and therefore, it is an elementary abelian q-group for some prime q. Moreover, $q \neq p$, because otherwise, since I a trivial elementary abelian p-brace, any subgroup of $(I, +)$ of order p would be a sub-skew brace of B, contrary to our assumption.

We claim that $|B/I| = p$. Since $q \neq p$ and p divides $|B|$, the prime p divides $|B/I|$. The image of P in $(B/I, \circ)$ is therefore a normal Sylow p-subgroup. Hence, by the minimality of B, the quotient B/I has a sub-skew brace $\overline{K}$ of order p. Let K be the inverse image of $\overline{K}$ in B. If $|B/I| > p$, then K is a proper sub-skew brace of B. Since $|K/I| = p$ and $|I|$ is a q-number with $q \neq p$, the subgroup $K \cap P$ is a normal Sylow p-subgroup of $(K, \circ)$. Thus, K satisfies the hypotheses of the corollary and, by the minimality of B, it contains a sub-skew brace of order p, again a contradiction. Therefore, $|B/I| = p$.

Since every skew brace of prime order is trivial, we get $B * B \leq I$. In particular, $(B * B, +)$ is a q-group, and so its Hall p-subgroup is trivial and normal. Theorem 12, applied with $\pi = \{p\}$, implies that P is a Hall p-sub-skew brace of B. But $|B/I| = p$ and $q \neq p$, so $|P| = p$. Hence, P itself is a sub-skew brace of order p, the final contradiction. ∎

Next, we deal with two-sided skew braces, and here the argument makes great use of some nontrivial facts about their automorphism group.

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Theorem 14 *Let p be a prime (resp. π be a set of primes), and let $(B, +, \circ)$ be a finite (resp. finite soluble) two-sided skew brace. Then the Sylow p-sub-skew braces (resp. Hall π-sub-skew braces) of B coincide with the Sylow p-subgroups (resp. Hall π-subgroups) of $(B, \circ)$.*

Proof We prove the result for a set π of primes, the case of a single prime p being easier. Since the multiplicative inner automorphisms of B are skew brace automorphisms (see [17, Proposition 2.3]) and $(B, +)$ is soluble, we only need to prove the existence of the Hall π-sub-skew braces (because of the well-known theorems of Philip Hall). Thus, let B be a minimal counterexample to the existence, and set $H = (B * B) \cap (B *^{\mathrm{opp}} B)$. Then H is an ideal of B and $(H, +)$ is abelian by Theorem 4.3 of [17].

Suppose first that $H = \{0\}$. Clearly, neither $B * B$ nor $B *^{\mathrm{opp}} B$ can be $\{0\}$ (because otherwise B has Hall π-sub-skew braces). By minimality of B, both the orders of $B/B * B$ and $B/B *^{\mathrm{opp}} B$ are π-numbers. Thus, $|B|$ is a π-number itself, and we have a contradiction.

Thus, $H \neq \{0\}$. Let I be a minimal ideal of B contained in H. Then $(I, +)$ is an elementary abelian q-group for some prime q. By minimality, the statement holds for B/I, so we have that the order of B/I is a π-number and $q \notin \pi$. Also $(H, +)$ is abelian, so it contains a unique Hall π-subgroup R, which is then an ideal of H (see [17, Corollary 2.4]). But this means that $H = I \times R$, so R is actually both additively and multiplicatively characteristic, implying that R is an ideal of B. By minimality of B, $R = \{0\}$. Thus, $H = I$ is contained in $Z(B * B, +)$ by [17, Theorem 4.3]. Let J/I be a minimal ideal of B/I contained in $B * B/I$, if any. Then J is an ideal of B of nilpotent type. Again, if S is a Hall π-subgroup of $(J, +)$, then S is an ideal of B which must then be $\{0\}$ by minimality of B. Therefore, $I = H = B * B$.

Let P be a Hall π-subgroup of $(B, \circ)$. Then there is a Hall π-subgroup P_1 of $(B, +)$ such that $P_1 \rtimes_\lambda P$ is a Hall π-subgroup of $G = (B, +) \rtimes_\lambda (B, \circ)$. Clearly, $P * P_1 \subseteq I \cap P_1 = \{0\}$. Now, let U be the largest subgroup of $(B, +)$ such that $P * U = \{0\}$. Clearly, $P_1 \leq U$. We claim that $P_1 < U$. Suppose not, so $P_1 = U$. Now, since B is two-sided, so $a * y = 0$ implies $a * y^{-1} = 0$ for every $a, y \in B$. Thus, $P_1^{-1} \subseteq U = P_1$. Similarly, if $a * b = a * c = 0$ for some $a, b, c \in B$, then $a * (b \circ c) = 0$. Consequently, $P_1 \circ P_1 \subseteq U = P_1$. This implies that P_1 is a multiplicative subgroup and so a Hall π-sub-skew brace of B, a contradiction.

This last contradiction shows that $U > P_1$. Then $V = U \cap I$ is not $\{0\}$ and is additively normal because U contains an additive complement to I. Now, $P * V = \{0\}$, and so also $B * V = \{0\}$ because $(B, \circ) = I \circ P$ and $I * V = \{0\}$.

Since the multiplicative inner automorphisms are skew brace automorphisms, we have that $B * \left(x \circ V \circ x^{-1}\right) = \{0\}$ for all $x \in (B, \circ)$, so $x \circ V \circ x^{-1} \subseteq U$ and hence $V = x \circ V \circ x^{-1}$. Therefore, V is multiplicatively normal. Since $B * V = \{0\}$, so V is an ideal of B, which must coincide with the minimal ideal I. Thence, $B * I = \{0\}$.

Finally, if a, b are multiplicative π-elements of B, then $\lambda_a(b) = b + u$ for some $u \in I$ (because $I = B * B$). But $\lambda_a(u) = u$ and u has additive order dividing q, so $u = 0$ (because $q \notin \pi$), and consequently $a * b = 0$. This shows that every Hall π-subgroup of $(B, \circ)$ is a (Sylow π-)sub-skew brace of B and gives the last contradiction. ∎

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Corollary 15 *Let π be a set of primes and let $(B, +, \circ)$ be a finite soluble skew brace such that $(B, \circ)$ is abelian. Then the Hall π-sub-skew braces of B coincide with the Hall π-subgroups of $(B, \circ)$.*

The argument we used for two-sided skew braces can be somewhat modified to prove a Hall-type theorem whenever the λ_b's are skew brace automorphisms instead of only automorphisms of the additive group.

Theorem 16 *Let π be a set of primes and $(B, +, \circ)$ a finite soluble skew brace such that λ_b is a skew brace isomorphism for every $b \in B$. If Q is any left ideal of B, then Q is contained in a Hall π-sub-skew brace of B.*

Proof Let B be a minimal counterexample to the statement, and let I be a minimal ideal of B. Then $(I, +)$ is an elementary abelian q-group for some prime q. By minimality of B, we have that B/I has order p^n for some $p \neq q$.

Let P be a Sylow p-subgroup of $(B, \circ)$. Then there is a Sylow p-subgroup P_1 of $(B, +)$ such that $P_1 \rtimes_\lambda P$ is a Sylow p-subgroup of $G = (B, +) \rtimes_\lambda (B, \circ)$. Thus, $\lambda_x(P_1) = P_1$ for every $x \in P$. Since λ_x is a skew brace isomorphism for every $x \in P$, we have that $\lambda_x(P_1) = P_1$ for every $x \in \lambda_b(P)$ with $b \in P$.

Let U be the normalizer of $(P_1, +)$ in $(B, \circ)$. Clearly, U contains every $\lambda_x(U)$ with $x \in P$ because λ_x is a skew brace isomorphism, and we may write $U = P \circ (I \cap U)$. Moreover, since P is not a sub-skew brace, $I \cap U \neq \{0\}$. If $c \in I \cap U$, then $\lambda_c(P_1) = P_1$. On the other hand, $(I, +) \rtimes_\lambda (I, \circ)$ is a normal Sylow q-subgroup of G, so $\lambda_c(u) = u$ for every $u \in P_1$. Therefore, $c * P_1 = \{0\}$. But obviously, $c * I = \{0\}$, and hence $c \in K = \mathrm{Ker}(\lambda)$. It follows that $I \cap U \leq K$, which means that $I \cap U = K \cap I$ because $K \leq U$.

Now, if $a \in P$, $b \in I \cap U \leq K$, and $u \in U$, then $\lambda_{a \circ b}(u) = \lambda_a(u) \in U$. Thus, $\lambda_u(v) \in U$ for every $u, v \in U$, and consequently, $u + v = u \circ \lambda_{u^{-1}}(v) \in U$ for every $u, v \in U$. By choosing $w = \lambda_u(u^{-1})$, we see that $u + w = 0$. Therefore, U is a sub-skew brace of B. Since $|P|$ divides $|U|$, we have by minimality that $U = B$, which implies that $I \subseteq K$. Lemma 7 provides the final contradiction. ∎

In order to deal with supersoluble skew braces, the following results are crucial. The first provides a complete description of automorphisms of the semidirect product of two groups under specific conditions, while the second is a special case of [8, Proposition 2.24], which, for the reader's convenience, we restate for skew *left* braces.

Theorem 17 (Special case of [10, Theorem 1]) *Let $G = M \rtimes P$ be a semidirect product, where M is abelian and characteristic in G. Then* $\mathrm{Aut}(G)$ *can be described in a natural way via the set of matrices*

$$\left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : \begin{array}{l} a \in \mathrm{Aut}(M),\ d \in \mathrm{Aut}(P), \\ a({}^k h) = {}^{d(k)} a(h) \quad \textit{for all } h \in M,\ k \in P, \\ b : P \to M \textit{ satisfies } b(xy) = b(x)\, {}^{d(x)} b(y) \quad \textit{for all } x, y \in P \end{array} \right\}.$$

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Here, an automorphism

$$\theta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

acts as $\theta(hk) = a(h)\, b(k)\, d(k)$, *for* $h \in M$ *and* $k \in P$.

Proposition 18 *Let* $(G, \cdot)$ *be a finite non-abelian group. Let* $M \neq \{1\}$ *be a characteristic subgroup of* G *of prime order* p *such that* $M \cap Z(G) = \{1\}$. *Let* $\lambda\colon a \in G \mapsto \lambda_a \in \mathrm{Aut}(G)$ *be a function satisfying* $\lambda_{a\cdot\lambda_a(b)} = \lambda_a\lambda_b$, *and suppose that for every* $m \in M$, *we have* $\lambda(m) = \iota(m^{-\sigma})$, *for some function* σ *on* M. *Then* $\sigma \in \mathrm{End}(M) = \mathbb{Z}/p\mathbb{Z}$, *and we have that either* $\sigma = 0$ *or* $\sigma = 1$.

Theorem 19 *Let* π *be a set of primes and* $(B, +, \circ)$ *a finite supersoluble skew brace. If* Q *is any left ideal of* B, *then* Q *is contained in a Hall* π*-sub-skew brace of* B.

Proof Let B be a counterexample of minimal order, and M a minimal ideal of B. By minimality, $|B/M|$ is a π-number, and M has prime order $q \notin \pi$. In particular, $(B, +) = (P, +) + (M, +)$, where P is a Hall π-subgroup of $(B, +)$. If P is normal in $(B, +)$, then P is characteristic in $(B, +)$, and so it is a Hall π-sub-skew brace of B, a contradiction. Thus, P is not additively normalized by M.

For $m \in M$, write

$$\lambda_m = \begin{bmatrix} a_m & b_m \\ 0 & d_m \end{bmatrix},$$

where a_m, b_m, d_m are defined as in Theorem 17. Since M is additively characteristic, it is also a subgroup of $(B, \circ)$, so $\lambda(M)$ is a finite q-group, and each $\lambda(m)$ is a q-element, for $m \in M$. In particular, a_m is an automorphism of order a power of q of the group M of order q, and thus $a_m = 1$ for all $m \in M$. We also have $d_m = 1$ for $m \in M$, because M is an ideal.

In order to obtain the final contradiction, we prove that there exists a function τ on M such that $b_m(x) = [m^\tau, x]$, for $x \in P$. The existence of such a function obviously implies that $\lambda_m = \iota(m^\tau)$, for $m \in M$, where ι denotes additive conjugation (on the left). Writing $\tau = -\sigma$, Proposition 18 now applies to $G = (B, +)$, so either $\sigma = 0$ and $M \leq \mathrm{Ker}(\lambda)$, or $\sigma = 1$ and M is easily seen to lie in $\mathrm{Ker}(\lambda^{\mathrm{opp}})$. In any case, we may apply Lemma 7 and obtain a contradiction.

It remains to prove the existence of such a function τ. We provide two proofs of this fact, one using cohomology results, and the other cohomology-free. As for the former, $b_m : P \to M$ is a derivation for the action of P on M by conjugation in $(B, +)$. Since P and M have coprime orders, we have $H^1(P, M) = 0$ (see [6, III.10.2]), that is, all such derivations are inner, so $b_m(x) = [a, x]_+ = a + x - a - x$ for some $a \in M$ and for all $x \in P$. This obviously proves our claim.

The latter proof is as follows. Let $c_1, c_2 \in C = C_{(P,+)}(M, +)$ and $m \in M$. Then

$$b_m(c_1 + c_2) = b_m(c_1) + {}^{+,c_1}b_m(c_2) = b_m(c_1) + b_m(c_2).$$

Therefore, $b_{m\restriction C} : (C, +) \longrightarrow (M, +)$ is a group homomorphism, and since $\gcd(|P|, |M|) = 1$, it is trivial. Let $t \in P$ be such that $(P/C, +) = \langle t + C \rangle_+$. Then every element

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of P is of the form $\ell t + c$, for some nonnegative integer ℓ and some $c \in C$, so that

$$b_m(\ell t + c) = b_m(\ell t) = {}^{1+t+\cdots+t^{\ell-1}}(b_m(t)).$$

It follows that b_m is completely determined by its value $b_m(t) \in M$, so there are at most q possibilities for b_m. But for $m \in M$, the map $\beta_m : x \in P \mapsto [m, x] \in M$ satisfies

$$\beta_m(x + y) = [m, x + y]_+ = [m, x]_+ + {}^x[m, y]_+ = \beta_m(x) + {}^x\beta_m(y),$$

and they are q distinct maps, as for $m, n \in M$ and $x \in P$, we have

$$[m - n, x]_+ = [m, x]_+ - [n, x]_+,$$

as M is abelian, and $[m - n, x]_+ = 0$ for all $x \in P$ if and only if $m - n = 0$, as P acts nontrivially on M. This again proves the existence of the map τ. ∎

4 An insoluble example

In this section, we show that two relevant examples of a simple skew brace, given by Byott in [7, Theorem 1.1], have Hall sub-skew braces for every set of primes dividing its order. Since each one of these skew braces is the opposite of the other, we may just work with one of them, call it $(B, +, \circ)$. By construction, $(B, +) \simeq P \rtimes C$, where P is an elementary abelian p-group, C is a cyclic group of order q, and p, q are distinct primes such that q divides $(p^p - 1)/(p - 1)$; moreover, $(B, \circ) \simeq D \rtimes R$, where D is cyclic of order q, and R is a p-group. Clearly, P is a Sylow p-sub-skew brace of G, and every p-sub-skew brace of B is contained in P. We need to show that B has sub-skew braces of order q. To see this, we consider the natural semidirect product $G = (B, +) \rtimes_\lambda (B, \circ)$. Then we extend the unique Sylow q-subgroup D of $(B, \circ)$ to a Sylow q-subgroup $T = S \rtimes_\lambda D$ of G. But T has order p^2, so is abelian, and hence $D * S = \{0\}$. Let g be any element of $(B, \circ)$. Since $D \trianglelefteq (B, \circ)$, we have that $D * \lambda_g(S) = \{0\}$. Thus, if we call N the subgroup of $(B, +)$ generated by the $\lambda_g(S)$, with $g \in (B, \circ)$, then $D * N = \{0\}$. But N is an additive subgroup of $(B, +)$ containing S, so either S is λ-invariant or N contains B because the action of C on P is irreducible. In the former case, S is a Sylow q-sub-skew brace of B, while in the latter, D is a Sylow q-sub-skew brace because $D * D = \{0\}$. Thus, in any case, B has Sylow q-sub-skew braces.

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University of Trento, Italy
e-mail: andrea.caranti@unitn.it

Università di Pisa, Italy
e-mail: ilaria.delcorso@unipi.it

Università degli Studi della Campania "Luigi Vanvitelli," Italy
e-mail: massimiliano.dimatteo@unicampania.it

Università Pegaso, Italy
e-mail: maria.ferrara1@unipegaso.it

Università di Napoli Federico II, Italy
e-mail: marco.trombetti@unina.it

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