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ON A PROBLEM ABOUT ADDITIVE REPRESENTATION FUNCTIONS

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Abstract

For a set A of positive integers and any positive integer n, let $R_1(A,n)$, $R_2(A,n)$ and $R_3(A,n)$ denote the number of solutions of $a+a'=n$ with $a,a'\in A$ and the additional restriction that $a<a'$ for R_2 and $a\le a'$ for R_3. We consider Problem 6 of Erdős *et al.* ['On additive properties of general sequences', *Discrete Math.* **136** (1994), 75–99] about locally small and locally large values of R_1, R_2 and R_3.

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1. Introduction

Let $\mathbb{N}$ denote the set of all the positive integers. For a set $A\subset\mathbb{N}$, the counting function $A(n)=|\{a: 1\le a\le n,\ a\in A\}|$ is the number of elements of A not exceeding n. For $n=1,2,\dots$, define the additive representation functions:

$$R_1(A,n)=|\{a+a'=n,\ a \text{ and } a'\in A\}|,$$
$$R_2(A,n)=|\{a+a'=n,\ a \text{ and } a'\in A,\ a<a'\}|,$$
$$R_3(A,n)=|\{a+a'=n,\ a \text{ and } a'\in A,\ a\le a'\}|.$$

In a series of papers [4, 5, 6], Erdős, Sárközy and Sós investigated the behaviour of these functions. Erdős *et al.* [7, 8] and Balasubramanian [1] studied the monotonicity properties of the functions $R_1(n), R_2(n)$ and $R_3(n)$. For other related problems, see [2, 3, 12].

In 1994, Erdős *et al.* posed the following problem [9, Problem 6] (see also [10, Problem 6] and [11, Problem 8]).

PROBLEM 1.1. What condition (on A) is needed to ensure that the function $R_i(A,n)$ $(i=1,2,3)$ assumes infinitely many 'locally small' or 'locally large' values, that is,

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$$R_i(A,n) < \min\{R_i(A,n-1), R_i(A,n+1)\} \quad \text{for infinitely many } n, \text{ or}$$
$$R_i(A,n) > \max\{R_i(A,n-1), R_i(A,n+1)\} \quad \text{for infinitely many } n?$$

Throughout this paper, the characteristic function of the set B is denoted by

$$\chi_B(i) = \begin{cases} 1 & \text{if } i \in B, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain the following results related to the problem of Erdős, Sárközy and Sós.

THEOREM 1.2. *Let A, B be infinite sets such that $\mathbb{N} = A \cup B$ and $A \cap B = \emptyset$. Then for all $n \geq 2$:*

(i) $R_1(A,n) < \min\{R_1(A,n-1), R_1(A,n+1)\}$ *if and only if*

$$R_1(B,n) - R_1(B,n-1) \leq 2\chi_B(n-1) - 2, \quad R_1(B,n+1) - R_1(B,n) \geq 2\chi_B(n);$$

(ii) $R_1(A,n) > \max\{R_1(A,n-1), R_1(A,n+1)\}$ *if and only if*

$$R_1(B,n) - R_1(B,n-1) \geq 2\chi_B(n-1), \quad R_1(B,n) - R_1(B,n+1) \geq 2 - 2\chi_B(n).$$

COROLLARY 1.3. *Let $A = \{n \in \mathbb{N} : n \equiv 1, 2, 3 \pmod 4\}$. Then for all $k \geq 2$,*

$$R_1(A,4k+1) < \min\{R_1(A,4k), R_1(A,4k+2)\},$$
$$R_1(A,4k) > \max\{R_1(A,4k-1), R_1(A,4k+1)\}.$$

THEOREM 1.4. *Let A, B be infinite sets such that $\mathbb{N} = A \cup B$ and $A \cap B = \emptyset$. Then for all $n \geq 1$:*

(i) *if n is even, then $R_2(A,n) > \max\{R_2(A,n-1), R_2(A,n+1)\}$ if and only if*

$$\begin{cases} R_2(B,n) - R_2(B,n-1) \geq 1 + \chi_B(n-1) - \chi_B(n/2), \\ R_2(B,n) - R_2(B,n+1) \geq 2 - \chi_B(n) - \chi_B(n/2); \end{cases}$$

(ii) *if n is even, then $R_2(A,n) < \min\{R_2(A,n-1), R_2(A,n+1)\}$ if and only if*

$$\begin{cases} R_2(B,n+1) - R_2(B,n) \geq \chi_B(n) + \chi_B(n/2), \\ R_2(B,n-1) - R_2(B,n) \leq 1 - \chi_B(n-1) + \chi_B(n/2); \end{cases}$$

(iii) *if n is odd, then $R_2(A,n) > \max\{R_2(A,n-1), R_2(A,n+1)\}$ if and only if*

$$\begin{cases} R_2(B,n) - R_2(B,n+1) \geq 1 + \chi_B((n+1)/2) - \chi_B(n), \\ R_2(B,n) - R_2(B,n-1) \geq \chi_B(n-1) + \chi_B((n-1)/2); \end{cases}$$

(iv) *if n is odd, then $R_2(A,n) < \min\{R_2(A,n-1), R_2(A,n+1)\}$ if and only if*

$$\begin{cases} R_2(B,n+1) - R_2(B,n) \geq 1 + \chi_B(n) - \chi_B((n+1)/2), \\ R_2(B,n) - R_2(B,n-1) \leq \chi_B(n-1) + \chi_B((n-1)/2) - 2. \end{cases}$$

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COROLLARY 1.5. *Let* $A = \{n \in \mathbb{N} : n \equiv 1, 2, 3 \pmod 4\}$. *By Theorem 1.4(i),*

$$R_2(A, 8k) > \max\{R_2(A, 8k-1), R_2(A, 8k+1)\} \quad \text{for all } k \geq 1.$$

Let $A = \{n \in \mathbb{N} : n \equiv 0, 2, 3 \pmod 4\}$. *By Theorem 1.4(ii), for all* $k \geq 1$,

$$R_2(A, 4k) < \min\{R_2(A, 4k-1), R_2(A, 4k+1)\}.$$

THEOREM 1.6. *Let* A, B *be infinite sets such that* $\mathbb{N} = A \cup B$ *and* $A \cap B = \emptyset$. *Then for all* $n \geq 1$*:*

(i) *if* n *is even, then* $R_3(A, n) > \max\{R_3(A, n-1), R_3(A, n+1)\}$ *if and only if*

$$\begin{cases} R_3(B, n) - R_3(B, n-1) \geq \chi_B(n-1) + \chi_B(n/2), \\ R_3(B, n) - R_3(B, n+1) \geq 1 - \chi_B(n) + \chi_B(n/2); \end{cases}$$

(ii) *if* n *is even, then* $R_3(A, n) < \min\{R_3(A, n-1), R_3(A, n+1)\}$ *if and only if*

$$\begin{cases} R_3(B, n+1) - R_3(B, n) \geq 1 + \chi_B(n) - \chi_B(n/2), \\ R_3(B, n-1) - R_3(B, n) \geq 2 - \chi_B(n-1) - \chi_B(n/2); \end{cases}$$

(iii) *if* n *is odd, then* $R_3(A, n) > \max\{R_3(A, n-1), R_3(A, n+1)\}$ *if and only if*

$$\begin{cases} R_3(B, n) - R_3(B, n+1) \geq 2 - \chi_B((n+1)/2) - \chi_B(n), \\ R_3(B, n) - R_3(B, n-1) \geq 1 + \chi_B(n-1) - \chi_B((n-1)/2); \end{cases}$$

(iv) *if* n *is odd, then* $R_3(A, n) < \min\{R_3(A, n-1), R_3(A, n+1)\}$ *if and only if*

$$\begin{cases} R_3(B, n+1) - R_3(B, n) \geq \chi_B(n) + \chi_B((n+1)/2), \\ R_3(B, n) - R_3(B, n-1) \leq \chi_B(n-1) - \chi_B((n-1)/2) - 1. \end{cases}$$

COROLLARY 1.7. *Let* $A = \{n \in \mathbb{N} : n \equiv 1, 2, 3 \pmod 4\}$. *Then for all* $k \geq 1$, *by Theorem 1.6(i) and (iv),*

$$R_3(A, 8k) > \max\{R_3(A, 8k-1), R_3(A, 8k+1)\} \quad \text{for all } k \geq 1,$$
$$R_3(A, 8k+1) < \min\{R_3(A, 8k), R_3(A, 8k+2)\} \quad \text{for all } k \geq 1.$$

2. Proof of Theorem 1.2

Write $f(z) = \sum_{a \in A} z^a$ and $g(z) = \sum_{b \in B} z^b$. Then,

$$f(z) + g(z) = z + z^2 + \cdots = \frac{z}{1-z}, \quad g^2(z) = \sum_{n=1}^{\infty} R_1(B, n) z^n.$$

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Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}(R_1(A,n)-R_1(A,n-1))z^n &= \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{a_i+a_j=n\\ a_i,a_j\in A}}1-\sum_{\substack{a_i+a_j=n-1\\ a_i,a_j\in A}}1\Bigg)z^n\\ &= \sum_{a_i,a_j\in A} z^{a_i+a_j}-\sum_{a_i,a_j\in A} z^{a_i+a_j+1}\\ &= (1-z)\Big(\frac{z}{1-z}-g(z)\Big)^2 = \frac{z^2}{1-z}-2zg(z)+(1-z)g^2(z)\\ &= z^2\sum_{n=0}^{\infty}z^n-2z\sum_{b\in B}z^b+(1-z)\sum_{n=1}^{\infty}R_1(B,n)z^n. \qquad (2.1)\end{aligned}$$

Comparing the coefficients of $z^n (n \geq 2)$ on both sides of (2.1), we see that

$$R_1(A,n)-R_1(A,n-1) = 1-2\chi_B(n-1)+R_1(B,n)-R_1(B,n-1).$$

Thus,

$$R_1(A,n)-R_1(A,n+1) = -1+2\chi_B(n)+R_1(B,n)-R_1(B,n+1)$$

and, for all $n \geq 2$,

$$R_1(A,n) < \min\{R_1(A,n-1), R_1(A,n+1)\}$$

if and only if

$$R_1(B,n)-R_1(B,n-1) \leq 2\chi_B(n-1)-2, \quad R_1(B,n+1)-R_1(B,n) \geq 2\chi_B(n).$$

Moreover,

$$R_1(A,n) > \max\{R_1(A,n-1), R_1(A,n+1)\}$$

if and only if

$$R_1(B,n)-R_1(B,n-1) \geq 2\chi_B(n-1), \quad R_1(B,n)-R_1(B,n+1) \geq 2-2\chi_B(n).$$

This completes the proof of Theorem 1.2.

3. Proof of Theorem 1.4

We have

$$\begin{aligned}R_2(A,n) &= \sum_{\substack{n=i+j\\ i,j\in A, i<j}}1 = \sum_{1\leq i<n/2}(1-\chi_B(i))(1-\chi_B(n-i))\\ &= \sum_{1\leq i<n/2}1-|\{i: 1\leq i\leq n-1, i\in B\}|+\chi_B(n/2)\\ &\quad +|\{(b,b'): b,b'\in A, b<b', b+b'=n\}|\\ &= \sum_{1\leq i<n/2}1-B(n-1)+\chi_B(n/2)+R_2(B,n). \qquad (3.1)\end{aligned}$$

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If n is even, then (3.1) gives

$$\begin{aligned} R_2(A,n) &= \sum_{1\le i\le n/2-1} 1 - B(n-1) + \chi_B(n/2) + R_2(B,n),\\ R_2(A,n+1) &= \sum_{1\le i\le n/2} 1 - B(n) + R_2(B,n+1),\\ R_2(A,n-1) &= \sum_{1\le i\le n/2-1} 1 - B(n-2) + R_2(B,n-1). \end{aligned}$$

Thus,

$$R_2(A,n) - R_2(A,n+1) = -1 + \chi_B(n) + \chi_B(n/2) + R_2(B,n) - R_2(B,n+1), \tag{3.2}$$

$$R_2(A,n) - R_2(A,n-1) = -\chi_B(n-1) + \chi_B(n/2) + R_2(B,n) - R_2(B,n-1). \tag{3.3}$$

By (3.2) and (3.3),

$$R_2(A,n) > \max\{R_2(A,n-1), R_2(A,n+1)\}$$

if and only if

$$\begin{cases} R_2(B,n) - R_2(B,n-1) \ge 1 + \chi_B(n-1) - \chi_B(n/2),\\ R_2(B,n) - R_2(B,n+1) \ge 2 - \chi_B(n) - \chi_B(n/2). \end{cases}$$

Again, by (3.2) and (3.3),

$$R_2(A,n) < \min\{R_2(A,n-1), R_2(A,n+1)\}$$

if and only if

$$\begin{cases} R_2(B,n+1) - R_2(B,n) \ge \chi_B(n) + \chi_B(n/2),\\ R_2(B,n-1) - R_2(B,n) \le 1 - \chi_B(n-1) + \chi_B(n/2). \end{cases}$$

If n is odd, then (3.1) gives

$$\begin{aligned} R_2(A,n) &= \sum_{1\le i\le (n-1)/2} 1 - B(n-1) + R_2(B,n),\\ R_2(A,n+1) &= \sum_{1\le i\le (n-1)/2} 1 - B(n) + \chi_B((n+1)/2) + R_2(B,n+1),\\ R_2(A,n-1) &= \sum_{1\le i\le (n-1)/2-1} 1 - B(n-2) + \chi_B((n-1)/2) + R_2(B,n-1). \end{aligned}$$

The remainder of the proof is similar to that for even n.

This completes the proof of Theorem 1.4.

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4. Proof of Theorem 1.6

We have

$$\begin{aligned} R_3(A,n) &= \sum_{\substack{n=i+j \\ i,j\in A, i\le j}} 1 = \sum_{1\le i\le n/2} (1-\chi_B(i))(1-\chi_B(n-i)) \\ &= \sum_{1\le i\le n/2} 1 - |\{i : 1\le i\le n-1, i\in B\}| - \chi_B(n/2) \\ &\quad + |\{(b,b') : b,b'\in B, b\le b', b+b'=n\}| \\ &= \sum_{1\le i\le n/2} 1 - B(n-1) - \chi_B(n/2) + R_3(B,n). \end{aligned} \tag{4.1}$$

If n is even, then by (4.1),

$$\begin{aligned} R_3(A,n) &= \sum_{1\le i\le n/2} 1 - B(n-1) - \chi_B(n/2) + R_3(B,n), \\ R_3(A,n+1) &= \sum_{1\le i\le n/2} 1 - B(n) + R_3(B,n+1), \\ R_3(A,n-1) &= \sum_{1\le i\le n/2-1} 1 - B(n-2) + R_3(B,n-1). \end{aligned}$$

If n is odd, then by (4.1),

$$\begin{aligned} R_3(A,n) &= \sum_{1\le i\le (n-1)/2} 1 - B(n-1) + R_3(B,n), \\ R_3(A,n+1) &= \sum_{1\le i\le (n+1)/2} 1 - B(n) - \chi_B\left(\frac{n+1}{2}\right) + R_3(B,n+1), \\ R_3(A,n-1) &= \sum_{1\le i\le (n-1)/2} 1 - B(n-2) - \chi_B\left(\frac{n-1}{2}\right) + R_3(B,n-1). \end{aligned}$$

The remainder of the proof is similar to the proof of Theorem 1.4.

This completes the proof of Theorem 1.6.

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