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Tangential contact problem for a transversely isotropic elastic layer bonded to a rigid foundation

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Abstract

A contact problem is called tangential when arbitrary tangential displacements are prescribed over a part of the boundary of a transversely isotropic layer, while the tangential stress is zero over the rest of the boundary; the normal stress vanishes all over the boundary. The Generalized Images method is used to give a complete elementary solution to the problem. A new governing integral equation is derived. A particular case of a circular domain of contact is studied in detail. The governing integral equation can be inverted in this case. Approximate formulae are derived for the resultant force and the torque. A direct relationship is established between the integral transform method and the Generalized Images method. A limiting case of general solution gives the solution for an isotropic layer.

1. *Introduction*

Lebedev and Ufliand [**9**] were apparently the first to publish the solution of an axisymmetric normal contact problem. Almost simultaneously, Vorovich and Ustinov [**11**] did the same. Thousands of articles have been published since then on this subject. They use integral transforms, and the solutions are generally very convoluted, difficult for numerical implementation and for estimation of the solution accuracy. The most recent publications can be quoted [**1**], [**8**], [**12**], where numerous other references can be found. There seem to be no complete solutions for *tangential* contact problems for an elastic layer, except for a torsional problem for a layer [**10**].

The Generalized Images method was first introduced and applied to solving crack problems in [**5**]. It uses the potential functions for exact solutions in an elastic space or half-space, supplemented by similar functions shifted in the z-direction in order to satisfy the boundary conditions in more complicated bodies, like layers, for example. The word "generalized" is used, because the mathematical structure used is not equivalent to an image crack or an image punch (except for the case of torsional deformation).

A general formulation in terms of generalized images of a tangential contact problem for a transversely isotropic elastic layer bonded to a rigid foundation is given in Section 2. A complete set of potential functions is presented and used to derive

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a new governing integral equation of the contact problem. The integral equation can be inverted exactly in the case of a circular domain of contact. The relationship between the traditionally used integral transform approach and the present method is discussed in the final section.

2. *Formulation of the contact problem and its solution*

The idea of the Generalized Images method comes from the following considerations. The general solution for the elastic field in a transversely isotropic body can be expressed through three potential functions F_k, $k = 1, 2, 3$ as follows [**2**]:

$$u = \Lambda(F_1 + F_2 + iF_3), \qquad w = m_1\frac{\partial F_1}{\partial z} + m_2\frac{\partial F_2}{\partial z} \tag{1}$$

$$\sigma_1 = 2A_{66}\frac{\partial^2}{\partial z^2}\left\{\left[\gamma_1^2 - (1+m_1)\gamma_3^2\right]F_1 + \left[\gamma_2^2 - (1+m_2)\gamma_3^2\right]F_2\right\} \tag{2}$$

$$\sigma_2 = 2A_{66}\Lambda^2(F_1 + F_2 + iF_3) \tag{3}$$

$$\sigma_z = A_{44}\frac{\partial^2}{\partial z^2}\left[(1+m_1)\gamma_1^2F_1 + (1+m_2)\gamma_2^2F_2\right] \tag{4}$$

$$\tau_z = A_{44}\Lambda\frac{\partial}{\partial z}[(1+m_1)F_1 + (1+m_2)F_2 + iF_3]. \tag{5}$$

Here $u = u_x + iu_y$ is the complex tangential displacement, w denotes the normal displacement; $m_1, m_2, \gamma_1, \gamma_2, \gamma_3, A_{44}, A_{66}$ are the elastic constants, introduced in [**2**], and the following notation is used:

$$\sigma_1 = \sigma_x + \sigma_y, \qquad \sigma_2 = \sigma_x - \sigma_y + 2i\tau_{xy}, \qquad \tau_z = \tau_{zx} + i\tau_{yz} \tag{6}$$

$$\Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = e^{i\varphi}\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\varphi}\right), \qquad i = \sqrt{-1}. \tag{7}$$

We consider a transversely isotropic elastic layer of thickness c bonded to a rigid foundation. An arbitrary tangential displacement u is prescribed in the domain S in the plane $z = 0$. The mathematical formulation of the boundary conditions can be presented as follows:

$$u = u(\rho,\varphi), \quad (\rho,\varphi) \subseteq S, \quad \tau_z = 0, \quad (\rho,\varphi) \notin S, \quad z = 0; \quad \sigma_z = 0, \quad \rho < \infty, \quad z = 0 \tag{8}$$

$$w = 0, \quad u = 0, \quad \text{for } \rho < \infty, \quad z = c. \tag{9}$$

It is known [**2**], that in the case of an arbitrary tangential stress τ applied to a transversely isotropic half-space, the potential functions take the form:

$$F_1 = \frac{H\gamma_1\gamma_2}{m_1 - 1}X(z_1), \qquad F_2 = \frac{H\gamma_1\gamma_2}{m_2 - 1}X(z_2), \qquad F_3 = \frac{i\gamma_3}{2\pi A_{44}}Y(z_3) \tag{10}$$

where

$$X(\rho,\varphi,z) \equiv X(z) = \frac{1}{2}\left[\bar{\Lambda}\int_S\int[z\ln(R_0+z) - R_0]\tau\,dS + \Lambda\int_S\int[z\ln(R_0+z) - R_0]\bar{\tau}\,dS\right] \tag{11}$$

$$Y(\rho,\varphi,z) \equiv Y(z) = \frac{1}{2}\left[\bar{\Lambda}\int_S\int[z\ln(R_0+z) - R_0]\tau\,dS - \Lambda\int_S\int[z\ln(R_0+z) - R_0]\bar{\tau}\,dS\right] \tag{12}$$

$$R_0 = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\varphi - \varphi_0) + z^2}, \quad z_k = z/\gamma_k, \quad k = 1, 2, 3. \tag{13}$$

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The Generalized Images method allows us to present the solution for an elastic layer as a series of potential functions for a half-space shifted along the z-axis as follows:

$$F_1 = \frac{H\gamma_1\gamma_2}{m_1 - 1}\left[X(z_1) + \sum_{n,k}\left(A_{nk}^{(1)}X(z_1 + c_{nk}) + B_{nk}^{(1)}X(c_{nk} - z_1)\right)\right] \tag{14}$$

$$F_2 = \frac{H\gamma_1\gamma_2}{m_2 - 1}\left[X(z_2) + \sum_{n,k}\left(A_{nk}^{(2)}X(z_2 + c_{nk}) + B_{nk}^{(2)}X(c_{nk} - z_2)\right)\right] \tag{15}$$

$$F_3 = \frac{i\gamma_3}{2\pi A_{44}}\left[Y(z_3) + \sum_{k=1}^{\infty}\left(A_kY(z_3 + kc_3) + B_kY(kc_3 - z_3)\right)\right] \tag{16}$$

where A_{nk}, B_{nk}, A_k and B_k are the yet unknown constants to be defined from the boundary conditions and

$$c_{nk} = nc_1 + kc_2, \qquad c_1 = c/\gamma_1, \qquad c_2 = c/\gamma_2, \tag{17}$$

$$\sum_{n,k} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}. \tag{18}$$

Since the case of $n = k = 0$ in (14–16) corresponds to the first term in each formula, we presume that this case is excluded from the sums in order to avoid duplication.

Presuming at this stage that the tangential stress at the boundary $z = 0$ is known, namely, $\tau_z = -\tau$, we can satisfy this condition by substituting (14–16) into (5), with the result

$$\begin{aligned}\tau_z = {} & A_{44}(1 + m_1)\frac{H\gamma_2}{m_1 - 1}\Lambda\left[X'(0) + \sum_{n,k}\left(A_{nk}^{(1)}X'(c_{nk}) - B_{nk}^{(1)}X'(c_{nk})\right)\right] \\ & + A_{44}(1 + m_2)\frac{H\gamma_1}{m_2 - 1}\Lambda\left[X'(0) + \sum_{n,k}\left(A_{nk}^{(2)}X'(c_{nk}) - B_{nk}^{(2)}X'(c_{nk})\right)\right] \\ & - \frac{1}{2\pi}\Lambda\left[Y'(0) + \sum_{k=1}^{\infty}\left(A_kY'(kc_3) - B_kY'(kc_3)\right)\right] = -\tau.\end{aligned} \tag{19}$$

Here the symbol $'$ denotes the first derivative with respect to the argument in the parentheses; we also need to use the identity from [**2**]:

$$\frac{A_{44}H(m_1 + 1)}{m_1 - 1} = -\frac{A_{44}H(m_2 + 1)}{m_2 - 1} = \frac{1}{2\pi(\gamma_1 - \gamma_2)}. \tag{20}$$

Equation (19) is satisfied if

$$\gamma_2\left(A_{nk}^{(1)} - B_{nk}^{(1)}\right) - \gamma_1\left(A_{nk}^{(2)} - B_{nk}^{(2)}\right) = 0 \tag{21}$$

and

$$A_k = B_k. \tag{22}$$

The next condition to satisfy is $\sigma_z = 0$ for $z = 0$. Substitution of (14–16) in (4) yields

$$\begin{aligned}\sigma_z = {} & A_{44}(1 + m_1)\frac{H\gamma_1\gamma_2}{m_1 - 1}\left[X''(0) + \sum_{n,k}\left(A_{nk}^{(1)}X''(c_{nk}) + B_{nk}^{(1)}X''(c_{nk})\right)\right] \\ & + A_{44}(1 + m_2)\frac{H\gamma_1\gamma_2}{m_2 - 1}\left[X''(0) + \sum_{n,k}\left(A_{nk}^{(2)}X''(c_{nk}) + B_{nk}^{(2)}X''(c_{nk})\right)\right] = 0.\end{aligned} \tag{23}$$

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Equation (23) is satisfied if

$$A_{nk}^{(1)} + B_{nk}^{(1)} - \left(A_{nk}^{(2)} + B_{nk}^{(2)}\right) = 0. \tag{24}$$

The fact that the tangential displacement vanishes at $z = c$ leads to

$$\begin{aligned} 0 = {} & \frac{H\gamma_1\gamma_2}{m_1 - 1}\Lambda\left[X(c_1) + \sum_{n,k}\left(A_{nk}^{(1)}X(c_1 + c_{nk}) + B_{nk}^{(1)}X(c_{nk} - c_1)\right)\right] \\ & + \frac{H\gamma_1\gamma_2}{m_2 - 1}\Lambda\left[X(c_2) + \sum_{n,k}\left(A_{nk}^{(2)}X(c_2 + c_{nk}) + B_{nk}^{(2)}X(c_{nk} - c_2)\right)\right] \\ & - \frac{\gamma_3}{2\pi A_{44}}\Lambda\left[Y(c_3) + \sum_{k=1}^{\infty}\left(A_kY(c_3 + kc_3) + B_kY(kc_3 - c_3)\right)\right]. \end{aligned} \tag{25}$$

Equation (25) is satisfied if

$$1 + B_{20}^{(1)} - m_1B_{11}^{(2)} = 0, \qquad B_{11}^{(1)} - m_1\left(1 + B_{02}^{(2)}\right) = 0 \tag{26}$$

$$A_{n-1,k}^{(1)} + B_{n+1,k}^{(1)} - m_1\left(A_{n,k-1}^{(2)} + B_{n,k+1}^{(2)}\right) = 0 \tag{27}$$

$$1 + B_2 = 0, \qquad A_{k-1} + B_{k+1} = 0. \tag{28}$$

Here we use the property established in [**2**] $m_1m_2 = 1$.

Satisfaction of the boundary condition $w = 0$ at $z = c$ requires substitution of (14–16) into the second formula (1), with the result

$$\begin{aligned} 0 = {} & \frac{H\gamma_2m_1}{m_1 - 1}\left[X'(c_1) + \sum_{n,k}\left(A_{nk}^{(1)}X'(c_1 + c_{nk}) - B_{nk}^{(1)}X'(c_{nk} - c_1)\right)\right] \\ & + \frac{H\gamma_1m_2}{m_2 - 1}\left[X'(c_2) + \sum_{n,k}\left(A_{nk}^{(2)}X'(c_2 + c_{nk}) - B_{nk}^{(2)}X'(c_{nk} - c_2)\right)\right]. \end{aligned} \tag{29}$$

Equation (29) is satisfied if

$$m_1\gamma_2\left(1 - B_{20}^{(1)}\right) + \gamma_1B_{11}^{(2)} = 0, \qquad m_1\gamma_2B_{11}^{(1)} + \gamma_1\left(1 - B_{02}^{(2)}\right) = 0 \tag{30}$$

$$m_1\gamma_2\left[A_{n-1,k}^{(1)} - B_{n+1,k}^{(1)}\right] - \gamma_1\left[A_{n,k-1}^{(2)} - B_{n,k+1}^{(2)}\right] = 0. \tag{31}$$

Solution of (22), (28) gives

$$B_2 = A_2 = -1, \qquad B_4 = A_4 = 1, \qquad B_6 = A_6 = -1, \ldots \tag{32}$$

with all A_k and B_k vanishing when k is odd.

Equations (26) and (30) give

$$B_{11}^{(1)} = \frac{2\gamma_1}{m_2\gamma_1 - m_1\gamma_2}, \qquad B_{20}^{(1)} = -\frac{m_2\gamma_1 + m_1\gamma_2}{m_2\gamma_1 - m_1\gamma_2} \tag{33}$$

$$B_{11}^{(2)} = -\frac{2\gamma_2}{m_2\gamma_1 - m_1\gamma_2}, \qquad B_{02}^{(2)} = \frac{m_2\gamma_1 + m_1\gamma_2}{m_2\gamma_1 - m_1\gamma_2}. \tag{34}$$

All the coefficients B_{nk} with odd $n + k$ vanish.

Equations (21) and (24), relate A_{nk} through B_{nk}:

$$A_{nk}^{(1)} = \frac{2\gamma_1B_{nk}^{(2)} - (\gamma_1 + \gamma_2)B_{nk}^{(1)}}{\gamma_1 - \gamma_2}, \qquad A_{nk}^{(2)} = \frac{(\gamma_1 + \gamma_2)B_{nk}^{(2)} - 2\gamma_2B_{nk}^{(1)}}{\gamma_1 - \gamma_2}. \tag{35}$$

We conclude that all A_{nk} with $n + k$ odd vanish as well.

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The recurrence relations can be obtained from (27) and (31):

$$B^{(1)}_{n+1,k} = \frac{2\gamma_1 A^{(2)}_{n,k-1} - (m_2\gamma_1 + m_1\gamma_2)A^{(1)}_{n-1,k}}{m_2\gamma_1 - m_1\gamma_2} \tag{36}$$

$$B^{(2)}_{n,k+1} = \frac{(m_2\gamma_1 + m_1\gamma_2)A^{(2)}_{n,k-1} - 2\gamma_2 A^{(1)}_{n-1,k}}{m_2\gamma_1 - m_1\gamma_2}. \tag{37}$$

Combination of (35) and (37) gives

$$\begin{aligned} B^{(1)}_{n,k} = {} & \frac{2\gamma_1}{m_2\gamma_1 - m_1\gamma_2}\,\frac{(\gamma_1+\gamma_2)B^{(2)}_{n-1,k-1} - 2\gamma_2 B^{(1)}_{n-1,k-1}}{\gamma_1 - \gamma_2} \\ & + \frac{m_2\gamma_1 + m_1\gamma_2}{m_2\gamma_1 - m_1\gamma_2}\,\frac{(\gamma_1+\gamma_2)B^{(1)}_{n-2,k} - 2\gamma_1 B^{(2)}_{n-2,k}}{\gamma_1 - \gamma_2} \end{aligned} \tag{38}$$

$$\begin{aligned} B^{(2)}_{n,k} = {} & \frac{m_2\gamma_1 + m_1\gamma_2}{m_2\gamma_1 - m_1\gamma_2}\,\frac{(\gamma_1+\gamma_2)B^{(2)}_{n,k-2} - 2\gamma_2 B^{(1)}_{n,k-2}}{\gamma_1 - \gamma_2} \\ & + \frac{2\gamma_2}{m_2\gamma_1 - m_1\gamma_2}\,\frac{(\gamma_1+\gamma_2)B^{(1)}_{n-1,k-1} - 2\gamma_1 B^{(2)}_{n-1,k-1}}{\gamma_1 - \gamma_2}. \end{aligned} \tag{39}$$

One can verify that formulae (33–34) can be obtained from (38–39) by setting

$$B^{(1)}_{00} = B^{(2)}_{00} = 1$$

and all B with negative subindex equal to zero.

Similarly, we can get

$$\begin{aligned} A^{(1)}_{nk} = {} & -\frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}\,\frac{2\gamma_1 A^{(2)}_{n-1,k-1} - (m_2\gamma_1 + m_1\gamma_2)A^{(1)}_{n-2,k}}{m_2\gamma_1 - m_1\gamma_2} \\ & + \frac{2\gamma_1}{\gamma_1-\gamma_2}\,\frac{(m_2\gamma_1 + m_1\gamma_2)A^{(2)}_{n,k-2} - 2\gamma_2 A^{(1)}_{n-1,k-1}}{m_2\gamma_1 - m_1\gamma_2} \end{aligned} \tag{40}$$

$$\begin{aligned} A^{(2)}_{nk} = {} & -\frac{2\gamma_2}{\gamma_1-\gamma_2}\,\frac{2\gamma_1 A^{(2)}_{n-1,k-1} - (m_2\gamma_1 + m_1\gamma_2)A^{(1)}_{n-2,k}}{m_2\gamma_1 - m_1\gamma_2} \\ & + \frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}\,\frac{(m_2\gamma_1 + m_1\gamma_2)A^{(2)}_{n,k-2} - 2\gamma_2 A^{(1)}_{n-1,k-1}}{m_2\gamma_1 - m_1\gamma_2}. \end{aligned} \tag{41}$$

Substitution of (33–34) into (35) yields

$$A^{(1)}_{11} = -\frac{2\gamma_1(\gamma_1+3\gamma_2)}{(m_2\gamma_1 - m_1\gamma_2)(\gamma_1-\gamma_2)}, \qquad A^{(1)}_{20} = \frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}\,\frac{m_2\gamma_1 + m_1\gamma_2}{m_2\gamma_1 - m_1\gamma_2} \tag{42}$$

$$A^{(1)}_{02} = \frac{2\gamma_1}{\gamma_1-\gamma_2}\,\frac{m_2\gamma_1 + m_1\gamma_2}{m_2\gamma_1 - m_1\gamma_2}, \qquad A^{(2)}_{20} = \frac{2\gamma_2}{\gamma_1-\gamma_2}\,\frac{m_2\gamma_1 + m_1\gamma_2}{m_2\gamma_1 - m_1\gamma_2} \tag{43}$$

$$A^{(2)}_{02} = \frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}\,\frac{m_2\gamma_1 + m_1\gamma_2}{m_2\gamma_1 - m_1\gamma_2}, \qquad A^{(2)}_{11} = -\frac{2\gamma_2(3\gamma_1+\gamma_2)}{(m_2\gamma_1 - m_1\gamma_2)(\gamma_1-\gamma_2)}. \tag{44}$$

One can verify that (42–44) can be formally obtained from (40–41) by taking $A^{(1)}_{00} = A^{(2)}_{00} = 1$.

We call the coefficients for a fixed $n + k = 2$ the first reflection, and in general, the mth reflection corresponds to $n + k = 2m$. The coefficients for the second

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reflection are

$$A_{40}^{(1)} = \frac{(\gamma_1+\gamma_2)^2(m_2\gamma_1+m_1\gamma_2)^2}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2}, \qquad A_{04}^{(1)} = \frac{2\gamma_1(\gamma+\gamma_2)(m_2\gamma_1+m_1\gamma_2)^2}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \tag{45}$$

$$A_{31}^{(1)} = -\frac{2\gamma_1(\gamma_1+\gamma_2)(\gamma_1+7\gamma_2)(m_2\gamma_1+m_1\gamma_2)}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \tag{46}$$

$$A_{22}^{(1)} = \frac{2\gamma_1\{2\gamma_2[5\gamma_1^2+10\gamma_1\gamma_2+\gamma_2^2]+(\gamma_1+3\gamma_2)(m_2\gamma_1+m_1\gamma_2)^2\}}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \tag{47}$$

$$A_{13}^{(1)} = -\frac{2\gamma_1(m_2\gamma_1+m_1\gamma_2)[\gamma_1^2+12\gamma_1\gamma_2+3\gamma_2^2]}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \tag{48}$$

$$A_{40}^{(2)} = \frac{2\gamma_2(\gamma_1+\gamma_2)(m_2\gamma_1+m_1\gamma_2)^2}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2}, \qquad A_{04}^{(2)} = \frac{(\gamma_1+\gamma_2)^2(m_2\gamma_1+m_1\gamma_2)^2}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \tag{49}$$

$$A_{31}^{(2)} = -\frac{2\gamma_2(m_2\gamma_1+m_1\gamma_2)[3\gamma_1^2+12\gamma_1\gamma_2+\gamma_2^2]}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \tag{50}$$

$$A_{22}^{(2)} = \frac{2\gamma_2\{2\gamma_1[\gamma_1^2+10\gamma_1\gamma_2+5\gamma_2^2]+(3\gamma_1+\gamma_2)(m_2\gamma_1+m_1\gamma_2)^2\}}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \tag{51}$$

$$A_{13}^{(2)} = -\frac{2\gamma_2(\gamma_1+\gamma_2)(7\gamma_1+\gamma_2)(m_2\gamma_1+m_1\gamma_2)}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \tag{52}$$

$$B_{40}^{(1)} = -\frac{(\gamma_1+\gamma_2)(m_2\gamma_1+m_1\gamma_2)^2}{(\gamma_1-\gamma_2)(m_2\gamma_1-m_1\gamma_2)^2}, \qquad B_{31}^{(1)} = \frac{2\gamma_1(\gamma_1+5\gamma_2)(m_2\gamma_1+m_1\gamma_2)}{(\gamma_1-\gamma_2)(m_2\gamma_1-m_1\gamma_2)^2} \tag{53}$$

$$B_{22}^{(1)} = -\frac{2\gamma_1[2\gamma_2(3\gamma_1+\gamma_2)+(m_2\gamma_1+m_1\gamma_2)^2]}{(\gamma_1-\gamma_2)(m_2\gamma_1-m_1\gamma_2)^2} \tag{54}$$

$$B_{13}^{(1)} = \frac{2\gamma_1(\gamma_1+\gamma_2)(m_2\gamma_1+m_1\gamma_2)}{(\gamma_1-\gamma_2)(m_2\gamma_1-m_1\gamma_2)^2}, \qquad B_{04}^{(1)} = 0 \tag{55}$$

$$B_{40}^{(2)} = 0, \qquad B_{31}^{(2)} = -\frac{2\gamma_2(\gamma_1+\gamma_2)(m_2\gamma_1+m_1\gamma_2)}{(\gamma_1-\gamma_2)(m_2\gamma_1-m_1\gamma_2)^2} \tag{56}$$

$$B_{22}^{(2)} = \frac{2\gamma_2[2\gamma_1(\gamma_1+3\gamma_2)+(m_2\gamma_1+m_1\gamma_2)^2]}{(m_2\gamma_1-m_1\gamma_2)^2(\gamma_1-\gamma_2)} \tag{57}$$

$$B_{13}^{(2)} = -\frac{2\gamma_2(5\gamma_1+\gamma_2)(m_2\gamma_1+m_1\gamma_2)}{(m_2\gamma_1-m_1\gamma_2)^2(\gamma_1-\gamma_2)}, \qquad B_{04}^{(2)} = \frac{(\gamma_1+\gamma_2)(m_2\gamma_1+m_1\gamma_2)^2}{(\gamma_1-\gamma_2)(m_2\gamma_1-m_1\gamma_2)^2}. \tag{58}$$

The solution is in principle finished, since all the coefficients are defined. Now we can derive the governing integral equation of the contact problem from the first condition (8). The result is

$$\begin{aligned} u = {} & \frac{H\gamma_1\gamma_2}{m_1-1}\Lambda\left[X(0)+\sum_{n,k}\left(A_{nk}^{(1)}+B_{nk}^{(1)}\right)X(c_{nk})\right] \\ & -\frac{m_1H\gamma_1\gamma_2}{m_1-1}\Lambda\left[X(0)\sum_{n,k}\left(A_{nk}^{(2)}+B_{nk}^{(2)}\right)X(c_{nk})\right] \\ & -\frac{\gamma_3}{2\pi A_{44}}\Lambda\left[Y(0)+\sum_{k=1}^{\infty}(A_k+B_k)Y(kc_3)\right]. \end{aligned} \tag{59}$$

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Equation (59) can be simplified by using (22) and (24), with the result

$$u(\rho,\varphi) = \frac{1}{2}G_1\int_S\int \frac{\tau(\rho_0,\varphi_0)\,dS_0}{R} + \frac{1}{2}G_2\int_S\int \frac{q^2\bar{\tau}(\rho_0,\varphi_0)\,dS_0}{R^3}$$
$$+ \frac{G_1-G_2}{4}\sum_{n,k}Q_{nk}\left(\int_S\int \frac{\tau(\rho_0,\varphi_0)\,dS_0}{R_{nk}} - \int_S\int \frac{q^2\bar{\tau}(\rho_0,\varphi_0)\,dS_0}{R_{nk}(R_{nk}+c_{nk})^2}\right)$$
$$+ \frac{G_1+G_2}{2}\sum_{m=1}^{\infty}(-1)^m\left(\int_S\int \frac{\tau(\rho_0,\varphi_0)\,dS_0}{R_{mc3}} + \int_S\int \frac{q^2\bar{\tau}(\rho_0,\varphi_0)\,dS_0}{R_{mc3}(R_{mc3}+2mc_3)^2}\right). \quad (60)$$

Here

$$Q_{nk} = A_{nk}^{(1)} + B_{nk}^{(1)} = A_{nk}^{(2)} + B_{nk}^{(2)}, \qquad G_1 = \beta + \gamma_1\gamma_2 H, \qquad G_2 = \beta - \gamma_1\gamma_2 H \quad (61)$$

$$H = \frac{(\gamma_1+\gamma_2)A_{11}}{2\pi\left(A_{11}A_{33}-A_{13}^2\right)}, \quad \beta = \frac{\gamma_3}{2\pi A_{44}}, \qquad c_3 = c/\gamma_3 \quad (62)$$

$$q = \rho e^{i\varphi} - \rho_0 e^{i\varphi_0}, \qquad \bar{q} = \rho e^{-i\varphi} - \rho_0 e^{-i\varphi_0}, \qquad R^2 = q\bar{q} \quad (63)$$

$$R_{nk} = \sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\varphi-\varphi_0)+c_{nk}^2}, \qquad dS_0 = \rho_0 d\rho_0 d\varphi_0 \quad (64)$$

$$R_{mc3} = \sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\varphi-\varphi_0)+(2mc_3)^2}. \quad (65)$$

The first line in (60) corresponds to the governing integral equation for a transversely isotropic elastic half-space, while the remaining part of (60) describes the influence of the bonded layer boundary.

Here are the expressions for the first and the second reflections Q_{nk}

$$Q_{11} = -\frac{8\gamma_1\gamma_2}{(m_2\gamma_1-m_1\gamma_2)(\gamma_1-\gamma_2)}, \qquad Q_{20} = \frac{2\gamma_2}{\gamma_1-\gamma_2}\frac{m_2\gamma_1+m_1\gamma_2}{m_2\gamma_1-m_1\gamma_2} \quad (66)$$

$$Q_{02} = \frac{2\gamma_1}{\gamma_1-\gamma_2}\frac{m_2\gamma_1+m_1\gamma_2}{m_2\gamma_1-m_1\gamma_2} \quad (67)$$

$$Q_{40} = \frac{2\gamma_2(\gamma_1+\gamma_2)(m_2\gamma_1+m_1\gamma_2)^2}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2}, \qquad Q_{31} = -\frac{8\gamma_1\gamma_2(\gamma_1+3\gamma_2)(m_2\gamma_1+m_1\gamma_2)}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \quad (68)$$

$$Q_{22} = \frac{8\gamma_1\gamma_2\left[\gamma_1^2+6\gamma_1\gamma_2+\gamma_2^2+(m_2\gamma_1+m_1\gamma_2)^2\right]}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2} \quad (69)$$

$$Q_{13} = -\frac{8\gamma_1\gamma_2(m_2\gamma_1+m_1\gamma_2)(3\gamma_1+\gamma_2)}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2}, \qquad Q_{04} = \frac{2\gamma_1(\gamma+\gamma_2)(m_2\gamma_1+m_1\gamma_2)^2}{(\gamma_1-\gamma_2)^2(m_2\gamma_1-m_1\gamma_2)^2}. \quad (70)$$

The governing integral equation (60) is valid for the case of isotropy as well. We just need to compute the limit for $\gamma_1 \to \gamma_2 \to 1$.

3. *The case of a circular domain of contact*

In this section, we still consider the prescribed tangential displacements to be arbitrary, but the domain of contact is presumed to be a circle of radius a. The exact solution to this kind of a problem for an elastic half-space was published in [**6**], so we can use all the results obtained there. For example, equation (60) can be inverted exactly. We need to rewrite it as follows

$$\frac{1}{2}G_1\int_S\int \frac{\tau(\rho_0,\varphi_0)\,dS_0}{R} + \frac{1}{2}G_2\int_S\int \frac{q^2\bar{\tau}(\rho_0,\varphi_0)\,dS_0}{R^3} = U(\rho,\varphi) \quad (71)$$

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where

$$U(\rho,\varphi)=u(\rho,\varphi)-\frac{G_1-G_2}{4}\sum_{n,k}Q_{nk}\left(\int_S\int\frac{\tau(\rho_0,\varphi_0)\,dS_0}{R_{nk}}-\int_S\int\frac{q^2\bar{\tau}(\rho_0,\varphi_0)\,dS_0}{R_{nk}(R_{nk}+c_{nk})^2}\right)$$
$$-\frac{G_1+G_2}{2}\sum_{m=1}^{\infty}(-1)^m\left(\int_S\int\frac{\tau(\rho_0,\varphi)\,dS_0}{R_{mc3}}+\int_S\int\frac{q^2\bar{\tau}(\rho_0,\varphi_0)\,dS_0}{R_{mc3}(R_{mc3}+2mc_3)^2}\right). \quad (72)$$

The exact solution to (71) was obtained in [**6**] as

$$\tau(\rho,\varphi)=-\frac{1}{\pi^3\left(G_1^2-G_2^2\right)}\Lambda\int_0^{2\pi}\int_0^a\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right)[G_1\bar{\Lambda}_0U(\rho_0,\varphi_0)$$
$$+G_2\Lambda_0\bar{U}(\rho_0,\varphi_0)]\rho_0d\rho_0d\varphi_0+\frac{1}{\pi^3G_1}\int_0^{2\pi}\int_0^a\left(\frac{a^2+\bar{\delta}^2}{\eta(a^2-\bar{\delta}^2)^2}\right)U(\rho_0,\varphi_0)\rho_0d\rho_0d\varphi_0 \quad (73)$$

where the overbar denotes the complex conjugate value and the following notation was introduced:

$$\delta^2=\rho\rho_0e^{i(\varphi-\varphi)_0},\qquad\eta=\frac{\sqrt{a-\rho^2}\sqrt{a-\rho_0^2}}{a},\qquad\Lambda_0=e^{i\varphi_0}\left(\frac{\partial}{\partial\rho_0}+\frac{i}{\rho_0}\frac{\partial}{\partial\varphi_0}\right). \quad (74)$$

The inversion of (71–72) using (73) yields

$$\tau(\rho,\varphi)=\tau_\infty(\rho,\varphi)-\frac{1}{4\pi^3}\sum_{n,k}Q_{nk}\left(\int_S\int\Phi_3(c_{nk})\tau\,dS+\int_S\int\Phi_4(c_{nk})\bar{\tau}\,dS\right)$$
$$-\frac{1}{2\pi^3}\sum_{m=1}^{\infty}(-1)^m\left(\int_S\int\Phi_3(2mc_3)\tau\,dS-\int_S\int\Phi_4(2mc_3)\bar{\tau}\,dS\right)$$
$$-\frac{G_1-G_2}{4\pi^3G_1}\sum_{n,k}Q_{nk}\left(\int_S\int\Phi_1(c_{nk})\tau\,dS-\int_S\int\Phi_2(c_{nk})\bar{\tau}\,dS\right)$$
$$-\frac{G_1+G_2}{2\pi^3G_1}\sum_{m=1}^{\infty}(-1)^m\left(\int_S\int\Phi_1(2mc_3)\tau\,dS+\int_S\int\Phi_2(2mc_3)\bar{\tau}\,dS\right). \quad (75)$$

Here τ_∞ is the stress distribution on the surface of a transversely isotropic half-space where domain S is subjected to a tangential displacement u; the remaining terms describe the influence of the bonded layer boundary.

The relevant integrals were originally computed in [**4**] and are

$$\Phi_1(z)=\int_0^{2\pi}\int_0^a\left(\frac{a^2+\vartheta^2}{\eta_r(a^2-\bar{\vartheta}^2)^2}\right)\frac{r\,dr\,d\psi}{R_r}$$
$$=\frac{2\pi}{\sqrt{a^2-\rho^2}}\left[\frac{a^2}{(a^2-\bar{\delta}^2)^{3/2}}\tan^{-1}\left(\frac{\sqrt{a^2-\bar{\delta}^2}}{\sqrt{l_2^2(\rho_0)-a^2}}\right)-\frac{\bar{\delta}^2\sqrt{l_2^2(\rho_0)-a^2}}{(a^2-\bar{\delta}^2)\left[l_2^2(\rho_0)-\bar{\delta}^2\right]}\right]. \quad (76)$$

Here

$$\vartheta^2=\rho re^{i(\varphi-\psi)},\quad\eta_r=\frac{\sqrt{a-\rho^2}\sqrt{a-r^2}}{a},\quad R_r=\sqrt{\rho_0^2+r^2-2\rho_0r\cos(\varphi_0-\psi)+z^2} \quad (77)$$

$$l_2(\rho_0)=\frac{1}{2}(\sqrt{(a+\rho_0)^2+z^2}+\sqrt{(a-\rho_0)^2+z^2}) \quad (78)$$

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$$\Phi_2(z) = \int_0^{2\pi}\int_0^a \left(\frac{a^2+\bar{\vartheta}^2}{\eta_r(a^2-\bar{\vartheta}^2)^2}\right)\frac{(re^{i\psi}-\rho_0 e^{i\varphi_0})^2 r\,dr\,d\psi}{R_r(R_r+z)^2}$$
$$= \frac{2\pi a^2 e^{i\varphi}}{\rho\sqrt{a^2-\rho^2}}\left\{-\frac{e^{i\varphi_0}}{\rho_0(\kappa^2-a^2)^2}\left[z(2\kappa^2+a^2)-\sqrt{l_2^2(\rho_0)-a^2}\left(2\kappa^2+\frac{\left[a^2-l_1^2(\rho_0)\right]l_2^2(\rho_0)}{l_2^2(\rho_0)-\bar{\delta}^2}\right)\right]\right.$$
$$-\frac{\rho_0 e^{i\varphi_0}\sqrt{l_2^2(\rho_0)-a^2}}{\kappa^2\left[l_2^2(\rho_0)-\bar{\delta}^2\right]}+\frac{3a^3 z e^{i\varphi}}{\rho(\kappa^2-a^2)^{5/2}}\left[\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2(\rho_0)}}{\sqrt{\kappa^2-a^2}}\right)\right.$$
$$\left.\left.-\tan^{-1}\left(\frac{a}{\sqrt{\kappa^2-a^2}}\right)\right]\right\}. \tag{79}$$

Here

$$\kappa^2 = a^2\frac{\rho_0}{\rho}e^{i(\varphi-\varphi_0)}, \qquad l_1(\rho_0) = \frac{1}{2}(\sqrt{(a+\rho_0)^2+z^2}-\sqrt{(a-\rho_0)^2+z^2}) \tag{80}$$

$$\Phi_3(z) = \Lambda\int_0^{2\pi}\int_0^a \frac{1}{R_N}\tan^{-1}\left(\frac{\eta_r}{R_N}\right)\frac{re^{-i\psi}-\rho_0 e^{-i\varphi_0}}{R_r^3}r\,dr\,d\psi$$
$$= 2\pi\left\{\frac{z}{R_0^3}\tan^{-1}\left(\frac{h_0}{R_0}\right)-\frac{a^2}{\sqrt{a^2-\rho^2}(a^2-\bar{\delta}^2)^{3/2}}\tan^{-1}\left(\frac{\sqrt{a^2-\bar{\delta}^2}}{\sqrt{l_2^2(\rho_0)-a^2}}\right)\right.$$
$$\left.+\frac{\sqrt{l_2^2(\rho_0)-a^2}}{\sqrt{a^2-\rho^2}}\left(\frac{\bar{\delta}^2}{\left[l_2^2(\rho_0)-\bar{\delta}^2\right](a^2-\bar{\delta}^2)}+\frac{1}{R_0^2}\right)\right\}. \tag{81}$$

Here R_0 is defined by (13) and

$$R_N = \sqrt{\rho^2+r^2-2r\rho\cos(\varphi-\psi)}, \qquad h_0 = \frac{\sqrt{a^2-\rho^2}\sqrt{a^2-l_1^2(\rho_0)}}{a} \tag{82}$$

$$\Phi_4(z) = \Lambda\int_0^{2\pi}\int_0^a \frac{1}{R_N}\tan^{-1}\left(\frac{\eta_r}{R_N}\right)\frac{re^{i\psi}-\rho_0 e^{i\varphi_0}}{R_r^3}r\,dr\,d\psi$$
$$= \frac{2\pi}{\bar{q}}\left[\frac{z\left(3R_0^2-z^2\right)}{\bar{q}R_0^3}\tan^{-1}\left(\frac{h_0}{R_0}\right)+\frac{zh_0}{R_0^2+h_0^2}\left(\frac{\rho e^{i\varphi}}{a^2-\rho^2}+\frac{q}{R_0^2}\right)\right.$$
$$\left.-\frac{1}{\bar{q}}\left(\frac{\sqrt{a^2-\rho^2}}{\sqrt{a^2-\delta^2}}+\frac{\sqrt{a^2-\delta^2}}{\sqrt{a^2-\rho^2}}\right)\tan^{-1}\left(\frac{\sqrt{a^2-\delta^2}}{\sqrt{l_2^2(\rho_0)-a^2}}\right)\right]. \tag{83}$$

We can get the estimation of the total force T, without solving the integral equation (60). The following formula was derived in [**4**]:

$$T = \frac{2}{\pi^2 G_1}\int_0^{2\pi}\int_0^a \frac{u(\rho,\varphi)\rho\,d\rho\,d\varphi}{\sqrt{a^2-\rho^2}}. \tag{84}$$

Application of (84) to (60) yields

$$\frac{2}{\pi^2 G_1}\int_0^{2\pi}\int_0^a \frac{u(\rho,\varphi)\rho\,d\rho\,d\varphi}{\sqrt{a^2-\rho^2}} = T+\frac{G_1-G_2}{\pi G_1}\sum_{n,k}Q_{nk}\left\{\int_0^{2\pi}\int_0^a \sin^{-1}\left(\frac{a}{l_{2nk}}\right)\tau\,dS\right.$$
$$\left.-2\int_0^{2\pi}\int_0^a \frac{e^{2i\varphi}}{\rho^2}\left[\sqrt{l_{2nk}^2-a^2}\left(a-\frac{l_{1nk}^2}{2a}\right)-c_{nk}a\right]\bar{\tau}\,dS\right\}$$
$$+2\frac{G_1+G_2}{\pi G_1}\sum_{m=1}^{\infty}(-1)^m\left\{\int_0^{2\pi}\int_0^a \sin^{-1}\left(\frac{a}{l_{2m3}}\right)\tau\,dS\right.$$
$$\left.+2\int_0^{2\pi}\int_0^a \frac{e^{2i\varphi}}{\rho^2}\left[\sqrt{l_{2m3}^2-a^2}\left(a-\frac{l_{1m3}^2}{2a}\right)-2mc_3 a\right]\bar{\tau}\,dS\right\}. \tag{85}$$

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Here

$$l_{1nk} = \frac{1}{2}\left(\sqrt{(a+\rho)^2 + c_{nk}^2} - \sqrt{(a-\rho)^2 + c_{nk}^2}\right) \tag{86}$$

$$l_{2nk} = \frac{1}{2}\left(\sqrt{(a+\rho)^2 + c_{nk}^2} + \sqrt{(a-\rho)^2 + c_{nk}^2}\right) \tag{87}$$

$$l_{1m3} = \frac{1}{2}\left(\sqrt{(a+\rho)^2 + (2mc_3)^2} - \sqrt{(a-\rho)^2 + (2mc_3)^2}\right) \tag{88}$$

$$l_{2m3} = \frac{1}{2}\left(\sqrt{(a+\rho)^2 + (2mc_3)^2} + \sqrt{(a-\rho)^2 + (2mc_3)^2}\right). \tag{89}$$

Here we use integrals from [**3**].

The left-hand side of (85) represents the resultant force T_∞ for the case of a transversely isotropic half-space. We can use the mean value theorem and get

$$\begin{aligned} T_\infty = T + \frac{G_1 - G_2}{\pi G_1} \sum_{n,k} Q_{nk} \left\{ T \sin^{-1}\left(\frac{a}{l_{2nk}}\right) - 2\bar{T}\frac{e^{2i\varphi}}{\rho^2}\left[\sqrt{l_{2nk}^2 - a^2}\left(a - \frac{l_{1nk}^2}{2a}\right) - c_{nk}a\right]\right\} \\ + 2\frac{G_1 + G_2}{\pi G_1} \sum_{m=1}^{\infty} (-1)^m \left\{ T \sin^{-1}\left(\frac{a}{l_{2m3}}\right) \right. \\ \left. + 2\bar{T}\frac{e^{2i\varphi}}{\rho^2}\left[\sqrt{l_{2m3}^2 - a^2}\left(a - \frac{l_{1m3}^2}{2a}\right) - 2mc_3 a\right]\right\}. \end{aligned} \tag{90}$$

Here we use integrals from [**3**].

We can easily express T from (90). Indeed, equation (90) can be schematically presented as

$$T_\infty = C_1 T + C_2 \bar{T} \tag{91}$$

from which we can obtain

$$T = \frac{C_1 T_\infty - C_2 \bar{T}_\infty}{C_1^2 - C_2 \bar{C}_2}. \tag{92}$$

The torque M_0 can be obtained in a similar fashion. The following formula was obtained in [**4**] for the case of a transversely isotropic half-space

$$M_0 = \mathfrak{F} \int_0^{2\pi} \int_0^a \rho e^{-i\varphi} \tau(\rho, \varphi)\, dS = \frac{4}{\pi^2 (G_1 + G_2)} \mathfrak{F} \int_0^{2\pi} \int_0^a \frac{u(\rho, \varphi) e^{-i\varphi}}{\sqrt{a^2 - \rho^2}} \rho^2 d\rho\, d\varphi, \tag{93}$$

where $\mathfrak{F}$ denotes the imaginary part of the expression to follow. Application of (93) to (60) yields

$$\begin{aligned} \frac{4}{\pi^2 (G_1 + G_2)} \mathfrak{F} \int_0^{2\pi} \int_0^a \frac{u(\rho, \varphi) e^{-i\varphi}}{\sqrt{a^2 - \rho^2}} \rho^2 d\rho\, d\varphi = \mathfrak{F} \int_0^{2\pi} \int_0^a \rho e^{-i\varphi} \tau\, dS \\ + \frac{4}{\pi} \sum_{m=1}^{\infty} (-1)^m \int_0^{2\pi} \int_0^a \rho \left[\sin^{-1}\left(\frac{a}{(l_{2m3})}\right) - \frac{a\sqrt{l_{2m3}^2 - a^2}}{l_{2m3}^2} \right] \mathfrak{F}(e^{-i\varphi}\tau)\, dS. \end{aligned} \tag{94}$$

Here we use integrals from [**3**]. Application of the mean value theorem to (94) gives

$$M_0 = \frac{(M_0)_\infty}{1 + \frac{4}{\pi} \sum_{m=1}^{\infty} (-1)^m \left[\sin^{-1}\left(\frac{a}{l_{2m3}}\right) - \frac{a\sqrt{l_{2m3}^2 - a^2}}{l_{2m3}^2} \right]}. \tag{95}$$

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Here $(M_0)_\infty$ is the torque which produces tangential displacement u when applied to a transversely isotropic half-space:

$$(M_0)_\infty = \frac{4}{\pi^2(G_1+G_2)}\Im\int_0^{2\pi}\int_0^a \frac{u(\rho,\varphi)e^{-i\varphi}}{\sqrt{a^2-\rho^2}}\rho^2\,d\rho\,d\varphi. \tag{96}$$

Formulae (92) and (95) look very effective and elementary, but there is one problem: due to the use of the mean value theorem, we do not know the exact value of ρ, we only know that it is in the interval between zero and a. The numerical study of (95) shows that the result does not vary much with ρ, when the ratio $a/c \leqslant 1$. Ufliand [**10**] has given the results $M_0/(M_0)_\infty$ for the case of isotropy: for a/c equal to 1 and 2 his results are 1.06 and 1.26 respectively. Formula (95) for $\rho = 0.8a$ and $\gamma_3 = 1$ gives 1.06 and 1.28, with maximum deviation not exceeding 2%.

It is important to notice, that in the case of isotropy, each Q_{nk} tends to infinity, while their sum remains finite. We can use the L'Hopital rule to find the necessary isotropic formula. The following limits can be computed

$$\begin{aligned}
&\lim_{\gamma_1\to\gamma_2\to 1}[Q_{20}f(c_{20})+Q_{11}f(c_{11})+Q_{02}f(c_{02})]\\
&\quad = -\frac{f(2c)[1+(3-4\nu)^2]+4cf'(2c)+4c^2f''(2c)}{3-4\nu}\\
&\quad = -(3-4\nu)f(2c)-\frac{4c[f'(2c)+cf''(2c)]+f(2c)}{3-4\nu},
\end{aligned} \tag{97}$$

$$\begin{aligned}
&\lim_{\gamma_1\to\gamma_2\to 1}[Q_{40}f(c_{40})+Q_{31}f(c_{31})+Q_{22}f(c_{22})+Q_{13}f(c_{13})+Q_{04}f(c_{04})]\\
&\quad = 8c^2f''(4c)+4cf'(4c)+(3-4\nu)^2f(4c)\\
&\qquad +\frac{16c^4f''''(4c)+16c^3f'''(4c)+8c^2f''(4c)+4cf'(4c)+f(4c)}{(3-4\nu)^2}.
\end{aligned} \tag{98}$$

Here f is an arbitrary function; $f'(\cdot)$ denotes the first derivative with respect to the argument, $f''(\cdot)$ stands for the second derivative, etc. In the case of

$$f(z) = \sin^{-1}\left(\frac{a}{l_2}\right) \tag{99}$$

the first 2 derivatives can be found in [**3**]:

$$\frac{\partial}{\partial z}\left[\sin^{-1}\left(\frac{a}{l_2}\right)\right] = -\frac{\sqrt{a^2-l_1^2}}{l_2^2-l_1^2} \tag{100}$$

$$\frac{\partial^2}{\partial z^2}\left[\sin^{-1}\left(\frac{a}{l_2}\right)\right] = \frac{z\left[a^2(2a^2+2z^2-\rho^2)-l_1^4\right]}{\sqrt{a^2-l_1^2}\left(l_2^2-l_1^2\right)^3}. \tag{101}$$

In the above formulae (99–101), z should be replaced by $2c$ or $4c$ respectively.

4. *Discussion*

It is of interest to establish a relationship between the Generalized Images method used in this paper and the integral transform method usually applied in this case. We modify the integral transform method by presenting the three potential functions as a sum of the potential functions for a half-space, derived in [**2**] and an integral

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transform term, which is used to satisfy the boundary conditions, namely,

$$F_1 = \frac{H\gamma_1\gamma_2}{m_1 - 1}\left[X(z_1) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[A_1(\xi,\eta)\cosh(\zeta z_1) + B_1(\xi,\eta)\sinh(\zeta z_1)]\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right] \tag{102}$$

$$F_2 = \frac{H\gamma_1\gamma_2}{m_2 - 1}\left[X(z_2) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[A_2(\xi,\eta)\cosh(\zeta z_2) + B_2(\xi,\eta)\sinh(\zeta z_2)]\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right] \tag{103}$$

$$F_3 = \frac{i\gamma_3}{2\pi A_{44}}\left[Y(z_3) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_3(\xi,\eta)\cosh(\zeta z_3)\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right] \tag{104}$$

where X and Y are defined by (11) and (12) and

$$\zeta = \sqrt{\xi^2 + \eta^2}. \tag{105}$$

Satisfying condition $\sigma_z = 0$ for $z = 0$, we arrive at

$$0 = \frac{H\gamma_1\gamma_2(m_1 + 1)A_{44}}{m_1 - 1}\left[X''(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta^2 A_1(\xi,\eta)\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right] + \frac{H\gamma_1\gamma_2(m_2 + 1)A_{44}}{m_2 - 1}\left[X''(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta^2 A_2(\xi,\eta)\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right]$$

which can be simplified as

$$0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta^2[A_1(\xi,\eta) - A_2(\xi,\eta)]\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta. \tag{106}$$

Equation (106) is satisfied if

$$A_1(\xi,\eta) - A_2(\xi,\eta) = 0. \tag{107}$$

The next condition to satisfy is $\tau_z = -\tau$ for $z = 0$. Substitution of (102–104) into (5) gives

$$-\tau = \frac{H\gamma_2(m_1 + 1)A_{44}}{m_1 - 1}\Lambda\left[X'(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta B_1(\xi,\eta)\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right] + \frac{H\gamma_1(m_2 + 1)A_{44}}{m_2 - 1}\Lambda\left[X'(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta B_2(\xi,\eta)\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right] - \frac{1}{2\pi}\Lambda[Y'(0)] \tag{108}$$

which can be simplified as

$$0 = \frac{\gamma_2}{\gamma_1 - \gamma_2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta B_1(\xi,\eta)\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right] - \frac{\gamma_1}{\gamma_1 - \gamma_2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta B_2(\xi,\eta)\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right]. \tag{109}$$

Equation (109) will be satisfied if

$$\gamma_2 B_1(\xi,\eta) - \gamma_1 B_2(\xi,\eta) = 0. \tag{110}$$

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The condition of bondage $u = 0$ for $z = c$ leads to

$$0 = \frac{H\gamma_1\gamma_2}{m_1 - 1}\Lambda\left[X(c_1) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[A_1(\xi,\eta)\cosh(\zeta c_1) + B_1(\xi,\eta)\sinh(\zeta c_1)]\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right]$$
$$+ \frac{H\gamma_1\gamma_2}{m_2 - 1}\Lambda\left[X(c_2) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[A_2(\xi,\eta)\cosh(\zeta c_2) + B_2(\xi,\eta)\sinh(\zeta c_2)]\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right]$$
$$- \frac{\gamma_3}{2\pi A_{44}}\Lambda\left[Y(c_3) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_3(\xi,\eta)\cosh(\zeta c_3)\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right]. \quad (111)$$

which can be re-written as

$$0 = \frac{H\gamma_1\gamma_2}{m_1 - 1}\left[-\frac{1}{2}\int_S\int\frac{\tau\,dS}{R_1} + \frac{1}{2}\int_S\int\frac{q^2\bar{\tau}\,dS}{R_1(R_1 + c_1)^2} + \frac{1}{2\pi}\Lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[A_1(\xi,\eta)\cosh(\lambda_1) + B_1(\xi,\eta)\sinh(\lambda_1)]\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right]$$
$$+ \frac{H\gamma_1\gamma_2}{m_2 - 1}\left[-\frac{1}{2}\int_S\int\frac{\tau\,dS}{R_2} + \frac{1}{2}\int_S\int\frac{q^2\bar{\tau}\,dS}{R_2(R_2 + c_2)^2} + \frac{1}{2\pi}\Lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[A_2(\xi,\eta)\cosh(\lambda_2) + B_2(\xi,\eta)\sinh(\lambda_2)]\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right]$$
$$- \frac{\gamma_3}{2\pi A_{44}}\left[-\frac{1}{2}\int_S\int\frac{\tau\,dS}{R_3} - \frac{1}{2}\int_S\int\frac{q^2\bar{\tau}\,dS}{R_3(R_3 + c_3)^2} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i(\xi + i\eta)A_3(\xi,\eta)\cosh(\lambda_3)\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right]. \quad (112)$$

Here

$$R_k = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\varphi - \varphi_0) + c_k^2}, \qquad \lambda_k = \zeta c_k, \qquad k = 1, 2, 3. \quad (113)$$

The inverse transformation of (112) gives

$$\frac{1}{2}\frac{e^{-\lambda_1}}{\zeta}\tilde{\tau} + \frac{1}{2}\frac{e^{-\lambda_1}(\xi + i\eta)^2}{\zeta^3}\tilde{\bar{\tau}} - m_1\left(\frac{1}{2}\frac{e^{-\lambda_2}}{\zeta}\tilde{\tau} + \frac{1}{2}\frac{e^{-\lambda_2}(\xi + i\eta)^2}{\zeta^3}\tilde{\bar{\tau}}\right)$$
$$= i(\xi + i\eta)\{-[A_1(\xi,\eta)\cosh(\lambda_1) + B_1(\xi,\eta)\sinh(\lambda_1)] + m_1[A_2(\xi,\eta)\cosh(\lambda_2) + B_2(\xi,\eta)\sinh(\lambda_2)]\} \quad (114)$$

and

$$0 = -\frac{\gamma_3}{2\pi A_{44}}\left[-\frac{1}{2}\frac{e^{-\lambda_3}}{\zeta}\tilde{\tau} + \frac{1}{2}\frac{e^{-\lambda_3}(\xi + i\eta)^2}{\zeta^3}\tilde{\bar{\tau}} - i(\xi + i\eta)A_3(\xi,\eta)\cosh(\lambda_3)\right]. \quad (115)$$

Here the property $m_1 m_2 = 1$ and the integrals from the Appendix are used. The following notation was introduced

$$\tilde{\tau}(\xi,\eta) = \int_S\int\tau(x_0,y_0)\exp[i(x_0\xi + y_0\eta)]\,dx_0\,dy_0 \quad (116)$$

$$\tilde{\bar{\tau}}(\xi,\eta) = \int_S\int\bar{\tau}(x_0,y_0)\exp[i(x_0\xi + y_0\eta)]\,dx_0\,dy_0. \quad (117)$$

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The last condition to satisfy is $w = 0$ for $z = c$. Proper substitution of (102–104) in the second formula (1) yields

$$0 = \frac{Hm_1\gamma_2}{m_1 - 1}\left[X'(c_1) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta[A_1(\xi,\eta)\sinh(\zeta c_1) + B_1(\xi,\eta)\cosh(\zeta c_1)]\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right] + \frac{H\gamma_1 m_2}{m_2 - 1}\left[X'(c_2) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta[A_2(\xi,\eta)\sinh(\zeta c_2) + B_2(\xi,\eta)\cosh(\zeta c_2)]\exp[-i(x\xi + y\eta)]\,d\xi\,d\eta\right]. \tag{118}$$

The inverse transformation of (118) and simplification gives us

$$0 = m_1\gamma_2\left[\frac{e^{-\lambda_1}}{2\zeta^2}[i(\xi - i\eta)\tilde{\tau} + i(\xi + i\eta)\tilde{\tilde{\tau}}] + \zeta[A_1(\xi,\eta)\sinh(\zeta c_1) + B_1(\xi,\eta)\cosh(\zeta c_1)]\right] - \gamma_1\left[\frac{e^{-\lambda_2}}{2\zeta^2}[i(\xi - i\eta)\tilde{\tau} + i(\xi + i\eta)\tilde{\tilde{\tau}}] + \zeta[A_2(\xi,\eta)\sinh(\lambda_2) + B_2(\xi,\eta)\cosh(\lambda_2)]\right]. \tag{119}$$

Again, we use the integrals from the Appendix. Now we have a set of four linear algebraic equations to solve, namely, (107), (110), (114) and (119). We rewrite them below as

$$A_1(\xi,\eta) - A_2(\xi,\eta) = 0 \tag{120}$$

$$\gamma_2 B_1(\xi,\eta) - \gamma_1 B_2(\xi,\eta) = 0 \tag{121}$$

$$[A_1(\xi,\eta)\cosh\lambda_1 + B_1(\xi,\eta)\sinh\lambda_1] - m_1[A_2(\xi,\eta)\cosh\lambda_2 + B_2(\xi,\eta)\sinh\lambda_2] = i(e^{-\lambda_1} - m_1 e^{-\lambda_2})\frac{(\xi - i\eta)\tilde{\tau} + (\xi + i\eta)\tilde{\tilde{\tau}}}{2\zeta^3} \tag{122}$$

$$m_1\gamma_2[A_1(\xi,\eta)\sinh\lambda_1 + B_1(\xi,\eta)\cosh\lambda_1] - \gamma_1[A_2(\xi,\eta)\sinh\lambda_2 + B_2(\xi,\eta)\cosh\lambda_2] = i(\gamma_1 e^{-\lambda_2} - m_1\gamma_2 e^{-\lambda_1})\frac{(\xi - i\eta)\tilde{\tau} + (\xi + i\eta)\tilde{\tilde{\tau}}}{2\zeta^3}. \tag{123}$$

The solution of the set (120–123) is

$$A_1(\xi,\eta) = A_2(\xi,\eta) = \frac{i[(\xi - i\eta)\tilde{\tau} + (\xi + i\eta)\tilde{\tilde{\tau}}]\left[4\gamma_1\gamma_2 e_1 e_2 + \gamma_- m_- e_1^2 e_2^2 - m_+\left(\gamma_1 e_2^2 + \gamma_2 e_1^2\right)\right]}{\zeta^3\left[8\gamma_1\gamma_2 e_1 e_2 + \left(e_1^2 e_2^2 + 1\right)\gamma_- m_- - \left(e_1^2 + e_2^2\right)\gamma_+ m_+\right]} \tag{124}$$

$$B_1(\xi,\eta) = \frac{i\gamma_1\left[m_+\left(e_1^2 + e_2^2\right) - 2\gamma_+ e_1 e_2\right][(\xi - i\eta)\tilde{\tau} + (\xi + i\eta)\tilde{\tilde{\tau}}]}{\left[8\gamma_1\gamma_2 e_1 e_2 + \left(e_1^2 e_2^2 + 1\right)\gamma_- m_- - \left(e_1^2 + e_2^2\right)\gamma_+ m_+\right]\zeta^3}. \tag{125}$$

$$B_2(\xi,\eta) = \frac{i\gamma_2\left[m_+\left(e_1^2 + e_2^2\right) - 2\gamma_+ e_1 e_2\right][(\xi - i\eta)\tilde{\tau} + (\xi + i\eta)\tilde{\tilde{\tau}}]}{\left[8\gamma_1\gamma_2 e_1 e_2 + \left(e_1^2 e_2^2 + 1\right)\gamma_- m_- - \left(e_1^2 + e_2^2\right)\gamma_+ m_+\right]\zeta^3}. \tag{126}$$

We have used the notation:

$$e_1 = e^{-\lambda_1}, \qquad e_2 = e^{-\lambda_2}, \qquad \gamma_+ = \gamma_1 + \gamma_2, \qquad \gamma_- = \gamma_1 - \gamma_2 \tag{127}$$

$$m_+ = \gamma_2 m_1 + \gamma_1 m_2, \qquad m_- = \gamma_2 m_1 - \gamma_1 m_2. \tag{128}$$

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From (115) we get

$$A_3(\xi,\eta)=\frac{i[(\xi-i\eta)\tilde{\tau}-(\xi+i\eta)\tilde{\bar{\tau}}]e^{-\lambda_3}}{2\zeta^3\cosh(\lambda_3)}=\frac{i[(\xi-i\eta)\tilde{\tau}-(\xi+i\eta)\tilde{\bar{\tau}}]e^{-2\lambda_3}}{\zeta^3(1+e^{-2\lambda_3})}. \tag{129}$$

The governing integral equation of the tangential contact problem can be obtained from the condition $u=u(\rho,\varphi)$ for $z=0,(\rho,\varphi)\subseteq S$. Substitution of (102–104) in the first formula (1) leads to

$$\begin{aligned}u=&\frac{H\gamma_1\gamma_2}{m_1-1}\Lambda\left[X(0)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_1(\xi,\eta)\exp[-i(x\xi+y\eta)]\,d\xi\,d\eta\right]\\&-\frac{H\gamma_1\gamma_2 m_1}{m_1-1}\Lambda\left[X(0)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_1(\xi,\eta)\exp[-i(x\xi+y\eta)]\,d\xi\,d\eta\right]\\&-\frac{\gamma_3}{2\pi A_{44}}\Lambda\left[Y(0)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_3(\xi,\eta)\exp[-i(x\xi+y\eta)]\,d\xi\,d\eta\right]\end{aligned} \tag{130}$$

which can be simplified as

$$\begin{aligned}u=&\frac{G_1}{2}\int_S\int\frac{\tau\,dS}{R}+\frac{G_2}{2}\int_S\int\frac{q^2\bar{\tau}\,dS}{R^3}\\&+\frac{i(G_1-G_2)}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\xi+i\eta)A_1(\xi,\eta)\exp[-i(x\xi+y\eta)]\,d\xi\,d\eta\\&+\frac{i(G_1+G_2)}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\xi+i\eta)A_3(\xi,\eta)\exp[-i(x\xi+y\eta)]\,d\xi\,d\eta.\end{aligned} \tag{131}$$

Now we can use the general property

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[-i(\xi+i\eta)]\tilde{\tau}(\xi,\eta)\Psi(\zeta)\exp[-i(x\xi+y\eta)]\,d\xi\,d\eta\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[-i(\xi+i\eta)]\Psi(\zeta)\exp[-i(x\xi+y\eta)]\,d\xi\,d\eta\int_S\int\tau(N)\exp[i(x_0\xi+y_0\eta)]\,dS_N\\&=2\pi\Lambda\int_S\int\tau(x_0,y_0)\,dx_0\,dy_0\int_0^{\infty}\Psi(\zeta)J_0(\zeta\sqrt{(x-x_0)^2+(y-y_0)^2})\zeta\,d\zeta\\&=-2\pi\int_S\int\frac{q}{R}\tau(\rho_0,\varphi_0)\rho_0\,d\rho_0\,d\varphi_0\int_0^{\infty}\Psi(\zeta)J_1(\zeta R)\zeta^2\,d\zeta.\end{aligned} \tag{132}$$

We also note that

$$\Lambda J_0(\zeta R)=-\zeta\frac{q}{R}J_1(\zeta R),\qquad \Lambda^2J_0(\zeta R)=\zeta^2\frac{q^2}{R^2}J_2(\zeta R). \tag{133}$$

Utilization of (132–133) in (131) yields

$$\begin{aligned}u=&\frac{G_1}{2}\int_S\int\frac{\tau\,dS}{R}+\frac{G_2}{2}\int_S\int\frac{q^2\bar{\tau}\,dS}{R^3}\\&-\frac{G_1-G_2}{2c}\int_S\int\left[K_1\left(\frac{R}{c}\right)\tau(\rho_0,\varphi_0)-\frac{q^2}{R^2}K_2\left(\frac{R}{c}\right)\bar{\tau}(\rho_0,\varphi_0)\right]\rho_0\,d\rho_0\,d\varphi_0\\&-\frac{G_1+G_2}{2c}\int_S\int\left[K_3\left(\frac{R}{c}\right)\tau(\rho_0,\varphi_0)+\frac{q^2}{R^2}K_4\left(\frac{R}{c}\right)\bar{\tau}(\rho_0,\varphi_0)\right]\rho_0\,d\rho_0\,d\varphi_0.\end{aligned} \tag{134}$$

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Here

$$K_1\left(\frac{R}{c}\right) = \int_0^\infty \Theta(\lambda) J_0\left(\lambda\frac{R}{c}\right) d\lambda \tag{135}$$

$$K_2\left(\frac{R}{c}\right) = \int_0^\infty \Theta(\lambda) J_2\left(\lambda\frac{R}{c}\right) d\lambda \tag{136}$$

$$K_3\left(\frac{R}{c}\right) = \int_0^\infty \frac{e^{-2\lambda_3}}{1+e^{-2\lambda_3}} J_0\left(\lambda\frac{R}{c}\right) d\lambda \tag{137}$$

$$K_4\left(\frac{R}{c}\right) = \int_0^\infty \frac{e^{-2\lambda_3}}{1+e^{-2\lambda_3}} J_2\left(\lambda\frac{R}{c}\right) d\lambda \tag{138}$$

$$\Theta(\lambda) = \frac{4\gamma_1\gamma_2 e^{-\lambda_1-\lambda_2} + \gamma_- m_- e^{-2\lambda_1-2\lambda_2} - m_+(\gamma_1 e^{-2\lambda_2} + \gamma_2 e^{-2\lambda_1})}{8\gamma_1\gamma_2 e^{-\lambda_1-\lambda_2} + \gamma_- m_-(e^{-2\lambda_1-2\lambda_2}+1) - \gamma_+ m_+(e^{-2\lambda_1}+e^{-2\lambda_2})}. \tag{139}$$

One can easily show that the governing integral equation (60) is equivalent to (134). Indeed, the expression for Θ(139) can be formally expanded as

$$\Theta(\lambda) = \frac{1}{\gamma_- m_-}\{4\gamma_1\gamma_2 e^{-\lambda_1-\lambda_2} + \gamma_- m_- e^{-2\lambda_1-2\lambda_2} - m_+(\gamma_1 e^{-2\lambda_2} + \gamma_2 e^{-2\lambda_1})\} \times \sum_{j=0}^{\infty}\left(\frac{\gamma_+ m_+}{\gamma_- m_-}(e^{-2\lambda_1}+e^{-2\lambda_2}) - \frac{8\gamma_1\gamma_2}{\gamma_- m_-} e^{-\lambda_1-\lambda_2} - e^{-2\lambda_1-2\lambda_2}\right)^j. \tag{140}$$

Now grouping together the coefficients at similar exponents, we get

$$\Theta(\lambda) = -\frac{1}{2}\sum_{n,k} Q_{nk}\exp(-n\lambda_1 - k\lambda_2) = -\frac{1}{2}\sum_{n,k} Q_{nk}\exp(-\lambda_{nk}). \tag{141}$$

By using the integrals from Gradshteyn and Ryzhik ([7], formula 6.611.1, page 728)

$$\int_0^\infty e^{-\lambda_{nk}} J_0\left(\lambda\frac{R}{c}\right) d\lambda = \frac{c}{\sqrt{c_{nk}^2 + R^2}} \tag{142}$$

$$\int_0^\infty e^{-\lambda_{nk}} J_2\left(\lambda\frac{R}{c}\right) d\lambda = \frac{cR^2}{\left(\sqrt{c_{nk}^2+R^2}+c_{nk}\right)^2\sqrt{c_{nk}^2+R^2}} \tag{143}$$

we can confirm the identity of the relevant parts of (60) and (139), as expected. Similarly, one can use the expansion

$$\frac{e^{-2\lambda_3}}{1+e^{-2\lambda_3}} = e^{-2\lambda_3}\sum_{j=0}^{\infty}(-e^{-2\lambda_3})^j \tag{144}$$

and the integrals (142–143) to prove the identity of the remaining parts of equations (60) and (139).

The analogy between the coefficients in (14–16) and the parameters in (102–104) can be proven by using formal expansion of $A_1(\xi,\eta) - B_1(\xi,\eta)$ to get $-A_{nk}^{(1)}$; while an expansion of $A_1(\xi,\eta) + B_1(\xi,\eta)$; gives $-B_{nk}^{(1)}$; a similar expansion of $A_2(\xi,\eta) - B_2(\xi,\eta)$ yields $-A_{nk}^{(2)}$; and finally $A_2(\xi,\eta) + B_2(\xi,\eta)$ gives $-B_{nk}^{(2)}$. This way we can prove the following general formulae of summation:

$$\sum_{n,k} A_{nk}^{(1)} x^n y^k = -\frac{2\gamma_1(\gamma_1+3\gamma_2)xy + \gamma_- m_- x^2y^2 - m_+(2\gamma_1 y^2 + \gamma_+ x^2)}{8\gamma_1\gamma_2 xy + \gamma_- m_-(x^2y^2+1) - \gamma_+ m_+(x^2+y^2)} \tag{145}$$

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$$\sum_{n,k} B_{nk}^{(1)} x^n y^k = -\frac{\gamma_-(m_- x^2y^2 + m_+ x^2 - 2\gamma_1 xy)}{8\gamma_1\gamma_2 xy + \gamma_- m_-(x^2y^2+1) - \gamma_+ m_+(x^2+y^2)} \tag{146}$$

$$\sum_{n,k} A_{nk}^{(2)} x^n y^k = -\frac{2\gamma_2(3\gamma_1+\gamma_2)xy + \gamma_- m_- x^2y^2 - m_+(\gamma_+ y^2 + 2\gamma_2 x^2)}{8\gamma_1\gamma_2 xy + \gamma_- m_-(x^2y^2+1) - \gamma_+ m_+(x^2+y^2)} \tag{147}$$

$$\sum_{n,k} B_{nk}^{(2)} x^n y^k = -\frac{\gamma_-(m_- x^2y^2 - m_+ y^2 + 2\gamma_2 xy)}{8\gamma_1\gamma_2 xy + \gamma_- m_-(x^2y^2+1) - \gamma_+ m_+(x^2+y^2)} \tag{148}$$

$$\sum_{n,k} Q_{nk} x^n y^k = -\frac{4\gamma_1\gamma_2 + \gamma_- m_- x^2y^2 - m_+(\gamma_1 y^2 + \gamma_2 x^2)}{8\gamma_1\gamma_2 xy + \gamma_- m_-(x^2y^2+1) - \gamma_+ m_+(x^2+y^2)}. \tag{149}$$

In the case of isotropy, expression (139) simplifies as

$$\Theta(\lambda) = -e^{-2\lambda}\frac{(1-2\lambda)^2 + (3-4\nu)^2 + 2(3-4\nu)e^{-2\lambda}}{(3-4\nu)(1+e^{-4\lambda}) + [4\lambda^2 + 1 + (3-4\nu)^2]e^{-2\lambda}} \tag{150}$$

which is in agreement with the result of Ufliand [**10**].

Expression (150) can be formally expanded in a series

$$\Theta(\lambda) = -\frac{e^{-2\lambda}}{3-4\nu}[(1-2\lambda)^2 + (3-4\nu)^2 + 2(3-4\nu)e^{-2\lambda}] \times \sum_{n=0}^{\infty}\left(-\frac{4\lambda^2 + 1 + (3-4\nu)^2}{3-4\nu}e^{-2\lambda} - e^{-4\lambda}\right)^n. \tag{151}$$

This has not been investigated rigorously, but it looks like each exponent term in (151) has a sign opposite to the previous, and this means that each truncation gives either the upper or the lower bound for Θ. By using the integral

$$\int_0^\infty \lambda^n \exp(-\lambda z) J_0(\lambda\rho)\, d\lambda = \left(-\frac{d}{dz}\right)^n \frac{1}{\sqrt{\rho^2+z^2}} \tag{152}$$

the kernels in (134) can be computed as a sum of inverse distances and their derivatives.

The method developed in this paper can be applied to solution of more complicated problems of an elastic layer bonded to a dissimilar elastic foundation. We can also try to use reflections in the Ox and Oy directions to solve elastic problems in quarter-space and smaller partitions of the space. These results will be published separately.

Appendix

The integrals needed for the inverse two-dimensional Fourier transformation are presented here.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp[i(x\xi + y\eta)]\, dx\, dy}{[(x-x_0)^2 + (y-y_0)^2 + z^2]^{1/2}} = 2\pi\frac{\exp[i(x_0\xi + y_0\eta) - z\zeta]}{\zeta} \tag{153}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z\exp[i(x\xi + y\eta)]\, dx\, dy}{[(x-x_0)^2 + (y-y_0)^2 + z^2]^{3/2}} = 2\pi\exp[i(x_0\xi + y_0\eta) - z\zeta] \tag{154}$$

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$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[x-x_0-i(y-y_0)]\exp[i(x\xi+y\eta)]\,dx\,dy}{[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}}$$
$$=2\pi\frac{i(\xi-i\eta)\exp[i(x_0\xi+y_0\eta)-z\zeta]}{\zeta} \tag{155}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[x-x_0+i(y-y_0)]\exp[i(x\xi+y\eta)]}{\{[(x-x_0)^2+(y-y_0)^2+z^2]^{1/2}+z\}}\,dx\,dy$$
$$=\frac{2\pi i(\xi+i\eta)\exp[i(x_0\xi+y_0\eta)-z\zeta]}{\zeta^3} \tag{156}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[x-x_0+i(y-y_0)]^2\exp[i(x\xi+y\eta)]}{[(x-x_0)^2+(y-y_0)^2+z^2]^{1/2}\{[(x-x_0)^2+(y-y_0)^2+z^2]^{1/2}+z\}^2}\,dx\,dy$$
$$=-\frac{2\pi(\xi+i\eta)^2\exp[i(x_0\xi+y_0\eta)-z\zeta]}{\zeta^3} \tag{157}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[x-x_0-i(y-y_0)]\exp[i(x\zeta+y\eta)]}{[(x-x_0)^2+(y-y_0)^2+z^2]^{1/2}\{[(x-x_0)^2+(y-y_0)^2+z^2]^{1/2}+z\}}\,dx\,dy$$
$$=\frac{2\pi i(\xi+i\eta)\exp[i(x_0\xi+y_0\eta)-z\zeta]}{\zeta^2} \tag{158}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[x-x_0+i(y-y_0)]\exp[i(x\zeta+y\eta)]}{[(x-x_0)^2+(y-y_0)^2+z^2]^{1/2}\{[(x-x_0)^2+(y-y_0)^2+z^2]^{1/2}+z\}}\,dx\,dy$$
$$=\frac{2\pi i(\xi-i\eta)\exp[i(x_0\xi+y_0\eta)-z\zeta]}{\zeta^2} \tag{159}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\{2[(x-x_0)^2+(y-y_0)^2+z^2]^{1/2}+z\}[(x-x_0)+i(y-y_0)]^2\exp[i(x\xi+y\eta)]}{[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}\{[(x-x_0)^2+(y-y_0)^2+z^2]^{1/2}+z\}^2}\,dx\,dy$$
$$=-\frac{2\pi(\xi+i\eta)^2\exp[i(x_0\xi+y_0\eta)-z\zeta]}{\zeta^2}. \tag{160}$$

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