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On Vertex-Triads in 3-Connected Binary Matroids

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A cocircuit C^* in a matroid M is said to be *non-separating* if and only if $M\backslash C^*$, the deletion of C^* from M, is connected. A *vertex-triad* in a matroid is a three-element non-separating cocircuit. Non-separating cocircuits in binary matroids correspond to vertices in graphs. Let C be a circuit of a 3-connected binary matroid M such that $|E(M)| \geqslant 4$ and, for all elements x of C, the deletion of x from M is not 3-connected. We prove that C meets at least two vertex-triads of M. This gives direct binary matroid generalizations of certain graph results of Halin, Lemos, and Mader. For binary matroids, it also generalizes a result of Oxley. We also prove that a minimally 3-connected binary matroid M which has at least four elements has at least $\frac{1}{2}r^*(M)+1$ vertex-triads, where $r^*(M)$ is the corank of the matroid M. An immediate consequence of this result is the following result of Halin: a minimally 3-connected graph with n vertices has at least $\frac{2n+6}{5}$ vertices of degree three. We also generalize Tutte's Triangle Lemma for general matroids.

1. Introduction

The 3-connected graphs and the existence of vertices of degree three in certain graphs are extensively studied in the literature. There is much interest in extending graph results to matroids, especially to binary matroids. This includes Tutte's pioneering work on matroids (see, for example, [17] and [18], as well as Robertson and Seymour's project in extending graph results to binary matroids. Matroid results are much more general than graph results and often lead to new consequences for graphs. In this paper, we give direct generalizations of some results of Halin and Mader for 3-connected graphs to binary matroids. Familiarity with matroids is assumed. The graph and matroid terminology will generally follow Bondy and Murty [3] and Oxley [12], respectively. For example, we use $M\backslash x, y$ and $M/x, y$ to denote, respectively, the deletion and contraction of $\{x, y\}$ from a matroid M. Mader [8] proved the following theorem.

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Theorem 1.1. *Let C be a cycle of a simple 3-connected graph G such that, for all edges x of C, the deletion $G\backslash x$ is not 3-connected. Then C meets a vertex of degree three.* □

Theorem 1.1 is a very useful result and has been extended by Lemos [6] and Mader [9], respectively. Halin [4] proved the following result.

Theorem 1.2. *Let G be a minimally 3-connected graph. Then G has at least $\frac{2|V(G)|+6}{5}$ vertices of degree three.* □

A matroid is *connected* if and only if any two elements are in a circuit. A cocircuit C^* in a matroid M is said to be *non-separating* if $M\backslash C^*$ is connected. Clearly, if M is a 3-connected graphic matroid, then a cocircuit C^* is non-separating if and only if C^* is the set of all edges incident to some vertex of the graph. Therefore, non-separating cocircuits in matroids are natural generalizations of vertices in graphs. Indeed, Kelmans [5] defines a *vertex* in a matroid as a non-separating cocircuit. A *vertex-triad* in a matroid is a non-separating three-element cocircuit. Clearly, in a 3-connected graphic matroid, vertex-triads are in one-to-one correspondence to vertices of degree three in the graph. Therefore, vertex-triads in matroids are direct generalizations of vertices of degree three in 3-connected graphs. Note, however, that a triad in a graphic matroid may not correspond to a vertex of degree three.

The following are the main results of this paper. The first two theorems give direct binary matroid generalizations of Theorem 1.1 as well as the corresponding results of Halin [4, Satz 5], Lemos [6, Theorem 2], Mader [9, Proposition 1], and Oxley [10, Theorem 2.5], respectively.

Theorem 1.3. *Let C be a circuit of a 3-connected binary matroid M such that $|E(M)| \geqslant 4$, and suppose that e is an element of C. If, for all elements $x \neq e$ of C, the matroid $M\backslash x$ is not 3-connected, then C meets a vertex-triad of M.*

Theorem 1.4. *Let C be a circuit of a 3-connected binary matroid M such that $|E(M)| \geqslant 4$. If, for all elements x of C, the matroid $M\backslash x$ is not 3-connected, then C meets at least two vertex-triads of M.*

Corollary 1.5. *Let C be a circuit of a minimally 3-connected binary matroid M such that $|E(M)| \geqslant 4$. Then C meets at least two vertex-triads of M.*

From Corollary 1.5, we will get the following analogue of Theorem 1.2. It will be shown in Section 4 that Theorem 1.2 is an immediate consequence of this result. In the binary matroid case, it also generalizes a theorem of Oxley [11]. We will use $r^*(M)$ to denote the corank of M.

Theorem 1.6. *Let M be a minimally 3-connected binary matroid with at least four elements. Then M has at least $\frac{1}{2}r^*(M) + 1$ vertex-triads.*

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Note that, if $n \geqslant 3$, then the uniform matroid $U_{n,n+2}$ is minimally 3-connected. However, it has no vertex-triads at all. Therefore, the main theorems above are best-possible and cannot be generalized to general matroids. However, we will prove several results stronger than Theorems 1.3, 1.4, and 1.6 in Section 3. Section 2 contains some definitions and preliminary results. Section 3 contains the proofs of the main theorems. In Section 4 we will derive some consequences including generalizations of Tutte's Triangle Lemma and the Fan Theorem of Oxley and Wu.

2. Preliminaries

We will present some definitions and preliminary results in this section. A matroid M is *binary* if and only if M has no $U_{2,4}$-minor. In a matroid, a *triangle* is a three-element circuit; a *triad* is a three-element cocircuit. A *loop* is a single-element circuit; a *coloop* is a single-element cocircuit. A circuit and a cocircuit in a matroid cannot have exactly one common element. We will refer to this useful property as *orthogonality*. Let M be a matroid and k be a positive integer. A k-separation of M is a partition $\{X, Y\}$ of $E(M)$ such that

$$\min\{|X|, |Y|\} \geqslant k \tag{2.1}$$

and

$$r(X) + r(Y) - r(M) \leqslant k - 1. \tag{2.2}$$

For all $n \geqslant 2$, M is n-connected if, for all k in $\{1, 2, \ldots, n-1\}$, M has no k-separation. Hence M is 2-connected if and only if M is connected.

Let M be a 3-connected matroid. An element e is *essential* if neither the contraction M/e nor the deletion $M\backslash e$ is 3-connected; e is *deletable* if $M\backslash e$ is 3-connected; e is *contractible* if M/e is 3-connected. The following fact about 3-connected matroids will be used frequently: *a 3-connected matroid with at least four elements cannot have circuits or cocircuits of size less than three*. From this fact, it is easy to see that, if an element e is in both a triangle and a triad of a 3-connected matroid that has more than four elements, then e is essential. To *simplify* a matroid means to delete all loops and all but one element from each parallel class. To *cosimplify* is to contract all coloops and all but one element from each series class. We use $\widetilde{M}$ and $\underset{\sim}{M}$ to denote, respectively, the simplified and cosimplified matroids obtained from M.

Suppose that $r \geqslant 2$. The *wheel* $\mathscr{W}_r$ *of rank* r is a graph having $r+1$ vertices, r of which lie on a cycle (the *rim*); the remaining vertex is joined by a single edge (a *spoke*) to each of the other vertices. The *rank-r whirl* $\mathscr{W}^r$ is a matroid on $E(\mathscr{W}_r)$ having as its circuits all cycles of $\mathscr{W}_r$ other than the rim as well as all sets of edges formed by adding a single spoke to the edges of the rim. The terms 'rim' and 'spoke' will be applied in the obvious way in both $M(\mathscr{W}_r)$ and $\mathscr{W}^r$. Moreover, we shall usually refer to the cycle matroid of a wheel as just a wheel.

The first lemma, due to Tutte [18], is often referred to as Tutte's Triangle Lemma.

Lemma 2.1. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M with at least four*

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elements. If neither $M\backslash a$ nor $M\backslash b$ is 3-connected, then the element a is in a triad with exactly one of b and c. □

Lemma 2.2 (Bixby [1]). *Let M be a 3-connected matroid and e be an element of M. Then either M/e or $M\backslash e$ is 3-connected.* □

The next two lemmas are well known and can be found in, for example, Oxley [12].

Lemma 2.3. *Let M be a matroid and let e be a loop or a coloop of M. Then $M\backslash e = M/e$.* □

Lemma 2.4. *If a triangle is also a triad in a 3-connected matroid M, then $M \cong U_{2,4}$.* □

Lemma 2.5 (Oxley [10]). *Let M be a 3-connected matroid having at least four elements. Suppose that x and y are two elements of M. If $M\backslash x$ is not 3-connected and $M\backslash x/y$ is 3-connected, then $\{x, y\}$ is contained in a triad T^* of M.* □

Lemma 2.6. *Let $T^* = \{x, y, z\}$ be a triad of a 3-connected matroid. If $M\backslash x/y$ is 3-connected, then the triad T^* is a vertex-triad.*

Proof. As T^* is a triad, y is a coloop of the matroid $M\backslash x, z$. By Lemma 2.3, $M\backslash T^* = M\backslash x, y, z = M\backslash x, z/y = M\backslash x/y\backslash z$. As $M\backslash x/y$ is 3-connected, we conclude that $M\backslash T^* = (M\backslash x/y)\backslash z$ is connected. Thus T^* is a vertex-triad. □

The next result can be found in [12].

Lemma 2.7. *Let M be a matroid. Suppose that e is an element of M such that M/e is connected but M is not. Then e is a loop or a coloop of M.* □

Lemma 2.8. *Let M be a 3-connected matroid with at least four elements. Suppose that T^* is a triad and e is an element of M not in T^*. If $M\backslash T^*/e$ is connected, but $M\backslash T^*$ is not connected, then $M^*|(T^* \cup e) \cong U_{2,4}$.*

Proof. By Lemma 2.7, e is a loop or a coloop of $M\backslash T^*$. As M is 3-connected having at least four elements, e cannot be a loop of $M\backslash T^*$. Thus e is a coloop of $M\backslash T^*$. Therefore, there is a subset $T_1 \subseteq T^*$, such that $T_1 \cup e$ is a cocircuit of M. Since T^* is a cocircuit of M and M is 3-connected, we deduce that $T_1 \cup e$ has exactly three elements. Thus $T_1 \cup e$ is also a triad of M. We conclude that $M^*|(T^* \cup e) \cong U_{2,4}$. □

Lemma 2.9 ([13], Lemma 2.6). *Let M be a 3-connected matroid. If $X \subseteq E(M)$ and $M|X \cong U_{2,4}$, then no element of X is essential in M.* □

Let $T_1, T_2, \ldots, T_k$ be a non-empty sequence of sets, each of which is a triangle or a triad of a matroid M such that, for all i in $\{1, 2, \ldots, k-1\}$,

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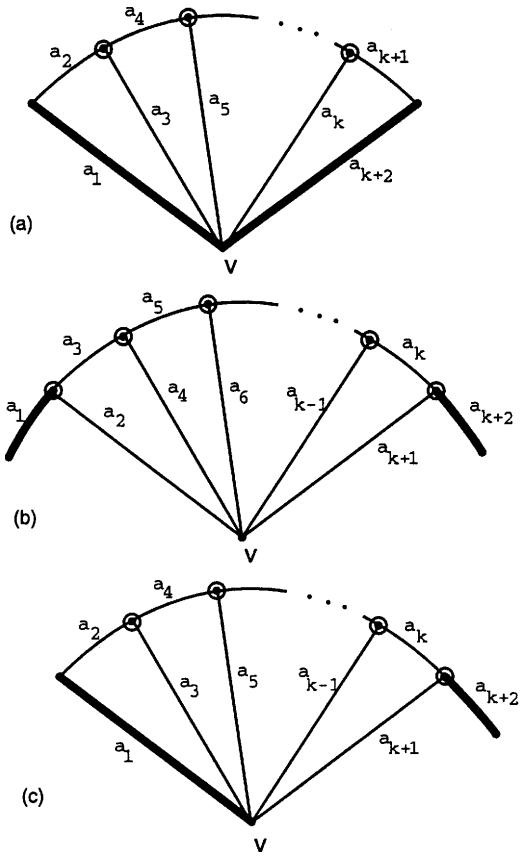


Figure 1

(i) exactly one of T_i and T_{i+1} is a triangle,
(ii) $|T_i \cap T_{i+1}| = 2$, and
(iii) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \ldots \cup T_i)$ is empty.

Then we call $T_1, T_2, \ldots, T_k$ a *chain* of M of *length* k with *links* $T_1, T_2, \ldots, T_k$. Evidently $T_1, T_2, \ldots, T_k$ is a chain of M if and only if it is a chain of M^*.

The next result [13, Lemma 1.8], shows that a maximal chain in a 3-connected matroid other than a wheel or a whirl has non-essential elements at both ends.

Lemma 2.10. *Let M be a 3-connected matroid with at least four elements and suppose that M is neither a wheel nor a whirl. Let $T_1, T_2, \ldots, T_k$ be a maximal chain in M. Then the elements of $T_1 \cup T_2 \cup \ldots \cup T_k$ can be labelled so that neither a_1 nor a_{k+2} is essential where $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all i.* □

Although chains can certainly occur in both non-graphic and graphic matroids, we follow Tutte [18] in keeping track of the triangles and triads in a chain by using graphs as in Figure 1. In each case, the chain is $T_1, T_2, \ldots, T_k$ where $T_i = \{a_i, a_{i+1}, a_{i+2}\}$. In (a), k

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is odd and T_1 is a triangle; hence T_k is also a triangle. In (b), k is odd and T_1 is a triad; hence T_k is also a triad. In (c), k is even, T_1 is a triangle and T_k is a triad. The remaining case, when k is even, T_1 is a triad, and T_k is a triangle, is, up to relabelling, the same as (c). In each of (a)–(c), every triangle in the graph is a triangle in the chain, while the triads in the chain correspond to circled vertices.

Now suppose that $T_1, T_2, \ldots, T_k$ is a maximal chain of a 3-connected matroid M where M is neither a wheel nor a whirl. We call this maximal chain a *fan* of M with *links* $T_1, T_2, \ldots, T_k$. Let $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all i. Then $\{a_1, a_2, \ldots, a_{k+2}\}$ is the *ground set* of the fan, and $a_1, a_2, \ldots, a_{k+2}$ are the *elements* of the fan. For $k \geqslant 2$, there are exactly two non-essential elements in $T_1 \cup T_2 \cup \ldots \cup T_k$, namely a_1 and a_{k+2}, for each of $a_2, a_3, \ldots, a_{k+1}$ is in both a triangle and a triad. We call a_1 and a_{k+2} the *ends* of the fan. When $k = 1$, the fan has T_1 as its ground set and contains either two or three non-essential elements of M. In the first case, the non-essential elements in T_1 are the ends of the fan; in the second case, we arbitrarily choose two of the elements of T_1 to be the ends of the fan. Figure 1(a), (b) and (c) show the three types of chains. Maximal chains of these three types will be called *type*-1, *type*-2, and *type*-3 *fans*, respectively. Two fans are *equal* if they have the same sets of links.

Let M be a 3-connected matroid other than a wheel or a whirl and suppose that M has at least four elements. Suppose that $\mathscr{F}$ is a fan of M. If $\mathscr{F}$ is a type-1 fan, then both ends of the fan are deletable but not contractible. If $\mathscr{F}$ is type-2, then both ends are contractible but not deletable. If $\mathscr{F}$ is type-3, then one end is deletable but not contractible, and the other end is contractible but not deletable.

The following result, called the Fan Theorem, was proved by Oxley and Wu [13, Theorem 1.9].

Lemma 2.11. *Let M be a 3-connected matroid that is neither a wheel nor a whirl. Suppose that e is an essential element of M. Then e is in a fan, both ends of which are non-essential. Moreover, this fan is unique unless exactly one of the following conditions holds.*

(a) *Every fan containing e consists of a single triangle and any two such triangles meet in $\{e\}$.*
(b) *Every fan containing e consists of a single triad and any two such triads meet in $\{e\}$.*
(c) *e is in exactly three fans of type 1 or type 2. These three fans are of the same type, each has five elements, and together they contain a total of six elements. Depending on whether these fans are of type 1 or type 2, the restriction or contraction, respectively, of M to this set of six elements is isomorphic to $M(K_4)$.* □

Lemma 2.12 ([13], Lemma 3.4). *Let e_1, e_2, e_3, e_4, and e_5 be distinct elements of a 3-connected matroid M that is not isomorphic to $M(\mathscr{W}_3)$. Suppose that $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are triangles and $\{e_2, e_3, e_4\}$ is a triad of M. Then these two triangles and this one triad are the only triangles and triads of M containing e_3.* □

Lemma 2.13. *Let M be a 3-connected matroid other than a wheel or a whirl. Suppose $T^* = \{a, b, c\}$ is a triad containing only essential elements. Then T^* is a vertex-triad.*

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Proof. Clearly, M has at least four elements. Since T^* is a triad containing only essential elements, it follows from Lemma 2.10 that the maximum chain containing T^* is a fan having at least five elements. Without loss of generality, assume that, in the fan, b is in two triangles, say, $\{a, b, d\}$ and $\{b, c, e\}$. By Lemma 2.12, b is in exactly one triad T^* and exactly two triangles $\{a, b, d\}$ and $\{b, c, e\}$. Thus $\widetilde{M\backslash b} \cong M/a\backslash b$ and $\widetilde{M/b} \cong M/b\backslash a, c$. If $\widetilde{M\backslash b}$ is 3-connected, then $M/a\backslash b$ is 3-connected. Hence $M\backslash T^* = M/a\backslash b, c$ is connected and the theorem holds. Therefore, we may assume that $\widetilde{M\backslash b}$ is not 3-connected. By Lemma 2.2, $\widetilde{M/b}$ is 3-connected. Therefore $M\backslash a, b, c = M/b\backslash a, c \cong \widetilde{M/b}$ is 3-connected. We conclude that T^* is a vertex-triad. □

Lemma 2.14 (Lemos [6]). *Let C be a circuit of a 3-connected matroid M such that $|E(M)| \geqslant 4$. If, for all elements x of C, the matroid $M\backslash x$ is not 3-connected, then C meets at least two triads of M.* □

Lemma 2.15 (Leo [7]). *Let C be a circuit of a 3-connected matroid M such that $|E(M)| \geqslant 4$. Suppose that e is an element of C. If, for all elements $x \neq e$ of C, the matroid $M\backslash x$ is not 3-connected, then C meets a triad of M.* □

3. Proofs of the main theorems

In this section we will prove our main theorems. First we give a sufficient condition under which a triad is a vertex-triad. This result is crucial to the proof of our main results. Then we use the result to prove three theorems that are more general than Theorems 1.3, 1.4 and 1.6, respectively.

Theorem 3.1. *Let M be a 3-connected matroid. Suppose that T^* is a triad which contains an essential element of M. Then T^* is a vertex-triad.*

Proof. Suppose there exists an essential element b in the triad $T^* = \{a, b, c\}$. First, suppose that M has at most five elements. Since M has a triad, $M \cong U_{1,3}, U_{2,4}$, or $U_{3,5}$. None of these matroids contains an essential element. Thus the theorem holds in this case. From now on, we will assume that M has at least six elements. If M is a wheel or a whirl, then the theorem holds. Therefore, we may assume that M is neither a wheel nor a whirl. By Lemma 2.13, if all the elements in T^* are essential, then T^* is a vertex-triad. Hence we may assume that T^* contains a non-essential element x, say $x = a$. By Lemma 2.10, a maximal chain containing T^* is a fan having two non-essential elements as its ends. Denote this fan by $\mathscr{F}$. Clearly, x is one end of $\mathscr{F}$.

Since an end of a type-1 fan is non-essential and is in some triangle, it is not in any triad. As x is in a triad T^*, hence $\mathscr{F}$ is not of type 1. Thus $\mathscr{F}$ is a type-2 or a type-3 fan.

Case 1: *$\mathscr{F}$ has at least four elements.*
Assume first that $\mathscr{F}$ is a type-2 fan having x as one of its ends. Note that $T^* = \{x, b, c\}$. The fan $\mathscr{F}$ has at least five elements. Without loss of generality, we may assume that $\{b, c, d\}$ is a triangle, while $\{c, d, e\}$ is a triad. Here e may be another end of the fan. By the dual of

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Lemma 2.12, c is in exactly two triads and one triangle. Therefore, $\widetilde{M\backslash c} \cong M\backslash c/b/d$ and $\widetilde{M/c} \cong M/c\backslash b$. By Lemma 2.2, either $M/c\backslash b$ is 3-connected, or $M\backslash c/b/d$ is 3-connected. In the former case, we deduce that $M\backslash T^* = (M/c\backslash b)\backslash x$ is connected and the theorem holds. In the latter case, we deduce that $M\backslash T^*/d = M\backslash b, c, x/d = (M\backslash c, x/b)/d = (M\backslash c/b/d)\backslash x$ is connected. If $M\backslash T^*$ is not connected, by Lemma 2.8, $M^*|(T^* \cup d) \cong U_{2,4}$. By Lemma 2.9, d is non-essential in M^* and thus is non-essential in M. This is a contradiction as d is in both a triangle and a triad and thus certainly essential. This completes the case when the fan $\mathscr{F}$ is of type 2.

Next we assume that $\mathscr{F}$ is a type-3 fan having x as one of its ends. The fan $\mathscr{F}$ has at least four elements. Thus b and c are in a common triangle, denoted by $\{b, c, d\}$. First suppose that d is the other end of the fan. Thus d is not in any triad and x is not in any triangle since x and d are the ends of the fan $\mathscr{F}$. By Lemma 2.11, b is in a unique fan. Thus b is in a unique triangle and a unique triad (or one can prove this easily by using orthogonality and Lemma 2.9). Thus $\widetilde{M\backslash b} \cong M\backslash b/c$ and $\widetilde{M/b} \cong M/b\backslash c$. By Lemma 2.2, either $M\backslash b/c$ or $M/b\backslash c$ is 3-connected. By Lemma 2.6, T^* is a vertex-triad. Hence we may assume that d is essential and $\mathscr{F}$ has at least six elements. By relabelling if necessary, we assume that $\{c, d, e\}$ is a triad. The result now follows as in the previous paragraph, since we use the fact that $\mathscr{F}$ is of type 2 there to guarantee only the existence of the triad $\{c, d, e\}$.

Case 2: *$\mathscr{F}$ has only three elements $x = a, b,$ and c.*
Thus $E(\mathscr{F}) = T^* = \{a, b, c\}$. As T^* is a triad, $\mathscr{F}$ is a type-2 fan. Since x is non-essential and b is essential, $y = c$ is the other end of the fan $\mathscr{F}$. Hence both x and y are contractible but not deletable. By Lemma 2.11, each fan containing the essential element b is a triad and every two such fans meet in b. Let $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k, \mathscr{F}$ be all the fans containing b where $k \geqslant 0$. Next we show that b is not in any triangle and is in exactly $k+1$ triads. If b is in a triangle T, then by Lemma 2.10 there is a fan containing T. But such a fan must be a triad. Thus M has a triangle T which is also a triad. Therefore, by Lemma 2.4, $M \cong U_{2,4}$, a contradiction. If b is in a triad R, then R is contained in a fan, which has exactly three elements, by Lemma 2.11. Hence R itself is a fan containing b. Thus e is not in a triad other than $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k, \mathscr{F}$. We conclude that b is in exactly $k+1$ triads. As b is essential and is not in any triangle, $\widetilde{M/b} = M/b$ is not 3-connected. By Lemma 2.2, $\widetilde{M\backslash b}$ is 3-connected. Assume that $E(\mathscr{F}_i) = \{b, x_i, y_i\}$ for all $i \in \{1, 2, \ldots, k\}$ and let $P = \{x_1, x_2, \ldots, x_k, x\}$. Consider the dual matroid M^*. Clearly, $\{b, x, y\}$ and $\{b, x_i, y_i\}$ where $i \in \{1, 2, \ldots, k\}$ are all the triangles of M^* containing b. Thus b is in exactly $k+1$ triangles of M^*. First assume that $k = 0$. Then b is in a unique triangle $\{b, x, y\}$ of M^*. Since x is a loop of $M^*/b, y$, we deduce that $M^*/T^* = M^*/b, x, y = M^*/b\backslash x/y \cong (\widetilde{M^*/b})/y$, as b is in a unique triangle. Note that $\widetilde{M^*/b} = \widetilde{M\backslash b}$ is 3-connected, thus $M^*/T^* = (\widetilde{M^*/b})/y$ is connected. Therefore $M\backslash T^*$ is connected: done. Therefore, we may assume that $k > 0$. Since $\widetilde{M^*/b} = \widetilde{M\backslash b}$ is 3-connected, $M^*/b\backslash P \cong \widetilde{M^*/b}$ is 3-connected. Therefore, $M^*/T^*\backslash(P - x) = M^*/b, x, y\backslash(P - x) = M^*/b, y\backslash x\backslash(P - x) = M^*/b, y\backslash P = (M^*/b\backslash P)/y$ is connected.

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Next we show that

(1) M^*/T^* has no loops, and
(2) in the matroid M^*/T^*, each element in $P\backslash x$ is parallel to some element of $M^*/T^*\backslash(P-x)$.

In fact, if z is a loop of M^*/T^*, then $T^* \cup z$ contains a circuit C^* of M^* containing z. Since T^* is also a circuit of M^*, and M^* is 3-connected, we deduce that $|C^*| = 3$ and thus $M^*|(T^* \cup z) \cong U_{2,4}$. This is a contradiction to the dual of Lemma 2.9 as b is essential. Thus (1) holds. In particular, each of $x_1, x_2, \ldots,$ and x_k is not a loop of M^*/T^*. Since $\{x_i, y_i\}$ is a circuit of M^*/b and M^*/T^* has no loops, we deduce that $\{x_i, y_i\}$ is also a circuit of M^*/T^*. Thus (2) holds. Therefore, M^*/T^* can be derived from the connected matroid $M^*/T^*\backslash(P-x)$ by adding k elements $x_1, x_2, \ldots, x_k$ parallel to $y_1, y_2, \ldots, y_k$ in $M^*/T^*\backslash(P-x)$, respectively. Hence M^*/T^* is also connected. We conclude that $M\backslash T^*$ is connected. This completes the proof of Case 2 and thus the proof of the theorem. □

An equivalent statement of Theorem 3.1 is that, given a triad T^* in a 3-connected matroid, either $M\backslash T^*$ is connected, or M/e is 3-connected for each element e in the triad. If M is a binary matroid, then M^* certainly has no $U_{2,4}$-restriction. Thus Theorem 1.3 is a direct consequence of the following result.

Theorem 3.2. *Let C be a circuit of a 3-connected matroid M such that $|E(M)| \geqslant 4$ and that the dual matroid M^* has no $U_{2,4}$-restriction. Suppose that e is an element of C. If, for all elements $x \neq e$ of C, the matroid $M\backslash x$ is not 3-connected, then C meets a vertex-triad of M.*

Proof. If M has exactly four elements, then $M \cong U_{2,4}$, a contradiction to our assumption. Thus M has at least five elements. We shall use induction on $|C|$. First, suppose that $|C| = 3$, say $C = \{a, b, e\}$. Since neither $M\backslash a$ nor $M\backslash b$ is 3-connected, by Lemma 2.1, a is in a triad T_0^*. Since a is in both a triangle and a triad, it is essential. By Theorem 3.1, T_0^* is a vertex-triad. This completes the case when $|C| = 3$.

Suppose that the theorem is true for all circuits of size less than $|C|$. Now we consider a circuit C where $|C| \geqslant 4$. First we assume that there exists an element e_1 of $C - e$, such that M/e_1 is 3-connected. Suppose that there is an element $f_1 \neq e$ in $C - e_1$ such that $M/e_1\backslash f_1$ is 3-connected. Then, since $M\backslash f_1$ is not 3-connected, by Lemmas 2.5 and 2.6, there is a vertex-triad containing e_1 and f_1 and the theorem holds. Thus we may assume that, for all $x \neq e$ in $C - e_1$, $M/e_1\backslash x$ is not 3-connected. Note that $C - e_1$ is a circuit of M/e_1, and $|C - e_1| < |C|$. Moreover, $(M/e_1)^*$ has no $U_{2,4}$-restriction also. Otherwise, $M^*\backslash e_1 = (M/e_1)^*$ has a $U_{2,4}$-restriction. Hence M^* will also have a $U_{2,4}$-restriction. This is a contradiction to our assumption. By induction, $C - e_1$ and thus C meets a triad T^* of M/e_1, such that $M/e_1\backslash T^*$ is connected. Clearly, T^* is also a triad of M. By Lemma 2.8, since M^* has no $U_{2,4}$-restriction, we deduce that $M\backslash T^*$ is connected. Therefore T^* is a vertex-triad of M meeting C.

From the above argument, we may assume that, for each $x \in C - e$, the matroid M/x is not 3-connected. Thus each element of $C - e$ is essential in M. By Lemma 2.15, there is

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a triad T_1^* of M meeting C. By orthogonality, $|T_1^* \cap C| \geqslant 2$. Thus T_1^* contains an essential element of M. By Theorem 3.1, T_1^* must be a vertex-triad meeting C. This completes the proof of Theorem 3.2. □

Theorem 1.4 is a direct consequence of the following theorem.

Theorem 3.3. *Let C be a circuit of a 3-connected matroid M such that $|E(M)| \geqslant 4$ and that M^* has no $U_{2,4}$-restriction. If, for all elements x of C, the matroid $M\backslash x$ is not 3-connected, then C meets at least two vertex-triads of M.*

Proof. Clearly, M has at least five elements. We shall use induction on $|C|$. For $|C| = 3$, the proof is similar to that in Theorem 3.2. Suppose that the theorem is true for all circuits of size less than $|C|$ where $|C| \geqslant 4$. Now we consider a circuit C. By Theorem 3.2, there exists a vertex-triad T_0^* meeting C. First we assume that there exists an element e_1 of $C\backslash T_0^*$, such that M/e_1 is 3-connected. Suppose that there is an element $f_1 \in C - e_1$ such that $M/e_1\backslash f_1$ is 3-connected. Then, since $M\backslash f_1$ is not 3-connected, by Lemmas 2.5 and 2.6, there is a vertex-triad R^* containing e_1 and f_1. Since $e_1 \notin T_0^*$, we deduce that $R^* \neq T_0^*$ and the theorem holds. Thus we may assume that, for all $x \in C - e_1$, $M/e_1\backslash x$ is not 3-connected. Note that $C - e_1$ is a circuit of M/e_1, and $|C - e_1| < |C|$. Moreover, $(M/e_1)^*$ has no $U_{2,4}$-restriction as M^* has no $U_{2,4}$-restriction. By induction, $C - e_1$ and thus C meets two triads T_1^* and T_2^* of M/e_1, such that both $M/e_1\backslash T_1^*$ and $M/e_1\backslash T_2^*$ are connected. Clearly, both T_1^* and T_2^* are also triads of M. By Lemma 2.8, since M^* has no $U_{2,4}$-restriction, we deduce that both T_1^* and T_2^* are vertex-triads of M meeting C.

From the above argument, we may assume that, for each $x \in C\backslash T_0^*$, the matroid M/x is not 3-connected. Thus each element of $C\backslash T_0^*$ is essential in M. Note that $C\backslash T_0^*$ is not empty, as C has at least four elements. By Lemma 2.14, there is a triad T^* of M meeting C and T^* is different from T_0^*. Suppose that $T^* \cap C \subseteq T_0^* \cap C$. By orthogonality, $|T^* \cap C| \geqslant 2$. Thus $|T^* \cap T_0^*| \geqslant 2$. Since $T^* \neq T_0^*$, we conclude that $M^*|(T_0^* \cup T^*) \cong U_{2,4}$, a contradiction. Therefore, there is an element $a \in (T^*\backslash T_0^*) \cap C$. Since a is an essential element of M on T^*, by Theorem 3.1, T^* must be a vertex-triad. Thus C meets two vertex-triads T^* and T_0^*, as required. □

Theorem 1.6 is a direct consequence of the following result. Its proof is almost identical to the proof of a result of Oxley in [11].

Theorem 3.4. *Let M be a minimally 3-connected matroid such that $|E(M)| \geqslant 4$ and that M^* has no $U_{2,4}$-restriction. Then M has at least $\frac{1}{2}r^*(M) + 1$ vertex-triads.*

Proof. Let X be the set of elements of M which are in some vertex-triads. Denote the number of vertex-triads of M by v_3. Then, by Theorem 3.3, X meets every circuit of M. Thus X contains a cobase B^* of M. Choose an element a of B^*. Then a is in some vertex-triad C_1^*. Now let C_a be the fundamental circuit of a respect to $E(M)\backslash B^*$. Then, by Theorem 3.3, C_a meets a vertex-triad C_2^* which is different from C_1^*. Since $|C_i^* \cap C_a| \geqslant 2$ for $i = 1, 2$, we deduce that $|B^* \cap (C_1^* \cup C_2^*)| \leqslant 3$. Thus, as B^* contains at

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most two elements of any vertex-triad, $v_3 \geqslant \frac{1}{2}|B^*\backslash(C_1^* \cup C_2^*)| + 2$. That is, $v_3 \geqslant \frac{1}{2}(|B^*| + 1)$ with equality holding only if $|B^* \cap C_1^*| = 2$ and $|B^* \cap (C_1^* \cup C_2^*)| = 3$. Thus we may assume that there is an element b in $(B^* \cap C_2^*)\backslash C_1^*$, for otherwise our theorem holds. Now let C_b be the fundamental circuit of b respect to $E(M)\backslash B^*$. Then C_b meets a vertex-triad C_3^* different from C_2^*. Moreover, since $b \notin C_1^*$, we deduce that $C_3^* \neq C_1^*$. Now $|B^*\backslash(C_1^* \cup C_2^* \cup C_3^*)| \geqslant |B^*| - 4$, so $v_3 \geqslant \frac{1}{2}(|B^*| - 4) + 3$. Therefore, $v_3 \geqslant \frac{1}{2}r^*(M) + 1$. This completes the proof of the theorem. □

4. Consequences and concluding remarks

In this section we derive some consequences of the results proved in the previous sections. The next result of Oxley [11] is an immediate consequence of Theorem 1.6.

Corollary 4.1. *Let G be a minimally 3-connected graph with v_3 vertices of degree three. Then $v_3 \geqslant \frac{1}{2}(|E(G)| - |V(G)| + 3)$.* □

The author is grateful to D. Vertigan for pointing out that Theorem 1.2 is an immediate consequence of Corollary 4.1 noting that $2|E(G)| \geqslant 3v_3 + 4(|V(G)| - v_3)$. Hence Theorem 1.2 is a corollary of Theorem 1.6. The next corollary is an extension of Tutte's Triangle Lemma (Lemma 2.1) for general matroids.

Corollary 4.2. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M with at least four elements. If neither $M\backslash a$ nor $M\backslash b$ is 3-connected, then the element a is in a vertex-triad with exactly one of b and c.*

Proof. Clearly, M has more than four elements. Since a is in a triangle, M/a is not 3-connected. Thus a is essential. By Lemma 2.1, a is in a triad with exactly one of b and c. By Theorem 3.1, this triad must be a vertex-triad. □

Corollary 4.2 gives a direct generalization of the following result of Tutte for 3-connected graphs. Note that this result is not a special case of Tutte's Triangle Lemma (Lemma 2.1).

Corollary 4.3 (Tutte [15]). *Let $\{a, b, c\}$ be a triangle in a 3-connected graph G. If neither $G\backslash a$ nor $G\backslash b$ is 3-connected, then the edge a meets a vertex of degree three with exactly one of b and c.* □

Tutte [18] proved that an essential element in a 3-connected matroid is in a triangle or a triad. The following result strengthens this useful result.

Corollary 4.4. *Let e be an essential element in a 3-connected matroid. Then e is in a triangle or a vertex-triad.*

Proof. By Tutte's result [18], e is in a triangle or a triad. By Theorem 3.1, if it is in a triad, this triad must be a vertex-triad, since e is essential. □

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Using Theorem 3.1, we immediately get the following corollary, which strengthens the Fan Theorem of Oxley and Wu (Lemma 2.11).

Corollary 4.5. *Let M be a 3-connected matroid which is neither a wheel nor a whirl. Suppose that e is an essential element of M. Then e is in a fan, both ends of which are non-essential and all triads in the fan are vertex-triads. Moreover, this fan is unique unless conditions* (a), (b) *or* (c) *of Lemma 2.11 hold.*

Proof. Each triad in any fan containing e has at least one essential element of M. By Theorem 3.1, this triad must be a vertex-triad. The result now follows from Lemma 2.11. □

Non-separating circuits, the dual of non-separating cocircuits, are very useful in studying the structures of graphs (see, for example, Thomassen and Toft [14] and Tutte [16]). The next theorem, due to Bixby and Cunningham [2] and, independently, Kelmans (see [5]), gives a good characterization of when a binary matroid is graphic. It shows that non-separating cocircuits in binary matroids resemble vertices in graphs and are very useful in studying binary matroids.

Theorem 4.6. *Let M be a 3-connected binary matroid on E, where $|E| \geqslant 4$, and let $e \in E$. Then e is an element of two distinct non-separating cocircuits of M. Moreover, M is graphic if and only if each element e is in at most two non-separating cocircuits.* □

Theorem 4.6 ensures the existence of 'vertices' in 3-connected binary matroids. Our main results in this paper show that certain 3-connected binary matroids (for example, minimally 3-connected binary matroids) have many 'vertices' of size three. These results directly generalize several well-known graph results involving the existence of vertices of degree three.

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