Math. Proc. Camb. Phil. Soc. (1982), **91**, 225–258
Printed in Great Britain

Subgroup closed Fitting classes are formations

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(*Received* 5 *March* 1981)

1. *Introduction.* Since their introduction by Fischer (12) and Fischer, Gaschütz and Hartley (13) Fitting classes of soluble groups have attracted attention on two fronts (all groups considered in this paper will be finite and soluble). On the one hand is their important role in the structure of finite soluble groups, a good account of which can be found in Gaschütz (14), and on the other is their intrinsic interest as classes of groups. This paper falls into the second category, and is a continuation and completion of (8). There we proved that a subgroup closed Fitting class is a formation if it consists of groups of nilpotent length at most three. Happily, at last, we can remove this qualification.

THEOREM 1·1. *A subgroup closed Fitting class of finite soluble groups is a formation.*

A consequence of (6) is then that subgroup closed Fitting classes are primitive saturated formations and hence completely classifiable.

This paper draws heavily on a number of sources and must be read in conjunction with them. Firstly, of course, there is (8), where the reduction in section 3 is crucial. Then there is Carter, Fischer and Hawkes (9) for structural results about $\mathscr{L}$-critical groups, and Isaacs (18) for the character theory of extensions of extra-special groups. In addition the standard reference works Curtis and Reiner (10) for representation theory and Isaacs (19) for character theory are used often.

We must draw attention to an error in (8): the entire p. 203 is vacuous and needs replacing by the appropriate special case of Lemma 7·2 of the present paper. The details are spelt out immediately after the statement of Lemma 7·2.

To complete this introduction we recall the long gestation of this result. In retrospect it can be seen that the story begins with Peter Neumann's result (23) that a formation of nilpotent groups is subgroup closed. It was an attempt to obtain a direct, non-variety theory proof of this that led to lemma 1·5 in (3) (this result says that a formation containing a group G contains every supplement of the Fitting subgroup of G) which in turn was crucial in Hawkes (15). Our subsequent papers (7), (8) were both inspired by this, dealing with the cases of subgroup closed Fitting classes of nilpotent length two and three.

Special thanks are due to Trevor Hawkes, who patiently listened to our attempts to explain the nilpotent length four case during his visit to Canberra in 1979. Most of the arguments in this paper appeared in that proof in a very primitive form.

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2. *The basic reduction.* The basic results about Fitting classes may be obtained from (14) and we assume familiarity with them. The notation and terminology of (8) will be carried on.

Let $\mathfrak{X}$ be a subgroup closed Fitting class, and suppose that, for every $n \geqslant 1$, $\mathfrak{X} \cap \mathfrak{N}^n$ is a formation. Then $\mathfrak{X}$ is a formation. For we need only show that $\mathfrak{X}$ is Q-closed; and if $G \in \mathfrak{X}$ and $N \trianglelefteq G$ then, for some $n \geqslant 1$, $G \in \mathfrak{X} \cap \mathfrak{N}^n$ whence $G/N \in \mathfrak{X} \cap \mathfrak{N}^n \subseteq \mathfrak{X}$ as required. It follows that to prove Theorem 1·1 it suffices to show that, for each $n \geqslant 1$, subgroup closed Fitting classes of nilpotent length n are formations. This we do by induction on n as explained in section 3 of (8).

To be precise suppose that for some positive integer n there is a subgroup closed Fitting class of nilpotent length n which is not a formation whilst every subgroup closed Fitting class of nilpotent length at most $n-1$ is a formation. Denote by $\mathscr{U}$ the set of all finite sequences $(p_1, p_2, \ldots, p_m)$ of primes where $p_i \neq p_{i+1}$ $(1 \leqslant i \leqslant m-1)$. For $u \in \mathscr{U}$, say $u = (p_1, p_2, \ldots, p_m)$, put $l(u) = m$ and

$$\mathfrak{S}_u = \mathfrak{S}_{p_1}\mathfrak{S}_{p_2} \ldots \mathfrak{S}_{p_m},$$

the usual class product. The message of Lemma 2·9 and section 3 of (8) is that for some $u \in \mathscr{U}$ with $l(u) = n$ there is a subgroup closed Fitting class $\mathfrak{X}$ such that

(i) $\mathfrak{X} \subseteq S_u$,

(ii) $\mathfrak{S}_u \cap \mathfrak{N}^{n-1} \subseteq \mathfrak{X}$,

(iii) for every $\mathscr{L}(n-1)$-critical group $G \in \mathfrak{X}$, $G/\Phi(G) \notin \mathfrak{X}$.

Theorem 1·1 will be proved by producing a contradiction to the existence of such a Fitting class $\mathfrak{X}$ in the following form.

THEOREM 2·1. *Let n be a positive integer, $u \in \mathscr{U}$ with $l(u) = n$ and suppose that $\mathfrak{F}$ is a subgroup closed Fitting class contained in $\mathfrak{S}_u$ satisfying $\mathfrak{S}_u \cap \mathfrak{N}^{n-1} \subseteq \mathfrak{F}$. Then there exists in $\mathfrak{F}$ an $\mathscr{L}(n-1)$-critical group G for which $G/\Phi(G) \in \mathfrak{F}$.*

On account of (7), (8) we may assume if necessary that $n \geqslant 4$ in proving this result.

3. *Preliminaries.* We collect together here some results which will be used later on, including a discussion of the twisted wreath product construction which is at the core of most of the later results.

First a result we know as Fitting's Lemma: a proof may be found in Higman (16).

LEMMA 3·1. *Let A be an abelian group on which G acts as a group of operators. Let N be a normal subgroup of G of order coprime to the order of A. Then $A = B \times C$ where B and C are G-subgroups of A and $[B, N] = 1$, $[C, N] = C$.*

LEMMA 3·2. *Suppose that p is a prime, and that P is a special p-group on which G acts as a group of operators; and suppose, moreover, that $[Z(P), G] = 1$ and that $P/\Phi(P)$ is completely reducible as a G-module. Let N be a normal subgroup of G of order prime to p such that $[P, N] = P$.*

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If C is a G-invariant normal subgroup of P then

$$P/C = A/C \times B/C,$$

where A, B are G-invariant, A/C is special and B/C is elementary abelian. Further if $A/C \neq 1$ then $[Z(A/C), G] = 1, (A/C)/\Phi(A/C)$ is completely reducible as a G-module and $[A/C, N] = A/C$; and if $B/C \neq 1$ then it is completely reducible and $[B/C, N] = B/C$.

Proof. First note that $(P/C)/Z(P)C/C \cong P/Z(P)C$, a quotient of $P/\Phi(P)$, is completely reducible as a G-module. Hence $Z(P/C)/Z(P)C/C$ has a G-admissible complement, $(A/C)/Z(P)C/C$ say, in $(P/C)/Z(P)C/C$. Hence

$$P/C = (A/C)Z(P/C), \quad A/C \cap Z(P/C) = Z(P)C/C.$$

But N acts trivially on $Z(P)C/C$ and fixed point free on $Z(P/C)/Z(P)C/C$, and consequently, by Fitting's Lemma,

$$Z(P/C) = Z(P)C/C \times B/C,$$

where B/C is G-isomorphic to a subgroup of $P/\Phi(P)$. Therefore

$$P/C = A/C \times B/C.$$

If P/C is abelian then $A/C = 1$, and the requirements of the lemma are satisfied. Hence suppose P/C is not abelian. If $B/C \neq 1$ it clearly satisfies the requirements. Next note that

$$Z(P)C/C \leqslant Z(A/C) \leqslant Z(P/C) \cap A/C = Z(P)C/C,$$

so

$$Z(A/C) = Z(P)C/C.$$

Also

$$Z(P)C/C = \Phi(P/C) = \Phi(A/C),$$

since the Frattini subgroup of a p-group is generated by pth powers and commutators. It follows that A/C is special; and the rest is easy.

The twisted wreath product was introduced in (21) by B. H. Neumann. We have found it useful to think of it as a group-theoretic version of induced representations. This point of view makes the existence of a Mackey Subgroup Theorem for twisted wreath products obvious, and is our main reason for developing the construction in the manner given below.

Let A be a group on which the group B acts as a group of operators. That is, there is a homomorphism $\alpha: B \to \mathrm{Aut}\, A$. Denote by AB the semidirect product of A by B: AB is the set product $A \times B$ with multiplication defined by

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2 (b_1^{-1}\alpha), b_1 b_2), \quad a_1, a_2 \in A, b_1, b_2 \in B.$$

As usual we identify A and B with the subgroups $\{(a, 1) : a \in A\}$ and $\{(1, b) : b \in B\}$ of AB, and in this spirit we have $a(b\alpha) = a^b$.

If C is a subgroup of B then the C-group determined by $\alpha|_C: C \to \mathrm{Aut}\, A$ is denoted $A|_C$, the *restriction* of A to C.

If B is a subgroup of D then we define a D-group $A|^D$, the *induction* of A to D, as follows. Let T be a right transversal of D to B, and let A^T be the cartesian product: this

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is simply the set of all functions $T \to A$ made into a group by componentwise multiplication. For $d \in D$ we define a function $d\beta: A^T \to A^T$ by $f \to f^d$ where, for all $f \in A^T$,

$$f^d(t) = f(t_1)^{b_1^{-1}},$$

where $td^{-1} = b_1 t_1 (b_1 \in B, t_1 \in T)$. It is a straightforward matter to check that $d\beta$ is an automorphism of A^T and that $\beta: D \to \operatorname{Aut} A^T$ given by $d \to d\beta$ is a homomorphism. $A|^D$ is A^T with D acting on it via β.

This definition of $A|^D$ is independent of T in the sense that if T_0 is another right transversal of D to B then there exists a D-isomorphism $A^T \to A^{T_0}$. For example if $\lambda: T_0 \to T$ is the bijection determined by $(t_0\lambda)t_0^{-1} = b_0 \in B$ $(t_0 \in T_0)$ then the function $A^T \to A^{T_0}$ defined by $f \to f_0$ where

$$f_0(t_0) = f(t_0\lambda)^{b_0}$$

is a D-isomorphism.

Note that β extends α in the sense that, for all $b \in B, f^b(1) = f(1)^b$. Note also that, if A is a kB-module for some field k, then $A|^D$ is just the induced module in the usual sense.

The semidirect product $(A|^D)D$ is usually called the twisted wreath product of A and D over B (the action of B on A being understood), and denoted $A \operatorname{twr}_B D$. $A|^D$ is called the *base group* of the twisted wreath product.

The following result is an analogue of Mackey's Subgroup Theorem for induced representations (see for example 44·2 of (10)).

LEMMA 3.3. *Let B, C be subgroups of D and let A be a B-group. Suppose that S is a complete set of (B, C) double coset representatives in D. Then*

$$A|^D|_C \cong \underset{s \in S}{\times} A^{(s)}|_{B^s \cap C}|^C,$$

where $A^{(s)}$ is A made into a B^s-group under the action $\gamma_s: B^s \to \operatorname{Aut} A$ where

$$a(b^s\gamma_s) = a^b.$$

Proof. For each $s \in S$ let T_s be a right transversal of C to $B^s \cap C$. Then

$$T = \bigcup_{s \in S} sT_s$$

is a right transversal of D to B. The subgroups A_s $(s \in S)$ of A^T defined by

$$A_s = \{f \in A^T : f(t) = 1 \text{ for } t \notin sT_s\},$$

are C-subgroups and provide a direct decomposition of A^T. We will have done therefore if we can show that A_s and $A^{(s)}|_{B^s \cap C}|^C$ are C-isomorphic. Indeed the mapping $f \to \bar{f}$ from A_s to $A^{(s)}|_{B^s \cap C}|^C$ defined by

$$\bar{f}(t) = f(st), \quad t \in T_s,$$

is an isomorphism, clearly, and is a C-isomorphism since for $c \in C$ and $t \in T_s$

$$\begin{aligned} \overline{f^c}(t) &= f^c(st) \\ &= f(st_1)^{b_1^{-1}}, \end{aligned}$$

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writing $tc^{-1} = b_1^s t_1$ where $b_1^s \in B^s \cap C$ and $t_1 \in T_s$, whence

$$\begin{aligned}\overline{f^c}(t) &= \bar{f}(t_1)(b_1^{-s}\gamma_s)\\ &= \bar{f}(t_1)^{b_1^{-s}}\\ &= \bar{f}^c(t),\end{aligned}$$

as required. This completes the proof of Lemma 3·3.

The next result provides a criterion for isomorphism of twisted wreath products.

LEMMA 3·4. *Suppose that D_1, D_2 are groups with subgroups B_1, B_2. Let A be a group for which there exist homomorphisms $\alpha_i: B_i \to \operatorname{Aut} A$ $(i = 1, 2)$, and suppose that there is an isomorphism $\mu: D_1 \to D_2$ such that $B_1\mu = B_2$ and*

$$\alpha_1 = \mu\alpha_2.$$

Then $A \operatorname{twr}_{B_1} D_1 \cong A \operatorname{twr}_{B_2} D_2$.

Proof. Choose T_1 to be a right transversal of D_1 to B_1. Then $T_1\mu$ is a right transversal of D_2 to B_2. Define a function $f \to \bar{f}$ from A^{T_1} to $A^{T_1\mu}$ by

$$\bar{f}(t\mu) = f(t), \quad t \in T_1.$$

Clearly this is an isomorphism. It also has the property that for all $d \in D_1$

$$\overline{f^d} = \bar{f}^{d\mu}.$$

For, if $t \in T_1$,

$$\begin{aligned}\overline{f^d}(t\mu) &= f^d(t)\\ &= f(t_1)(b_1^{-1}\alpha_1), \quad \text{where} \quad td^{-1} = b_1 t_1, \quad b_1 \in B_1, \quad t_1 \in T_1,\\ &= \bar{f}(t_1\mu)(b_1^{-1}\mu\alpha_2)\\ &= \bar{f}^{d\mu}(t\mu),\end{aligned}$$

as required. Finally one checks that

$$(f, d) \to (\bar{f}, d\mu)$$

is an isomorphism.

LEMMA 3·5. *Let A be a D-group and B a subgroup of D. There is an embedding of AD into $A \operatorname{twr}_B D$.*

Proof. Let T be a transversal of D to B. For each $a \in A$ define $f_a \in A^T$ by

$$f_a(t) = a^{t^{-1}}.$$

It is easy to check that $a \to f_a$ is an embedding of A into A^T, and a D-homomorphism at that. The result then follows.

The next result is in some sense a paradigm of the methods of this paper, and will be relied on heavily.

LEMMA 3·6. *Suppose that $\mathfrak{F}$ is a subgroup closed Fitting class with $\mathfrak{F} \subseteq \mathfrak{S}_u$ and, for some m, $\mathfrak{S}_u \cap \mathfrak{N}^m \subseteq \mathfrak{F}$. Let A be a D-group and N a normal subgroup of D such that the sum of the nilpotent lengths of A and D/N is at most m.*

If B is a subgroup of D containing N then $AD \in \mathfrak{F}$ if and only if $A \operatorname{twr}_B D \in \mathfrak{F}$.

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Proof. That $AD \in \mathfrak{F}$ if $A \operatorname{twr}_B D \in \mathfrak{F}$ follows from Lemma 3·5.

Suppose therefore that $AD \in \mathfrak{F}$. Consider the subgroups

$$\bar{D} = \{(d, Nd) : d \in D\}$$

and

$$\bar{B} = \{(b, Nb) : b \in B\}$$

of $D \times D/N$. The map $\mu : D \to \bar{D}$ defined by

$$d\mu = (d, dN)$$

is an isomorphism and $B\mu = \bar{B}$.

Let $\alpha : D \to \operatorname{Aut} A$ be the action implied by the hypothesis. Define an action of $D \times D/N$ on A by

$$a^{(d, Nd)} = a^d,$$

and then let $\bar{\alpha}$ be the restriction of this to $\bar{D}$. Since $\alpha = \mu\bar{\alpha}$ it follows from Lemma 3·4 that

$$A \operatorname{twr}_B D \cong A \operatorname{twr}_{\bar{B}} \bar{D}.$$

Now $(D \times B/N)\,\bar{D} = D \times D/N$ and hence Lemma 3·3 yields

$$\begin{aligned} A|_{D\times B/N}|^{D\times D/N}|_{\bar{D}} &\cong A|_{(D\times B/N)\cap\bar{D}}|^{\bar{D}} \\ &= A|_{\bar{B}}|^{\bar{D}}. \end{aligned}$$

It follows that $A \operatorname{twr}_B \bar{D}$ is isomorphic to a subgroup of $A \operatorname{twr}_{D\times B/N} (D \times D/N)$, and to complete the proof it is enough to show that

$$A \operatorname{twr}_{D\times B/N} (D \times D/N) \in \mathfrak{F}.$$

This comes easily: on the one hand

$$\begin{aligned} A|_{D\times B/N}|^{D\times D/N}|_D &\cong \underset{t\in T}{\times} A^{(t)}|_{(D\times B/N^t\cap D}|^D \\ &\cong \underset{t\in T}{\times} A|_D, \end{aligned}$$

where T is a right transversal of D/N to B/N, since $A \cong A^{(t)}$ as D-groups, t commuting with D, and it follows that $(A|^{D\times D/N})\,D$ is a subdirect product of copies of AD and therefore is in $\mathfrak{F}$; and on the other

$$(A|^{D\times D/N})\,(D/N) \in \mathfrak{F},$$

since it has nilpotent length at most m by hypothesis. Finally

$$\begin{aligned} A \operatorname{twr}_{D\times B/N}(D \times D/N) &= (A|^{D\times D/N})\,D\,.\,(A|^{D\times D/N})\,(D/N) \\ &\in \mathfrak{N}_0\mathfrak{F} \\ &= \mathfrak{F}. \end{aligned}$$

4. *$\mathscr{L}$-critical groups.* Let $u = (p_1, p_2, \ldots, p_n)$, in the notation of section 2, and suppose that G is an $\mathscr{L}(n-1)$-critical group in $\mathfrak{S}_u$. We say that G is in *expanded form* $P_1 P_2 \ldots P_n$ if G has subgroups $P_i\,(1 \leqslant i \leqslant n)$ such that

(i) P_i is a p_i-group, $1 \leqslant i \leqslant n$,

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(ii) $P_i \leqslant N_G(P_j)$ for $i \geqslant j$, and

(iii) $P_i \cap \prod_{j \neq i} P_j = 1$, $1 \leqslant i \leqslant n$.

Since G has nilpotent length n, $P_i \neq 1$ $(1 \leqslant i \leqslant n)$.

The structure of $\mathscr{L}$-critical groups given by Carter, Fischer and Hawkes in Theorems 5·1 and 5·3 of (9) has the following result as an easy consequence. Some notation will help simplify things. If $P_1P_2 \dots P_n$ is an expanded form of the $\mathscr{L}(n-1)$-critical group G write $A_i(G) = P_1P_2 \dots P_i$ and $B_i(G) = P_iP_{i+1} \dots P_n (1 \leqslant i \leqslant n)$.

LEMMA 4·1. *Let $G = P_1P_2 \dots P_n$ be an $\mathscr{L}(n-1)$-critical group in expanded form.*

(i) *P_i has a unique maximal $B_i(G)$-invariant subgroup, M_i say $(1 \leqslant i \leqslant n)$,*

(ii) $[P_i, P_{i+1}] = P_i$, $1 \leqslant i \leqslant n-1$.

Set $C_i = C_{P_i}(P_{i-1})$ for $i > 1$, and $C_1 = 1$.

(iii) $[M_i, P_{i+1}] \leqslant C_i$, $1 \leqslant i \leqslant n-1$.

(iv) *$C_i \leqslant M_i$ and P_i/C_i is a special p_i-group with $\Phi(P_i/C_i) = M_i/C_i$, $1 \leqslant i \leqslant n$.*

An $\mathscr{L}$-critical group in expanded form is essentially one that splits over each term of its lower nilpotent series. The existence of such groups in a subgroup closed Fitting class is proved below and we rely on it heavily. As with a number of steps in our proof of Theorem 1·1 it is not clear to us that the use of this fact is essential. However we have not been able to do without it.

LEMMA 4·2. *Let $\mathfrak{F} \subseteq \mathfrak{S}_u$ be a subgroup closed Fitting class of nilpotent length $n = l(u)$. Then $\mathfrak{F}$ contains an $\mathscr{L}(n-1)$-critical group in expanded form.*

Proof. Now $\mathfrak{F}$ certainly contains an $\mathscr{L}(n-1)$-critical group since among the subgroups of a group in $\mathfrak{F}$ of nilpotent length n there is one, G say, which is also of nilpotent length n but with every proper subgroup of G having nilpotent length less than n. Of course $G \in \mathrm{s}\mathfrak{F} \subseteq \mathfrak{F}$.

The rest of the proof involves a trick from the theory of varieties of groups, a good general reference for which is Hanna Neumann (22). Let $\mathfrak{V}$ be the variety generated by G, and let H be the projective cover of G in $\mathfrak{V}$ (see Bryant (2)). By 51·1 of (22) H is a quotient of a subgroup of a finite direct power of G and therefore, being projective, is a subgroup of a finite direct power of G. Consequently $H \in \mathfrak{F}$.

Next observe that H is $\mathscr{L}(n-1)$-critical. This follows from Lemma 3·4 of Bryant (1).

Finally to show that H is in expanded form let P_1 be the $\mathfrak{S}_{p_2}\mathfrak{S}_{p_3} \dots \mathfrak{S}_{p_n}$-residual of H, where $u = (p_1, p_2, \dots, p_n)$. P_1 is complemented in H, by H_1 say, because of (3·1) in (5). Also H_1 is projective in $\mathfrak{V}$ on account of Lemma 1 in (2). Then let P_2 be the $\mathfrak{S}_{p_3} \dots \mathfrak{S}_{p_n}$-residual of H_1 and H_2 a complement for it in H_1; and so on. Then $H = P_1P_2 \dots P_n$ is an expanded form of H.

While Lemma 4·2 gives some control over the structure of $\mathscr{L}(n-1)$-critical groups in $\mathfrak{S}_u$ a wide degree of arbitrariness remains, particularly as to the centralizers C_i. An $\mathscr{L}(n-1)$-critical group in expanded form is said to be in *standard* form if the centralizers C_i are all trivial (so that the building blocks of the expanded form are all special groups). One local goal along the way to proving Theorem 2·1 is the following result.

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LEMMA 4·3. *If $\mathfrak{F} \subseteq \mathfrak{S}_u$ is an* s*-closed Fitting class satisfying the hypotheses of Theorem 2·1 then it contains an $\mathscr{L}(n-1)$-critical group in standard form.*

The proof of this will be given in section 7. One property of standard forms is worth noting.

LEMMA 4·4. *If $G = P_1 P_2 \dots P_n$ is an $\mathscr{L}(n-1)$-critical group in standard form then*

$$C_{B_{i+1}(G)}(P_i/\Phi(P_i)) = 1, \quad 1 \leqslant i \leqslant n-1.$$

Proof. For let $N_{i+1} = C_{B_{i+1}}(P_i/\Phi(P_i))$ $(1 \leqslant i \leqslant n-1)$. Now $N_n = C_n = 1$; and, for $i < n-1$, $N_{i+1} \cap P_{i+1} = C_{i+1} = 1$ so $P_{i+1} \leqslant N_{i+1}P_{i+1} \leqslant C_{B_{i+1}}(P_{i+1}/\Phi(P_{i+1}))$, whence $N_{i+1}P_{i+1} \cap B_{i+2}(G) \leqslant N_{i+2} = 1$ by induction. But then

$$N_{i+1}P_{i+1} = P_{i+1}(N_{i+1}P_{i+1} \cap B_{i+2}(G)) = P_{i+1}$$

so $N_{i+1} \leqslant P_{i+1} \cap N_{i+1} = 1$. This completes the induction and the proof.

5. *Representation theory.* Here and in the next section we collect the facts from representation theory needed later. Our notation is mostly standard: we have tended to follow that of Curtis and Reiner(10) for modules and Isaacs(19) for characters. Proofs of results unavailable in the form we want from the literature are included.

Let $\phi: G \to H$ be an epimorphism, and M a kH-module for some field k. The *inflation* of M to G is that kG-module based on M defined by $mg = m(g\phi)$ $(m \in M, g \in G)$. This idea will be used even when ϕ is one-to-one and we are interested merely in the isomorphism class of the semidirect product MG: all such inflations produce isomorphic semidirect products and we will usually not specify the isomorphism ϕ. Another typical context will have $G = AB$, a semidirect product of A by B and M a B-module; here ϕ will always be the natural retraction of G onto B.

Suppose that U is an absolutely irreducible kG-module and that U^* is its contragredient (43·7 in (10)). It follows from 43·14 of (10) that the dimension of the space of fixed points of G in $U \otimes U^*$ is 1. The next result generalizes this slightly.

LEMMA 5.1. *Let U be an irreducible kG-module where k is a field of characteristic $p > 0$. Let K be the algebriac closure of k, then*

$$U^K = U_1 \oplus U_2 \oplus \dots \oplus U_m,$$

where each U_i is irreducible and $U_i \cong U_j$ if and only if $i = j$.

The dimension of the space of fixed points of G in $U \otimes U^*$ is m.

Proof. That U^K is completely reducible with distinct irreducible summands being non-isomorphic follows from 9·21 of (19).

Now

$$\begin{aligned}(U \otimes U^*)^K &= U^K \otimes U^{*K} \\ &= \bigoplus_{i,j} U_i \otimes U_j^*.\end{aligned}$$

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By 43·14 of (10) the space of fixed points of G in $U_i \otimes U_j^*$ is 0 if $i \neq j$ and 1 if $i = j$. Hence the dimension of the fixed point space of G in $(U \otimes U^*)^K$ is m. Since

$$m = \dim_K \mathrm{Hom}_{KG}(T^K, (U \otimes U^*)^K),$$

where T is the trivial irreducible kG-module, and

$$\mathrm{Hom}_{KG}(T^K, (U \otimes U^*)^K) \cong (\mathrm{Hom}_{kG}(T, U \otimes U^*))^K,$$

by 29·5 in (10), m is also the dimension of the fixed point space in $U \otimes U^*$, as required.

LEMMA 5·2. *Let $G = AB$ where $A \trianglelefteq G$ and $A \cap B = 1$. Suppose that $N \trianglelefteq G$ with $N \leqslant B$. Let U be an irreducible kG-module where the characteristic of k is non-zero and prime to $|N|$, and suppose that $C_A(U) = 1$. Let V be the inflation to G of kB.*

There is an irreducible submodule W of $U \otimes V$ for which $C_A(W) = 1$ and $C_B(W) \geqslant N$.

Proof. Let S be an irreducible submodule of $U|_N$. Since $V|_N$ contains a copy of kN it contains a copy of S^*. Hence the space of fixed points, Y say, of N in $U \otimes V$ is non-zero by Lemma 5·1. Also Y is a G-submodule of $U \otimes V$. However $(U \otimes V)|_A$ is a direct sum of copies of U and hence A has no fixed points in $U \otimes V$. We can choose W therefore to be any irreducible kG-submodule of Y.

Next let M be a faithful irreducible module of dimension d for a group G over $GF(q)$, the field of q elements, where q is a prime. $E = \mathrm{End}_{GF(q)G} M$ is a field, of order q^k say and, if $\dim_E M = t$, $d = kt$.

LEMMA 5·3. *In $\mathrm{End}_{GF(q)} M$ there is a finite field F of order q^d containing E.*

Proof. Since M may be regarded as the additive group of $GF(q^d)$ the multiplicative group of $GF(q^d)$ acts on it. Since $q^k - 1 | q^d - 1$, $GF(q^d)^\times$ has a subgroup H of order $q^k - 1$. In their actions on M, $E^\times$ and H are isomorphic as linear groups. That is, for some $g \in GL(M)$, $g^{-1}Hg = E^\times$. Thus $F = g^{-1}GF(q^d)g$ will do.

Let us denote by M^v the direct sum of $v (\geqslant 1)$ copies of M.

LEMMA 5·4. *The semidirect product M^vG has a faithful irreducible representation over a field k of characteristic prime to q if and only if $v \leqslant t$.*

Proof. This is just a special case of Satz 7 of Kochendörffer (20).

LEMMA 5·5. *M^t has a subspace V with the property that for every irreducible submodule W of M^t*

$$M^t = V \oplus W,$$

(where this is the vector space direct sum).

Proof. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ be an E-basis for F, with $\sigma_1 = 1$. Define the subspace V of M^t by

$$V = 0 \quad \text{if} \quad t = 1;$$

and for $t > 1$ set

$$V_i = \{f \in M^t : f(i) = f(1)\sigma_i^{-1}; f(j) = 0, j \neq i, j \neq 1\}, \quad 2 \leqslant i \leqslant t,$$

and then set

$$V = \bigoplus_{i=2}^{t} V_i.$$

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Suppose that W is an arbitrary irreducible submodule of M^t. Let k $(1 \leqslant k \leqslant t)$ be minimal with the property that $f(k) \neq 0$ for $f \in W$. Then

$$W = \{f \in M^t : f(i) = f(k)\,\alpha_i,\ k+1 \leqslant i \leqslant t\},$$

for some endomorphisms $\alpha_{k+1}, \ldots, \alpha_t \in E$. Define $\alpha_1 = \alpha_2 = \ldots = \alpha_{k-1} = 0$. Since V has dimension $d(t-1)$ and W has dimension d to show that $M^t = V \oplus W$ it suffices to show that $V \cap W = 0$. Hence suppose that $f \in V \cap W$. Then

$$f = f_2 + f_3 + \ldots + f_t,$$

where $f_i \in V_i$ $(2 \leqslant i \leqslant t)$. Now for $i \neq k$,

$$\begin{aligned} f(k)\,\alpha_i &= f(i) \\ &= f_i(i) \\ &= f_i(1)\,\sigma_i^{-1}, \end{aligned}$$

so that $f_i(1) = f(k)\,\alpha_i\,\sigma_i$. $(i \neq k)$. Then

$$\begin{aligned} f(1) &= f_2(1) + f_3(1) + \ldots + f_t(1) \\ &= f(k)\,(\alpha_2\sigma_2 + \ldots + \alpha_{k-1}\sigma_{k-1} + 1\,.\,\sigma_1 + \alpha_{k+1}\sigma_{k+1} + \ldots + \alpha_t\sigma_t), \end{aligned}$$

since $f_k(k) = f(k) = f(k)\,\sigma_1$. If $k > 1$ then $f(1) = 0$ whence $f(k) = 0$ since

$$1\,.\,\sigma_1 + \alpha_2\sigma_2 + \ldots + \alpha_{k-1}\sigma_{k-1} + \alpha_{k+1}\sigma_{k+1} + \ldots + \alpha_t\sigma_t \neq 0,$$

being a non-zero linear combination of a basis for F over E. If $k = 1$ then we have

$$f(1)\,(1\,.\,\sigma_1 - \alpha_2\sigma_2 - \ldots - \alpha_t\sigma_t) = 0,$$

and again $f(k) = 0$. In either case $f = 0$ and we are done.

LEMMA 5·6. *Let M and G be as in the preceding discussion, and let p be a prime different from q. Put $X = M^t$, $H = XG$. Choose V as in Lemma 5·5 and let L be a trivial irreducible $GF(p)$ V-module. Then*

$$L|^H = N_1 \oplus N_2,$$

where N_1 is the space of fixed points of X and where every composition factor of N_2 is faithful.

Moreover if W is an arbituary irreducible submodule of X then

$$L|^H|_{WG} \cong GF(p)\,WG.$$

Proof. The Mackey Subgroup Theorem gives us

$$\begin{aligned} L|^H|_X &\cong \bigoplus_{g \in G} L^{(g)}|_{V^g \cap X}|^X \\ &= \bigoplus_{g \in G} L^{(g)}|_{V^g}|^X. \end{aligned}$$

Each direct summand displayed here has one-dimensional fixed point space for X and so if N_1 is the fixed point space for X in $L|^H$ then $\dim N_1 = |G|$. By Fitting's Lemma N_1 has a complement N_2 in $L|^H$.

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Next let W be an arbitrary irreducible submodule of X. Then by Lemma 5·5 $X = V \oplus W$ and G is a complete set of (V, W) coset representatives in H. Then we have

$$L|^H|_W \cong \bigoplus_{g \in G} L^{(g)}|_{Vg \cap W}|^W$$
$$= \bigoplus_{g \in G} L^{(g)}|_1|^W$$

a direct sum of copies of $GF(p)\,W$. As before the dimension of the fixed point space for W in $L|^H$ is $|G|$, and so N_1 is precisely the fixed point space of W in $L|^H$. It then follows that every composition factor in N_2 must be faithful for H.

Finally note that $H = V(WG)$. Then

$$L|^H|_{WG} = L|_{V \cap WG}|^{WG}$$
$$= L|_1|^{WG}$$
$$\cong GF(p)\,WG,$$

and the proof of Theorem 5·6 is complete.

LEMMA 5·7. *Let k be a field, $G_1, \ldots, G_m$ groups and U_i $(1 \leqslant i \leqslant m)$ kG_i-modules. Suppose that, for $1 \leqslant i \leqslant m$, E_i is a subspace of codimension 1 in U_i which contains no non-zero submodules of U_i. In the $G_1 \times \ldots \times G_m$-module $U_1 \# U_2 \# \ldots \# U_m$ let*

$$F_i = U_1 \# \ldots \# U_{i-1} \# E_i \# U_{i+1} \# \ldots \# U_m$$

and put

$$F = F_1 + F_2 + \ldots + F_m.$$

Then F contains no non-zero submodules of $U_1 \# \ldots \# U_m$.

Proof. By induction it is enough to consider the case $m = 2$ since F has co-dimension 1. Suppose therefore that M is a non-zero submodule of $U_1 \# U_2$. $(U_1 \# U_2)|_{G_1}$ is a direct sum of kG_1-modules of the form $U_1 \# v$ $(0 \neq v \in U_2)$ and hence M has non-zero projection onto some $U_1 \# v$ where, moreover, we may suppose $v \notin E_2$. For, if $v \in E_2$ then, since E_2 contains no non-zero submodules of U_2, $vg \notin E_2$ for some $g \in G_2$, and M projects onto a non-zero submodule of $U_1 \# vg$. Consequently since E_1 contains no non-zero submodule of U_1, the projection of M onto $U_1 \# v$ contains an element $u \# v$ where $u \notin E_1$. However the projection of $U_1 \# E_2 + E_1 \# U_2$ onto $U_1 \# v$ is contained in $E_1 \# v$ so $M \nleqslant E_1 \# U_2 + U_1 \# E_2$.

6. *More representation theory.* Let G be a group, k a field and U a kG-module with basis $\{u_1, u_2, \ldots, u_m\}$. Let σ be an automorphism of k and α an automorphism of G. For $g \in G$ we have

$$u_i g = \sum_j k_{ij} u_j,$$

say. Define a kG-module U^σ on U by setting

$$u_i g = \sum_j (k_{ij})\,\sigma u_j,$$

and another kG-module U^α on U by setting $ug = u(g\alpha)$ $(u \in U)$.

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The following notation and conditions apply for the whole of this section. Let $G = QH$ where Q is a normal extraspecial q-subgroup of G and $Q \cap H = 1$. Further: $[Q, F(H)] = Q$ where $F(H)$ is the Fitting subgroup of H; $Z(Q) \leqslant Z(G)$; and H/H' has odd order. Let k be a finite field of characteristic p different from q and let K be the algebraic closure of k.

We make the following observation for future reference: standard arguments show that $H = N_G(O_{q'}(F(H)))$ modulo $Z(Q)$ and in consequence if α is an automorphism of G such that $Q = Q\alpha$ then H and $H\alpha$ are conjugate in G modulo $Z(Q)$.

This section is largely devoted to proving the following result which is crucial to the proof of Theorem 2·1.

THEOREM 6·1. *Let $U_1, U_2, \ldots, U_m$ be a complete set of faithful, non-isomorphic irreducible kQ-modules. There exist irreducible kG-modules $V_1, V_2, \ldots, V_m$ such that*

(i) $V_i|_Q \cong U_i$, $1 \leqslant i \leqslant m$,

(ii) *if* $U_j \cong U_i^*$ *then* $V_j \cong V_i^*$, *and*

(iii) *if α is an automorphism of G under which Q is invariant and if $U_j \cong U_i^\alpha$ then $V_j \cong V_i^\alpha$.*

Proof. Techniques capable of giving a direct proof of this result for an arbitrary field seem to be lacking. Our proof divides the cases $p = 2$ and p odd, though the broad outline is the same for each. Results of Dade and Isaacs enable appropriate extensions from *complex* irreducible Q-representations to G-representations to be made; then a Brauer character technique is used to transfer the results to K; and finally standard results on the behaviour of modules under field extension complete the proof.

It is worth noting that when q is odd the requirement that $|H/H'|$ be odd is unnecessary.

We begin observing that over K, and over $\mathbf{C}$ (the complex numbers), Q has $q-1$ distinct faithful irreducible modules, each of dimension a power of q: see Huppert(17), 5·16·14.

The case $q = 2$. Let χ be the unique faithful irreducible character of Q over $\mathbf{C}$. By a result of Dade(11) there is a character θ of G such that $\theta|_Q = \chi$. By 51·7 of(10) every other extension of χ to G is of the form $\lambda\theta$ where λ is a linear character of G. Now the group of linear characters of G is isomorphic to $G/G' \cong H/H'$ and so has odd order. Also $\theta(1)$, the degree of θ, is a power of 2; and

$$\det(\lambda\theta) = \lambda^{\theta(1)} \det\theta$$

(see p. 29 of(19) for the definition of the determinant of a character). Since unique $\theta(1)$-th roots exist in the group of linear characters of G there is a unique linear character λ' such that $\det(\lambda'\theta)$ is the principal character. Put $\phi = \lambda'\theta$. Then ϕ is uniquely determined by the properties

(i) $\phi|_Q = \chi$, and

(ii) $\det\phi = 1_G$.

We now transfer these results to K. By the preamble to 15·8 of(19) there is a local

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ring of integers $\tilde{R}$ and an $\tilde{R}G$-module Y affording ϕ. Then by 15·8 of (19) there is a KG-module Y' satisfying

(i)′ $Y'|_Q$ is isomorphic to the unique faithful irreducible KG-module and

(ii)′ each element of G, regarded as a linear transformation of Y', has determinant 1.

Moreover Y' is the unique KG-module satisfying (i)′ and (ii)′. For, if Z' is another then there is, by the Fong–Swan Theorem (Puttaswamaiah and Dixon (24)), a complex character ψ satisfying (i) and so $\psi = \lambda\phi$. But then $Z' = Y' \otimes M'$ where M' is a one-dimensional KG-module on which, of course, the elements of G are represented by scalars. Much as above it follows that there is a unique M' for which Z' satisfies (ii)′.

Finally we cut the field down to k. Since $(kG)^K = KG$ it follows that there is an irreducible kG-module V such that V^K has Y' as a composition factor. Then

$$V^K = V_1 \oplus V_2 \oplus \ldots \oplus V_m,$$

with $V_1 \cong Y'$ say. By 9·21 of (19) $V_1, V_2, \ldots, V_m$ are non-isomorphic Galois conjugates of V_1. However (i)′ and (ii)′ are clearly preserved under Galois conjugation and hence $V_i \cong Y'$ $(1 \leqslant i \leqslant m)$. This means that $m = 1$ and V is absolutely irreducible.

Let U be a faithful irreducible kQ-module. Then another application of 9·21 in (19) shows that U^K is irreducible. Since there is a unique irreducible faithful KQ-module it follows from 29·7 of (10) that U is also unique. Hence $U \cong V|_Q$. Also notice that (i)′ and (ii)′ are preserved under the forming of contragredient modules and the action of automorphisms of G leaving Q invariant. Hence

$$\begin{aligned}(V^*)^K &\cong (V^K)^* \\ &= V^K,\end{aligned}$$

and, for all $\alpha \in \mathrm{Aut}\, G$ leaving Q invariant,

$$\begin{aligned}(V^\alpha)^K &\cong (V^K)^\alpha \\ &= V^K,\end{aligned}$$

the last step in each case coming from the uniqueness of Y' in (i)′ and (ii)′. Then from 29·7 in (10) we obtain $V^* \cong V$ and $V^\alpha \cong V$. This completes the proof of the theorem when $q = 2$.

The case $q \neq 2$. Let $\chi_1, \chi_2, \ldots, \chi_{q-1}$ be the faithful irreducible complex characters of Q. Again by Dade (11) each χ_i has an extension to a character θ_i of G, and Dade observes that it follows from Isaacs (18) that this extension can be defined canonically: we set about showing how this is done.

From 5·16·14 (2) of (17) there is an irreducible character λ_i of $Z(Q)$ such that $\chi_i|_{Z(Q)}$ is a multiple of λ_i. Moreover $\lambda_1, \lambda_2, \ldots, \lambda_{q-1}$ are distinct so χ_i is determined uniquely by λ_i. Since $[Z(Q), H] = 1$ it follows from 51·13 of (10) that

$$\theta_i|_{Z(Q)H} = \lambda_i \psi_i, \quad 1 \leqslant i \leqslant q-1,$$

where λ_i is now regarded as a character for $Z(Q)\,H$ by inflation and ψ_i is a character for $Z(Q)\,H$ whose kernel contains $Z(Q)$.

We now apply the results of Isaacs (18), in particular 5·3: the reader is left to

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translate his results into our terminology. Observe that the relevant constellation is $(G, Q, Z(Q)H, Z(Q), \chi_i, \lambda_i)$ with prime q and that ψ_i is a character arising from the constellation. (The relevant definitions are on pp. 600, 606 of (18).) The proof of 5·1 in (18) ensures that by multiplying ψ_i by a suitable power of $\det \psi_i$ we get a character of determinantal order a power of q which still arises from the constellation. Hence we may suppose θ_i chosen so that $\det \psi_i$ has order a power of q and, moreover, by 5·3 of (18), that if H_q is a Sylow q-subgroup of H then the principal character is the unique irreducible constituent of $\psi_i|_{Z(Q)H_q}$ with odd multiplicity. Now $Q \leqslant \ker \det \theta_i$, $\lambda_i|_H$ is principal so $\theta_i|_H = \psi_i|_H$ whence $\det \theta_i$ has order a power of q.

To sum up so far: θ_i may be chosen so that

(*a*) $\theta_i|_Q = \chi_i$, $1 \leqslant i \leqslant q-1$;

(*b*) $\det \theta_i$ has order a power of q;

(*c*) $1_{Z(Q)H_q}$ is the unique irreducible constituent of $\psi_i|_{Z(Q)H_q}$ with odd multiplicity.

Moreover if ϕ_i is a character of G for which $\phi_i|_Q = \chi_i$ then, as in the case $q = 2$, $\phi_i = \mu_i \theta_i$ where μ_i is a linear character of G.

Suppose that X_i, Y_i, L_i, Z_i are modules over $\mathbf{C}$ affording the characters $\chi_i, \theta_i, \lambda_i, \chi_i$ respectively ($1 \leqslant i \leqslant q-1$). As explained in the preamble to 15·8 of (19) we may assume them to be over $\tilde{R}$ a suitable local ring of integers. Then by 15·8 in (19) we may transfer them to K, obtaining modules X_i', Y_i', L_i' and Z_i' with the properties

(*a*)′ $Y_i'|_Q = X_i'$, $\quad 1 \leqslant i \leqslant q-1$;

(*b*)′ for each $g \in G$ the determinant of g, as linear transformation of Y_i', is a q-power root of unity, and

(*c*)′ the trivial irreducible $K(Z(Q))H_q$)-module is the unique irreducible constituent of $Z_i'|_{Z(Q)H_q}$ with odd multiplicity.

The proof of (*a*)′ is immediate; (*b*)′ follows because, being linear, $\det \theta_i = 1$ on Q and on H is equal to $\det \psi_i$; and (*c*)′ follows since the process of transferring modules from $\tilde{R}$ to K preserves irreducibility.

Next we show that the conditions (*a*)′, (*b*)′, (*c*)′ determine Y_i' uniquely. For suppose that R_i' is a KG-module which satisfies (*a*)′, (*b*)′ and (*c*)′, where to satisfy (*c*)′ we mean that

$$R_i'|_{Z(Q)H_q} = L_i' \otimes S_i',$$

and in S_i' the unique irreducible constituent with odd multiplicity is trivial. Now the Fong–Swan Theorem ensures that there is an irreducible $\tilde{R}G$-module R_i which, transferred to K, gives R_i'. Then 15·13 of (19) yields that $R_i|_Q \cong X_i$. But 51·13 of (10) then ensures that

$$R_i|_{Z(Q)H_q} = L_i \otimes S_i,$$

(where the justification for writing the second factor as S_i is that its transference to K does give S_i').

Let ϕ_i be the character of R_i so that $\phi_i = \mu\theta_i$ where μ is a linear character of G. We aim to show that μ has order a power of p. Suppose that r is a prime different from p, q

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dividing the order of μ. Since $\det \psi_i$ does have order a power of q it follows that for some $h \in H$, $h \in \ker \psi_i$ and $\mu(h)$ is of order a power of r different from 1. But then

$$\begin{aligned}\det \phi_i(h) &= \mu(h)^{\theta_i(1)} \det \theta_i(h) \\ &= \mu(h)^{\theta_i(1)} \det \psi_i(h),\end{aligned}$$

since ${}_{\theta i}|_H = \psi_i$, so

$$\det \phi_i(h) = \mu(h)^{\theta_i(1)},$$

and the order of $\det \phi_i(h)$ is therefore a power of r not equal to 1, $\theta_i(1)$ being a power of q and therefore coprime to the order of $\mu(h)$. Transferring to R_i', we find that the determinant of h on it has order a power of q, contradicting $(b)'$.

Next let σ_i be the character of S_i. Then

$$\begin{aligned}\lambda_i \sigma_i = \phi_i|_{Z(Q)H_q} &= \theta_i|_{Z(Q)H_q} \mu|_{Z(Q)H_q} \\ &= (\lambda_i \psi_i)\, \mu|_{Z(Q)H_q}\end{aligned}$$

so that

$$\sigma_i = \psi_i(\mu|_{Z(Q)H_q}).$$

If the order of $\det \mu$ is divisible by a power of q not 1 then $\mu|_{Z(Q)H_q}$ is not the principal character but is an irreducible constituent of odd multiplicity of σ_i. This contradicts the property $(c)'$ of S_i'. Hence the order of μ is indeed a power of p.

However being a one-dimensional module over a field of characteristic p, M' is trivial. Therefore $R_i' = Y_i' \otimes M' \cong Y_i'$, so Y_i' is the unique module satisfying $(a)'$, $(b)'$ and $(c)'$. The importance of this fact is this: if, for some $i, j \in \{1, 2, \ldots, q-1\}$, $X_j' \cong X_i'^*$ then $Y_j' \cong Y_i'^*$; and if $X_j' \cong X_i'^{\alpha}$ then $Y_j' \cong Y_i'^{\alpha}$ where $\alpha \in \operatorname{Aut} G$ leaves Q invariant. The proof of these is easy since properties $(b)'$, $(c)'$ are preserved when contragredients are formed or such automorphisms α are applied.

Finally we get back to the field k. Let U be a faithful irreducible kQ-module. By 9·21 of (19) we have that U^K is a direct sum of a Galois conjugacy class of faithful irreducible KQ-modules one of which we may take to be X_i' for some $i \in \{1, 2, \ldots, q-1\}$. Now as in the case $q = 2$ there is an irreducible kG-module V such that Y_i' is a component of V^K. Then

$$V^K = S_1 \oplus S_2 \oplus \ldots \oplus S_r,$$

where $S_1, S_2, \ldots, S_r$ are non-isomorphic but Galois conjugate, and $S_1 \cong Y_i'$, say, using 9·21 of (19) again. Moreover since properties (b), (c) are preserved under field automorphisms $S_k|_Q \cong S_l|_Q$ entails $S_k \cong S_l$ whence $k = l$. Therefore the modules $S_k|_Q$ $(1 \leqslant k \leqslant r)$ are distinct and Galois conjugate. If W is an irreducible submodule of $V|_Q$, $S_i|_Q \leqslant W^K$ for some i, and then, since W^K contains all Galois conjugates of $S_i|_Q$, we have $W^K = V^K$, and so $V|_Q$ is irreducible. It now follows from 29·6 of (10) that U is isomorphic to $V|_Q$.

Hence if $U_1, U_2, \ldots, U_m$ are a complete set of non-isomorphic, faithful irreducible kQ-modules then there exist Galois conjugacy classes $\{X_{i,1}', \ldots, X_{i,m(i)}'\}$ $(1 \leqslant i \leqslant m)$ of $\{X_1', \ldots, X_{q-1}'\}$ and kG-modules $V_1, V_2, \ldots, V_m$ such that

$$V_i|_Q = U_i, \quad 1 \leqslant i \leqslant m,$$

$$U_i^K = X_{i,1} \oplus \ldots \oplus X_{i,m(i)},$$

and

$$V_i^K = Y_{i,1}' \oplus \ldots \oplus Y_{i,m(i)}',$$

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where $Y'_{i,j}$ is the unique extension of $X'_{i,j}$ satisfying properties $(a)'$, (b') and $(c)'$. Now $U_j \cong U_i^*$ means that there is a function $\tau\colon \{1, 2, \ldots, m(j)\} \to \{1, 2, \ldots, m(i)\}$, one-to-one and onto, such that

$$X'_{j,l} \cong X'^{*}_{i,\tau(l)}.$$

From above we have that

$$Y'_{j,l} \cong Y'^{*}_{i,\tau(l)}, \quad 1 \leqslant l \leqslant m(j),$$

whence $V_j^K \cong (V_i^K)^*$, and so $V_j^K \cong (V_i^*)^K$, and therefore $V_j \cong V_i^*$.

Finally take $\alpha \in \mathrm{Aut}\, G$, with Q α-invariant, and suppose $U_i^\alpha \cong U_j$. Clearly, $(Y'_{i,l})^\alpha$ satisfies property $(b)'$, and also satisfies property $(c)'$ with respect to $Z(Q)H_q\alpha$. But, as observed at the start of this section, $Z(Q)H/Z(Q)$ and $Z(Q)H\alpha/Z(Q)$ are conjugate in $G/Z(G)$. Since $Z(Q)H_q/Z(Q)$ and $Z(Q)H_q\alpha/Z(Q)$ are Sylow q-subgroups of these, it follows that $Z(Q)H_q$ and $Z(Q)H_q\alpha$ are conjugate in G. Thus $(Y'_{i,l})^\alpha$ also satisfies property $(c)'$. As in the previous paragraph, we see that $V_i^\alpha \cong V_j$. This completes the proof of Theorem 6·1.

The next result is needed later on and contains the crucial application of Theorem 6·1.

THEOREM 6·2. *Let G be an $\mathscr{L}(m-1)$-critical group in expanded form $P_1P_2\ldots P_m$, where*

(i) *P_1 is a non-abelian special p_1-group, and;*

(ii) *$P_1/\Phi(P_1)$ is faithful and irreducible as $B_2(G)$-module.*

Suppose that for some $k \in \{2, 3, \ldots, m\}$

(iii) *p_k is odd and*

(iv) *$Z(P_1) \leqslant Z(A_k(G))$.*

Let k be a finite field of characteristic $q \neq p_1$ and let M be a faithful irreducible G-module over k. Write

$$M|_{A_k(G)} = M_1 \oplus \ldots \oplus M_t,$$

for an unrefinable decomposition. Suppose that $P_1/C_{P_1}(M_1)$ is an extra-special group. Let N be the inflation of the regular module $kB_2(G)$ to G.

Then there exists an irreducible submodule W of $M \otimes N$ such that the fixed point spaces of P_1 and of $A_k(G)$ in $W \otimes W$ are identical.

Proof. For convenience write $P = P_1$, $B = P_2P_3\ldots P_k$ and $C = C_{P_1}(M_1)$. Since P/C is extra-special it follows that $Z(P/C) = Z(P)C/C$. Hence by (iv) $Z(P/C) \leqslant Z(PB/C)$. The group PB/C satisfies the requirements of the group in Theorem 6·1. Here $Q = P/C$, $H = BC/C$; that $|H/H'|$ is odd follows from (iii) and that $[Q, F(H)] = Q$ follows from (ii).

First note that $M_1|_P$ is homogeneous. For, if X, Y are irreducible components of $M_1|_P$ then they are conjugate so that $X|_{Z(P)} \cong Y|_{Z(P)}$, $Z(P)$ being central in PB; but then $X^K|_{Z(P)} \cong (X|_{Z(P)})^K \cong (Y|_{Z(P)})^K \cong Y^K|_{Z(P)}$ where K is the algebraic closure of k. However, by 9·21 of (19), X^K and Y^K is each a direct sum of non-isomorphic irreducible modules, faithful for P/C and since the isomorphism type of a faithful irreducible $K(P/C)$-module is determined by its restriction to $Z(P/C)$ (5·16·14 (2) (17)) we have that $X^K \cong Y^K$ and then that $X \cong Y$. Denote an irreducible component of $M_1|_P$ by U_1. The inertia subgroup H of U_1 contains PB.

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It follows from the last stages of the proof of Theorem 6·1 that there is an extension V_1 of U_1 to PB such that if

$$U_1^K = X_1 \oplus \ldots \oplus X_s,$$

is an unrefinable decomposition then there are extensions $Y_1, \ldots, Y_s$ of $X_1, \ldots, X_s$ to PB such that

$$V_1^K = Y_1 \oplus \ldots \oplus Y_s.$$

Moreover if

$$M_1^K = N_1 \oplus \ldots \oplus N_v,$$

is an unrefinable decomposition we may suppose that $X_1 \leqslant N_1|_P$. Of course $N_1|_P$ is homogeneous, and it follows from the Fong–Swan Theorem and from 51·7 of (10) that

$$N_1 \cong Y_1 \otimes S,$$

where S is the inflation of an irreducible KB-module to PB.

Now by 29·5 of (10) we have

$$\begin{aligned}(\mathrm{Hom}_{kPB}(V_1, M_1 \otimes N|_{PB}))^K &\cong \mathrm{Hom}_{KPB}(V_1^K, (M_1 \otimes N|_{PB})^K) \\ &\geqslant \mathrm{Hom}_{KPB}(Y_1, N_1 \otimes N|_{PB}^K) \\ &\cong \mathrm{Hom}_{KPB}(Y_1, Y_1 \otimes S \otimes N|_{PB}^K) \\ &\geqslant \mathrm{Hom}_{KPB}(Y_1, Y_1 \otimes S \otimes S^*),\end{aligned}$$

since $N|_{PB}^K$ is free as KB-module and hence contains a submodule isomorphic to S^*. But $S \otimes S^*$ contains a trivial submodule so $\mathrm{Hom}_{KPB}(Y_1, Y_1 \otimes S \otimes S^*) \neq 0$. It follows that $\mathrm{Hom}_{kPB}(V_1, M_1 \otimes N|_{PB}) \neq 0$ and hence that V_1 is isomorphic to a submodule of $(M \otimes N)|_{PB}$.

Next we show that there is an irreducible submodule W of $M \otimes N$ such that V_1 is isomorphic to a submodule of $W|_{PB}$. For we know that there is a submodule V_0 of $(M \otimes N)|_{PB}$ with $V_0 \cong V_1$, and the G-submodule T of $M \otimes N$ generated by V_0 is a sum of all the conjugates $V_0^g (g \in G)$. Let W be an irreducible submodule of T so that $W|_{PB}$ is a direct sum of conjugates of V_0. A suitable conjugate of W will contain V_0.

Next we dispose of a triviality. Suppose that $(M \otimes M)|_P$ contains no non-zero fixed points. Then since $P \leqslant \ker N$ it follows that $((M \otimes N) \otimes (M \otimes N))|_P$ has zero fixed point space for P, and the same is therefore true of every irreducible submodule W of $M \otimes N$. Hence for every such W the fixed point spaces of P and of PB in $W \otimes W$ are the same.

From now on, therefore, we assume that $(M \otimes M)|_P$ contains non-zero fixed points. Since M is a direct sum of conjugates of U_1 it follows that, for some $x \in G$, $U_1 \otimes U_1^x$ has non-zero fixed points. It then follows from 43·14 of (10) that

$$U_1^x \cong U_1^*.$$

Next choose W as in the antepenultimate paragraph, and write

$$W|_{PB} = L_1 \oplus L_2 \oplus \ldots \oplus L_r,$$

where L_i are the homogeneous components. In particular suppose that

$$L_1 = K_1 \oplus K_2 \oplus \ldots \oplus K_v,$$

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where $K_i \cong V_1$ $(1 \leqslant i \leqslant v)$. Now $K_i|_P \cong U_1$; and all conjugates of K_1 in G have the form V_1^α where α is an automorphism of PB induced by an inner automorphism of G. It follows from Theorem 6·1 therefore that H, the inertia subgroup of U_1, is also the inertia subgroup of L_1. We distinguish two cases.

First suppose that $x \in H$. U_1 is then self-contragredient. From Theorem 6·1 it follows that L_1 is also self-contragredient, and then each L_i is self-contragredient. Moreover $L_i|_P$ and $L_j|_P$ are not contragredient for $i \neq j$. Consequently for $i \neq j$ $(L_i \otimes L_j)|_P$ contains no non-zero fixed points (43·14 of (10)) and to establish the result we must show that

$$\bigoplus_{i=1}^{r} L_i \otimes L_i$$

has the same fixed point space for P as for PB. Clearly it suffices to show this for $L_1 \otimes L_1$. But

$$L_1 \otimes L_1 = \bigoplus_{i,j} K_i \otimes K_j,$$

and it follows that we need only show that $V_1 \otimes V_1$ has the same fixed point space for P as for PB. This follows from Lemma 5·1; both fixed point spaces have dimension s.

Finally suppose that $x \notin H$. Since

$$U_1^{x^2} = (U_1^x)^x \cong (U_1^*)^x \cong (U_1^x)^* \cong U_1^{**} \cong U_1,$$

we have $x^2 \in H$. Also since $L_1^x \cong L_1^*$ the inertia subgroup of L_1^* is H^x. But it is also H since L_1, L_1^* have the same inertia subgroup. Hence x normalizes H. Put $J = \langle H, x \rangle$ and let T be a right transversal of G to J. Since $L_1 \oplus L_1^x$ is self-contragredient, so also is $(L_1 \oplus L_1^x)^t$ for all $t \in T$; and if t_1, $t_2 \in T$ with $t_1 \neq t_2$ then $L_1^{t_1}$, $L_1^{xt_1}$, $L_2^{tx_2}$, $L_2^{xt_2}$ are non-isomorphic. Consequently for $t_1 \neq t_2$

$$(L_1 \oplus L_1^x)^{t_1} \otimes (L_1 \oplus L_1^x)^{t_2}$$

has no non-zero fixed points for P. Hence to finish the proof of Theorem 6·2 we need only show that in

$$(L_1 \oplus L_1^x) \otimes (L_1 \oplus L_1^x),$$

P and PB have the same fixed points. This reduces to showing that in $L_1 \otimes L_1^x$ this is true. Now

$$L_1 \otimes L_1^x = \bigoplus_{i,j} K_i \otimes K_j^x,$$

and $K_i^* \cong K_j^x$ $(1 \leqslant i, j \leqslant v)$. As before the fixed point spaces of P and PB are equal in $K_i \otimes K_j^x$, both having dimension s. This concludes the proof of Theorem 6·2.

7. *Constructing groups in standard form in* s*-closed Fitting classes.* Throughout this section $\mathfrak{F}$ is an s-closed Fitting class satisying

$$\mathfrak{F} \subseteq \mathfrak{S}_u \quad \text{where} \quad l(u) = n,$$

and

$$\mathfrak{S}_u \cap \mathfrak{N}^{n-1} \subseteq \mathfrak{F}.$$

A number of techniques for modifying $\mathscr{L}(n-1)$-critical groups in $\mathfrak{F}$ are collected

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together here: they will be needed in the proof of Theorem 2·1. The present aim is to prove Lemma 4·3.

The first contains a technique we will use many times. Roughly speaking the technique involves replacing a group G in $\mathfrak{F}$ by another group in $\mathfrak{F}$ 'similar' to it in which a prescribed chief factor has been changed. We have been unable to refine this technique to a point where all its uses can be covered by a single lemma. A precise description of the technique covering most applications is given in Lemma 7·1. For other uses we will describe the technique and suppress most of the detail.

LEMMA 7·1. *Let $G = P_1P_2 \dots P_n$ be an $\mathscr{L}(n-1)$-critical group in expanded form in $\mathfrak{F}$. For some $i \in \{1, 2, \dots, n-2\}$ let $U = P_i/M_i$ as $B_{i+1}(G)$-module and let V be an irreducible submodule of*

$$U|_{P_{i+1}}{}^{B_{i+1}(G)}.$$

There exists in $\mathfrak{F}$ an $\mathscr{L}(n-1)$-critical group H in expanded form $Q_1Q_2 \dots Q_n$ such that

(i) $B_{i+1}(G) \cong B_{i+1}(H)$;

(ii) *if N_i is the unique maximal $B_i(H)$-invariant subgroup of Q_i then Q_i/N_i as $B_{i+1}(H)$-module is the inflation of V under the isomorphism in* (1); *and*

(iii) *if $D_k = C_{Q_k}(Q_{k-1})$ $(2 \leqslant k \leqslant n)$ and $D_1 = 1$, then Q_k/D_k is abelian if P_k/C_k is abelian.*

Proof. For convenience write $B = B_{i+1}(G)$. By Lemma 3·6

$$K = A_i(G) \operatorname{twr}_{P_{i+1}} B \in \mathfrak{F}.$$

The group H will be constructed as a subgroup of K and will therefore be in $\mathfrak{F}$.

To begin set $Q_j = P_j$, $N_j = M_j$ for $i+1 \leqslant j \leqslant n$.

The following notation will be convenient: for any subgroup L of $A_i(P)$, invariant under P_{i+1}, denote $L|^B$ by L^*. This notation is fairly standard but it does conflict with that for contragredient modules which, however, do not occur in the present proof.

The subgroups Q_j $(1 \leqslant j \leqslant i)$ of K are chosen inductively, beginning with Q_i, with the following properties:

- $Q_j \leqslant P_j^*$ and Q_j is normalized by $Q_{j+1}, \dots, Q_n$;
- $[Q_j, Q_{j+1}] = Q_j$;
- $Q_j \nleqslant M_j^*$ and $N_j = Q_j \cap M_j^*$ is the unique maximal $Q_jQ_{j+1} \dots Q_n$-invariant subgroup of Q_j;
- Q_i/N_i is the inflation of V to a $Q_{i+1} \dots Q_n$-module given by the natural isomorphism of $Q_{i+1} \dots Q_n$ to $B_{i+1}(G)$.

To begin note that as B-module $P_i^*/M_i^* \cong U_{P_{i+1}}|^B$ so one can find a B-subgroup X_i of P_i^* containing M_i^* so that $X_i/M_i^* \cong V$ as B-module. Let Q_i be a minimal B-invariant supplement for M_i^* in X_i. Then, if $N_i = Q_i \cap M_i^*$, $Q_i/N_i \cong V$ as required. Now by the Mackey Subgroup Theorem $U|_{P_{i+1}}|^B|_{P_{i+1}}$ has zero fixed point space and hence so does $V|_{P_{i+1}}$.

Consequently Fitting's Lemma and the minimality of Q_i ensures that P_{i+1} has no non-trivial fixed points in $Q_i/\Phi(Q_{i+1})$. Hence $[Q_i, Q_{i+1}] = [Q_i, P_{i+1}] = Q_i$. The

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minimality of Q_i ensures that N_i is the unique maximal B-invariant normal subgroup of Q_i. This completes case $j = i$.

Next suppose that $j < i$ and that for $j+1 \leqslant k \leqslant i$ subgroups Q_k satisfying the inductive hypotheses have been chosen. Choose Q_j as follows. Since $C_{P_{j+1}}(P_j/M_j)$ is a $B_j(G)$-invariant subgroup of P_{j+1} which cannot be the whole of P_{j+1}, since $[P_j, P_{j+1}] = P_j$, we have that $[Q_{j+1}, P_j^*] \nleqslant M_j^*$. Hence by Fitting's Lemma there is a non-trivial complement Y/M_j^* for the fixed point space of Q_{j+1} in P_j^*/M_j^*. Let X_j/M_j^* be an irreducible $Q_{j+1} \dots Q_n$-submodule of Y/M_j^*. Note that Q_{j+1} has no fixed points in X_j/M_j^*. Now let Q_j be a minimal $Q_{j+1} \dots Q_n$-invariant supplement for M_j^* in X_j. Put $N_j = Q_j \cap M_j^*$ and then N_j is the unique maximal $Q_j Q_{j+1} \dots Q_n$-invariant subgroup of Q_j; Q_j/N_j is irreducible and has no fixed points for Q_{j+1}. As for the case $j = i$ we conclude that

$$[Q_j, Q_{j+1}] = Q_j.$$

This concludes the induction.

Put $H = Q_1 Q_2 \dots Q_n$. We need to show that H is $\mathscr{L}(n-1)$-critical and that condition (iv) holds. To this end note that for $1 < k \leqslant i$

$$\begin{aligned}[N_k, Q_{k+1}] &\leqslant [M_k^*, P_{k+1}^*] \cap Q_k \\ &\leqslant C_{P_k^*}(P_{k-1}^*) \cap Q_k \\ &\leqslant C_{Q_k}(P_{k-1}^*) \\ &\leqslant C_{Q_k}(Q_{k-1}) \\ &= D_k,\end{aligned}$$

and that

$$[N_1, Q_2] \leqslant [M_1^*, P_2^*] = 1.$$

Writing $D_k = C_k$ for $i+1 \leqslant k \leqslant n$ we have in any case $[N_k, Q_{k+1}] = [M_k, P_{k+1}] \leqslant C_k = D_k$. Also $D_k \leqslant N_k$ $(1 \leqslant k \leqslant n)$. For $i+1 \leqslant k \leqslant n$ this follows because $C_k \leqslant M_k$; and for $1 \leqslant k \leqslant i$ simply observe that D_k is a proper $Q_k \dots Q_n$-invariant subgroup of Q_k and hence must be in N_k.

Finally note that if K/L is a chief factor of $Q_k Q_{k+1} \dots Q_n$ between D_k and N_k then

$$\begin{aligned}Q_{k+1} &\leqslant C_{Q_k \dots Q_n}(D_k/N_k) \\ &\leqslant C_{Q_k \dots Q_n}(K/L),\end{aligned}$$

so that the automizer of K/L has nilpotent length at most $n-k-1$. Also H has nilpotent length n; and the chief factors

$$(Q_1 \dots Q_j)/(Q_1 \dots Q_{j-1} N_j), \quad 1 \leqslant j \leqslant n,$$

are complemented in H whilst the remaining chief factors in any refinement of the chain

$$\dots < Q_1 \dots Q_{j-1} N_j < Q_1 \dots Q_j < Q_1 \dots Q_j N_{j+1} < \dots$$

are Frattini, so that in any chief series of H there are precisely n complemented chief factors. It now follows from Theorem 5·1 of Carter, Fischer and Hawkes (9) that H is $\mathscr{L}(n-1)$-critical; and clearly it is in expanded form. As for (iv) of the Lemma: if P_i/C_i, for some $i > 1$, is abelian then by (iv) of Lemma 4·1, $M_i = C_i$ whence

$$[Q_{i-1}, N_i] \leqslant [P_{i-1}^*, M_i^*] = 1,$$

so $N_i \leqslant D_i$. But $D_i \leqslant N_i$ so $N_i = D_i$ and therefore Q_i/D_i is abelian.

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The next result allows us to factor out certain normal subgroups of an $\mathscr{L}(n-1)$-critical group in $\mathfrak{F}$. It is the result we need to repair the error in (8) mentioned in section 1 and the way this is done is explained below.

LEMMA 7·2. *Let $G = P_1P_2 \dots P_n$ be an $\mathscr{L}(n-1)$-critical group in $\mathfrak{F}$ in expanded form and suppose that, for some $i > 1$, there is a normal subgroup N of G contained in P_i. Then $G/N \in \mathfrak{F}$.*

The error in (8) lies in the assumption in the last paragraph on p. 203 that the group $P(F_0/\Phi(F_0))R$ has nilpotent length three precisely. However it could be of nilpotent length two, indeed in particular examples is so. This last page needs to be replaced by the following argument. By Lemmas 4·5 and 2·9 of (8) it suffices to show that $G/C_{QR}(P) \in \mathfrak{X}$, in the notation of section 4 of (8); and Lemma 7·2 above allows one to conclude that this is true. One should note that the proof of Lemma 7·2 about to be given in no way depends on the truth of the nilpotent length three case.

Proof of Lemma 7·2. Write $A = A_{i-1}(G)$ and $B = B_i(G)$. Since $[A, N] = 1$, $N \leqslant C_i$. Hence G/N satisfies all the conditions of Theorem 5·1 of (9) and so is $\mathscr{L}(n-1)$-critical. We may therefore assume that N is minimal normal in G and then induction on $|G|$ will take care of the general case.

Hence suppose that N is a minimal normal subgroup of G. Put $C = B/N$. Since N centralizes A, C acts on A naturally and G/N is isomorphic to the semidirect product AC.

Let U be a $GF(p_i)G$-module isomorphic to N regarded as $GF(p_i)G$-module. Also let V be the projective cover of the trivial irreducible $GF(p_i)C$-module. Put

$$W = U \not\# V,$$

and if T is the trivial irreducible submodule of V write $X = U \not\# T$. Let H be the semidirect product $W(G \times C)$. Note that N, X are minimal normal subgroups of H; indeed they are isomorphic as H-modules. Suppose that D is a diagonal minimal normal subgroup of H in $N \times X$. The first step of the proof is to show that $H/D \in \mathfrak{F}$.

To this end observe that H is a normal product of WG and WC. Hence it suffices to prove that WG/D and WCD/D are both in $\mathfrak{F}$. The second is easy:

$$\begin{aligned} WCD/D &\cong WC/WC \cap D \\ &\cong WC, \end{aligned}$$

since $WC \cap D = 1$. But $WC \in \mathfrak{S}_{(p_i, \dots, p_n)} \subseteq \mathfrak{F}$ since $i > 1$. Now $W|_G$ is completely reducible so X has a complement Z in $W|_G$. Note that $A \trianglelefteq H$ since it centralizes W and that $DA \cap DZ = D$. Hence WG/D is a subdirect product of WG/DA and WG/DZ. But

$$\begin{aligned} WG/DA &= (DA)(WB)/DA \\ &\cong WB/WB \cap DA \\ &\in \mathfrak{S}_{(p_i, \dots, p_n)} \\ &\subseteq \mathfrak{F}; \end{aligned}$$

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and

$$\begin{aligned} WG/DZ &= (DZ)G/DZ \\ &\cong G/G \cap DZ \\ &\cong G \\ &\in \mathfrak{F}, \end{aligned}$$

since $G \cap DZ = 1$, and therefore $WG/D \in \mathfrak{F}$. This concludes the proof that $H/D \in \mathfrak{F}$.

Next we put

$$B_0 = \{(b, Nb) : b \in B\}.$$

Observe that $N \leqslant B_0$ and that under the action of B_0 on A inherited in the group $G \times C$ $AB_0 \cong G$ and $AB_0/N \cong G/N$.

Now regarding U, V as B_0/N-modules in the obvious way, we see that W as B_0/N-module is isomorphic to $U \otimes V$ which is projective, V being projective. Hence W, as B_0/N-module, is cohomologically trivial so every extension of W by B_0/N splits; in particular since $WD/D \cong W$ as B_0/N-module, and

$$\begin{aligned} (WB_0/D)/(WD/D) &\cong WB_0/WD \\ &= (WD)B_0/WD \\ &\cong B_0/B_0 \cap WD \\ &= B_0/B_0 \cap WN \\ &= B_0/N, \end{aligned}$$

WD/D has a complement C_0/D in WB_0/D. Hence $WB_0 = WC_0$; and of course $WN = WD$. Therefore on the one hand

$$\begin{aligned} AWB_0/WN &\cong AWNB_0/WN \\ &\cong AB_0/AB_0 \cap WN \\ &= AB_0/N \\ &\cong G/N, \end{aligned}$$

and on the other

$$\begin{aligned} AWB_0/WN &= AWC_0/WD \\ &= AWDC_0/WD \\ &\cong AC_0/AC_0 \cap WD \\ &= AC_0/D, \end{aligned}$$

so $G/N \cong AC_0/D$. However AC_0/D is isomorphic to a subgroup of H/D and so is in $\mathfrak{F}$. This concludes the proof of Lemma 7·2.

The next lemma and its corollary redeem the promise made in section 4 to prove the existence of $\mathscr{L}(n-1)$-critical groups in standard form. In these results we begin to use the technical machinery we have built up.

LEMMA 7·3. *Let $H = P_1P_2 \dots P_n$ be an $\mathscr{L}(n-1)$-critical group in expanded form in $\mathfrak{F}$. Let $k \in \{2, 3, \dots, n\}$. There is an $\mathscr{L}(n-1)$-critical group $G = Q_1Q_2 \dots Q_n$ in expanded form in $\mathfrak{F}$ such that*

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(i) $C_{Q_i}(Q_{i-1}) = 1, 2 \leqslant i \leqslant k$;

(ii) *if* $k < n$ *then there is an isomorphism* $\phi: B_{k+1}(H) \to B_{k+1}(G)$ *with* $P_i\phi = Q_i$ $(k+1 \leqslant i \leqslant n)$ *and the inflation of* $Q_k/\Phi(Q_k)$ *under* ϕ *is isomorphic to* P_k/M_k; *and*

(iii) *if* P_i/C_i *is abelian for some* $i \in \{2, \ldots, k\}$ *then* Q_i *is abelian.*

Proof. For each $\mathscr{L}(n-1)$-critical group H in expanded form in $\mathfrak{F}$ and for each $k \in \{2, 3, \ldots, n\}$ let $\mathfrak{F}_k(H)$ denote the set of all $\mathscr{L}(n-1)$-critical groups G in expanded form in $\mathfrak{F}$ which satisfy conditions (i), (ii) and (iii) of the lemma. We will show by induction on k that $\mathfrak{F}_k(H)$ is never empty which will be enough to prove the result.

To begin, $\mathfrak{F}_2(H)$ is not empty. For, $C_2 \trianglelefteq H$ so $G = H/C_2 \in \mathfrak{F}$, by Lemma 7·2, is $\mathscr{L}(n-1)$-critical by Theorem 5·10 of (9), and with $Q_i = P_iC_2/C_2$ $(1 \leqslant i \leqslant n)$ is in expanded form $Q_1Q_2 \ldots Q_n$. Also G clearly satisfies (i), (ii) and (iii) and therefore $G \in \mathfrak{F}_2(H)$.

Suppose $k < n$. We show that, if $\mathfrak{F}_l(H)$ is not empty for $2 \leqslant l \leqslant k$ and for all $\mathscr{L}(n-1)$-critical groups H in $\mathfrak{F}$ in expanded form, then $\mathfrak{F}_{k+1}(H)$ is not empty. To this end choose $H = P_1P_2 \ldots P_n$ an arbitrary $\mathscr{L}(n-1)$-critical group in expanded form in $\mathfrak{F}$ and let $G = Q_1Q_2 \ldots Q_n$ in expanded form belong to $\mathfrak{F}_k(H)$.

First consider the case that $C_{Q_{k+1}}(Q_k) = 1$. Now there is an isomorphism

$$\phi: B_{k+1}(H) \to B_{k+1}(G)$$

which inflates $Q_k/\Phi(Q_k)$ to P_k/M_k. In particular, therefore,

$$\begin{aligned} C_{k+1} = C_{P_{k+1}}(P_k) \leqslant C_{P_{k+1}}(P_k/M_k) &= C_{Q_{k+1}}(Q_k/\Phi(Q_k)) \\ &= C_{Q_{k+1}}(Q_k) \\ &= 1, \end{aligned}$$

and it follows that if P_{k+1}/C_{k+1} is abelian then so is Q_{k+1}, as required in (iii). Also the isomorphism ϕ induces, in the case $k+1 < n$, an isomorphism $B_{k+2}(H) \to B_{k+2}(G)$ satisfying (ii).

Hence we may suppose that $D = C_{Q_{k+1}}(Q_k) \neq 1$ for every $G \in \mathfrak{F}_k(H)$ and for every H. Define $\delta(G)$ as follows:

$$\delta(G) = 0, \quad \text{if} \quad [Q_i, D] = 1 \quad \text{for} \quad 1 \leqslant i \leqslant k,$$

and $$\delta(G) = j, \quad \text{if} \quad [Q_j, D] \neq 1 \quad \text{but} \quad [Q_i, D] = 1 \quad \text{for} \quad j+1 \leqslant i \leqslant k.$$

If there exists a G for which $\delta(G) = 0$ then $D \trianglelefteq G$ and, much as in the case $k = 2$, we see that $G/D \in \mathfrak{F}_{k+1}(H)$. So we may suppose that $\delta(G)$ is never 0. Let H and G be chosen so that $\delta(G)$ is minimal, say $\delta(G) = l \geqslant 1$.

First we discuss the case $p_{k+1} = p_l$. For in this event $[Q_l, D]$ is a proper subgroup of Q_l which is, moreover, $B_{l+1}(G)$-invariant (since $[Q_i, D] = 1$ for $l+1 \leqslant i \leqslant k$). Hence

$$[Q_l, D] \leqslant \Phi(Q_l).$$

However for each $x \in D$ the function $Q_l/\Phi(Q_l) \to \Phi(Q_l)$ defined by

$$\Phi(Q_l)\, y \to [y, x],$$

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is a Q_{l+1}-module homomorphism since D is centralized by Q_{l+1}. Since $Q_l/\Phi(Q_l)$ contains no trivial Q_{l+1}-submodules while $\Phi(Q_l)$ is a trivial Q_{l+1}-module we have by Schur's Lemma that $[y, x] = 1$ for all $y \in Q_l$ and all $x \in D$. That is

$$[Q_l, D] = 1.$$

But this contradicts the definition of l, so $p_{k+1} \neq p_l$.

It follows easily from 44·1 of (10) that, as $B_{l+1}(G)$-module,

$$Q_l/\Phi(Q_l)|_{Q_{l+1}}|^{B_{l+1}(G)} \cong Q_l/\Phi(Q_l) \otimes U,$$

where U is the inflation to $B_{l+1}(G)$ of the regular module $GF(p)\,B_{l+2}(G)$. From Lemmas 5·2 and 7·1 it follows that $\mathfrak{F}$ contains an $\mathscr{L}(n-1)$-critical group $F = R_1 R_2 \dots R_n$ in expanded form where $B_{l+1}(G) \cong B_{l+1}(F)$; where R_l/O_l is centralized by $E_{k+1} = C_{R_{k+1}}(R_k)$, O_l being the unique maximal $B_l(F)$-invariant subgroup of R_l; and where, for $i \in \{2, \dots, n\}$, R_i/E_i is abelian if Q_i/D_i is. The only impediment to $F \in \mathfrak{F}_k(G)$ is that $R_2, \dots, R_l$ need not be special since condition (i) for $l+1 \leqslant i \leqslant k$ and condition (ii) are assured by the isomorphism of $B_{l+1}(G)$ and $B_{l+1}(F)$ and condition (iii) has already been mentioned in the last sentence. Since $l < k$ we can choose $E \in \mathfrak{F}_l(F)$ by induction, say $E = S_1 S_2 \dots S_n$ in expanded form. We claim that $E \in \mathfrak{F}_k(G)$. For

$$C_{S_i}(S_{i-1}) = 1, \quad 2 \leqslant i \leqslant l,$$

and

$$C_{S_i}(S_{i-1}) = 1, \quad l+2 \leqslant i \leqslant k,$$

the first because $E \in \mathfrak{F}_l(F)$ and the second because $B_{l+1}(F) \cong B_{l+1}(E)$; and to fill in the gap note that since $C_{Q_{l+1}}(Q_l) = 1$ the application of Lemma 7·1 ensures that $C_{R_{l+1}}(R_l/O_l) = 1$ and then condition (ii) applied to $E \in \mathfrak{F}_l(F)$ gives in particular

$$\begin{aligned} C_{S_{l+1}}(S_l) &= C_{S_{l+1}}(S_l/\Phi(S_l)) \\ &\cong C_{R_{l+1}}(R_l/O_l) \\ &= 1. \end{aligned}$$

This takes care of (i). Condition (ii) follows from the composition of the isomorphisms $B_{l+1}(G) \cong B_{l+1}(F) \cong B_{l+1}(E)$, and (iii) is assured by the corresponding condition in Lemma 7·1 and the inductive application of the present lemma. Hence $E \in \mathfrak{F}_k(G)$.

It is easy to see that if $G \in \mathfrak{F}_k(H)$ then $\mathfrak{F}_k(G) = \mathfrak{F}_k(H)$, so $E \in \mathfrak{F}_k(H)$. However $\delta(E) < l$ since, by condition (ii), $C_{S_{k+1}}(S_k)$ centralizes $S_l/\Phi(S_l)$ because E_{k+1} centralizes R_l/O_l. This is a contradiction and it concludes the proof of Lemma 7·3.

COROLLARY 7·4. *$\mathfrak{F}$ contains an $\mathscr{L}(n-1)$-critical group in standard form.*

Proof. This follows from Lemma 7·3 with $k = n$.

Note that Lemma 4·3 is thereby now proved.

8. *Further modifications of $\mathscr{L}(n-1)$-critical groups.* Let $\mathfrak{F}$ be an s-closed Fitting class with the properties listed at the head of section 7, and let $G = P_1 P_2 \dots P_n$ be an $\mathscr{L}(n-1)$-critical group in standard form in $\mathfrak{F}$: G exists on account of Corollary 7·4. Recall from

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Lemma 4·4 that for $i \in \{1, \ldots, n-1\}$ $P_i/\Phi(P_i)$ is faithful and irreducible as module for $B_{i+1}(G)$. The next results are aimed at obtaining some control over two further features of G, namely the centralizers $C_G(\Phi(P_i))$ when P_i is non-abelian, and the sections P_i which can be abelian.

LEMMA 8·1. *Let $G = P_1 P_2 \ldots P_n$ be an $\mathscr{L}(n-1)$-critical group in $\mathfrak{F}$ in standard form. Suppose that for some pair of integers k, l with $1 \leqslant k < l \leqslant n$ we have*

(i) *$P_k, \ldots, P_{l-1}$ are non-abelian,*

(ii) *p_i is odd, and*

(iii) $C_{B_{k+1}(G)}(\Phi(P_{k+1})) \geqslant P_{k+1} \ldots P_l$.

Then there is in $\mathfrak{F}$ an $\mathscr{L}(n-1)$-critical group $H = Q_1 Q_2 \ldots Q_n$ in standard form such that either Q_{k+1} is abelian, or

(iv) *Q_k, Q_{k+1} are both non-abelian,*

(v) $B_{k+2}(H) \cong B_{k+2}(G)$, *and*

(vi) $C_{B_k(H)}(\Phi(Q_k)) \geqslant Q_k \ldots Q_l$.

Proof. If $l = k+1$ then G itself satisfies the conclusions of the lemma. We assume therefore that $l \geqslant k+2$.

Write $U = P_k/\Phi(P_k)$ as $B_{k+1}(G)$-module and consider the unrefinable decomposition

$$U|_{P_{k+1} \ldots P_l} = U_1 \oplus \ldots \oplus U_t.$$

Put $C = C_{P_{k+1}}(U_1)$. From Lemma 3·2

$$P_{k+1}/C = A/C \times B/C,$$

where A/C, B/C are $P_{k+2} \ldots P_l$-invariant with A/C special and B/C elementary abelian. Moreover since $Z(A/C) \leqslant Z(P_{k+1} \ldots P_l/C)$, and the centre of a faithfully and irreducibly represented group is cyclic, $Z(A/C)$ is cyclic. That is A/C is extra special.

There are two cases to consider: either B/C is trivial or it is not.

Suppose first that B/C is trivial, so P_{k+1}/C is extra special. Now $P_{k+1} \ldots P_n$ is an $\mathscr{L}(n-k-1)$-critical group by Theorem 5·1 of (9). We apply Theorem 6.2. If V is the inflation to $B_{k+1}(G)$ of the regular module $GF(p_k) B_{k+2}(G)$ then there exists in $U \otimes V$ an irreducible submodule W for which the fixed point spaces of P_{k+1} and $P_{k+1} \ldots P_l$ in $W \otimes W$ are the same. Recall from the proof of Lemma 7·3 that

$$U \otimes V \cong U|_{P_{k+1}}{}^{B_{k+1}(G)}.$$

Hence by Lemma 7·1 there exists in $\mathfrak{F}$ an $\mathscr{L}(n-1)$-critical group $H = Q_1 Q_2 \ldots Q_n$ in expanded form so that $B_{k+1}(H) \cong B_{k+1}(G)$ and under this isomorphism Q_k/N_k, as $B_{k+1}(H)$-module, is the inflation of W. A further use of Lemma 7·3 enables us to find in $\mathfrak{F}$ an $\mathscr{L}(n-1)$-critical group in standard form, say $K = R_1 R_2 \ldots R_n$ where $B_{k+1}(K) \cong B_{k+1}(H)$ and where $R_k/\Phi(R_k)$ is an inflation of W. Since $\Phi(R_k)$ is a homomorphic image of $R_k/\Phi(R_k) \otimes R_k/\Phi(R_k)$ as $B_{k+1}(K)$-modules and is fixed by R_{k+1} it follows that $\Phi(R_k)$ is fixed by $R_k R_{k+1} \ldots R_l$, using the analogous property of W. This completes the proof of Lemma 8·1 in the case B/C is trivial.

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Now suppose that B/C is not trivial. By Lemma 3·2 we see that, if D_0/C is an irreducible submodule of B/C, $[D_0/C, X] = D_0/C$ where $X = P_{k+2} \dots P_l$. Let D be a minimal X-invariant supplement for C in D_0. Now $\Phi(D) \leqslant \Phi(P_{k+1})$ and since $[\Phi(P_{k+1}), P_{k+2}] = 1$ we deduce from Fitting's Lemma and the minimality of D that $\Phi(D) = \Phi(P_{k+1}) \cap D$. Also $P_{k+1}/\Phi(P_{k+1})$ is a completely reducible X-module and $D/\Phi(D)$ is isomorphic to a submodule of it. It follows that since $\Phi(D) \leqslant C \cap D$, $D/C \cap D$ being elementary abelian, $\Phi(D) = C \cap D$ by the minimality of D again. Hence $D/\Phi(D) \cong D_0/C$ as X-modules; and $[D, X] = D$.

Let W be an irreducible submodule of U_1. Now $D_0 = DC$; and $D_0 \trianglelefteq P_{k+1} \dots P_l$ and so can have no fixed points in U. It follows that $[W, D] = W$. Choose E to be a minimal DX-invariant subgroup of P_k with $E\Phi(P_k)/\Phi(P_k) = W$. Then, much as above, $\Phi(E) = E \cap \Phi(P_k)$ and $E/\Phi(E) \cong W$ as DX-group. Also $[E, \Phi(D)] \leqslant [E, C] = 1$.

The group

$$L = (P_1 \dots P_{k-1} ED) \operatorname{twr}_X P_{k+2} \dots P_n$$

is in $\mathfrak{F}$ since it is a subgroup of $A_{k+1}(G) \operatorname{twr}_X B_{k+2}(G)$ which is in $\mathfrak{F}$ by Lemma 3·4. From L an $\mathscr{L}(n-1)$-critical subgroup $H = Q_1 Q_2 \dots Q_n$ may be chosen in the manner of the proof of Lemma 7·1. The details of this construction are left to the reader, but we observe its crucial properties. First $B_{k+2}(H) \cong B_{k+2}(G)$. Second, since $\Phi(D)|^{B_{k+2}(G)}$ centralizes $E|^{B_{k+2}(G)}$ then N_{k+1}, the unique maximal $B_{k+1}(H)$-invariant subgroup of H, is equal to $C_{Q_{k+1}}(Q_k)$. Now an application of Lemma 7·3 shows that $\mathfrak{F}$ contains an $\mathscr{L}(n-1)$-critical group $K = R_1 R_2 \dots R_n$ in standard form in which R_{k+1} is abelian. This concludes the proof of Lemma 8·1.

The next step is the proof of the following lemma.

LEMMA 8·2. *Let $G = P_1 P_2 \dots P_n$ be an $\mathscr{L}(n-1)$-critical group in standard form in $\mathfrak{F}$ and suppose that, for some $j \in \{2, 3, \dots, n-1\}$, P_j is abelian. Then there exists in $\mathfrak{F}$ an $\mathscr{L}(n-1)$-critical group $H = Q_1 Q_2 \dots Q_n$ in standard form in which $Q_j, Q_{j+1}, \dots, Q_n$ are all abelian and, for $i < j$, Q_i is abelian if P_i is.*

The proof of this relies on the following preliminaries.

LEMMA 8·3. *Let $B_1, B_2, \dots, B_m$ be groups on which C acts as a group of operators in such a way that $B_i C$ has greater nilpotent length than does C $(1 \leqslant i \leqslant m)$. Suppose moreover that A is a $(B_1 \times B_2 \times \dots \times B_m)$ C-group such that $AB_iC \in \mathfrak{F}$ $(1 \leqslant i \leqslant m)$.*

Then

$$A(B_1 \times B_2 \times \dots \times B_m) C \in \mathfrak{F}.$$

Proof. By induction it suffices to consider the case $m = 2$.

Let $D = \{(c, c) : c \in C\} \leqslant B_1 C \times B_2 C$ and set $H = (B_1 \times B_2) D$. Then $H \cong (B_1 \times B_2) C$; it supplements both direct factors in $B_1 C \times B_2 C$; and A may be regarded as an H-group in the obvious way. Let

$$W = A \operatorname{twr}_H (B_1 C \times B_2 C).$$

Since $A(B_1 \times B_2) C$ is isomorphic to a subgroup of W it will be enough to prove $W \in \mathfrak{F}$.

Write A^* for the base group of W. Then for $i = 1, 2$

$$\begin{aligned} A^*|_{B_iC} &= A|^{B_1C \times B_2C}|_{B_iC} \\ &\cong A|_{B_i}|^{B_iC}, \end{aligned}$$

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and therefore $A^*B_i \cong A \operatorname{twr}_{B_i} B_i C$ which is in $\mathfrak{F}$ by Lemma 3·6. Hence $W \in \mathfrak{F}$ since it is a normal product of A^*B_1C and A^*B_2C.

The next step in preparation for the proof of Lemma 8·2 needs some effort to set up. Let $G = APQB$ where A is normal and complemented by PQB in G, where P is normal and complemented by QB in PQB and where Q is normal and complemented by B in QB. Further suppose that, for distinct primes p, q, P is a p-group and Q is an elementary abelian q-group, that $[P, Q] = P$, and that Q is faithful and irreducible as B-module.

There exists P_0, a Q-subgroup of P, such that $P_0/\Phi(P_0)$ is irreducible and not trivial as Q-module. Put $E = C_Q(P_0/\Phi(P_0)) = C_Q(P_0)$, and then $|Q:E| = q$. Now $Q = E \times \langle \alpha \rangle$ for some $\alpha \in Q$; and since Q is irreducible $\cap_{b \in B} E^b = 1$. For m a natural number let $\theta_i : Q \to Q_i$ be $GF(q)$ B-isomorphisms and set $E_i = E\theta_i$, $\alpha_i = \alpha\theta_i$ $(1 \leqslant i \leqslant m)$. Write

$$Q_0 = Q_1 \# Q_2 \# \ldots \# Q_m,$$

and $B_0 = B^m$, the direct product of m copies of B, so that Q_0 is a B_0-module in the usual way. Denote the ith coordinate subgroup of B_0 by B_i $(1 \leqslant i \leqslant m)$. Next define

$$U_i = Q_1 \# \ldots \# Q_{i-1} \# E_i \# Q_{i+1} \# \ldots \# Q_m, \quad 1 \leqslant i \leqslant m,$$

$E_0 = U_1 + U_2 + \ldots + U_m$, and $\alpha_0 = \alpha_1 \# \alpha_2 \# \ldots \# \alpha_m$. Then, as vector spaces over $GF(q)$,

$$Q_0 = E_0 \oplus \langle \alpha_0 \rangle.$$

An action of Q_0 on P_0 may be defined with kernel E_0 and with α_0 acting on P_0 as α does.

Finally write $W_1 = P_0 \operatorname{twr}_{Q_0} Q_0 B_0$ with base group P_0^*. Then, as a group,

$$P_0^* = \underset{x \in B_0}{\times} P_0^x.$$

If we set $A^* = \times_{x \in B_0} A^x$ then P_0^* has a natural action on A^*, namely that inherited from the direct product $\times_{x \in B_0} (AP_0)^x$. With this action form

$$W = A^* \operatorname{twr}_{P_0^*} W_1.$$

LEMMA 8·4. *Suppose that the nilpotent length of A is less than that of AP_0. Then $W \in \mathfrak{F}$ if $G \in \mathfrak{F}$.*

Proof. The conditions imposed on G ensure that the nilpotent length of B is less than that of QB, and of QB than PQB.

Write Y for the base group of W. Then W is a normal product of its subgroups $YP_0^*Q_0B_i$ $(1 \leqslant i \leqslant m)$. To prove $W \in \mathfrak{F}$, therefore, it suffices by symmetry to prove $YP_0^*Q_0B_1 \in \mathfrak{F}$. Now

$$\begin{aligned} P_0^*|_{Q_0B_1} &= P_0|^{Q_0B_0}|_{Q_0B_1} \\ &\cong \underset{x \in \bar{B}_1}{\times} P_0^{(x)}|_{Q_0^x}|^{Q_0B_1} \end{aligned}$$

by Lemma 3·3, where $\bar{B}_1 = B_2 \times B_3 \times \ldots \times B_m$, say

$$P_0^*|_{Q_0B_1} \cong \underset{x \in \bar{B}_1}{\times} P_0(x).$$

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However the groups $YP_0(x)Q_0B_1$ are all conjugate in W and therefore they are isomorphic. It follows from Lemma 8·3 that $W \in \mathfrak{F}$ provided $YP_0(1)Q_0B_1 \in \mathfrak{F}$.

Next we consider Y as a $P_0(1)Q_0B_1$-group:

$$Y|_{P_0(1)Q_0B_1} = A^*|^{P_0^\sharp Q_0 B_0}|_{P_0(1)Q_0B_1} \cong \underset{x\in\bar{B}_1}{\times} A^{*(x)}|_{P_0(x^{-1})x}|^{P_0(1)Q_0B_1},$$

by Lemma 3·3, say

$$Y|_{P_0(1)Q_0B_1} \cong \underset{x\in\bar{B}_1}{\times} A_x^*.$$

However

$$(A_x^* P_0(1)Q_0B_1)^{x^{-1}} \cong A^* \operatorname{twr}_{P_0(x^{-1})} P_0(x^{-1})Q_0B_1,$$

and $YP_0(1)Q_0B_1$ is a subdirect product of the groups $A_x^*P_0(1)Q_0B_1$ $(x\in\bar{B}_1)$, so it suffices to prove that $A^* \operatorname{twr}_{P_0(x)} P_0(x)Q_0B_1 \in \mathfrak{F}$ whenever $x\in\bar{B}_1$.

Let

$$A(x) = \underset{b\in B_1}{\times} A^{bx}, \quad \bar{A}(x) = \underset{\substack{b\in B_1 \\ x\neq y\in\bar{B}_1}}{\times} A^{by},$$

so that $A^* = A(x)\times\bar{A}(x)$ $(x\in\bar{B}_1)$, and note that $[\bar{A}(x), P_0(x)] = 1$. Then since $\bar{A}(x)\operatorname{twr}_{P_0(x)} P_0(x)Q_0B_1$ has nilpotent length less than n and since $A^*\operatorname{twr}_{P_0(x)} P_0(x)Q_0B_1$ is a subdirect product of $A(x)\operatorname{twr}_{P_0(x)} P_0(x)Q_0B_1$ and $\bar{A}(x)\operatorname{twr}_{P_0(x)} P_0(x)Q_0B_1$ it suffices to prove that $A(x)\operatorname{twr}_{P_0(x)} P_0(x)Q_0B_1 \in \mathfrak{F}$ $(x\in\bar{B}_1)$; and conjugacy in W reduces this to proving that $A(1)\operatorname{twr}_{P_0(1)} P_0(1)Q_0B_1 \in \mathfrak{F}$.

Let Z be the base group of $A(1)\operatorname{twr}_{P_0(1)} P_0(1)Q_0B_1$. As B_1-module

$$Q_0 = V_0 \oplus V_1,$$

where $V_0 = Q_1 \# \alpha_2 \# \ldots \# \alpha_m$ and $V_1 = E_2 + \ldots + E_m$. Since $V_1 \leqslant E_0$, $[P_0(1), V_1] = 1$ and $ZP_0(1)V_1B_1$ has nilpotent length less than n, and so is in $\mathfrak{F}$. By Lemma 8·3 therefore, we need only prove that $ZP_0(1)V_0B_1 \in \mathfrak{F}$. Now

$$Z|_{P_0(1)V_0B_1} = A(1)|^{P_0(1)Q_0B_1}|_{P_0(1)V_0B_1} \cong \underset{x\in V_1}{\times} A(1)^{(x)}|_{P_0(1)x}|^{P_0(1)V_0B_1},$$

by Lemma 3·3, say

$$Z|_{P_0(1)V_0B_1} = \underset{x\in V_1}{\times} A_x.$$

But since V_1 centralizes $P_0(1)$ we have

$$A_x P_0(1)V_0B_1 \cong A_1 P_0(1)V_0B_1 \cong A(1)\operatorname{twr}_{P_0(1)} P_0(1)V_0B_1.$$

Consequently it is enough to prove that $A(1)\operatorname{twr}_{P_0(1)} P_0(1)V_0B_1 \in \mathfrak{F}$, $ZP_0(1)V_0B_1$ being a subdirect product of the groups $A_xP_0(1)V_0B_1$ $(x\in V_1)$.

Finally from $G = APQB \in \mathfrak{F}$ we deduce from Lemma 3·6 that $AP\operatorname{twr}_Q QB \in \mathfrak{F}$ whence $AP_0\operatorname{twr}_Q QB \in \mathfrak{S}\mathfrak{F} = \mathfrak{F}$. But $AP_0\operatorname{twr}_Q QB \cong A(1)P_0(1)QB \cong A(1)P_0(1)V_0B_1$ and then Lemma 3·6 again shows that $A(1)\operatorname{twr}_{P_0(1)} P_0(1)V_0B_1 \in \mathfrak{F}$, as required. This completes the proof of Lemma 8·4.

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Proof of Lemma 8·2. It suffices to show that if $\mathfrak{F}$ contains $G = P_1 P_2 \dots P_n$, an $\mathscr{L}(n-1)$-critical group in standard form in which P_j is abelian (and therefore elementary) and P_{j+1} non-abelian for some $j \in \{2, \dots, n-1\}$ then there exists an $\mathscr{L}(n-1)$-critical group $H = Q_1 Q_2 \dots Q_n$ in standard form in $\mathfrak{F}$ such that: Q_j, Q_{j+1} are abelian and for $i < j$, Q_i is abelian if P_i is.

For convenience let $A = A_{j-2}(G)$ if $j > 2$, and $A = 1$ otherwise, $B = B_{j+1}(G)$ and $N = \Phi(P_{j+1})$. Now B/N is monolithic and hence has a faithful irreducible representation over $GF(p_j)$ (for, if not, Maschke's Theorem would ensure that the regular module $GF(p_j)B$ was not faithful). By Bryant and Kovács (4) some tensor power of P_j, as B-module, contains an irreducible submodule whose kernel is precisely N: let it be the m-fold tensor power.

Now put $P = P_{j-1}$, $Q = P_j$ and let P_0 be a Q-subgroup of P for which $P_0/\Phi(P_0)$ is irreducible and non-trivial as Q-module. Recall the preamble to Lemma 8·4 where

$$W = A^* \operatorname{twr}_{P_0^*} W_1,$$

and $W_1 = A_0 \operatorname{twr}_{Q_0} Q_0 B_0$, the integer m being that chosen in the last paragraph, and where P_0^* is the base group of W_1. By Lemma 8·4 $W \in \mathfrak{F}$. The $\mathscr{L}(n-1)$-critical group H we seek is a subgroup of W, constructed as follows.

Let $D = \{(b, b, \dots, b) \in B_0 : b \in B\}$ so that Q_0 as D-module is the m-fold tensor power of Q. Hence there is an irreducible submodule U of Q_0 with

$$C_D(U) = \{(n, n, \dots, n) \in B_0 : n \in N\}.$$

We have

$$\begin{aligned} P_0^*/\Phi(P_0^*) &\cong P_0/\Phi(P_0)|^{Q_0 B_0} \\ &= \underset{x \in B_0}{\times} (P_0/\Phi(P_0))^x. \end{aligned}$$

Now

$$\begin{aligned} C_{Q_0}(P_0^*/\Phi(P_0^*)) &= \bigcap_{x \in B_0} C_{Q_0}(P_0/\Phi(P_0))^x \\ &= \bigcap_{x \in B_0} E_0^x \\ &\leqslant E_0, \end{aligned}$$

so by Lemma 5·7 $\cap_{x \in B_0} E_0^x$ is trivial being a B_0-submodule. Hence $P_0^*/\Phi(P_0^*)$ is faithful as $Q_0 B_0$-module and therefore as UD-module. By Fitting's Lemma there is an irreducible submodule $S_0/\Phi(P_0^*)$ of $P_0^*/\Phi(P_0^*)$ as UD-module such that

$$U \cap C_{UD}(S_0/\Phi(P_0^*)) = 1.$$

Choose S_{j-1} to be a minimal UD-invariant supplement for $\Phi(P_0^*)$ in S_0: note that S_{j-1} is abelian if P_{j-1} is. Also by Fitting's Lemma $S_{j-1} \cap \Phi(P_0^*) = \Phi(S_{j-1})$, and

$$S_{j-1}/\Phi(S_{j-1}) \cong S_0/\Phi(P_0^*)$$

as UD-modules. Put $S_j = U$ and, for $j+1 \leqslant i \leqslant n$, $S_i = \{(b, b, \dots, b) \in B_0 : b \in P_i\}$. Then proceed as in the proof of Lemma 7·1 to choose subgroups $S_{j-2}, \dots, S_1$ of W so that $K = S_1 S_2 \dots S_n$ is $\mathscr{L}(n-1)$-critical and, of course, in $\mathfrak{F}$. Observe that S_j is abelian; for $i < j$ S_i is abelian if P_i is, $S_{j+1}/C_{S_{j+1}}(S_j)$ is abelian; and $B_{j+1}(K) \cong B_{j+1}(G)$.

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Finally Lemma 7·3 ensures that $\mathfrak{F}$ contains an $\mathscr{L}(n-1)$-critical group $H = Q_1 Q_2 \dots Q_n$ in standard form in which Q_j, Q_{j+1} are both abelian; for $i < j$ Q_i is abelian if P_i is, and $B_{j+2}(H) \cong B_{j+2}(K) \cong B_{j+2}(G)$ as required. This completes the proof of Lemma 8·2.

The culmination of this section is the next result which gives enough control over the structure of at least one $\mathscr{L}(n-1)$-critical group in $\mathfrak{F}$ to enable the proof of Theorem 2·1 to be completed in the next section.

LEMMA 8·5. *There exists in $\mathfrak{F}$ an $\mathscr{L}(n-1)$-critical group $G = P_1 P_2 \dots P_n$ in standard form with the property that for some integer $l \in \{1, \dots, n-1\}$*

(i) *$P_1, P_2, \dots, P_{l-1}$ are non-abelian,*

(ii) *$P_l, P_{l+1}, \dots, P_n$ are abelian and*

(iii) *$A_{l-1}(G) \leqslant C_G(\Phi(P_1))$.*

Proof. In an arbitrary $\mathscr{L}(n-1)$-critical group in expanded form $Q_1 Q_2 \dots Q_n$, Q_n is abelian. Let $K = R_1 R_2 \dots R_n$ be an $\mathscr{L}(n-1)$-critical group in standard form in $\mathfrak{F}$ in which the number of R_i which are abelian is maximal. By the first sentence this number is at least one and by Lemma 8·2 there must be an integer l satisfying (i) and (ii).

If $1 \leqslant l \leqslant 3$ then (iii) is automatically satisfied by K, so suppose $l \geqslant 4$. Let k be minimal with respect to

$$C_G(\Phi(R_k)) \geqslant R_k R_{k+1} \dots R_{l-1}, \quad p_{l-1} \text{ odd},$$

or

$$C_G(\Phi(R_k)) \geqslant R_k R_{k+1} \dots R_l, \quad p_{l-1} = 2.$$

It follows from Lemma 8·1 that there exists in $\mathfrak{F}$ an $\mathscr{L}(n-1)$-critical group in standard form satisfying (i), (ii) and (iii) because $k = 1$.

9. *The proof of Theorem* 2·1. Let G be the group in $\mathfrak{F}$ ensured by Lemma 8·5. The aim of this section is to show that $G/\Phi(G) \in \mathfrak{F}$.

The crux of the matter is Lemma 9·1 below: all our previous results have been the outcome of attempts to find a group in $\mathfrak{F}$ to which we could apply this lemma. The lemma is a direct descendant of Lemma 3·5 of (7) and Lemma 2·3 of (8).

LEMMA 9.1. *Let H be a group of nilpotent length $n-1$ with the following structure. H has a normal subgroup A with complement B; and B has nilpotent length less than $n-1$ and has a unique chief series every chief factor of which is complemented in B.*

Suppose that P is a p-group on which H acts and suppose that N is a minimal H-invariant normal subgroup of P such that $C_H(N) = A$. Put $G = PH$. Then $G/N \in \mathfrak{F}$ if $G \in \mathfrak{F}$.

Proof. The proof is by induction on the nilpotent length of B, the cases with length at most one being covered by Lemma 3·5 of (7) and Lemma 2·3 of (8).

Let M be the unique minimal normal subgroup of B and let K be a complement for it in B. Note that M is a q-group for some prime $q \neq p$ and is faithful and irreducible as K-module. Let t be the dimension of M over $\mathrm{End}_K M$, $X = M^t$, the direct sum of t copies of M and V the subspace of X ensured by Lemma 5.5 containing no non-zero submodules of X.

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The group

$$W = P \operatorname{twr}_{AB \times V} (AB \times XK),$$

using trivial action of V on P and the action of AB on P inherited from G otherwise, belongs to $\mathfrak{F}$ for the usual reasons: if P^* is the base group of W then $P^*AB \in \mathrm{R}_0\{G\} \subseteq \mathfrak{F}$; $P^*XK \in \mathfrak{F}$ because it has nilpotent length less than n; and W is a normal product of P^*AB and P^*XK.

Now let N^* denote the normal closure of N in W. As a module for $AB \times XK$ over $GF(p)$ we have

$$N^* \cong N \# L|^{XK},$$

where L is a trivial irreducible $GF(p)\,V$-module. It follows from Lemma 5·6 that $L|^{XK} = N_1 \oplus N_2$ where N_1 is the fixed point space of X and all composition factors of N_2 are faithful, so

$$N^* = S_1 \oplus S_2,$$

where S_1 is precisely the fixed point space in N^* of X and where every composition factor of S_2 is faithful for XK.

Our immediate aim is to show that $W/S_2 \in \mathfrak{F}$. To this end note that A centralizes N^* which is therefore faithful as $(B \times XK)$-module. Set $K_0 = \{(k, k): k \in K\} \subseteq B \times XK$. Then, on account of the choice of t and Lemma 5·4, no composition factor of S_2 regarded as $(M \times X)\,K_0$-module can be faithful, though its kernel must intersect both M and X trivially. The last statement follows since $[N, M] = N$ and $[S_2, X] = S_2$. Hence if R/S is a composition factor of $S_2|_{(M \times X)\,K_0}$ there is an irreducible K_0-submodule D of $M \times X$ *projecting non-trivially onto* M with

$$D \leqslant C_{M \times X}(R/S).$$

A combination of Maschke's Theorem and Fitting's Lemma now yields that, as $(M \times X)\,K_0$-module,

$$S_2 = M(D_1) + M(D_2) + \ldots + M(D_m),$$

where $D_1, D_2, \ldots, D_m$ are the irreducible submodules of $M \times X$ centralizing some composition factor of $S_2|_{(M \times X)\,K_0}$ and where $M(D_i)$ is the fixed point space of D_i in S_2 $(1 \leqslant i \leqslant m)$.

Write $W_0 = P^*A(M \times X)\,K_0$ and note that, being a subgroup of W, it is in $\mathfrak{F}$. For $i \in \{1, 2, \ldots, m\}$ put

$$M_i = M(D_1) + \ldots + M(D_i),$$

$P_i = P^*/M_i$ and $W_i = P_i A(M \times X)\,K_0$, the natural semi-direct product. Of course $W_i \cong W_0/M_i$. We show that $W_i \in \mathfrak{F}$ $(0 \leqslant i \leqslant m)$ by induction on i (within the overall induction on the nilpotent length of B). If therefore $W_i \in \mathfrak{F}$ for some $i \in \{0, 1, \ldots, m-1\}$ let $M_{i+1}/M_i = \bar{M}(D_{i+1})$ so that in W_{i+1}

$$[\bar{M}(D_{i+1}), D_{i+1}] = 1.$$

Now $AD_{i+1}K_0 \cong H$ so it follows from the induction on the nilpotent length of B that

$$(P_i/\bar{M}(D_{i+1}))\,AD_{i+1}K_0 \cong P_i AD_{i+1}K_0/\bar{M}(D_{i+1}) \in \mathfrak{F}.$$

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That is, $P_{i+1}AD_{i+1}K_0 \in \mathfrak{F}$. However $P_{i+1}AXK_0 \in \mathfrak{F}$ since its nilpotent length is less than n and therefore Lemma 8·3 ensures that

$$W_{i+1} = P_{i+1}A(M \times X)K_0 = P_{i+1}A(D_{i+1} \times X)K_0 \in \mathfrak{F},$$

completing this induction. From $W_m \in \mathfrak{F}$ we deduce that $P_m AMK_0 \in \mathfrak{F}$ and hence that $P_m \operatorname{twr}_A AMK_0 \in \mathfrak{F}$ by Lemma 3·6.

There is an isomorphism $AB \cong AMK_0$ which is the identity on A and therefore by Lemma 3·4

$$P_m \operatorname{twr}_A AB \cong P_m \operatorname{twr}_A AMK_0 \in \mathfrak{F}.$$

Since S_2 admits $AB \times XK$ we have that $P_m = P^*/S_2$ is an AB-group. It then follows from Lemma 3·6 that $P_m AB \in \mathfrak{F}$. But $P_m XK \in \mathfrak{F}$ because its nilpotent length is less than n and

$$W/S_2 \in \mathfrak{N}_0\{P_m AB, P_m XK\},$$

so $W/S_2 \in \mathfrak{F}$ as required.

We next examine the structure of P_m as an AB-group. Let X_0 be an irreducible K-submodule of X. For $k \in K$ write

$$P_k = \bigtimes_{x \in X_0} P^{kx}$$

so that $P^* = \times_{k \in K} P_k$. First we show that $S_2 = \times_{k \in K} S_2 \cap P_k$. For as $AB \times X_0$-module

$$P_1 \cap N^* = \bigtimes_{x \in X_0} N^x \cong N \# GF(p) X_0,$$

and under this isomorphism $P_1 \cap S_2$ is mapped to the unique smallest submodule of $N \# GF(p) X_0$ whose quotient module is trivial for X_0, namely to $N \# I$ where I is the augmentation ideal of $GF(p) X_0$. Hence

$$\begin{aligned} P_1 \cap N^*/P_1 \cap S_2 &\cong N \# GF(\mathrm{p}) X_0/N \# I \\ &\cong N \# GF(p) X_0/I \\ &\cong N, \end{aligned}$$

as AB-module. Now $S_2 \cap P_k = (S_2 \cap P_1)^k$ so $P_k \cap N^*/S_2 \cap P_k \cong N$ as AB-modules for $k \in K$. Also $N^* = \times_{k \in K} P_k \cap N^*$ so

$$\begin{aligned} N^*/\bigtimes_{k \in K} S_2 \cap P_k &\cong \bigtimes_{k \in K} P_k \cap N^*/P_k \cap S_2 \\ &\cong \bigtimes_{k \in K} N, \end{aligned}$$

which is trivial for X_0. Since X_0 has no fixed points in S_2 it follows that $S_2 = \times_{k \in K} P_k \cap S_2$ as required.

Since $S_0 \cap P^{kx} = 1$ for all $k \in K$, $x \in X_0$ it follows that as AB-group $P_k/S_2 \cap P_k$ is a central product of copies of P amalgamating N, call it C_k say. Then as AB-group

$$\begin{aligned} P_m = P^*/S_2 &\cong \bigtimes_{k \in K} P_k/P_k \cap S_2 \\ &\cong \bigtimes_{k \in K} C_k. \end{aligned}$$

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That is, as AB-group, P_m is a direct product of central products amalgamating N of copies of P.

In particular $C_1 AB \in \mathfrak{F}$. The proof of Lemma 9·1 may now be completed using the method described in the second and third paragraphs of p. 197 of (8).

Proof of Theorem 2·1. We now merely observe that Lemma 8·5 ensures that $\mathfrak{F}$ contains an $\mathscr{L}(n-1)$-critical group in standard form $G = P_1 P_2 \dots P_n$ where $P_1, \dots, P_{l-1}$ are special and non-abelian for some $l \geqslant 1$, where $P_l, P_{l+1}, \dots, P_n$ are abelian, and where $A_{l-1}(G) \leqslant C_G((P_1))$. If $l = 1$ then $\Phi(G) = 1$ so we are done. Suppose $l > 1$ and note that $\Phi(G) = \Phi(P_1)$. If $U \leqslant \Phi(G)$ is a normal subgroup of G then G/U satisfies the hypotheses of Lemma 9·1 with $P = P_1/U$, $A = C_{B_2(G)}(\Phi(P))$ (which contains $P_2 U/U$ by Lemma 4·1 (iii)) and B is a complement for A in $B_2(G)\, U/U$. Hence if

$$1 < U_1 < U_2 < \dots < U_r < \Phi(G)$$

is part of a chief series of G we conclude from Lemma 9·1, by induction on r, that $G/\Phi(G) \in \mathfrak{F}$.

This completes the proof of Theorem 2·1 and with it the proof of Theorem 1·1.

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