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RETRACTION – MEASURING CLUB-SEQUENCES TOGETHER WITH THE CONTINUUM LARGE

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Abstract. *Measuring* says that for every sequence $(C_\delta)_{\delta<\omega_1}$ with each C_δ being a closed subset of δ there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$, a tail of $C \cap \delta$ is either contained in or disjoint from C_δ. We answer a question of Justin Moore by building a forcing extension satisfying measuring together with $2^{\aleph_0} > \aleph_2$. The construction works over any model of ZFC + CH and can be described as a finite support forcing iteration with systems of countable structures as side conditions and with symmetry constraints imposed on its initial segments. One interesting feature of this iteration is that it adds dominating functions $f : \omega_1 \longrightarrow \omega_1$ mod. countable at each stage.

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§1. **Introduction.** *Measuring* (see [6]) is the following very strong form of failure of Club Guessing at ω_1.

Definition 1.1. *Measuring* holds if and only if for every sequence $\vec{C} = (C_\delta : \delta \in \omega_1)$, if each C_δ is a closed subset of δ in the order topology, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta)\backslash\alpha \subseteq C_\delta$, or
- $(C\backslash\alpha) \cap C_\delta = \emptyset$.

In the above definition, we will say that C *measures* $\vec{C}$. Measuring is of course equivalent to its restriction to club-sequences $\vec{C}$ on ω_1, i.e., to sequences of the form $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$, where each C_δ is a club of δ and where $\mathrm{Lim}(\omega_1)$ denotes the set of countable limit ordinals.

In this paper we prove that measuring is consistent with $2^{\aleph_0}$ being arbitrarily large. This answers a question of Moore, who asked whether measuring is consistent with $2^{\aleph_0} > \aleph_2$ (see also [3]).

Recall that a partial order is $\aleph_2$-Knaster iff every collection of $\aleph_2$-many conditions contains a subcollection of cardinality $\aleph_2$ consisting of pairwise compatible conditions. Also, $\mathfrak{b}(\omega_1)$ denotes the minimal cardinality of an unbounded subset of ${}^{\omega_1}\omega_1$ mod. countable.

Our main theorem is the following.

Theorem 1.2. (CH) *Let κ be a cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$. There is then a partial order $\mathcal{P}$ with the following properties.*

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(1) $\mathcal{P}$ *is proper.*
(2) $\mathcal{P}$ *is* $\aleph_2$*-Knaster.*
(3) $\mathcal{P}$ *forces measuring.*
(4) $\mathcal{P}$ *forces* $2^{\aleph_0} = 2^{\aleph_1} = \kappa$.
(5) $\mathcal{P}$ *forces* $\mathfrak{b}(\omega_1) = \mathrm{cf}(\kappa)$.

Theorem 1.2 will be proved by means of what can be described as a finite support iteration incorporating systems of countable structures with symmetry requirements as side conditions. The actual construction is a variation of the constructions carried out in [2–5]. We refer the reader to [3] and [4] for the relevant background. The following two paragraphs are mostly aimed at readers familiar with some of the above references.

There are two main new ingredients in our present construction. Specifically, at any given stage $\beta < \kappa$ of the iteration,

(a) the set $\mathcal{N}^q_\beta$ of models N that are active at that stage, in the sense that $\beta \in N$ and that the marker associated to N at that stage is β, is actually a T–symmetric system (for a suitable predicate T), and
(b) if $\beta = \alpha + 1$, we use a separate symmetric system in the working part at α included in the above symmetric system corresponding to the previous stage, i.e., in $\mathcal{N}^q_\alpha$; these are the symmetric systems we denote $\mathcal{O}_{q,\alpha}$ in the definition of the forcing in Section 3.

This use of local symmetry is crucial in the verification that measuring holds in the final generic extension (see the proof of Lemma 3.9). Specifically, it is needed in the verification that the generic club C added at a stage α will be such that for every $\delta \in \mathrm{Lim}(\omega_1)$, a tail of $C \cap C_\delta$ be contained in C_δ in case we could not make the promise of avoiding C_δ (where C_δ is the δ-indexed member of the club-sequence picked at stage α). We should point out that adopting this added symmetry introduces some extra complications in the proof of properness (Lemma 3.5) with respect to the corresponding proofs in the above references; to be somewhat more precise, the main new complication comes up in the proof in the limit case α of the induction, where we need to assign markers to the 'copies' of the new models coming from inside the model N relative to which we are proving properness, in such a way that these (active) copies do not interfere with the information carried by the surrounding condition, and which N cannot see. Fortunately, this can be accomplished by picking the ordinal $\sigma < \alpha$, relative to which we will apply our induction hypothesis, sufficiently high within N, and by then carefully choosing these new markers.

In [1], Abraham and Shelah build, given any cardinal $\lambda \geq \aleph_2$, a forcing notion $\mathcal{P}$ which, if CH holds, preserves cardinals and is such that if G is $\mathcal{P}$-generic over $\mathbf{V}$, then in $\mathbf{V}[G]$ there is a family $\mathcal{C}$ of size λ consisting of clubs of ω_1 and with the property that, in any outer model M of $\mathbf{V}[G]$ in which ω_1 is not collapsed, the intersection of every uncountable subfamily of $\mathcal{C}$ is finite. In particular, if M is any outer model of $\mathbf{V}[G]$ with the same ω_1 and ω_2 as $\mathbf{V}[G]$, then there is no club C of ω_1 in M diagonalising $\mathcal{C}$ (where C diagonalising $\mathcal{C}$ means that $C \setminus D$ is bounded in ω_1 for each $D \in \mathcal{C}$). CH necessarily fails in the Abraham–Shelah model $\mathbf{V}[G]$ since, by a result of Galvin, CH implies that for every family $\mathcal{C}$ of size $\aleph_2$ consisting of clubs of ω_1 there is an uncountable $\mathcal{C}' \subseteq \mathcal{C}$ such that $\bigcap \mathcal{C}'$ is a club.

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It is not difficult to see that the generic club added at every stage α of our iteration diagonalises all clubs of ω_1 from $\mathbf{V}[G_\alpha]$ (where G_α is the generic filter at that stage). It follows, by the remarks in the previous paragraph, that it would be impossible to run anything like our iteration over the Abraham–Shelah model without collapsing ω_2. As a matter of fact, every known example of an ω_1-preserving forcing notion which adds a witness to measuring for a club-sequence which is not measured in the ground model also diagonalises all clubs of ω_1 from the ground model. Hence, every known instance of such a forcing, when performed over the Abraham–Shelah model, will necessarily collapse ω_2.

The above considerations make it implausible that a forcing construction preserving ω_1 and ω_2 succeeds in forcing measuring together with $2^{\aleph_0} > \aleph_2$ unless it starts from a ground model which is sufficiently different from the Abraham–Shelah model; for example, one in which CH holds (as is the case in our ground model).

It also follows from the above considerations that our generic extension is a model of $\mathfrak{b}(\omega_1) = \mathrm{cf}(\kappa)$, and in fact that it adds dominating functions $f : \omega_1 \longrightarrow \omega_1$ mod. countable over all intermediate models. In this context, the following question arises naturally.

QUESTION 1.3. *Is is consistent to have measuring together with* $\mathfrak{b}(\omega_1) = \aleph_2$ *and* $2^{\aleph_1} > \aleph_2$?

We should point out that we do not know if any of the constructions from [2, 3], [4] or [5] adds dominating functions $f : \omega_1 \longrightarrow \omega_1$ mod. countable at cofinally many stages; in fact, we do not know if any of those constructions produces models of $\mathfrak{b}(\omega_1) > \aleph_2$.

The following important open question in the context of measuring, also asked by Moore (s. [6]), remains open.

QUESTION 1.4. *Is measuring compatible with* CH?

This question poses an interesting challenge to the standard methods for iterating proper forcing without adding new reals. Indeed, the strongest failures of Club Guessing at ω_1 known to be within reach of the current forcing iteration methods for producing models of CH (s. [10]) seem to be only in the region of $\neg$ WCG, which is the statement that for every club–sequence $(C_\delta)_{\delta \in \mathrm{Lim}(\omega_1)}$ such that $\mathrm{ot}(C_\delta) = \omega$ for all δ there is a club $C \subseteq \omega_1$ which has finite intersection with each C_δ.

We will finish this introduction by mentioning that Moore has proved that our construction forces the set of ground model reals to be meagre (s. Fact 3.12). His argument actually shows that, letting $\mathcal{M}$ denotes the meagre ideal on the reals, $\mathbf{non}(\mathcal{M}) \geq \mathrm{cf}(\kappa)$ holds in our forcing extension.

QUESTION 1.5. *Is measuring consistent together with* $2^{\aleph_0} > \aleph_2$ *and* $\mathbf{non}(\mathcal{M}) = \aleph_1$?[1]

Much of the notation used in this paper follows the standards set forth in [8] and [9]. Other, less standard, pieces of notation will be introduced as needed.

[1]Measuring is consistent together with $2^{\aleph_0} = \aleph_2$ and $\mathbf{non}(\mathcal{M}) = \aleph_1$, as shown by the standard countable support iteration adding enough measuring sequences by initial segments (s. [6]). In fact, each of the forcings employed along the iteration fails to add new reals, and the countable support iteration of nonmeagre-preserving forcings is itself nonmeagre-preserving (s. [7]).

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Acknowledgment. We thank Justin Moore for his comments on this work. We especially thank him for proving Fact 3.12, and for calling our attention to the Abraham–Shelah model mentioned in the introduction and drawing the relevant connections with our construction. Our remarks in the introduction reflect his comments in this respect.

§2. Symmetric systems. Throughout the paper, if N is a set such that $N \cap \omega_1 \in \omega_1$, δ_N denotes $N \cap \omega_1$.

Let us fix an infinite cardinal θ for this section. Given $T \subseteq H(\theta)$ and $N \in [H(\theta)]^{\aleph_0}$, we will tend to write (N, T) instead of $(N, T \cap N)$. We will need the following notion of symmetric system from [3].

DEFINITION 2.1. Let $T \subseteq H(\theta)$ and let $\mathcal{N}$ be a finite set of countable subsets of $H(\theta)$. We will say that $\mathcal{N}$ *is a* T*-symmetric system* if and only if the following holds.

(*A*) For every $N \in \mathcal{N}$, $(N, \in, T)$ is an elementary substructure of $(H(\theta), \in, T)$.

(*B*) Given distinct N, N' in $\mathcal{N}$, if $\delta_N = \delta_{N'}$, then there is a unique isomorphism

$$\Psi_{N,N'} : (N, \in, T) \longrightarrow (N', \in, T)$$

Furthermore, $\Psi_{N,N'}$ is the identity on $N \cap N'$.

(*C*) For all N, N', M in $\mathcal{N}$, if $M \in N$ and $\delta_N = \delta_{N'}$, then $\Psi_{N,N'}(M) \in \mathcal{N}$.

(*D*) For all N, M in $\mathcal{N}$, if $\delta_M < \delta_N$, then there is some $N' \in \mathcal{N}$ such that $\delta_{N'} = \delta_N$ and $M \in N'$.

Strictly speaking, the phrase 'T-symmetric system' is ambiguous in general since $H(\theta)$ may not be determined by T. However, in all practical cases $\bigcup T = H(\theta)$, so T does determine $H(\theta)$ in these cases.

The following lemmas are proved in [3].

LEMMA 2.2. *Let $T \subseteq H(\theta)$ and let N and N' be countable elementary substructures of $(H(\theta), \in, T)$. Suppose $\mathcal{N} \in N$ is a T-symmetric system and $\Psi : (N, \in, T) \longrightarrow (N', \in, T)$ is an isomorphism. Then $\Psi(\mathcal{N}) = \Psi``\mathcal{N}$ is also a T-symmetric system.*

LEMMA 2.3. *Let $T \subseteq H(\theta)$, let $\mathcal{N}$ be a T-symmetric system and let $N \in \mathcal{N}$. Then the following holds.*

(1) *$\mathcal{N} \cap N$ is a T-symmetric system.*

(2) *Suppose $\mathcal{N}^* \in N$ is a T-symmetric system such that $\mathcal{N} \cap N \subseteq \mathcal{N}^*$. Let*

$$\mathcal{M} = \mathcal{N} \cup \bigcup \{\Psi_{N,N'}``\mathcal{N}^* : N' \in \mathcal{N}, \delta_{N'} = \delta_N\}$$

Then $\mathcal{M}$ is the $\subseteq$-minimal T-symmetric system $\mathcal{W}$ such that $\mathcal{N} \cup \mathcal{N}^* \subseteq \mathcal{W}$.

Given $T \subseteq H(\theta)$ and T-symmetric systems $\mathcal{N}_0, \mathcal{N}_1$, let us write $\mathcal{N}_0 \cong \mathcal{N}_1$ iff

- $|\mathcal{N}_0| = |\mathcal{N}_1| = m$, and
- there are enumerations $(N_i^0)_{i<m}$ and $(N_i^1)_{i<m}$ of $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively, together with an isomorphism between

$$\left\langle \bigcup \mathcal{N}_0, \in, T, N_i^0 \right\rangle_{i<m}$$

and

$$\left\langle \bigcup \mathcal{N}_1, \in, T, N_i^1 \right\rangle_{i<m}$$

which is the identity on $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1)$.

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LEMMA 2.4. *Let $T \subseteq H(\theta)$ and let $\mathcal{N}_0$ and $\mathcal{N}_1$ be T-symmetric systems. Suppose $\mathcal{N}_0 \cong \mathcal{N}_1$. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a T-symmetric system.*

§3. The construction. In this section we prove Theorem 1.2. It will be useful to consider the following notion of rank: Given sets N, $\mathcal{X}$ and an ordinal η, we define $\mathrm{rank}(\mathcal{X}, N) \geq \eta$ recursively by:

- $\mathrm{rank}(\mathcal{X}, N) \geq 1$ if and only if for every $a \in N$ there is some $M \in \mathcal{X} \cap N$ such that $a \in M$.
- If $\eta > 1$, then $\mathrm{rank}(\mathcal{X}, N) \geq \eta$ if and only if for every $\eta' < \eta$ and every $a \in N$ there is some $M \in \mathcal{X} \cap N$ such that $a \in M$ and $\mathrm{rank}(\mathcal{X}, M) \geq \eta'$.

Now let $\Phi : \kappa \longrightarrow H(\kappa)$ be such that $\Phi^{-1}(x)$ is unbounded in κ for all $x \in H(\kappa)$. Notice that Φ exists by $2^{<\kappa} = \kappa$. Let also $\lhd$ be a well–order of $H((2^\kappa)^+)$.

Let $(\theta_\alpha)_{\alpha<\kappa}$ be the sequence of cardinals defined by $\theta_0 = |H((2^\kappa)^+)|^+$ and $\theta_\alpha = (2^{<\sup_{\beta<\alpha}\theta_\beta})^+$ if $\alpha > 0$. For each $\alpha < \kappa$ let $\mathcal{M}^*_\alpha$ be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing Φ, $\lhd$ and $(\theta_\beta)_{\beta<\alpha}$, and let $\mathcal{M}_\alpha = \{N^* \cap H(\kappa) : N^* \in \mathcal{M}^*_\alpha\}$. Let T^α be the $\lhd$–first $T \subseteq H(\kappa)$ such that for every $N \in [H(\kappa)]^{\aleph_0}$, if $(N, \in, T \cap N) \preccurlyeq (H(\kappa), \in, T)$, then $N \in \mathcal{M}_\alpha$. Let also

$$\mathcal{T}^\alpha = \{N \in [H(\kappa)]^{\aleph_0} : (N, \in, T^\alpha \cap N) \preccurlyeq (H(\kappa), \in, T^\alpha)\}.$$

The following fact is immediate.

FACT 3.1. *Let $\alpha < \beta \leq \kappa$.*

(1) *If $N^* \in \mathcal{M}^*_\beta$ and $\alpha \in N^*$, then $\mathcal{M}^*_\alpha \in N^*$ and $N^* \cap H(\kappa) \in \mathcal{T}^\alpha$.*
(2) *If $N, N' \in \mathcal{T}^\beta$, $\Psi : (N, \in, T^\beta \cap N) \longrightarrow (N', \in, T^\beta \cap N')$ is an isomorphism, and $M \in N \cap \mathcal{T}^\beta$, then $\Psi(M) \in \mathcal{T}^\beta$.*

Our forcing $\mathcal{P}$ will be $\mathcal{P}_\kappa$, where $(\mathcal{P}_\beta : \beta \leq \kappa)$ is the sequence of posets to be defined next. In the following definition, and throughout the paper, if q is an ordered pair (F, Δ), we will denote F and Δ by F_q and Δ_q, respectively.

Let $\beta \leq \kappa$ and suppose $\mathcal{P}_\alpha$ has been defined for all $\alpha < \beta$. Conditions in $\mathcal{P}_\beta$ are ordered pairs $q = (F, \Delta)$ with the following properties.

(1) F is a finite function with $\mathrm{dom}(F) \subseteq \beta$.
(2) Δ is a finite set of pairs (N, γ) such that $N \in [H(\kappa)]^{\aleph_0}$ and γ is an ordinal such that $\gamma \leq \beta$ and $\gamma \leq \sup(N \cap \kappa)$.
(3) $\mathcal{N}^q_\beta := \{N : (N, \beta) \in \Delta,\ \beta \in N\}$ is a T^β-symmetric system.
(4) For every $\alpha < \beta$, the restriction of q to α,
$$q|_\alpha := (F \restriction \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\}),$$
is a condition in $\mathcal{P}_\alpha$.
(5) Suppose $\beta = \alpha + 1$. Let $\mathcal{N}^{\dot{G}_\alpha}$ be a $\mathcal{P}_\alpha$-name for
$$\bigcup\{\mathcal{N}^r_\alpha : r \in \dot{G}_\alpha\}$$
(where $\dot{G}_\alpha$ is the canonical $\mathcal{P}_\alpha$-name for the generic object). Let $\dot{C}^\alpha$ be a $\mathcal{P}_\alpha$-name for a club-sequence on ω_1 such that $\mathcal{P}_\alpha$ forces that
- $\dot{C}^\alpha = \Phi(\alpha)$ in case $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$-name for a club-sequence on ω_1, and that
- $\dot{C}^\alpha$ is some fixed club-sequence on ω_1 in the other case.

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If $\alpha \in \mathrm{dom}(F)$, then $F(\alpha) = (f, b, \mathcal{O})$ has the following properties.

(a) $f \subseteq \omega_1 \times \omega_1$ is a finite strictly increasing function.

(b) $\mathcal{O} \subseteq \mathcal{N}_\alpha^{q|\alpha}$ is a T^β-symmetric system.

(c) $\mathrm{range}(f) \subseteq \{\delta_N : N \in \mathcal{O}\}$.

(d) For every $\delta \in \mathrm{dom}(f)$, if $N \in \mathcal{O}$ is such that $p(\delta) = \delta_N$, then

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^\beta, N) \geq \delta.$$

(e) $\mathrm{dom}(b) \subseteq \mathrm{dom}(f)$ and $b(\delta) < f(\delta)$ for every $\delta \in \mathrm{dom}(b)$.

(f) For every $\delta \in \mathrm{dom}(b)$,

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{range}(f \restriction \delta) \cap \dot{C}^\alpha(f(\delta)) \subseteq b(\delta).$$

(g) For every $\delta \in \mathrm{dom}(b)$, if $N \in \mathcal{O}$ is such that $f(\delta) = \delta_N$, then

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{rank}(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^\beta : \delta_M \notin \dot{C}^\alpha(f(\delta))\}, N) \geq \delta.$$

(h) If $N \in \mathcal{N}_\beta^q$, then $N \in \mathcal{O}$, $\delta_N \in \mathrm{dom}(f)$ and $f(\delta_N) = \delta_N$.

Given $\mathcal{P}_\beta$-conditions $q_i = (F_i, \Delta_i)$, for $i = 0, 1$, q_1 extends q_0 if and only if

- $\mathrm{dom}(F_0) \subseteq \mathrm{dom}(F_1)$ and for all $\alpha \in \mathrm{dom}(F_0)$, if $F_0(\alpha) = (f, b, \mathcal{O})$ and $F_1(\alpha) = (f', b', \mathcal{O}')$, then $f \subseteq f'$, $b \subseteq b'$ and $\mathcal{O} \subseteq \mathcal{O}'$, and
- $\Delta_0 \subseteq \Delta_1$.

If q is a condition in $\mathcal{P}_\beta$ for some $\beta \leq \kappa$, we will use $\mathrm{supp}(q)$ to denote the domain of F_q ($\mathrm{supp}(q)$ stands for the support of q). Also, if $\alpha \in \mathrm{supp}(q)$ and $F_q(\alpha) = (f, b, \mathcal{O})$, then $f_{q,\alpha}$, $b_{q,\alpha}$ and $\mathcal{O}_{q,\alpha}$ denote f, b and $\mathcal{O}$, respectively.

3.1. The relevant facts. In this subsection we prove the relevant facts about $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ that, together, will yield a proof of Theorem 1.2.

Note that if $\alpha < \beta \leq \kappa$, then $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ and every $\mathcal{P}_\beta$-condition $q = (F, \{(N_j, \gamma_j) : j < m\})$ such that $\mathrm{supp}(q) \subseteq \alpha$ and $\gamma_j \leq \alpha$ for all j is also a $\mathcal{P}_\alpha$-condition and is in fact equal to its restriction to α. Also note that if β is a nonzero limit ordinal, then a pair $q = (F_q, \Delta_q)$ is a $\mathcal{P}_\beta$-condition if and only if it satisfies (1)–(4).

LEMMA 3.2. *Let $\alpha \leq \beta \leq \kappa$. If $q = (F_q, \Delta_q) \in \mathcal{P}_\alpha$, $r = (F_r, \Delta_r) \in \mathcal{P}_\beta$, and $q \leq_\alpha r|_\alpha$, then*

$$r \wedge_\alpha q := (F_q \cup (F_r \restriction [\alpha, \beta)), \Delta_q \cup \Delta_r)$$

is a condition in $\mathcal{P}_\beta$ extending r. Hence, $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$.

PROOF. The proof is essentially identical to the proof of the corresponding lemmas in [3] and [4], so we will not give many details here. Let us just point out that the proof depends on the use of the markers in the definition of the forcing. The fact that a marker γ is associated to a submodel N in a condition (F, Δ) (i.e., the fact that $(N, \gamma) \in \Delta$) tells us that N is 'active', for that condition, up to and including stage γ in the iteration. New side conditions (N, γ) appearing in Δ_q may well be such that $N \cap [\alpha, \beta)$ is nonempty, but they will not impose any problematic promises—corresponding to clauses (3) and (5) in the definition—on ordinals occurring in the interval $[\alpha, \beta)$ simply because $\gamma \leq \alpha$. ⊣

LEMMA 3.3. *For every ordinal $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ is $\aleph_2$-Knaster.*

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PROOF. Let us consider first the case $\alpha = 0$. Suppose $m < \omega$ and $q_\xi = \{N_i^\xi : i < m\}$ is a $\mathcal{P}_0$–condition for each $\xi < \omega_2$. For notational convenience we are identifying a $\mathcal{P}_0$–condition q with $\mathrm{dom}(\Delta_q)$, which is fine for the proof in this case. By CH we may assume that

$$\{\bigcup_{i<m} N_i^\xi : \xi < \omega_2\}$$

forms a Δ-system with root X. Furthermore, again by CH, we may assume that, for all $\xi, \xi' < \omega_2$, the structures

$$\langle \bigcup_{i<m} N_i^\xi, \in, X, T^0, N_i^\xi \rangle_{i<m}$$

and

$$\langle \bigcup_{i<m} N_i^{\xi'}, \in, X, T^0, N_i^{\xi'} \rangle_{i<m}$$

are isomorphic and that the corresponding isomorphism fixes X. This is true since there are only $\aleph_1$-many isomorphism types for such structures and since the only isomorphism between X and itself is the identity. Hence, by Lemma 2.4 we have, for all $\xi, \xi' < \omega_2$, that $q_\xi \cup q_{\xi'}$ extends both q_ξ and $q_{\xi'}$.

For general α, suppose that q_ξ is a $\mathcal{P}_\alpha$-condition for each $\xi < \omega_2$; we will show that there is an unbounded subset $I \subseteq \omega_2$ such that q_ξ and $q_{\xi'}$ are compatible for all ξ and ξ' in I. We may assume that there is some $m < \omega$ such that we may write

$$\mathrm{dom}(\Delta_{q_\xi}) = \{N_i^\xi : i < m\}$$

for each ξ. Let

$$\vec{T} = \{(a, \gamma) : \gamma \leq \alpha,\ a \in T^\gamma\}$$

By an argument as in the case $\alpha = 0$, we are allowed to adopt the point of view that $\{\bigcup_{i<m} N_i^\xi : \xi < \omega_2\}$, for $\xi < \omega_2$, forms a Δ–system with root X and that for all ξ, $\xi' < \omega_2$, the structures

$$\langle \bigcup_{i<m} N_i^\xi, \in, X, \vec{T}, N_i^\xi \rangle_{i<m}$$

and

$$\langle \bigcup_{i<m} N_i^{\xi'}, \in, X, \vec{T}, N_i^{\xi'} \rangle_{i<m}$$

are isomorphic. We may also assume that there is a finite set $x \subseteq X$ such that

- $\{\mathrm{supp}(q_\xi) : \xi \in \omega_2\}$ forms a Δ–system with root x, and
- for all $\xi, \xi' \in \omega_2$ and for all $\alpha \in x$, $(f_{q_\xi,\alpha}, b_{q_\xi,\alpha}) = (f_{q_{\xi'},\alpha}, b_{q_{\xi'},\alpha})$.

Finally, again by the same argument as above, we may assume that for all $\xi, \xi' \in \omega_2$ and all $\gamma \in x$, $\mathcal{O}_{\xi,\gamma} \cup \mathcal{O}_{\xi',\gamma}$ is a T^γ–symmetric system. It is now straightforward to check, using Lemma 2.4, that for all ξ, ξ', $(F_{q_\xi} \cup F_{q_{\xi'}}, \Delta_{q_\xi} \cup \Delta_{q_{\xi'}})$ is a condition in $\mathcal{P}_\alpha$ witnessing the compatibility of q_ξ and $q_{\xi'}$. ⊣

Given $\alpha < \kappa$, a condition $q \in \mathcal{P}_\alpha$, and a countable elementary substructure N of $H(\kappa)$, we will say that q *is* $(N, \mathcal{P}_\alpha)$*-pre-generic* in case $(N, \alpha) \in \Delta_q$. Also, given a countable elementary substructure N of $H(\kappa)$ and a $\mathcal{P}_\alpha$-condition q, we will say that q *is* $(N, \mathcal{P}_\alpha)$*-generic* iff q forces $\dot{G}_\alpha \cap A \cap N \neq \emptyset$ for every maximal antichain A of $\mathcal{P}_\alpha$ such that $A \in N$. Note that this is more general than the standard notion

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of $(N, \mathbb{P})$-genericity, for a forcing notion $\mathbb{P}$, which applies only if $\mathbb{P} \in N$. Indeed, in our situation $\mathcal{P}_\alpha$ is of course never a member of N if $N \subseteq H(\kappa)$.

The following technical lemma will be needed in the proof of Lemma 3.5.

LEMMA 3.4. *Let $\beta < \kappa$. Suppose q is $(M, \mathcal{P}_\beta)$-generic whenever q is $(M, \mathcal{P}_\beta)$-pre-generic and $M \in \mathcal{T}^{\beta+1}$.*[2] *Then for every $R \subseteq H(\kappa)$, if M is such that $\langle M, T^{\beta+1}, R\rangle \preccurlyeq \langle H(\kappa), T^{\beta+1}, R\rangle$, then $\mathcal{P}_\beta$ forces that if $M \in \mathcal{N}^{\dot{G}_\beta}$, then $\langle M[\dot{G}_\beta], \dot{G}_\beta, R\rangle \preccurlyeq \langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, R\rangle$.*

PROOF. Let us work in $V^{\mathcal{P}_\beta}$. It suffices to show that if $\varphi(x, y)$ is a formula in the language for the structure $\langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, R\rangle$, $a \in M[\dot{G}_\beta]$, and there is some $b \in H(\kappa)^{V[\dot{G}_\beta]}$ such that $\langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, R\rangle \models \varphi(b, a)$, then there is some $b \in M[\dot{G}_\beta]$ such that $\langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, R\rangle \models \varphi(b, a)$. For this, let $\dot{a} \in M$ be a $\mathcal{P}_\beta$-name for a and note that there is a maximal antichain A of $\mathcal{P}_\beta$ and a function g with domain A such that for every $r \in A$, $g(r)$ is a $\mathcal{P}_\beta$-name such that r forces in $\mathcal{P}_\beta$ that if

$$\langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, R\rangle \models \exists x \varphi(x, \dot{a}),$$

then

$$\langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, R\rangle \models \varphi(g(r), \dot{a}).$$

By the $\aleph_2$-chain condition of $\mathcal{P}_\beta$ together with $\kappa^{\aleph_1} = \kappa$, both A and g are in $H(\kappa)$, and by picking A and g to be the least such objects relative to some well-order of $H(\kappa)$ definable from Φ we may assume that A and g are in M and in fact that they are definable in $\langle M, T^{\beta+1}, R\rangle$ from $\dot{a}$ as parameter (this uses the fact that $\mathcal{P}_\beta$ is definable in $\langle H(\kappa), T^{\beta+1}\rangle$). But now, by our hypothesis, $\mathcal{P}_\beta$ forces that if M is as in the statement of the lemma, then $\dot{G}_\beta$ meets A in M and hence there is some $r \in A \cap M$ such that $\langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, R\rangle \models \varphi(g(r), \dot{a})$, which implies what we wanted to see since then $g(r) \in M$. ⊣

The properness of all $\mathcal{P}_\alpha$ is an immediate consequence of the following lemma, together with the $\aleph_2$-c.c. of $\mathcal{P}_\alpha$, for all α, and with $\kappa^{\aleph_1} = \kappa$.

LEMMA 3.5. *Suppose $\alpha < \kappa$ and $N \in \mathcal{T}^{\alpha+1}$. Then the following holds.*

$(1)_\alpha$ *For every $q \in N$ there is $q' \leq_\alpha q$ such that q' is $(N, \mathcal{P}_\alpha)$-pre-generic.*
$(2)_\alpha$ *If $q \in \mathcal{P}_\alpha$ is $(N, \mathcal{P}_\alpha)$-pre-generic, then q is $(N, \mathcal{P}_\alpha)$-generic.*

PROOF. The proof will be by induction on α. We start with the case $\alpha = 0$. For simplicity we are identifying a $\mathcal{P}_0$-condition $q = (\emptyset, \Delta_q)$ with $\mathrm{dom}(\Delta_q)$. The proof of $(1)_0$ is trivial: It suffices to set $q' = q \cup \{N\}$.

The proof of $(2)_0$ is also easy: Let A be a maximal antichain of $\mathcal{P}_0$ in N. It suffices to show that there is some condition in $A \cap N$ compatible with q. By extending q if necessary we may assume that it extends some condition in A. Notice that $q \cap N \in \mathcal{P}_0$ by Lemma 2.3(1). Hence, we may find a condition $q^\circ \in N$ extending $q \cap N$ and extending some condition in A. Now, by Lemma 2.3(2), there is a T^0-symmetric system $\mathcal{M}$ extending $q \cup q^\circ$. Then $\mathcal{M}$ is a condition in $\mathcal{P}_0$ extending q and q°.

Let us proceed to the case $\alpha = \sigma + 1$. We start by proving $(1)_\alpha$. By $(1)_\sigma$ we may assume, by extending $q|_\sigma$, that $q|_\sigma$ is $(N, \mathcal{P}_\sigma)$-pre-generic. So, if $\sigma \notin \mathrm{supp}(q)$, then

[2] We will see in Lemma 3.5 that this hypothesis is true.

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$q' = (F_q, \Delta_q \cup \{(N, \alpha)\}$ clearly witnesses $(1)_\alpha$. Assume now that $\sigma \in \mathrm{supp}(q)$. Since $q|_\sigma$ is $(N, \mathcal{P}_\sigma)$-pre-generic, $q|_\sigma$ forces in $\mathcal{P}_\sigma$ that $N \in \mathcal{N}^{\dot{G}_\sigma}$. In particular, $q|_\sigma$ forces that for every $x \in N$ there is $M \in \mathcal{N}^{\dot{G}^\sigma} \cap \mathcal{T}^{\sigma+1}$ such that $x \in M$ (as witnessed by N itself).

Let us work in $V^{\mathcal{P}_\sigma \restriction q|_\sigma}$. Since

$$\langle N[\dot{G}_\sigma], \dot{G}_\sigma, T^{\sigma+2}, H(\kappa)^V \rangle \preccurlyeq \langle H(\kappa)[\dot{G}_\sigma], \dot{G}_\sigma, T^{\sigma+2}, H(\kappa)^V \rangle$$

by Lemma 3.4, there exists an M as above in $N[\dot{G}_\sigma] \cap V$ (where V denotes the ground model). We may also assume that $M \in N$, since $N[\dot{G}_\sigma] \cap V = N$ (which follows from $(2)_\sigma$ applied to N and $q|_\sigma$). This shows that $q|_\sigma$ forces $\mathrm{rank}(\mathcal{N}^{\dot{G}_\sigma} \cap \mathcal{T}^{\sigma+1}, N) \geq 1$. In fact, by iterating this argument we can show, by induction on μ, that $q|_\sigma$ forces $\mathrm{rank}(\mathcal{N}^{\dot{G}_\sigma} \cap \mathcal{T}^{\sigma+1}, N) \geq \mu$ for every $\mu < \delta_N$. In view of these considerations, it suffices to define q' as the condition $(F', \Delta_q \cup \{(N, \alpha)\})$, where F' extends F_q and

$$F'(\sigma) = (f_{q,\sigma} \cup \{\langle \delta_N, \delta_N \rangle\}, b_{q,\sigma}, \mathcal{O}_{q,\sigma} \cup \{N\}).$$

For the proof of $(2)_\alpha$ when $\alpha = \sigma + 1$, suppose $A \in N$ is a maximal antichain of $\mathcal{P}_\alpha$ and suppose, without loss of generality, that q extends a condition $r^* \in A$. We may further assume that $\sigma \in \mathrm{supp}(q)$. We want to show that $r^* \in N$, and for this it will suffice to show that there is a condition in $N \cap A$ compatible with q. We will start by slightly altering A into a more informative object.

Let $\leq$ be some well-order of $H(\kappa)$ definable from Φ. For each $r \in A$ and each $\eta < \omega_1$ let $\psi(r, \eta)$ be the $\leq$-least $\mathcal{P}_\sigma$-name for a condition $\dot{r}$ in $\mathcal{P}_\alpha$ such that $r|_\sigma$ forces the following.

(1) $\dot{r}|_\sigma \in \dot{G}_\sigma$.
(2) $\dot{r}$ extends $\check{r}$.
(3) Suppose there is some condition $s \in \mathcal{P}_\alpha$ such that $s|_\sigma \in \dot{G}_\sigma$, s extends r, $\mathcal{N}^q_\alpha \cap N \subseteq \mathcal{N}^s_\alpha$, $\sigma \in \mathrm{supp}(s)$, and such that $F_s(\sigma)$ satisfies the following properties.
   (a) $f_{s,\sigma}$ is a proper end-extension of $f_{q,\sigma} \restriction \delta_N$.
   (b) $\eta < \min(\mathrm{dom}(f_{s,\sigma} \setminus f_{q,\sigma}))$.
   (c) $b_{s,\sigma}$ end-extends $b_{q,\sigma} \restriction \delta_N$.
   (d) $\mathcal{O}_{q,\sigma} \cap N \subseteq \mathcal{O}_{s,\sigma}$.
   Then $\dot{r}$ is such a condition s.

Since A is in N and by correctness of N with respect to the predicate $T^{\alpha+1}$ and the parameters $\mathcal{N}^q_\alpha \cap N$, $f_{q,\sigma} \restriction \delta_N$, $b_{q,\sigma} \restriction \delta_N$, $\mathcal{O}_{q,\sigma} \cap N \in N$, it is clear that $\bar{A} := \{\psi(r, \eta) : r \in A, \eta < \omega_1\}$ is in N (since $\bar{A}$ is definable over $(H(\kappa), \in, T^{\alpha+1})$ from the above parameters and $\bar{A} \in H(\kappa)$ as $|\bar{A}| \leq |A| \cdot \aleph_1$). Let G_σ be a $\mathcal{P}_\sigma$-generic filter over V with $q|_\sigma \in G_\sigma$ (by $(2)_\sigma$ we have that G_σ is also generic over N) and let $\mathcal{N}^{G_\sigma}$ and $(C(\delta))_{\delta \in \mathrm{Lim}(\omega_1)}$ be the interpretations, via G_σ, of $\mathcal{N}^{\dot{G}_\sigma}$ and $\Phi(\sigma)$ respectively. In the following we will assume that $\delta_N \in \mathrm{dom}(b_{q,\sigma})$ (the proof in the other case is considerably easier).

By condition (5) (h) in the definition of our forcing, we have that $f_{q,\sigma}(\delta_N) = \delta_N$. Note that by condition (5) (f) we can find $\mu < \delta_N$ such that if $\delta \in \mathrm{dom}(b_{q,\sigma}) \setminus (\delta_N + 1)$ and $b_{q,\sigma}(\delta) < \delta_N$, then $C(f_{q,\sigma}(\delta)) \cap [\mu, \delta_N) = \emptyset$. This μ exists because, for such a δ, δ_N does not belong to the closed set $C(f_{q,\sigma}(\delta))$.

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Let $\nu = \max(\mathrm{range}(f_{q,\sigma} \restriction \delta_N)) + 1$ (which we may assume exists), and let $M \in N \cap \mathcal{N}^{G_\sigma} \cap \mathcal{T}^{\sigma+1}$ be such that all the relevant objects—namely A, $\bar{A}$, μ, ν, $\mathrm{dom}(\Delta_q) \cap N$ and $\mathcal{O}_{q,\sigma} \cap N$—are in M, and such that $\delta_M \notin C(\delta_N)$. The existence of such an M is ensured by (5) (g) in the definition of the iteration. Using the openness of $\delta_N \backslash C(\delta_N)$ we can also find $\eta < \delta_M$, $\eta > \mu, \nu$, such that $[\eta, \delta_M] \cap C(\delta_N) = \emptyset$. But now, by correctness of $M[\dot{G}_\sigma]$ within $H(\kappa)^{V[G_\sigma]}$ and since $M[\dot{G}_\sigma]$ contains all relevant objects, there is some $r \in A \cap M[G_\sigma]$ such that $r|_\sigma$ is in G_σ and forces that $f_{\psi(r,\eta),\sigma}$ is a proper end-extension of $f_{q,\sigma} \restriction \delta_N$, $\eta < \min(\mathrm{dom}(\psi(r,\eta) \backslash f_{q,\sigma}))$, $b_{\psi(r,\eta),\sigma}$ end-extends $b_{q,\sigma} \restriction \delta_N$, and $\mathcal{O}_{q,\sigma} \cap N \subseteq \mathcal{O}_{\psi(r,\eta),\sigma}$.

By, if necessary, extending $q|_\sigma$, we may assume that $q|_\sigma$ decides all relevant facts above and that it extends $t|_\sigma$, where t is the condition in $\mathcal{P}_\alpha$ that it forces $\psi(r,\eta)$ to be. We may also assume that $M \in \mathcal{N}_\sigma^{q|_\sigma}$. In particular, we may assume that $\mathcal{N}_\alpha^t \cup \mathcal{O}_{t,\sigma} \subseteq \mathcal{N}_\sigma^{q|_\sigma}$.

By $(2)_\sigma$ applied to $q|_\sigma$ and M, we then have that t is in M, and hence also in N. The proof of $(2)_\alpha$ in this case will therefore be finished once we prove that the natural amalgamation q^* of t and q is a condition in $\mathcal{P}_\alpha$. The construction of q^* is as follows. By Lemma 2.3(2), there are $\subseteq$-minimal T^α-symmetric systems $\mathcal{N}^*$ and $\mathcal{O}^*$ extending $\mathcal{N}_\alpha^t \cup \mathcal{N}_\alpha^q$ and $\mathcal{O}_{t,\sigma} \cup \mathcal{O}_{q,\sigma}$ respectively (as given in the statement of Lemma 2.3(2)). Let $f^* = f_{t,\sigma} \cup f_{q,\sigma}$ and $b^* = b_{t,\sigma} \cup b_{q,\sigma}$, and let $q^* = (F^*, \Delta^*)$, where F^* is the function with $\mathrm{dom}(F^*) \subseteq \alpha$ such that $F^* \restriction \sigma = F_q \restriction \sigma$ and $F^*(\sigma) = (f^*, b^*, \mathcal{O}^*)$, and where $\Delta^* = \Delta_q \cup \{(Q, \alpha) : Q \in \mathcal{N}^*\}$. Next we will verify that q^* is a condition extending both q and t, which finishes the proof in this case.

Note that the assumption $\mathcal{N}_\alpha^t \cup \mathcal{O}_{t,\sigma} \subseteq \mathcal{N}_\sigma^{q|_\sigma}$, the minimality of $\mathcal{N}^*$ and $\mathcal{O}^*$ and the fact that $\mathcal{N}_\sigma^{q|_\sigma}$ is a T^σ-symmetric system together imply that $\mathcal{N}^* \cup \mathcal{O}^* \subseteq \mathcal{N}_\sigma^{q|_\sigma}$. In particular, $q|_\sigma = q^*|_\sigma \leq_\sigma t|_\sigma$ and q^* clearly satisfies all conditions in the definition of our forcing except for, possibly, some of (5) (d), (f), (g) or (h). By invoking the inclusion $\mathrm{range}(f^*) \backslash \mathrm{range}(f_{q,\sigma}) \subseteq (\eta, \delta_M)$, one can check that q^* satisfies (5) (f) since this holds for both q and t. Similarly, (5) (h) for q^* follows from $\{\delta_W : W \in \mathcal{N}^*\} = \{\delta_W : W \in \mathcal{N}_\alpha^t \cup \mathcal{N}_\alpha^q\}$. Finally, note that the two remaining verifications (i.e., (5) (d) and (5) (g)) are easy consequences of Fact 3.1(2), using an inductive argument.

Suppose now that α is a nonzero limit ordinal. The proof of $(1)_\alpha$ is easy: Let $\sigma \in N \cap \alpha$ be such that $\mathrm{supp}(q) \subseteq \sigma$, let r be an $(N, \mathcal{P}_\sigma)$-pre-generic extension of $q|_\sigma$ (which exists by $(1)_\sigma$), and let

$$q' = (F_r, \Delta_q \cup \Delta_r \cup \{(N, \alpha)\}).$$

For $(2)_\alpha$, let $A \in N$ be a maximal antichain of $\mathcal{P}_\alpha$, and suppose that q extends a condition in A. We will show that q is compatible with a condition in $N \cap A$. We may fix $\sigma \in N \cap \alpha$ such that for every $\xi \in \mathrm{supp}(q)$ the following holds:

($\star$) If there is some $\gamma > \xi$ such that $\gamma \in N \cap \alpha$, then $\sigma > \gamma$ for some such γ.

Claim 3.6. *We can find a condition $r \in N$ such that*

(i) *r extends a condition in A,*
(ii) $(\mathrm{supp}(r) \backslash \sigma) \cap \bigcup(\mathrm{dom}(\Delta_q) \cap N) = \emptyset$,
(iii) $\Delta_q \cap N \subseteq \Delta_r$, *and such that*
(iv) *there is a common extension $t \in \mathcal{P}_\sigma$ of $r|_\sigma$ and $q|_\sigma$.*

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PROOF. Fix a $\mathcal{P}_\sigma$-generic G_σ over $\mathbf{V}$ with $q|_\sigma \in G$ and let us work for a moment in $\mathbf{V}[G_\sigma]$. Note that, by correctness of $N[G_\sigma]$ and since $N[G_\sigma] \cap \mathbf{V} = N$ by $(2)_\sigma$, we may find $\dot{r} \in N$ satisfying (i) and (iii) and such that $r|_\sigma \in G_\sigma$. The fact that $\dot{r}$ can be found so that (ii) holds too is clear if $\mathrm{cf}(\alpha) = \omega$ since then σ is in fact above ξ for every $\xi \in \mathrm{supp}(q)$, and in the case $\mathrm{cf}(\alpha) \geq \omega_1$ it is true again by the choice of σ and since $\bigcup(\mathrm{dom}(\Delta_q) \cap N) \in N$ is countable. Finally we can extend $q|_\sigma$ to a condition deciding $\dot{r}$ to be some $r \in N$ and extending $r|_\sigma$. This r is then as required. ⊣

We want to amalgamate t, r, and q into a condition q^* in $\mathcal{P}_\alpha$. In particular we need to guarantee that clause (3) in the definition of $\mathcal{P}_\alpha$-condition holds for q^*. For this, given $(M, \gamma) \in \Delta_r$ such that $\gamma > \sigma$ and N' in

$$\mathcal{N}(M,\gamma) := \{N' \in \mathcal{N}_\beta^{q|_\beta} \,:\, \sigma < \beta \leq \gamma,\ \beta \in M, \delta_{N'} = \delta_N\},$$

we need to decide on a marker $\tilde{\gamma}$ to assign to $\Psi_{N,N'}(M)$ in such a way that the symmetry expressed in clause (3) holds ultimately for $\mathcal{N}_\beta^{q^*|_\beta}$ for all $\beta \leq \alpha$. For this, for all (M, γ) and N' as above we let $\rho_{N'}^\gamma$ be the least ordinal ρ such that $\beta \leq \rho$ whenever

(1) $\beta \in M \cap (\gamma + 1)$, and
(2) $N' \in \mathcal{N}_\beta^{q|_\beta}$.

Let now

$$\tilde{\Delta}_r = \{(\Psi_{N,N'}(M), \rho_{N'}^\gamma) \,:\, (M,\gamma) \in \Delta_r,\ N' \in \mathcal{N}(M,\gamma)\}$$

and

$$q^* = (F^*, \Delta_q \cup \Delta_r \cup \tilde{\Delta}_r\},$$

where F^* is a function with domain $\mathrm{dom}(F_t) \cup \mathrm{dom}(F_r \backslash \sigma)$ such that for every $\xi \in \mathrm{dom}(F_r \backslash \sigma)$,

$$F^*(\xi) = (f_{r,\xi} \cup \{(\delta_Q, \delta_Q) \,:\, Q \in \mathcal{N}_\xi^{q_\xi},\ \delta_Q \geq \delta_N\}, b_{r,\xi}, \mathcal{O}_\xi^*),$$

where $\mathcal{O}_\xi^*$ is the $\subseteq$-minimal $T^{\xi+1}$-symmetric system $\mathcal{O}$ such that $\mathcal{O}_{r,\xi} \cup \{Q \in \mathcal{N}_{\xi+1}^{q|_{\xi+1}} \,:\, \delta_Q \geq \delta_N\} \subseteq \mathcal{O}$.

Note that by Lemma 2.3(2), $\mathcal{O}_\xi^*$ exists for each $\xi \in \mathrm{dom}(F_r \backslash \sigma)$, and is precisely

$$\mathcal{O}_\xi^* = \mathcal{N}_{\xi+1}^{q|_{\xi+1}} \cup \{\Psi_{N,N'}(M) \,:\, M \in \mathcal{O}_{r,\xi} \,:\, N' \in \mathcal{N}_{\xi+1}^{q|_{\xi+1}},\ \delta_{N'} = \delta_N\}.$$

Now one proves by induction on $\beta \leq \alpha$ that $q^*|_\beta$ is a condition in $\mathcal{P}_\beta$. For $\beta \leq \sigma$ this is obvious since then $q_\beta^* = t|_\beta$. For $\beta > \sigma$ one just needs to prove that $\mathcal{N}_\beta^{q^*|_\beta}$ is a T^β-symmetric system and that $\mathcal{O}_\xi^* \subseteq \mathcal{N}_\xi^{q_\xi^*}$ if $\beta = \xi + 1$, where $\xi \in \mathrm{dom}(F_r \backslash \sigma)$. This will be enough by the following.

- No instance of (5) (d) or (g) in the definition of $\mathcal{P}_{\xi+1}$ will cause any problems thanks to the symmetry imposed by clause (3) on $\mathcal{N}_\xi^{q'}$ for every $q' \in \mathcal{P}_\xi$ extending $q^*|_\xi$.
- By the choice of $\mathcal{O}_\xi^*$, all relevant instances of clause (5) (h) are satisfied.
- There is no danger that $\Psi_{N,N'}(M)$, for some $(\Psi_{N,N'}(M), \rho_{N'}^\gamma)$ as above, interferes with any $\xi \in \mathrm{supp}(q) \backslash \sigma$: Suppose $\xi \in \Psi_{N,N'}(M)$ and $\xi + 1 \leq \rho_{N'}^\gamma$, where $(M,\gamma) \in \Delta_r$ and $N' \in \mathcal{N}(M,\gamma)$. Since $\xi > \sigma$, it follows from $(\star)$ that there is

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no $\rho > \xi$ such that $\rho \in N \cap \alpha$. But then $\rho^{\gamma}_{N'} \leq \xi$ since the set of β' such that $\beta' \in M \cap (\gamma+1)$ and $N' \in \mathcal{N}^{q|_{\beta'}}_{\beta'}$ is bounded by ξ, which is a contradiction.

Since $\mathcal{N}^{q|_\beta}_\beta$ and $\mathcal{N}^{r|_\beta}_\beta \in N$ are T^β-symmetric systems and $\mathcal{N}^{r|_\beta}_\beta \supseteq \mathcal{N}^{q|_\beta}_\beta \cap N$, to prove that $\mathcal{N}^{q^*|_\beta}_\beta$ is a T^β-symmetric system it suffices, by Lemma 2.3(2), to prove that if $(M,\gamma) \in \Delta_r$, $\beta \leq \gamma$, $\beta \in M$, $N' \in \mathcal{N}^{q|_\beta}_\beta$, and $\delta_{N'} = \delta_N$, then $(\Psi_{N,N'}(M),\rho) \in \tilde{\Delta}_r$ for some $\rho \geq \beta$. But this is exactly what our choice of $\rho^{\gamma}_{N'}$ guarantees in this case.

Finally, suppose $\beta = \xi+1$, where $\xi \in \mathrm{dom}(F_r \backslash \sigma)$. Let $\mathcal{N} = \mathcal{O}_{r,\xi} \cup \{Q \in \mathcal{N}^{q|_{\xi+1}}_{\xi+1} : \delta_Q \geq \delta_N\}$, and note that $\mathcal{N}_0 := \{Q \in \mathcal{N}^{q|_{\xi+1}}_{\xi+1} : \delta_Q \geq \delta_N\}$ and $\mathcal{O}_{r,\xi}$ are T^ξ-symmetric systems such that $\mathcal{O}_{r,\xi} \supseteq \mathcal{N}_0 \cap N$ (since $\mathcal{N}_0 \cap N = \emptyset$). Hence, by Lemma 2.3(2), $\mathcal{O}^*_\xi$ is the $\subseteq$-minimal T^ξ-symmetric system $\mathcal{O}$ such that $\mathcal{N} \subseteq \mathcal{O}$. Since $\mathcal{N} \subseteq \mathcal{N}^{q^*|_\xi}_\xi$ and $\mathcal{N}^{q^*|_\xi}_\xi$ is a T^ξ-symmetric system, we have from this that $\mathcal{O}^*_\xi \subseteq \mathcal{N}^{q^*|_\xi}_\xi$ as required. $\dashv$

Corollary 3.7 follows immediately from Lemma 3.5, together with the $\aleph_2$-c.c. of all $\mathcal{P}_\kappa$ (Lemma 3.3) and with $\kappa^{\aleph_1} = \kappa$.

COROLLARY 3.7. *For every $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ is proper.*

There are many ways to prove that forcing with $\mathcal{P}_\kappa$ adds κ-many new reals. One way to do this is given in the proof of the following lemma.

LEMMA 3.8. *$\mathcal{P}_\kappa$ adds κ-many reals, each one of which is Cohen-generic over* $\mathbf{V}$.

PROOF. Let $\alpha < \kappa$. Let H be $\mathcal{P}_{\alpha+1}$-generic, $F = \bigcup\{f_{q,\alpha} : q \in H\}$, $\delta = F(\omega)$, and $G = H \restriction \mathcal{P}_\alpha$. We show that H adds Cohen reals over $V[G]$. For this, suppose $\omega \in \mathrm{dom}(b_{q,\alpha})$ for some $q \in H$ (the proof in the case when $\omega \notin \mathrm{dom}(b_{q,\alpha})$ for any $q \in H$ is similar and notationally slightly simpler). Let $(C_\epsilon : \epsilon \in \mathrm{Lim}(\omega_1)) = \Phi(\alpha)_G$. Let $S \subseteq \delta$, $S \in V[G]$, be such that for some (equivalently, for all) $q \in H$ and $N \in \mathcal{O}_{q,\alpha}$ such that $\delta_N = \delta$,

$$\mathrm{rank}(\{M \in \mathcal{N}^G \cap \mathcal{T}^{\alpha+1} : \delta_M \notin C_\delta, \delta_M \in S\}, N) \geq \omega$$

and

$$\mathrm{rank}(\{M \in \mathcal{N}^G \cap \mathcal{T}^{\alpha+1} : \delta_M \notin C_\delta, \delta_M \notin S\}, N) \geq \omega.$$

Now it is straightforward to see, by a standard density argument, that $\{n < \omega : F(n) \in S\}$ is a Cohen real over $V[G]$. $\dashv$

On the other hand, since $\mathcal{P}_\kappa$ has cardinality κ and has the $\aleph_2$-chain condition, there are $\kappa^{\aleph_1} = \kappa$ many nice names (s. [9]) for subsets of ω_1, and hence $\mathcal{P}_\kappa$ forces $2^{\aleph_0} = 2^{\aleph_1} = \kappa$.

LEMMA 3.9. *$\mathcal{P}_\kappa$ forces measuring.*

PROOF. Let $\alpha < \kappa$, let G be $\mathcal{P}_\alpha$-generic, and suppose $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$-name for a club-sequence on ω_1. Let $\vec{C} = \Phi(\alpha)_G = (C_\epsilon : \epsilon \in \mathrm{Lim}(\omega_1))$. Let H be a $\mathcal{P}_{\alpha+1}$-generic filter such that $H \restriction \mathcal{P}_\alpha = G$, and let $C = \bigcup \mathrm{range}\{f_{q,\alpha} : q \in H\}$. By the $\aleph_2$-c.c. of $\mathcal{P}_\kappa$ and the choice of Φ, the conclusion will follow, by standard arguments, if we show that C is a club of ω_1 measuring $\vec{C}$.

By condition (5) (d) in the definition of our iteration it follows, using standard density arguments, that C is a club of ω_1.[3] Also, if $\epsilon \in C$ is such that there is

[3] See [3] and [4] for similar arguments.

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some $q \in H$ such that $\epsilon = f_{q,\alpha}(\delta)$ for some $\delta \in \mathrm{dom}(b_{q,\alpha})$, then a tail of $C \cap \epsilon$ is disjoint from C_ϵ (by (5) (e), (f) in the definition of the iteration). Hence, it suffices to show that if $\delta \in \omega_1$ is such that $\delta \notin \mathrm{dom}(b_{q,\alpha})$ for every $q \in H$ and ϵ is such that $f_{q,\alpha}(\delta) = \epsilon$ for some $q \in H$, then a tail of $C \cap \epsilon$ is contained in C_ϵ. But this implies, by (5) (g) and the usual density argument, that there is some $q \in H$ and some $N \in \mathcal{O}_{q,\alpha}$ such that $f_{q,\alpha}(\delta) = \delta_N$ and such that $q|_\alpha$ forces, in $\mathcal{P}_\alpha$, that $\mathrm{rank}(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1} : \delta_M \notin \Phi(\alpha)(\epsilon)\}, N) = \delta_0$ for some given $\delta_0 < \delta$. It will now be enough to find some $\eta \in [\delta_0, \delta)$ and some extension q^* of q such that every extension q' of q^* is such that $q'|_\alpha$ forces that $f_{q',\alpha}(\delta') \in \Phi(\alpha)(\delta)$ for every $\delta' \in \mathrm{dom}(f_{q',\alpha}) \cap [\eta, \delta)$.

Claim 3.10. *By extending $q|_\alpha$ if necessary we may assume that there is some $a \in N$ such that $q|_\alpha$ forces that if $M \in N \cap \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}$ is such that $a \in M$ and* $\mathrm{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) \geq \delta_0$, *then* $\delta_M \in \Phi(\alpha)(\epsilon)$.

Proof. Let us work in $\mathbf{V}^{\mathcal{P}_\alpha \restriction q|_\alpha}$. If the conclusion fails, then for every $a \in N$ there is some $M \in N \cap \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}$ such that $a \in M$, $\delta_M \notin \Phi(\alpha)(\epsilon)$ and $\mathrm{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) \geq \delta_0$. Fix any such M. By the openness of $\epsilon \backslash \Phi(\alpha)(\epsilon)$ there is some $\rho < \delta_M$ such that $[\rho, \delta_M) \cap \Phi(\alpha)(\epsilon) = \emptyset$. Now, if $\mathrm{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) = \delta^*$, then for every $\gamma < \delta^*$ and every $b \in M$ there is some $M' \in M \cap \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}$ such that $\{b, \rho\} \in M'$ and $\mathrm{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M') \geq \gamma$, and of course $\delta_{M'} \notin \Phi(\alpha)(\epsilon)$ by the above choice of ρ since $\delta_{M'} \in [\rho, \delta_M)$. Iterating this argument we then have that $\mathrm{rank}(\{M' \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1} : \delta_{M'} \notin \Phi(\alpha)(\epsilon)\}, M) = \delta^*$. This shows that $\mathrm{rank}(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1} : \delta_{M'} \notin \Phi(\alpha)(\epsilon)\}, N) > \delta_0$ since $\delta_M \notin \Phi(\alpha)(\epsilon)$, which is a contradiction. ⊣

Again by extending $q|_\alpha$ if necessary, we may also assume that there is some $M \in N \cap \mathcal{N}_\alpha^{q|_\alpha} \cap \mathcal{T}^{\alpha+1}$ containing all relevant objects, where this includes a, and such that $q|_\alpha$ forces $\mathrm{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) = \delta_1$, where $\delta_1 < \delta$ is such that $\delta_1 > \max(\mathrm{dom}(f_{q,\alpha} \restriction \delta))$ and $\delta_1 \geq \delta_0$. Let now q^* be any extension of q such that $M \in \mathcal{O}_{q^*,\alpha}$ and such that $f_{q^*,\alpha}(\delta_1) = \delta_M$. This extension can be easily found thanks to Lemma 2.3. Now it is easy to verify that $\eta = \delta_1$ and q^* are as desired. Indeed, it suffices to note that if q' is any condition extending q^* and $R \in \mathcal{O}_{q',\alpha}$ is such that $\delta_R > \delta_M$ and $\delta_R < \delta_N$, then $q'|_\alpha \Vdash_{\mathcal{P}_\alpha} \delta_R \in \Phi(\alpha)(\epsilon)$. But by symmetry of $\mathcal{O}_{q',\alpha}$ there is some $R' \in \mathcal{O}_{q',\alpha} \cap N$ such that $M \in R'$ and $\delta_{R'} = \delta_R$. Since $a \in R'$ and $q'|_\alpha$ extends $q^*|_\alpha$, it follows then that $q'|_\alpha \Vdash_{\mathcal{P}_\alpha} \delta_R = \delta_{R'} \in \Phi(\alpha)(\epsilon)$. ⊣

The following lemma will conclude the proof of Theorem 1.2.

Lemma 3.11. *$\mathcal{P}_\kappa$ forces* $\mathfrak{b}(\omega_1) = \mathrm{cf}(\kappa)$.

Proof. This proof is essentially contained in the proof of Lemma 3.9. As in that proof, let $\alpha < \kappa$, let H be a $\mathcal{P}_{\alpha+1}$-generic filter, and let $C = \bigcup \mathrm{range}\{f_{q,\alpha} : q \in H\}$. By the $\aleph_2$-c.c. of $\mathcal{P}_\kappa$, it is enough to show that C diagonalises all clubs of ω_1 in $\mathbf{V}[G]$, where $G = H \cap \mathcal{P}_\alpha$.

For this, let us step back into $\mathbf{V}$. Let $\dot{D} \in H(\kappa)$ be a $\mathcal{P}_\alpha$-name for a club of ω_1, let $q \in \mathcal{P}_{\alpha+1}$, and suppose $\dot{D} \in M$ for some $M \in \mathcal{O}_{q,\alpha}$. It suffices to show that if $N \in \mathcal{O}_{q,\alpha}$ is such that $\delta_M < \delta_N$, then $q \Vdash_{\mathcal{P}_{\alpha+1}} \delta_N \in \dot{D}$. By symmetry of $\mathcal{O}_{q,\alpha}$ there is some $N' \in \mathcal{O}_{q,\alpha}$ such that $M \in N'$ and $\delta_{N'} = \delta_N$. But $q|_\alpha$ is $(N', \mathcal{P}_\alpha)$-generic by Lemma 3.5 and hence forces that $\delta_{N'} = \delta_N$ is a limit point of ordinals in $\dot{D}$ and therefore in $\dot{D}$. ⊣

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We finish the paper by mentioning the following result due to Moore.

FACT 3.12 (Moore). *$\mathcal{P}_\kappa$ forces that the set of ground model reals is meagre, and in fact that* $\mathbf{non}(\mathcal{M}) \geq \mathrm{cf}(\kappa)$, *where $\mathcal{M}$ denotes the meagre ideal on the reals.*

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