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Stably continuous frames

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In the lattice theory that underlies topology, that is, in the study of frames, a class of frames arising naturally is that of the stably continuous frames (see §0 for definitions). On the one hand, they correspond to the most reasonable not necessarily Hausdorff compact spaces, and on the other, they are precisely the retracts of coherent frames. Moreover, an important special case of stably continuous frames are the compact regular frames which correspond to compact Hausdorff spaces.

This work owes its motivation to a desire to understand better why the functor from compact regular T_0-bispaces to topological spaces which forgets the second topology is monadic, making the compact regular T_0-bispaces the category of algebras for the open prime filter space monad in the category of all topological spaces (Salbany [**9**], Wyler [**11**]). The outcome of this enquiry is a single proposition, showing that stably continuous frames are exactly the first parts of compact regular biframes. In a sense this means that the most obvious example of stably continuous frames, the compact regular frames, is indeed generic provided one lifts the latter notion to biframes.

The paper is arranged to provide as short a path as we know to the culminating Proposition which asserts that the category **KR2Frm** of compact regular biframes and the category **StCont** of stably continuous frames and proper frame homomorphisms are equivalent by the functor **F** taking first parts of biframes. §0 recalls the main concepts that make up the background to our problem, and §1 deals with four lemmas concerning the two major tools employed later on, the Lawson dual and the congruence lattice. §2 then presents, in a sequence of five further lemmas, the proof of the Proposition, and concludes with a number of remarks giving some additional observations on the subject. Finally, §3 provides a few further comments, fitting our result into some other contexts, including the one that originally led to this enquiry.

Concerning our methods of proof it should be emphasized that *they are specifically chosen to be independent of any choice principles.* In particular, this means that, in contrast with the spirit of Hofmann [**6**], the spectrum of a stably continuous frame is not an admissible tool in our context, given that the spatiality of these frames is equivalent to the Boolean Ultrafilter Theorem (Banaschewski [**1**]).

As basic references, we list Johnstone [**7**] for topics related to frames, the

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Compendium [4] for general facts concerning continuous lattices, and Mac Lane [8] for the few technicalities on adjoint functors needed here.

0. *Background*

A *frame* (or locale) is a complete lattice L in which $x \wedge \bigvee x_i = \bigvee x \wedge x_i$ for binary meet $\wedge$, arbitrary join $\bigvee$, and any $x, x_i \in L$, and a *frame homomorphism* $h: L \to M$ is a map between frames preserving finite meets, including the unit (= top) e and arbitrary joins, including the zero (= bottom) 0. The resulting category is denoted **Frm**. For a detailed account of the major aspects of frames see Johnstone [7].

On any complete lattice L, $a \ll b$ (a is much below b) means that $b \leqslant \bigvee x_i$, for any $x_i \in L$, implies $a \leqslant x_{i_1} \vee \ldots \vee x_{i_n}$ for suitable $i_1, \ldots, i_n$. An element $c \in L$ for which $c \ll c$ is called *compact* (Johnstone [7]: finite). A *continuous* lattice is a complete lattice L such that $a = \bigvee x\,(x \ll a)$ for each $a \in L$. A *continuous frame* (Johnstone [7]: locally compact frame) is a frame L which is a continuous lattice, and L is called *stably* continuous (Johnstone [7]: stably locally compact) whenever, in addition, the relation $\ll$ is closed under finite meets in $L \times L$. The latter amounts to the requirement that $a \ll b, c$ implies $a \ll b \wedge c$ and $e \ll e$. The important maps between stably continuous frames are the *proper* frame homomorphisms $h: L \to M$, that is, the frame homomorphisms preserving the $\ll$ relation. The corresponding category will be denoted **StCont**. For the general theory of continuous lattices see the Compendium [4].

Particular examples of stably continuous frames are the *compact regular* frames, compact meaning that the unit is compact and regular that each element a is the join of all $x \prec a$ (x rather below a) where the latter means that $x \wedge y = 0$ and $a \vee y = e$ for some y. In a compact regular frame L, one easily sees that $a \ll b$ if and only if $a \prec b$, and L is stably continuous because $a \prec b, c$ implies $a \prec b \wedge c$ by a simple calculation based on distributivity. In a similar way, one obtains that any frame homomorphism between compact regular frames is proper.

Associated with any frame L, or more generally any distributive lattice with zero and unit, one has the frame $\mathfrak{J}L$ of all ideals of L, that is, all non-empty $J \subseteq L$ such that $a, b \in J$ implies $a \vee b \in J$, and $x \leqslant a \in J$ implies $x \in J$. The correspondence $L \rightsquigarrow \mathfrak{J}L$ is functorial, any lattice homomorphism $h: L \to M$ giving rise to $\mathfrak{J}h: \mathfrak{J}L \to \mathfrak{J}M$ which takes each ideal in L to the ideal in M generated by its image under h. These ideal lattices $\mathfrak{J}L$ are stably continuous frames, $I \ll J$ in $\mathfrak{J}L$ meaning that $I \subseteq \downarrow a \subseteq J$ for some $a \in L$, where $\downarrow a = \{x \mid x \leqslant a\}$ is the principal ideal generated by a. The frames isomorphic to $\mathfrak{J}L$ for some distributive lattice with zero and unit are called *coherent*. They are intrinsically characterized as those frames which are generated by their compact elements and in which any finite meet of such elements (including e) is again compact. The properties of $\mathfrak{J}L$ immediately imply that $\mathfrak{J}h: \mathfrak{J}L \to \mathfrak{J}M$ is proper for any lattice homomorphism $h: L \to M$. In particular, $\mathfrak{J}$ transforms **Frm** into **StCont**.

For any stably continuous frame L, there is a proper frame homomorphism $\kappa_L: L \to \mathfrak{J}L$, natural in L, given by $\kappa_L(a) = \downarrow a = \{x \mid x \ll a\}$. The functor $\mathfrak{J}: \mathbf{Frm} \to \mathbf{StCont}$ is right adjoint to the identical embedding $\mathbf{StCont} \to \mathbf{Frm}$ with unit κ and counit given by the maps $\sigma_L: \mathfrak{J}L \to L$ sending each ideal to its join (Banaschewski [1]). On the other hand, the identity $\sigma_L \kappa_L = \mathrm{id}_L$ for stably continuous frames proves

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one part of the characterization (Johnstone [7]) that the stably continuous frames are exactly the retracts of coherent frames.

The original motivation for the concept of frame came from topological spaces, and some of the particular notions mentioned above have obvious topological counterparts; indeed, the continuous frames correspond to the locally compact spaces, the compact regular frames to the compact Hausdorff spaces, and the stably continuous frames to the spaces, called spectral retracts, which are the retracts of spectral spaces (Johnstone [7]: coherent spaces). Moreover, given the Axiom of Choice (or, more economically, the Boolean Ultrafilter Theorem), one has that these categories of frames are actually dually equivalent to the corresponding categories of spaces, by the obvious functor from spaces to frames.

In much the same way as topological spaces lead to the notion of frame, bitopological spaces (i.e. sets together with *two* specified topologies) motivate the notion of biframe. A *biframe* then is a triple $L = (L_0, L_1, L_2)$ of frames where L_1 and L_2 are subframes of L_0 such that together they generate L_0 (Banaschewski, Brümmer & Hardie [2]). We call L_1 and L_2 the first and second part of L and L_0 its total frame. In accordance with the notion of continuous maps between bitopological spaces, a biframe homomorphism $h: L \to M$ is a frame homomorphism $h: L_0 \to M_0$ which preserves first and second parts. One then has obvious functors **F** and **S** from the category **2Frm** of biframes and biframe homomorphisms to **Frm**, assigning to each $L \in$ **2Frm** its first and its second part, respectively, and to each biframe homomorphism the appropriate frame homomorphism induced by it. We shall be concerned here with the category **KR2Frm** of compact regular biframes (Banaschewski, Brümmer & Hardie [2]) which is the full subcategory of **2Frm** determined by those L for which

(1) the unit is compact in L_0,

(2) for each $a \in L_1$ one has $a = \bigvee x\,(x \prec_1 a)$, where $x \prec_1 a$ means that $x \in L_1$ and there exists an element $y \in L_2$ such that $x \wedge y = 0$ and $a \vee y = e$, and

(3) for each $b \in L_2$ one has $b = \bigvee z\,(z \prec_2 b)$, where $z \prec_2 b$ means that $z \in L_2$ and there exists an element $t \in L_1$ such that $z \wedge t = 0$ and $b \vee t = e$.

We note that the compact regular biframes are exactly the frame analogue of compact regular bitopological T_0-spaces. Indeed, given the Boolean Ultrafilter Theorem, the category $\mathbf{KR2TOP_0}$ of the latter is dually equivalent to **KR2Frm**: see [2].

Finally, it should be noted that, for any compact regular biframe L, the total frame L_0 is compact regular because $x \prec_i y$ in L_i $(i = 1, 2)$ clearly implies $x \prec y$ in L_0. Thus compact regular biframes may be viewed as compact regular frames with additional structure provided by the specification of two particular subframes.

1. *Tools*

In this section we collect the facts needed later on concerning two notions which are of importance generally but which play a particularly significant role in our treatment of stably continuous frames: the *Lawson dual* and the *congruence lattice*.

Recall that, for any complete lattice L, a subset $U \subseteq L$ is called *Scott open* whenever U is an up-set (i.e. if $x \geqslant y$ and $y \in U$ then $x \in U$) and $\bigvee D \in U$ implies

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$D \cap U \neq \emptyset$ for any up-directed $D \subseteq L$. The Lawson dual ΛL is then defined as the partially ordered set of all Scott open filters in L. For general L, ΛL has only rather weak properties but it is obviously always closed under directed unions and finite intersections. Further, ΦL will be the frame of *all* filters in L.

For a continuous frame L, it is clear that an up-set $U \subseteq L$ is Scott open if and only if, for each $c \in U$, there exists an $a \ll c$ in U. This is especially important in the case of stably continuous frames as is shown by the proof of

LEMMA 1. *The correspondence* $L \rightsquigarrow \Lambda L$ *determines a covariant functor* Λ: **StCont** $\to$ **StCont**.

Proof. (i) For any stably continuous frame L, ΛL is a subframe of the frame ΦL of all filters in L: if F and G are Scott open filters in L then $F \cap G$ is trivially a Scott open filter, but so is the filter $F \vee G$ generated by F and G, that is, the filter generated by all $a \wedge b$ with $a \in F$ and $b \in G$, since if $x \in F$ and $y \in G$ satisfy $x \ll a$ and $y \ll b$, then $x \wedge y \ll a \wedge b$, precisely by stability. The latter implies that the filter join $\bigvee F_i$ of any Scott open filters F_i in L, for $i \in I \neq \emptyset$, is Scott open, as the union of its finite subjoins. Regarding $I = \emptyset$, the join is then the smallest filter $\{e\}$ which is Scott open since e is compact.

(ii) ΛL is stably continuous. Note first that, for any $a \in L$, $\Uparrow a = \{x \mid a \ll x\}$ is Scott open since L is continuous ($\ll$ interpolates) and a filter since it is stably so. For any Scott open filter G in L, the $\Uparrow a$ with $a \in G$ are up-directed because G is a filter and $G = \bigcup \Uparrow a \, (a \in G)$ because G is Scott open. Hence $F \ll G$ implies $F \subseteq \Uparrow a$ for some $a \in G$, and thus also $F \subseteq \uparrow a \subseteq G$ for some $a \in L$. Conversely, the latter implies $F \ll G$ in ΦL and hence also in ΛL. It follows immediately that ΛL is continuous: for any Scott open filter G in L, $a \in G$ implies $\Uparrow a \subseteq \uparrow a \subseteq G$ and hence $\Uparrow a \ll G$, whereas G is the union of these $\Uparrow a$. Moreover, if $F \ll G, H$ in ΛL then $F \ll G \cap H$ since $F \subseteq \uparrow a \subseteq G$ and $F \subseteq \uparrow b \subseteq H$ implies $F \subseteq \uparrow(a \vee b) \subseteq G \cap H$. Since, further, the unit $\uparrow 0$ of ΛL is compact, this shows that ΛL is indeed stably continuous.

(iii) If $h: L \to M$ is a map in **StCont**, that is, any frame homomorphism preserving $\ll$, then the filter $[h(F)]$ generated by the image of any $F \in \Lambda L$ is Scott open, and hence the map $F \rightsquigarrow [h(F)]$ is a frame homomorphism $\Lambda L \to \Lambda M$. Also, $F \ll G$ in ΛL means that $F \subseteq \uparrow a \subseteq G$ for some $a \in L$, which implies $[h(F)] \subseteq \uparrow h(a) \subseteq [h(G)]$ and therefore $[h(F)] \ll [h(G)]$. Therefore the map $\Lambda h: \Lambda L \to \Lambda M$ given by $F \rightsquigarrow [h(F)]$ indeed belongs to **StCont**, and it is clear that we have thus obtained a functor Λ: **StCont** $\to$ **StCont**.

Remark. In other settings, the Lawson dual (as the name indicates) provides contravariant functors; hence it is noteworthy that, for stably continuous frames, one has covariance.

We now turn to the other notion to be discussed. For any frame L, we let $\mathfrak{C}L$ be the lattice of *congruences* on L, that is, the elements of $\mathfrak{C}L$ are the equivalence relations Θ on L which are subframes of $L \times L$, and its partial order is inclusion. $\mathfrak{C}L$ is evidently a complete lattice, with intersection as meet, $\Delta = \{(x, x) \mid x \in L\}$ as zero, and $\nabla = L \times L$ as unit. $\mathfrak{C}L$ is isomorphic to the lattice NL of nuclei of L which are special closure operators $k: L \to L$, via the map $\mathfrak{C}L \to NL$ taking $\Theta \in \mathfrak{C}L$ to $k \in NL$ defined by $k(x) = \bigvee y \, ((x, y) \in \Theta)$. It follows from this and the known facts about nuclei (Johnstone [7]) that $\mathfrak{C}L$ is again a frame. An elegant independent proof of this

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fact was recently given by Frith [3]. The constructions described later on could clearly also be expressed in terms of nuclei but we found it advantageous to employ congruences, especially since the proof of the crucial Lemma 3 requires the consideration of lattice congruences.

The significance of frame congruences is the same as that of congruences on any algebraic system: if $h: L \to M$ is a frame homomorphism, then its kernel $\ker h = \{(x, y) \mid h(x) = h(y)\}$ is a congruence, and all congruences on L arise in this way. Particularly important congruences are, for each $a \in L$, the kernels ∇_a and Δ_a of the frame homomorphisms $L \to \uparrow a$ by $x \rightsquigarrow x \vee a$ and $L \to \downarrow a$ by $x \rightsquigarrow x \wedge a$, respectively. Note that $\nabla_0 = \Delta$, $\nabla_e = \nabla$, $\Delta_0 = \nabla$, and $\Delta_e = \Delta$.

Routine calculations, also known for the congruences of a (finitary) distributive lattice, show that, for any $a \in L$, ∇_a and Δ_a are the congruences generated by $(0, a)$ and (a, e), respectively, and for any $a \leqslant b$, $\Delta_a \cap \nabla_b$ is generated by (a, b). The latter shows that, for any congruence Θ on L, one has $\Theta = \bigvee \Delta_a \cap \nabla_b \, (a \leqslant b, (a, b) \in \Theta)$.

In addition, one easily checks that the map $a \rightsquigarrow \nabla_a$ is a frame embedding $L \to \mathfrak{C}L$, whereas $a \rightsquigarrow \Delta_a$ is a dual lattice embedding taking finite meets to joins and arbitrary joins to meets (= intersections). Thus $\mathfrak{C}L$ accommodates a copy of both L and its dual. Moreover, the relationship of these two copies to each other is especially useful: for any $a \in L$ we have $\nabla_a \cap \Delta_a = \Delta$ and $\nabla_a \vee \Delta_a = \nabla$, that is, ∇_a and Δ_a are complementary to each other.

For the following, recall that ΦL is the frame of all filters in L.

LEMMA 2. *For any frame L, the map $F \rightsquigarrow \Delta_F = \bigvee \Delta_a \, (a \in F)$ is a frame homomorphism $\Phi L \to \mathfrak{C}L$.*

Proof. Since $\uparrow e = \{e\}$ is the zero and $\uparrow 0 = L$ the unit of ΦL, and $\Delta_{\uparrow e} = \Delta_e = \Delta$ and $\Delta_{\uparrow 0} = \Delta_0 = \nabla$, this map preserves zero and unit. Further, for any filters F and G in L,

$$\Delta_F \cap \Delta_G = \bigvee_{\substack{a \in F \\ b \in G}} \Delta_a \cap \Delta_b = \bigvee_{\substack{a \in F \\ b \in G}} \Delta_{a \vee b} = \Delta_{F \cap G},$$

the final identity holding since $F \cap G = \{a \vee b \mid a \in F, b \in G\}$. Finally, if $F = \bigvee F_i$ in ΦL then F is generated by all $a_1 \wedge \ldots \wedge a_n$ with arbitrary $a_k \in F_{i_k}$ and arbitrary $i_1 \ldots, i_n$, and hence

$$\Delta_F = \bigvee \Delta_{a_1 \wedge \ldots \wedge a_n} = \bigvee \Delta_{a_1} \vee \ldots \vee \Delta_{a_n},$$

these unions being over all $a_1 \in F_{i_1}, \ldots, a_n \in F_{i_n}$ and all $(i_1, \ldots, i_n)$, so that

$$\Delta_F = \bigvee_i \bigvee_{a \in F_i} \Delta_a = \bigvee \Delta_{F_i}.$$

The following deals with a somewhat technical point needed later on. Recall that a frame L is *coherent* if and only if its compact elements generate L as a frame and are closed under finite (including the empty) meet. The latter implies that the compact elements of L form a sublattice K of L (including the unit $e \in L$), and it is a familiar fact that L is isomorphic to the ideal lattice $\mathfrak{J}K$ of K.

LEMMA 3. *For any coherent frame L, the subframe $\mathfrak{R}L$ of $\mathfrak{C}L$ generated by all $[a, b] = \nabla_a \cap \Delta_b$ for $a, b \in K$ is compact.*

Proof. Since $[a, b] \cap [c, d] = [a \wedge c, b \vee d]$ for any $a, b, c, d \in K$, $\mathfrak{R}L$ consists of all

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joins of such $[a,b]$, and hence it suffices to show that if $\bigvee[a_i,b_i]=\nabla$ then already $[a_{i_1},b_{i_1}]\vee\ldots\vee[a_{i_n},b_{i_n}]=\nabla$ for suitable $i_1,\ldots,i_n$.

Given such $[a_i,b_i]\,(i\in I)$, suppose their restrictions $[a_i,b_i]\,|\,K=[a_i,b_i]\cap(K\times K)$ to K are all contained in some $\Theta\neq\nabla$ in the congruence lattice $\mathfrak{C}K$ of the (finitary) distributive lattice K. Then one has the following commuting square

$$\begin{array}{ccc} K & \xrightarrow{\ i\ } & L\cong\mathfrak{J}K \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\bar{\nu}} \\ K/\Theta & \xrightarrow[\ j\ ]{} & \mathfrak{J}(K/\Theta) \end{array}$$

where ν is the quotient homomorphism, i the natural embedding, j the standard embedding of K/Θ into its ideal lattice (taking elements to principal ideals), and $\bar{\nu}$ resulting from the functoriality of the ideal lattice. Then $\bar{\nu}$ is a frame homomorphism, and since each $[a_i,b_i]$ is generated by $(b_i,a_i\vee b_i)\in K\times K$ it follows from $\Theta\subseteq\ker(\bar{\nu})$ that $[a_i,b_i]\subseteq\ker(\bar{\nu})$ for all i and hence $\nabla=\ker(\bar{\nu})$. This is a contradiction since $\bar{\nu}(0)\neq\bar{\nu}(e)$ because $(0,e)\notin\Theta$. As a result, $\bigvee[a_i,b_i]\,|\,K=\nabla$ in $\mathfrak{C}K$; but here ∇ is compact and hence finitely many of the $[a_i,b_i]\,|\,K$ have join ∇ in $\mathfrak{C}K$. Since the joins of any $[a,b]\,|\,K$ are clearly contained in the corresponding joins in $\mathfrak{C}L$, this yields the desired conclusion.

Remark. Assuming the Boolean Ultrafilter Theorem, any coherent frame L is the topology of a spectral space X (which may be taken as the spectrum of L), and the frame $\mathfrak{K}L$ of the above lemma is then readily identified as the topology of the well-known patch space of X. Since the latter is compact (see Johnstone [7], p. 72) this provides an easy proof of the compactness of $\mathfrak{K}L$, albeit one not admissible in our context. What is of interest, then, is that we have provided here a choice-free argument for the compactness of this frame. As a by-product, we have the observation that the compactness of the patch space of a spectral space only requires the Boolean Ultrafilter Theorem, in contrast with the remark on p. 72 in Johnstone [7].

Next, a known fact to be used repeatedly here which follows from a more general result; we include a short independent proof.

LEMMA 4. *For any compact regular frame L, if two congruences Θ and Ψ on L have the same e-blocks, that is $\Theta[e]=\{x\,|\,(x,e)\in\Theta\}=\Psi[e]$, then $\Theta=\Psi$.*

Proof. Let $(0,b)\in\Theta$ and $a\leqslant b$. Then, for any $x\prec b$ and $u\in L$ such that $x\wedge u=0$ and $b\vee u=e$, first $(a\vee u,e)=(a,b)\vee(u,u)$ belongs to Θ and hence to Ψ, next $(a\wedge x,x)=(a\vee u,e)\wedge(x,x)$ is in Ψ, and finally we have $(a,b)\in\Psi$ by taking the join over all $x\prec b$. This shows that $\Theta\subseteq\Psi$, and by symmetry we have equality.

Finally, we note that the correspondence $L\rightsquigarrow\mathfrak{C}L$ is functorial giving a functor $\mathfrak{C}:\mathbf{Frm}\to\mathbf{Frm}$. This results from the corresponding fact for nuclei (see Johnstone [7]) but can be checked directly easily enough. Any $h:L\to M$ in $\mathbf{Frm}$ determines the map $\mathfrak{C}M\to\mathfrak{C}L$ by taking inverse images, which preserves arbitrary intersections and hence has a left adjoint $\mathfrak{C}h:\mathfrak{C}L\to\mathfrak{C}M$ preserving arbitrary joins. Explicitly, one sees that, for any $\Theta\in\mathfrak{C}L$, $\mathfrak{C}h(\Theta)$ is the congruence generated by all $(h(a),h(b))$ with $(a,b)\in\Theta$. It is then obvious that $\mathfrak{C}h(\Theta)=\bigvee\Delta_{h(a)}\cap\nabla_{h(b)}\,(a\leqslant b,(a,b)\in\Theta)$, and from this one readily obtains that $\mathfrak{C}h$ also preserves finite meets.

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2. *The equivalence*

Recall that **F** and **S** are the functors from biframes to frames taking first and second part, respectively, and acting analogously on biframe homomorphisms. Our aim is to show that **F** induces an equivalence between **KR2Frm** and **StCont**.

As a first observation, we have

LEMMA 5. **F** *takes* **KR2Frm** *into* **StCont**.

Proof. We begin by showing that, for any compact regular biframe L, $a \ll b$ holds in L_1 if and only if $a \prec_1 b$ does. If $a \ll b$ for $a, b \in L_1$ then also $a \leqslant x$ for some $x \prec_1 b$ since b is the join of such x and the set of these is closed under finite joins, and therefore $a \prec_1 b$. Conversely, given the latter and $b \leqslant \bigvee x_i$ for some $x_i \in L_1$, take any $c \in L_2$ such that $a \wedge c = 0$ and $b \vee c = e$, and therefore also $\bigvee x_i \vee c = e$. Compactness then implies $x_{i_1} \vee \ldots \vee x_{i_n} \vee c = e$ for some $i_1, \ldots, i_n$, and by taking meet with a we obtain $a \leqslant x_{i_1} \vee \ldots \vee x_{i_n}$.

By the definition of compact regular biframes, it follows immediately from this, that L_1 is continuous. Further, $a \ll b, c$ implies that $a \prec_1 b, c$, but this implies that $a \prec_1 b \wedge c$, since

$$a \wedge x = 0, \quad b \vee x = e, \quad a \wedge y = 0, \quad c \vee y = 0 \quad (x, y \in L_2)$$

implies

$$a \wedge (x \vee y) = 0, \quad (b \wedge c) \vee (x \vee y) = e \quad (x \vee y \in L_2),$$

and therefore $a \ll b \wedge c$. This, together with the compactness of e in L_1 which it inherits from L_0, shows that L_1 is a stably continuous frame.

Finally, for any homomorphism $h: L \to M$ between compact regular biframes, the map $h_1: L_1 \to M_1$ induced by h preserves $\ll$, and thus is proper, because h obviously preserves $\prec_1$.

Note that, by the symmetry between L_1 and L_2 for $L \in$ **KR2Frm**, **S** also takes **KR2Frm** into **StCont**. The next result links the functors **S** and **F** on **KR2Frm** as follows:

LEMMA 6. *On* **KR2Frm**, **S** *and* Λ**F** *are naturally equivalent.*

Proof. For each $b \in L_2$ in a given compact regular biframe L, let

$$F_b = \{a \in L_1 \mid a \vee b = e\}.$$

This is a filter by distributivity and Scott open by the compactness of e; and hence $F_b \in \Lambda(L_1)$. The correspondence $b \rightsquigarrow F_b$ from L_2 to $\Lambda(L_1)$ clearly preserves order but it also reflects it: if $F_b \subseteq F_c$ for b and c in L_2, consider any $x \prec_2 b$ in L_2 and, accordingly, $a \in L_1$ such that $x \wedge a = 0$ and $b \vee a = e$. Then $a \in F_b$, hence $a \in F_c$, that is $c \vee a = e$, and we conclude that $x \leqslant c$ by taking meets with x. Since b is the join of these x we have $b \leqslant c$, as desired. It follows that we have an order embedding $L_2 \to \Lambda(L_1)$. To see that this is an isomorphism it has to be shown that it is onto. For this, consider any $F \in \Lambda(L_1)$ and take

$$S = \{c \in L_2 \mid x \wedge c = 0 \quad \text{for some} \quad x \in F\}.$$

It is easily checked that S is an ideal. Now let $b = \bigvee S$. Then $b \in L_2$ and we claim that $F = F_b$. If $a \in F_b$, that is $a \vee b = e$, then also $a \vee c = e$ for some $c \in S$ by compactness and for $x \in F$ such that $x \wedge c = 0$ one then has $x = x \wedge (a \vee c) = x \wedge a \leqslant a$, and henc

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$a \in F$. This shows that $F_b \subseteq F$. Conversely, if $a \in F$, then take $x \ll a$ in F and, accordingly, $c \in L_2$ such that $x \wedge c = 0$ and $a \vee c = e$. Then $c \in S$, hence also $a \vee b = e$, and thus $a \in F_b$.

As a result, we have an isomorphism $\phi_L : L_2 \to \Lambda(L_1)$ for each $L \in \mathbf{KR2Frm}$, and it remains to show that this determines a natural equivalence. For this, it has to be checked that the diagram

$$\begin{array}{ccc} L_2 & \xrightarrow{\phi_L} & \Lambda(L_1) \\ {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle \Lambda(h_1)} \\ M_2 & \xrightarrow[\phi_M]{} & \Lambda(M_1) \end{array}$$

resulting from any $h : L \to M$ in **KR2Frm** commutes. This amounts to saying that, for each $b \in L_2$, the Scott open filter $F_{h(b)}$ in M_1 is the same as the Scott open filter generated by all $h(a)$ with $a \in F_b$. By the fact that $a \vee b = e$ implies $h(a) \vee h(b) = e$, the latter is trivially contained in $F_{h(b)}$. Conversely, if $c \vee h(b) = e$ then compactness yields an element $x \ll b$ in L_2 such that $c \vee h(x) = e$, and we can find $d \in L_1$ with $x \wedge d = 0$ and $b \vee d = e$. Then $h(x) \wedge h(d) = 0$, hence $h(d) = c \wedge h(d) \ll c$. However $d \in F_b$, and thus c belongs to the Scott open filter generated by the image of F_b.

We now show that the functor $\mathbf{F}$ reflects isomorphisms on **KR2Frm**.

LEMMA 7. *For any $h : L \to M$ in* **KR2Frm**, *if $\mathbf{F}h$ is an isomorphism then so is h.*

Proof. Given that $\mathbf{F}h$ is an isomorphism, Lemma 6 shows the same for $\mathbf{S}h \cong \Lambda\mathbf{F}h$, and it follows that h is onto. Hence it remains to show h is one–one. We do this by proving that the congruence $\ker h = \{(x, y) \mid h(x) = h(y)\}$ has trivial e-block $\{e\}$ and must therefore be trivial by Lemma 4.

Thus let $h(x) = e$ for $x = \bigvee a_i \wedge b_i$, where $a_i \in L_1$ and $b \in L_2$. Then

$$\bigvee h(a_i) \wedge h(b_i) = e,$$

and by compactness there will be finitely many terms, with indices to be called $1, 2, \ldots, n$, such that

$$(h(a_1) \wedge h(b_1)) \vee \ldots \vee (h(a_n) \wedge h(b_n)) = e. \tag{1}$$

We want to conclude from this that

$$(a_1 \wedge b_1) \vee \ldots \vee (a_n \wedge b_n) = e. \tag{2}$$

Since the implication (1) $\Rightarrow$ (2) obviously holds for $n = 1$ (because then

$$h(a_1) = h(b_1) = e$$

and hence $a_1 = b_1 = e$, $\mathbf{F}h$ and $\mathbf{S}h$ being isomorphisms) this may be proved by induction. For this purpose, we use the general observation that, for any $c \in L_0$, the obviously compact biframe

$$(\uparrow c, \{x \vee c \mid x \in L_1\}, \{z \vee c \mid z \in L_2\})$$

is again regular because, for any $u, v \in L_i\,(i = 1, 2)$, $u \prec_i v$ implies $u \vee c \prec_i v \vee c$. Assuming (1) $\Rightarrow$ (2) for joins of fewer than n terms, for any homomorphism in **KR2Frm** whose first part is an isomorphism, note that (1) implies

$$(h(a_2 \vee a_1) \wedge h(b_2 \vee a_1)) \vee \ldots \vee (h(a_n \vee a_1) \wedge h(b_n \vee a_1)) = e, \tag{3}$$

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which can be viewed as the analogue of (1), with $n-1$ terms, for the homomorphism induced by h between the compact regular biframes

$$L' = (\uparrow a_1, L_1 \cap \uparrow a_1, \{x \vee a_1 \mid x \in L_2\})$$

and

$$M' = (\uparrow h(a_1), M_1 \cap \uparrow h(a_1), \{y \vee h(a_1) \mid y \in M_2\}).$$

Since this homomorphism again induces an isomorphism on first parts it follows by the induction hypothesis that

$$a_1 \vee (a_2 \wedge b_2) \vee \ldots \vee (a_n \wedge b_n) = e. \tag{4}$$

By using b_1 instead of a_1 we obtain the corresponding analogue of (4), and taking meets then produces (2) by distributivity.

Our next step is to produce a functor from **StCont** to **KR2Frm** which will eventually turn out to be an equivalence inverse of **F**. Let $\mathfrak{B}L = \mathfrak{L} = (\mathfrak{L}_0, \mathfrak{L}_1, \mathfrak{L}_2)$ where $\mathfrak{L}_1 = \{\nabla_a \mid a \in L\}$, $\mathfrak{L}_2 = \{\Delta_F \mid F \in \Lambda L\}$ and $\mathfrak{L}_0$ is the subframe of $\mathfrak{C}L$ generated by $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Then $\mathfrak{L}_1$ is a subframe of $\mathfrak{C}L$ as noted in § 1, and by Lemma 2 the same holds for $\mathfrak{L}_2$ since ΛL is a subframe of ΦL. Hence $\mathfrak{B}L$ is indeed a biframe, and the correspondence $L \rightsquigarrow \mathfrak{B}L$ is clearly functorial, the map $\mathfrak{B}L \to \mathfrak{B}M$ for any $h: L \to M$ in **StCont** being induced by the frame homomorphism $\mathfrak{C}h: \mathfrak{C}L \to \mathfrak{C}M$. Note further that, for any stably continuous frame L, there is the obvious isomorphism $\nu_L: L \to \mathbf{F}\mathfrak{B}L$ for which $\nu_L(a) = \nabla_a$ for all $a \in L$, which is clearly natural in L.

The crucial result now is

LEMMA 8. *For each* $L \in$ **StCont**, $\mathfrak{B}L$ *is a compact regular biframe.*

Proof. To see that each $\nabla_a \in \mathfrak{L}_1$ is the join of all $\nabla_x \prec_1 \nabla_a$ it is sufficient to prove that $\nabla_x \prec_1 \nabla_a$ whenever $x \ll a$ in L. Now $x \ll a$ implies that $a \in F \subseteq \uparrow x$ for some Scott open filter F, such as $\Uparrow x$. Thus $\Delta_a \subseteq \Delta_F \subseteq \Delta_x$, hence $\nabla_x \cap \Delta_F = \Delta$ and $\nabla_a \vee \Delta_F = \nabla$, which yields $\nabla_x \prec_1 \nabla_a$. Concerning the counterpart for $\mathfrak{L}_2$, it is again enough to show that $\Delta_F \prec_2 \Delta_G$ whenever $F \ll G$ in ΛL. Now $F \ll G$ implies $F \subseteq \uparrow a \subseteq G$ for some $a \in L$ by the proof of Lemma 1. Hence $\Delta_F \subseteq \Delta_a \subseteq \Delta_G$, and then $\Delta_F \cap \nabla_a = \Delta$ and $\Delta_G \vee \nabla_a = \nabla$, showing that $\Delta_F \prec_2 \Delta_G$.

To prove the compactness of $\mathfrak{L}_0$, we shall apply Lemma 3 via the embedding $\mathfrak{C}k: \mathfrak{C}L \to \mathfrak{C}(\mathfrak{J}L)$ resulting from the left invertible frame homomorphism $k = \kappa_L: L \to \mathfrak{J}L$ given by $k(a) = \Downarrow a = \{x \mid x \ll a\}$. This embedding takes any $\nabla_a \cap \Delta_F$ with $a \in L$ and $F \in \Lambda L$ to the congruence

$$\Theta = \bigvee [\downarrow x, \downarrow b] \, (x \ll a, b \in F)$$

on $\mathfrak{J}L$: it maps Δ_F to $\bigvee \Delta_{k(b)} \, (b \in F)$; but because $k(b) \subseteq \downarrow b$ one has

$$\bigvee \Delta_{k(b)} \, (b \in F) \supseteq \bigvee \Delta_{\downarrow b} \, (b \in F),$$

and the reverse inclusion holds since there exists an element $c \ll b$ in F for each $b \in F$. This shows that $\nabla_a \cap \Delta_F$ is mapped to

$$\bigvee_{b \in F} \nabla_{k(a)} \cap \Delta_{\downarrow b} = \bigvee_{x \ll a, \, b \in F} \nabla_{\downarrow x} \cap \Delta_{\downarrow b}$$

since $k(a) = \bigvee \downarrow x \, (x \ll a)$ in $\mathfrak{J}L$. The latter congruence, however, is Θ. Now the principal ideals of L are exactly the compact elements of $\mathfrak{J}L$, and hence $\mathfrak{C}k$ already embeds $\mathfrak{L}_0$ into the subframe of $\mathfrak{C}(\mathfrak{J}L)$ which is compact by Lemma 3. Thus $\mathfrak{L}_0$ is compact.

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As a result, we now have a functor $\mathfrak{B}$: **StCont** $\to$ **KR2Frm**. The last major step towards the desired goal is

LEMMA 9. *For each* $L \in$ **KR2Frm**, *there is an isomorphism* τ_L: $\mathfrak{B}\mathbf{F}L \to L$ *taking* ∇_a *to* a *for each* $a \in \mathbf{F}L$.

Proof. We first consider the frame homomorphism $\mathfrak{C}L_1 \to \mathfrak{C}L_0$ induced by the natural embedding $L_1 \to L_0$; it takes any congruence Θ on L_1 to the congruence $\bar{\Theta}$ which it generates on the larger frame L_0. For the special congruences Δ_a and ∇_a with $a \in L_1$, one then has

$$\bar{\Delta}_a = \Delta_a^\circ \quad \text{and} \quad \bar{\nabla}_a = \nabla_a^\circ,$$

where the superscript $^\circ$ indicates the analogous congruences on L_0. It follows that $\bar{\Delta}_F = \bigvee \Delta_a^\circ (a \in F)$ for any $F \in \Lambda L_1$. We claim this is just ∇_b° for that $b \in L_2$ such that $F = F_b$ as in the proof of Lemma 6. If $a \in F$ then $a \vee b = e$ in L_0, hence $\nabla_a^\circ \vee \nabla_b^\circ = \nabla^\circ$ in $\mathfrak{C}L_0$. Thus $\Delta_a^\circ = \Delta_a^\circ \cap \nabla_b^\circ \subseteq \nabla_b^\circ$, and therefore $\bar{\Delta}_F \subseteq \nabla_b^\circ$. On the other hand, $(x, e) \in \nabla_b^\circ$ implies that $x \vee b = e$ by definition, hence by compactness and 2-regularity also $x \vee c = e$ for some $c \prec_2 b$, and if $a \in L_1$ satisfies $c \wedge a = 0$ and $b \vee a = e$ then $x \wedge a = a = e \wedge a$, showing that $(x, e) \in \Delta_a^\circ$. Since $a \in F$, this implies that $(x, e) \in \bar{\Delta}_F$, and in all we then have the inclusion $\nabla_b^\circ[e] \subseteq \bar{\Delta}_F[e]$. Since also $\bar{\Delta}_F \subseteq \nabla_b^\circ$, as was just proved, this means that $\bar{\Delta}_F$ and ∇_b° have the same e-blocks, and by Lemma 4 it follows that they are equal.

In all, this shows that the frame homomorphism $\mathfrak{C}L_1 \to \mathfrak{C}L_0$ maps the subframe of $\mathfrak{C}L_1$ consisting of all $\Theta = \bigvee \nabla_{a_i} \cap \Delta_{F_i}$ where $a_i \in L_1$ and $F_i \in \Lambda L_1$, that is, the total frame of $\mathfrak{B}\mathbf{F}L$, onto the subframe of $\mathfrak{C}L_0$ consisting of all ∇_c° with $c \in L_0$, since

$$\bar{\Theta} = \bigvee \nabla_{a_i}^\circ \cap \nabla_{b_i}^\circ = \nabla_c^\circ, \quad c = \bigvee a_i \wedge b_i,$$

where $b_i \in L_2$ corresponds to $F_i \in \Lambda L_1$ as previously described.

Given that the map $c \rightsquigarrow \nabla_c^\circ$ for $c \in L_0$ embeds L_0 in $\mathfrak{C}L_0$, we therefore obtain a frame homomorphism τ_L from the total frame of $\mathfrak{B}\mathbf{F}L$ into L_0 such that $\tau_L(\Theta) = c$ whenever $\bar{\Theta} = \nabla_c^\circ$. Obviously $\tau_L(\nabla_a) = a$ for any $a \in L_1$ because $\bar{\nabla}_a = \nabla_a^\circ$; similarly $\tau_L(\Delta_F) = b$ if $F = F_b$ for $F \in \Lambda L_1$. Hence τ_L: $\mathfrak{B}\mathbf{F}L \to L$ is in **KR2Frm**, and since it is obviously an isomorphism for the first parts, Lemma 7 shows it is an isomorphism.

Note that, for any h: $L \to M$ in **KR2Frm**, the corresponding $\mathfrak{B}\mathbf{F}h$: $\mathfrak{B}\mathbf{F}L \to \mathfrak{B}\mathbf{F}M$ takes each ∇_a with $a \in L_1$ to $\nabla_{h(a)}$, and each Δ_F for $F = F_b$ with $b \in L_2$ to Δ_G where $G = \Lambda h_1(F_b)$, which is $F_{h(b)}$ by the naturality of ϕ_L in the proof of Lemma 6. This says that $(\mathfrak{B}\mathbf{F}h)\tau_L$ and $\tau_M h$ coincide on the first and the second part of L, and hence they are equal. Thus τ_L is natural in L. On the other hand, $\mathbf{F}\tau_L = \nu_{\mathbf{F}L}^{-1}$ since $\nu_{\mathbf{F}L}$ takes $a \in \mathbf{F}L$ to ∇_a. From this one immediately deduces by straightforward calculations that $\mathfrak{B}$ is left adjoint to $\mathbf{F}$ with the natural equivalence ν and τ as unit and counit, respectively (see Mac Lane [**8**]), and hence concludes

PROPOSITION. *The functor* $\mathbf{F}$ *induces a right adjoint equivalence between* **KR2Frm** *and* **StCont**.

Remark 1. By Lemma 6, this equivalence between **KR2Frm** and **StCont** makes Λ on **StCont** correspond to the obvious endofunctor D of **KR2Frm** which reverses first and second parts: $\mathbf{F}D = \mathbf{S} \cong \Lambda\mathbf{F}$. Since D^2 is plainly the identity functor, this implies that $\Lambda^2 \cong \mathrm{Id}$ on **StCont**. Tracing through the details one sees that the natural

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isomorphism $L \to \Lambda(\Lambda L)$ is given by the map $a \rightsquigarrow \{F \mid a \in F \in \Lambda L\}$. The crucial point in the necessary calculation is the following obvious symmetry observation: for $a \in L_1$ and $b \in L_2$ with corresponding $F_b \in \Lambda L_1$ and $F_a \in \Lambda L_2$, one has $a \in F_b$ if and only if $b \in F_a$. This holds since each statement is equivalent to $a \vee b = e$.

Remark 2. The symmetric $L \in \mathbf{KR2Frm}$, that is, the L with equal first and second (and hence total) parts, are precisely those for which the first part is regular. Hence the equivalence **F** makes the compact regular frames, an important subcategory of **StCont**, correspond exactly to the symmetric compact regular biframes.

Remark 3. As pointed out in §0, the topological counterpart of **KR2Frm** is precisely the category $\mathbf{KR2TOP_0}$ of compact regular T_0-bispaces. By the very definition, one has the contravariant functor $\mathbf{KR2TOP_0} \to \mathbf{KR2Frm}$ taking each $X \in \mathbf{KRTOP_0}$ to the biframe consisting of its total topology together with its first and second topology. On the other hand, the topological analogue of **StCont** is the category **SpRetr** of spectral retracts, which may be defined as the sober spaces X whose topology $\mathfrak{O}X$ is a stably continuous frame, and the maps of **SpRetr** are those continuous maps $f: X \to Y$ which induce a proper frame homomorphism $\mathfrak{O}Y \to \mathfrak{O}X$. **SpRetr** can be described in many different ways, and spectral retracts have been called many different names, but the point here is that one has an obvious contravariant functor $\mathbf{SpRetr} \to \mathbf{StCont}$. Moreover, by Lemma 5 one has the functor $\mathbf{KR2TOP_0} \to \mathbf{SpRetr}$ which forgets the second topologies.

As already mentioned, given the Axiom of Choice (or, slightly less restrictively, the Boolean Ultrafilter Theorem (*BUT*)), the first two functors just described are dual equivalences – and, indeed, either of these assertions is equivalent to *BUT*. Hence we conclude from the Proposition:

given BUT, the functor $\mathbf{KR2TOP_0} \to \mathbf{SpRetr}$ *forgetting the second topology is an equivalence.*

This result appears, in various equivalent forms, in Hofmann [**6**], Salbany [**9**], and Wyler [**11**].

Remark 4. One of the results that gives coherent frames their special importance is that (assuming the Axiom of Choice) they are precisely the frames $\mathrm{RId}\, A$ of radical ideals of commutative rings A with unit – which is just the frame version of the result of Hochster [**5**] that the spectral spaces, that is, the spectra of coherent frames are precisely the prime spectra of those rings, that is, the spaces of prime ideals with the Zariski topology. It is a long standing question whether there is a constructive proof of Hochster [**5**] that the spectral spaces, that is, the spectra of coherent frames, are be a proof that produces for any coherent frame L a commutative ring A with unit such that $L \cong \mathrm{RId}\, A$. Hochster uses the patch topology on a spectral space, essentially turning the space into a bitopological space. Now the frame counterpart of this is precisely our $\mathfrak{B}L$, and it may be this can be used to construct the desired ring. We hope to pursue this matter further.

Remark 5. Another natural problem concerning coherent frames is to characterize them in terms of compact regular biframes. It can be shown that the compact regular biframes L with coherent first part are exactly those which are 0-*dimensional* in the following biframe sense: each $L_i\,(i = 1, 2)$ is generated, as a frame, by those $x \in L_i$

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which are complemented in L_0, with complement belonging to $L_k, k \neq i$. The details, and certain ramifications of this notion, are considered elsewhere.

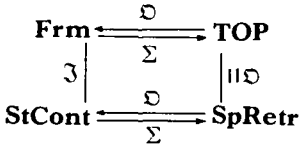
3. *Concluding remarks*

Here we add a number of observations, all quite straightforward to prove, concerning the role of **StCont** in **Frm** and various facts connected with that.

As pointed out in §0, **StCont** is coreflective in **Frm** with the ideal lattice functor $\mathfrak{J}$ as the coreflection functor, the join maps $\sigma_L: \mathfrak{J}L \to L$ as the counit and the maps $\kappa_L: L \to \mathfrak{J}L$ defined by $\kappa_L(a) = \mathop{\downarrow} a$ for $a \in L$ as the unit. The comonad associated with this adjointness is then the *ideal comonad* $(\mathfrak{J}, \sigma, \mathfrak{J}\kappa)$ where the comultiplication $\mathfrak{J}\kappa_L: \mathfrak{J}L \to \mathfrak{J}\mathfrak{J}L$ is explicitly described as the map $J \rightsquigarrow \{I \in \mathfrak{J}L \mid \bigvee I \in J\}$. The result that **StCont** is coreflective in **Frm** via $\mathfrak{J}$ is easily augmented to the assertion that **StCont** *is comonadic in* **Frm** *by the functor* $\mathfrak{J}$ *such that each* $L \in$ **StCont** *carries exactly one coalgebra structure.*

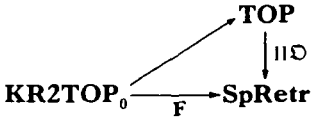
It trivially follows from this and our Proposition that the functor **KR2Frm** $\to$ **Frm** taking first parts is comonadic, the comonad involved again being the ideal comonad since the left adjoint is obviously $\mathfrak{B}\mathfrak{J}$: **Frm** $\to$ **StCont** $\to$ **KR2Frm**.

The topological counterpart of the comonadicity of **StCont** in **Frm** is the monadicity of **SpRetr** in **TOP**, and assuming *BUT* the latter becomes a consequence of the former since the bottom part of the diagram



(where Σ is frame spectrum and Π is lattice spectrum) is then an equivalence. The main point to note here is that, already for arbitrary bounded distributive lattices L, the lattice spectrum ΠL, that is, the space of prime filters of L, is the same as the frame spectrum $\Sigma\mathfrak{J}L$ of its ideal lattice, that is, the space of all completely prime filters in $\mathfrak{J}L$. The ideal comonad on the category **Frm** then corresponds to the monad given by the functor $\Pi\mathfrak{O}$, the open prime filter monad. Thus we recover the result of Simmons [**10**]: **SpRetr** *is monadic in* **TOP** *by the functor* $\Pi\mathfrak{O}$ *such that each spectral retract carries exactly one algebra structure.*

Replacing **StCont** by $\mathbf{KR2TOP_0}$ in the above consideration (still assuming *BUT*) one obtains the diagram



where the unnamed functor again forgets second topologies. This diagram exhibits $\mathbf{KR2TOP_0}$ as the category of algebras of the open prime filter monad in **TOP** via the functor forgetting second topologies. This result is also contained in Salbany [**9**] and Wyler [**11**].

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