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Measures with uniform scaling scenery

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Abstract. We introduce a property of measures on Euclidean space, termed 'uniform scaling scenery'. For these measures, the empirical distribution of the measure-valued time series, obtained by rescaling around a point, is (almost everywhere) independent of the point. This property is related to existing notions of self-similarity: it is satisfied by the occupation measure of a typical Brownian motion (which is 'statistically' self-similar), as well as by the measures associated to attractors of affine iterated function systems (which are 'exactly' self-similar). It is possible that different notions of self-similarity are unified under this property. The proofs trace a connection between uniform scaling scenery and Furstenberg's 'CP processes', a class of natural, discrete-time, measure-valued Markov processes, useful in fractal geometry.

1. *Introduction*

Consider a measure θ on $\mathbb{R}^d$ and a point x in its support. Use a 'microscope' with a continuous knob, centered on x, to expand θ spatially by a factor of e^t. At the same time, rescale θ to leave the measure of the unit cube around x unchanged. Denote the expanded and rescaled measure by $\theta_{x,t}$. We call the continuous-time, measure-valued time-series $t \mapsto \theta_{x,t}$ the *scaling scenery* of θ at x. As a time series, it may or may not have limiting empirical means. If it does, let us denote by $\eta[\theta, x]$ the corresponding ('empirical') measure-valued stationary process. The map $x \mapsto \eta[\theta, x]$ is an interesting structure attached to θ, pertaining to its fractal behavior. Of special interest is the case where the map $x \mapsto \eta[\theta, x]$ is θ-almost everywhere constant, namely when θ exhibits just a *single* process $\eta[\theta]$—regardless of where we place our microscope. Our main objective is to make precise and investigate this property of *uniform scaling scenery*. Do such measures exist? A typical Brownian occupation measure in $\mathbb{R}^d$ ($d \geq 3$) is a natural example. What is perhaps more surprising, the classical Cantor measure is another; this stems from an intriguing relation with a class of natural, discrete-time, measure-valued Markov processes, called *CP* (conditional probability) *processes*. These systems provide a general approach for attacking problems in fractal geometry [**F1, F2, FW**]. We show

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that 'almost any' measure in an ergodic CP process has uniform scaling scenery, thus obtaining a large collection of measures with this property. This raises natural questions regarding the extent of the connection between CP processes (which are discrete-time, and therefore possess inherent scale) and measures with uniform scaling scenery (a property that does not depend on scale). Finally, we combine our main result with the general CP system approach, mentioned above, to show that any self-similar measure (a natural measure associated with the attractor of an affine iterated function system) has uniform scaling scenery. The different notions of self-similarity in Brownian motion (considered 'statistical' in nature) and of self-similarity in, say, the Cantor measure (considered 'exact') are therefore unified under this property.

This text is organized as follows. In §§2 and 3, we define the uniform scaling scenery property and introduce its relation to existing notions of self-similarity. In §4, we describe CP processes following [**F2**]. Sections 5 and 6 contain our main result, Theorem 5.1, whereby almost any measure in an ergodic CP process has uniform scaling scenery. In §7 we invoke this result to prove that any self-similar measure has uniform scaling scenery (Theorem 3.2).

Straightforward propositions are stated without proof for brevity. All proofs can be found in [**G**] (available online).

2. *The uniform scaling scenery property*

We begin with some notation. We write $\mathcal{M}(\mathbb{R}^d)$ for the set of Radon measures on $\mathbb{R}^d$. A *cube* in $\mathbb{R}^d$ is a product of compact intervals. Suppose that $Q \subset \mathbb{R}^d$ is a cube. We write $\mathcal{P}(Q)$ for the space of probability measures on Q. Both $\mathcal{M}(\mathbb{R}^d)$ and $\mathcal{P}(Q)$ are considered with their natural weak-* topology. $\mathcal{P}(Q)$ is thus compact and metric. We write $|\theta|$ for the support of the measure $\theta \in \mathcal{M}(\mathbb{R}^d)$. For a Borel map $f : \mathbb{R}^d \to \mathbb{R}^d$, denote by $f_*(\theta)$ the push-forward of θ by f, namely $f_*(\theta)(A) = \theta(f^{-1}(A))$ for $A \subset \mathbb{R}^d$. For any two cubes $Q_1, Q_2 \subset \mathbb{R}^d$, denote by $\mathbf{H}\langle Q_1, Q_2\rangle : \mathbb{R}^d \to \mathbb{R}^d$ the unique homothety† inducing a bijection $\mathring{Q}_1 \twoheadrightarrow \mathring{Q}_2$. We remove redundant brackets and write $\mathbf{H}\langle Q_1, Q_2\rangle_*(\theta)$ for the push-forward of θ by the map $\mathbf{H}\langle Q_1, Q_2\rangle$.

Our first step is to make precise the notion of rescaling a Radon measure $\theta \in \mathcal{M}(\mathbb{R}^d)$ around a point $x \in \mathbb{R}^d$.

Definition 2.1. Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Define the family of cubes $\{Q^{\boxdot}_{x,t}\}_{t\in\mathbb{R}}$ by

$$Q^{\boxdot}_{x,t} = \prod_{i=1}^{d}\left[x_i - \frac{1}{2}e^{-t},\, x_i + \frac{1}{2}e^{-t}\right].$$

We abbreviate $Q^{\boxdot}_{x,0}$ (the cube of side length 1 centered at x) as $Q^{\boxdot}_{x}$.

Definition 2.2. Suppose that $\theta \in \mathcal{M}(\mathbb{R}^d)$ and $x \in |\theta|$. The ($\mathcal{P}(Q^{\boxdot}_x)$-valued, continuous-time) time series $\{\theta^{\boxdot}_{x,t}\}_{t\in\mathbb{R}}$ defined by

$$\theta^{\boxdot}_{x,t} = \left.\frac{\mathbf{H}\langle Q^{\boxdot}_{x,t},\, Q^{\boxdot}_{x}\rangle_*(\theta)}{\theta(Q^{\boxdot}_{x,t})}\right|_{Q^{\boxdot}_x}$$

is called the *scaling scenery* of θ around x‡.

† By homothety we mean a dilation composed with a translation in Euclidean space.

‡ Compare our definition with the notion of *scenery flow* in [**Fi1, Fi2**] and the notion of *tangential measure* in [**PrMö**].

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Definition 2.3. Let θ, x and $\mathbf{x} = \{\theta^{\boxdot}_{x,t}\}_{t\in\mathbb{R}}$ be as above. Define a net $\{\varphi_T\}_{T>0}$ of positive, linear and normalized† functionals $\varphi_T : \mathcal{C}(\mathcal{P}(Q^{\boxdot}_x)^{\mathbb{R}}) \to \mathbb{R}$ by

$$\varphi_T : g \longmapsto \frac{1}{T}\int_{t=0}^{t=T} g\circ\sigma_t(\mathbf{x})\,dt, \tag{2.1}$$

where σ_t is the time shift by t. Recall that $\mathbf{x}$ is a generic point for some shift-invariant measure on $\mathcal{P}(Q^{\boxdot}_x)^{\mathbb{R}}$ if and only if the net $\{\varphi_T\}_{T>0}$ converges weakly (as $T\to\infty$) to some positive, linear and normalized functional Φ. In this case, the time series $\mathbf{x} = \{\theta^{\boxdot}_{x,t}\}_{t\in\mathbb{R}}$ is said to have *limiting empirical means*, and the limiting functional Φ then corresponds to a probability distribution $\eta[x,\theta]$ on $\mathcal{P}(Q^{\boxdot}_x)^{\mathbb{R}}$ such that

$$\frac{1}{T}\int_{t=0}^{t=T} g\circ\sigma_t(\mathbf{x})\,dt \longrightarrow \int_{\mathcal{P}(Q^{\boxdot}_x)^{\mathbb{R}}} g\,d\eta[x,\theta] \quad \text{as } T\to\infty,$$

for all continuous g. When this holds, we will say that θ has *limiting scaling means* at x, and will refer to the stationary measure-valued process $\eta[x,\theta]$ as the *empirical scaling process* of θ at the point x.

An interesting structure is thus attached to a measure $\theta\in\mathcal{M}(\mathbb{R}^d)$: to any point $x\in|\theta|$ such that θ has limiting scaling means at x, we attach the $\mathcal{P}(Q^{\boxdot}_x)$-valued, continuous-time empirical scaling process $\eta[x,\theta]$. Of special interest is the case where the mapping $x\mapsto\eta[x,\theta]$ is θ-a.e defined and (in an appropriate sense) constant.

Note that for $x\neq y\in|\theta|$, we cannot directly compare the processes $\eta[x,\theta]$ and $\eta[y,\theta]$ as one is $\mathcal{P}(Q^{\boxdot}_x)$-valued and the other is $\mathcal{P}(Q^{\boxdot}_y)$-valued. To remedy this, suppose that η_x (η_y) are continuous-time, $\mathcal{P}(Q^{\boxdot}_x)$ ($\mathcal{P}(Q^{\boxdot}_y)$)-valued stationary processes for some x, y. If the processes $\mathbf{H}\langle Q^{\boxdot}_x, Q^{\boxdot}_y\rangle_*(\eta_x)$ and η_y have the same distribution, we say that η_x and η_y are equivalent, and write $\eta_x\sim\eta_y$. This is an equivalence relation. We write $\eta_\sim$ for the equivalence class of η. Let $\theta\in\mathcal{M}(\mathbb{R}^d)$ and $x, y\in|\theta|$. Observe that if the $\mathcal{P}(Q^{\boxdot}_0)$-valued time series $(\mathbf{H}\langle Q^{\boxdot}_x, Q^{\boxdot}_0\rangle_*(\theta^{\boxdot}_{x,t}))_{t\in\mathbb{R}}$ and $(\mathbf{H}\langle Q^{\boxdot}_y, Q^{\boxdot}_0\rangle_*(\theta^{\boxdot}_{y,t}))_{t\in\mathbb{R}}$ both have limiting empirical means with the same empirical process, then θ has limiting scaling means both at x and at y, and that in fact $\eta[x,\theta]\sim\eta[y,\theta]$.

The stage is now set for our key definition.

Definition 2.4. Let $\theta\in\mathcal{M}(\mathbb{R}^d)$ be a Radon measure. We say that θ has *uniform scaling scenery* if for θ-almost every $x\in Q$, θ has limiting scaling means at x, such that the mapping $x\mapsto\mu[x,\theta]_\sim$ is θ-a.e. constant on Q. In this case, the $\mathcal{P}(Q^{\boxdot}_0)$-valued representative of the (almost everywhere) constant value of this mapping is called the *uniform scaling process*, and denoted by $\eta[\theta]$.

The following useful criterion for uniform scaling scenery is an easy consequence of the Stone–Weierstrass theorem.

LEMMA 2.5. *Let $\theta\in\mathcal{M}(\mathbb{R}^d)$ and assume that $\mathcal{A}\subset\mathcal{C}(\mathcal{P}(Q^{\boxdot}_0))$ is a dense subalgebra with $\mathbf{1}_{Q^{\boxdot}_0}\in\mathcal{A}$. If there exists a set $H\subset Q$ with $\theta(Q\backslash H)=0$ such that for any $\ell\in\mathbb{N}$, any*

† In the sense that $\varphi(\mathbf{1})=1$.

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continuous functions $f_0, \ldots, f_\ell \in \mathcal{A}$ *and every* $(\tau_0, \ldots, \tau_\ell) \in \mathbb{Q}^{\ell+1}$ *the limit*

$$\lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{T} \prod_{i=0}^{\ell} f_i(\boldsymbol{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\tau_i}^{\boxdot}))\, dt$$

exists and is identical for each $x \in H$*, then* θ *has uniform scaling scenery, and the uniform scaling process* $\eta[\theta]$ *is uniquely determined by the values*

$$\int_{\mathcal{P}(Q_0^{\boxdot})^{\mathbb{R}}} \left(\prod_{i=0}^{\ell} f_i(\mathbf{y}_{\tau_i}) \right) d\eta[\theta](\mathbf{y}).$$

3. *Self-similarity and uniform scaling scenery*

In the context of fractal geometry, the notion of self-similarity is given different formal interpretations. For example, attractors of certain iterated function systems are said to demonstrate 'exact' self-similarity, while sample paths of certain stochastic processes are said to demonstrate 'statistical' self-similarity.

The uniform scaling scenery property appears to be related, in some sense, to the notion of self-similarity. This connection is now given formal content by two observations. The first is that a typical Brownian occupation measure (a prototype for 'statistical' self-similarity) enjoys the uniform scaling scenery property. The second is that so do self-similar measures, which are measures associated to attractors of affine iterated function systems, such as the Sierpiński triangle (prototypes for 'exact' self-similarity).

These observations raise the question of whether, and to what extent, the uniform scaling scenery property could be used to unify different formal interpretations of the term 'self-similarity'.

3.1. *Brownian motion.*

An ensemble of measures with uniform scaling scenery is generated by transient Brownian motion as follows. Abbreviate 'two-sided, standard Brownian motion through the origin in $\mathbb{R}^d$' by *BM*. Let $d \geq 3$ and denote by $(\Omega, \mathcal{B}, \mu_w)$ the probability space of continuous paths through the origin in $\mathbb{R}^d$ in the topology of convergence on compact sets, its corresponding Borel σ-algebra, and the Wiener measure induced by a BM process. For a given BM sample path† B_s, define the corresponding Brownian occupation measure θ_B on $\mathbb{R}^d$ to be the push-forward measure $B_*\lambda$, where λ is the Lebesgue measure on $\mathbb{R}$. The probability law of the random BM curve B thus turns θ_B into a random Radon measure on $\mathbb{R}^d$.

THEOREM 3.1. *There exists a set of BM paths* $\mathcal{G} \in \mathcal{B}$ *with* $\mu_w\mathcal{G} = 1$ *such that for every* $B \in \mathcal{G}$*,* θ_B *has uniform scaling scenery. Moreover, the mapping* $B \mapsto \eta[\theta_B]$ *is constant on* $\mathcal{G}$*, where* $\eta[\theta_B]$ *is the uniform scaling process of the measure* θ_B*.*

The proof is straightforward [**G**]. This is a consequence of the ergodicity of the scaling flow $\Psi_t : \Omega \to \Omega$, defined by $(\Psi_t f)(t) = e^t f(e^{-2t}s)$ [**Fi1**]. It is interesting to note that for $B \in \mathcal{G}$, the map $x \mapsto \eta[x, \theta_B]$ is θ_B-almost everywhere constant on Q, but not everywhere constant.

† Since the variable t is used to denote scaling, we write B_s for a Brownian path and s for the path parameter.

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3.2. *Self-similar measures.* The connection of the uniform scaling scenery property with CP processes, which we develop below, will yield in §7 the following result.

THEOREM 3.2. *Let $\phi_1, \ldots, \phi_n : Q \to Q$ be an iterated function system of contracting similitudes ($Q \subset \mathbb{R}^d$ is the unit cube and $n \geq 2$), satisfying the open set condition, with attractor $A = \bigcup_{i=1}^n \phi_i(A)$. Let θ be a self-similar probability measure on A corresponding to the strictly positive probability vector $(p_1, \ldots, p_n)$. Then θ has uniform scaling scenery.*

4. *CP processes*

Let $Q \subset \mathbb{R}^d$ be the unit cube, $Q = [0, 1]^d$. Choose a *grid-size parameter* $2 \leq p \in \mathbb{N}$ and divide each of the d axes into p equal subintervals of length $1/p$, indexed by $\{0, 1, \ldots, p-1\}$. This induces the 'p-ary decomposition' of Q into p^d disjoint subcubes† of side length $1/p$, indexed by the product set $\Lambda = \{0, 1, \ldots, p-1\}^d$, given by

$$Q = \biguplus_{\lambda \in \Lambda} Q_\lambda.$$

For $x \in Q$, there is a unique $\lambda(x) \in \Lambda$ with $x \in Q_{\lambda(x)}$. This partition is iterated in each subcube to yield a multi-scale partition of Q: for a word of length ℓ, $\lambda_1 \ldots \lambda_\ell \in \Lambda^\ell$, a subcube $Q_{\lambda_1, \lambda_2, \ldots, \lambda_\ell}$ of side length $p^{-\ell}$ is now defined inductively. Note that $Q_{\lambda_1 \lambda_2 \ldots \lambda_{\ell+1}} \subset Q_{\lambda_1 \lambda_2 \ldots \lambda_\ell}$ and that if $\{\lambda_i\}_{i \in \mathbb{N}}$ are such that $x \in \bigcap_{\ell \geq 1} Q_{\lambda_1 \lambda_2 \ldots \lambda_\ell}$, then

$$x = \sum_{n=1}^{\infty} \frac{\lambda_n}{p^n},$$

where each λ_i is a vector in $\mathbb{R}^d$.

We now construct a discrete-time Markov process on the state-space $\mathcal{P}(Q)$. Write θ^λ for the measure $\theta|_{Q_\lambda}$, stretched, translated and normalized to yield a probability measure on Q, namely

$$\theta^\lambda = \left. \frac{\mathbf{H}\langle Q_\lambda, Q\rangle_*(\theta)}{\theta(Q_\lambda)} \right|_Q.$$

The decomposition $Q = \biguplus_{\lambda \in \Lambda} Q_\lambda$ suggests a natural transition function (that depends implicitly on the parameter p): from the state $\theta \in \mathcal{P}(Q)$ the transition $\theta \to \theta^\lambda$ occurs with probability $\theta(Q_\lambda)$. In terms of a Markov operator $\Delta_p : \mathcal{C}(\mathcal{P}(Q)) \to \mathcal{C}(\mathcal{P}(Q))$,

$$(\Delta_p f)(\theta) = \sum_{\lambda \in \Lambda} \theta(Q_\lambda) f(\theta^\lambda). \tag{4.1}$$

Definition 4.1. Let $Q \subset \mathbb{R}^d$.

(1) If ν is a Δ_p-invariant distribution on $\mathcal{P}(Q)$, the corresponding stationary process is called a CP ('conditional probability') process. We denote it by (ν, p).

(2) A CP process (ν, p) is said to be *non-degenerate* if for ν-almost every $\theta \in \mathcal{P}(Q)$, we have $\theta(\partial Q) = 0$.

† Some half-open, half-closed. See [**F2**, p. 407] for details.

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Below, all *ergodic* CP processes are implicitly assumed to be non-degenerate. Indeed, as the set of measures that violate the condition above is invariant, ergodicity implies that it can always be assumed to hold ν-almost surely [**G**].

The CP process notion was suggested implicitly in [**F1**] and defined (in the equivalent form of a dynamical system) in [**F2**]. See [**F2, FW**] for applications of the CP process notion and an equivalent definition using trees. [**LP**, Ch. 16, **LPP1, LPP2, LPP3**] contain interesting applications to random walks on Galton–Watson trees. The abstract existence of CP processes can be shown as indicated in §7 below (see also [**F2**, Theorem 4.1]). Explicit examples of CP processes can be found in [**F2**, p. 409] and in [**G**].

4.1. *Extension of an ergodic CP process.* The state space of a CP process is $\mathcal{P}(Q)$, the probability measures on $Q \subset \mathbb{R}^d$. This space is not closed under translation of measures, a crucial property for our needs. We now show that for an ergodic CP process, this can be remedied by extending the CP process at hand to a process whose states are Radon measures on $\mathbb{R}^d$ that are normalized on Q. Denote this space by $\mathcal{M}_Q(\mathbb{R}^d)$.

Given a grid-size parameter p, define a Markov transition operator $\Delta_p : \mathcal{C}(\mathcal{M}_Q(\mathbb{R}^d)) \to \mathcal{C}(\mathcal{M}_Q(\mathbb{R}^d))$ as in (4.1), where for $\theta \in \mathcal{M}_Q(\mathbb{R}^d)$ and $\lambda \in \Lambda$,

$$\theta^\lambda = \frac{\mathbf{H}\langle Q_\lambda,\, Q\rangle_*(\theta)}{\theta(Q_\lambda)}.$$

If $\bar{\nu}$ is a Δ_p-invariant distribution on $\mathcal{M}_Q(\mathbb{R}^d)$, the corresponding stationary process is called an *extended* CP process, again denoted by $(\bar{\nu},\, p)$.

Clearly, if $(\bar{\nu},\, p)$ is an extended CP process, then the restriction

$$\mathcal{M}_Q(\mathbb{R}^d) \ni \theta \mapsto \theta|_Q \in \mathcal{P}(Q)$$

produces a CP process. We now reason that for ergodic CP processes, this correspondence can be reversed.

LEMMA 4.2. *Let $(\nu,\, p)$ be an ergodic (non-degenerate) CP process. Then there exists an ergodic extended CP process $(\bar{\nu},\, p)$†, from which the process $(\nu,\, p)$ can be recovered by the restriction $\theta \mapsto \theta|_Q$.*

Proof. Consider a slight revision of the CP process definition, for which the current state also contains the information about the last transition to have occurred. The state space here is $\mathcal{P}(Q) \times \Lambda$ and the transition

$$(\theta,\, \cdot) \to (\theta^\lambda,\, \lambda)$$

occurs with probability $\theta(Q_\lambda)$.

Next, recall that any stationary process $(X_n)_{n\in\mathbb{N}}$ may be naturally extended to a process $(X_n)_{n\in\mathbb{Z}}$ having the same finite-dimensional distributions. If $(\nu,\, p)$ is our ergodic CP process (assumed as above to be $\mathcal{P}(Q) \times \Lambda$-valued), and $(X_n)_{n\in\mathbb{N}}$ is the corresponding $\mathcal{P}(Q) \times \Lambda$-valued stationary process, let $(X_n)_{n\in\mathbb{Z}}$ denote its natural extension. Map any

† Here $\bar{\nu}$ is an invariant distribution on $\mathcal{M}_Q(\mathbb{R}^d)$.

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sample path of $(X_n)_{n\in\mathbb{Z}}$, which we shall denote by $(\theta_n, \lambda_n)_{n\in\mathbb{Z}}$, to a sequence $(\vartheta_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{M}_Q(\mathbb{R}^d)$ defined by $\vartheta_0 = \theta_0$ and

$$\vartheta_n = \frac{\mathbf{H}\langle Q_{w_n}, Q\rangle_*(\theta_{-n})}{\theta_{-n}(Q_{w_n})}$$

where $w_n = \lambda_{-n+1} \dots \lambda_0$ is a word in Λ^n. The non-degeneracy condition implies that almost surely the sequence $(\vartheta_n)_{n\in\mathbb{N}}$ converges in $\mathcal{M}(\mathbb{R}^d)$ to a measure $\vartheta \in \mathcal{M}_Q(\mathbb{R}^d)$.

Finally, Let $\bar{\nu}$ be the probability law induced from the process $(X_n)_{n\in\mathbb{Z}}$ on $\mathcal{M}_Q(\mathbb{R}^d)$ by this mapping. It is easy to verify that $(\bar{\nu}, p)$ is an ergodic extended CP process, which extends the original process (ν, p) in the required sense. □

Throughout the sequel, an ergodic CP process is implicitly assumed to be given in its extended form, with $\mathcal{M}_Q(\mathbb{R}^d)$ as state space. At our convenience, we can consider the same ergodic CP process with $\mathcal{P}(Q)$ as state space, by restricting the measures to Q.

5. *Discrete scaling scenery in CP processes*

We now aim to establish a connection between ergodic CP processes and the uniform scaling scenery property. We will prove the following theorem.

THEOREM 5.1. *For an ergodic CP process (ν, p), ν-almost every measure $\theta \in \mathcal{M}(\mathbb{R}^d)$ has uniform scaling scenery. In fact, the map $\theta \mapsto \eta[\theta]$ is* ν-almost everywhere constant.

The proof is developed over this section and §6.

Our first step is to reformulate (extended) CP processes as equivalent dynamical systems, and to relate their orbits to the scaling scenery. The standard way of passing from our stationary process formalism to a dynamical system is to consider the product space $\mathcal{M}_Q(\mathbb{R}^d) \times \mathcal{M}_Q(\mathbb{R}^d) \times \cdots$ with the shift operator and a probability measure, induced by the transition probabilities. However, [**F1, F2**] suggested a simpler alternative for attaching the 'future path' to an initial state $\theta \in \mathcal{M}_Q(\mathbb{R}^d)$, and thus admitting a dynamical system formalism: the sequence of transitions $\theta \mapsto \theta^{\lambda_1} \mapsto \theta^{\lambda_1\lambda_2} \mapsto \cdots$ corresponds to the choice of a sequence of subcubes $Q, Q_{\lambda_1}, Q_{\lambda_1\lambda_2}, \dots$ that is decreasing to some point $x \in Q$. It is easily verified that the probability law of the choice of the point x is none other than θ itself. Therefore, instead of considering the state θ along with the 'future path' $(\theta, \theta^{\lambda_1}, \theta^{\lambda_1\lambda_2}, \dots) \in \mathcal{M}_Q(\mathbb{R}^d) \times \mathcal{M}_Q(\mathbb{R}^d) \times \cdots$ we consider the couple (θ, x) where $x \in Q$ is chosen according to θ. This leads to the definition of a CP *system*. The measurable space is a subspace of $\mathcal{M}_Q(\mathbb{R}^d) \times Q$. The transformation maps $(\theta, \sum_{n=1}^{\infty}(\lambda_n/p^n))$ to $(\theta^{\lambda_1}, \sum_{n=1}^{\infty}(\lambda_{n+1}/p^n))$. Finally, if the initial distribution of the Markov process is ν, then the distribution on $\mathcal{M}_Q(\mathbb{R}^d) \times Q$ will describe the choice of a measure θ according to ν, followed by a choice of a point x according to θ. Formally, we have the following definition.

Definition 5.2. Let (ν, p) be an (extended) ergodic CP process.

(1) Given $\theta \in \mathcal{M}(\mathbb{R}^d)$, we define

$$|\theta|_0 = \{x \in \mathbb{R}^d \mid \forall n \geq 0. \theta(Q_n(x)) > 0\} \subset |\theta| = \text{support}(\theta).$$

(2) Denote $\Phi = \{(\theta, x) \mid \theta(Q) = 1, \text{ and } x \in |\theta|_0\} \subset \mathcal{M}_Q(\mathbb{R}^d) \times Q$.

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(3) Define a map $T : \Phi \to \Phi$ by

$$T(\theta, x) = \left(\frac{\mathbf{H}\langle Q_{\lambda(x)}, Q\rangle_*(\theta)}{\theta(Q_{\lambda(x)})}, px - \lambda(x) \right) \tag{5.1}$$

where $\lambda(x) \in \Lambda$ is such that $x \in Q_{\lambda(x)}$.

(4) Recall that ν is a distribution on $\mathcal{M}_Q(\mathbb{R}^d)$. The measure μ on $\mathcal{M}_Q(\mathbb{R}^d) \times Q$, defined by

$$\int_\Phi f(\theta, x)\, d\mu(\theta, x) = \int_{\mathcal{M}_Q(\mathbb{R}^d)} \left(\int_Q f(\theta, x)\, d\theta(x) \right) d\nu(\theta), \tag{5.2}$$

is called the distribution *adapted* to ν†.

(5) The system (Φ, T, μ) is called the CP system corresponding to the (extended) CP process (ν, p).

It is routine to check that (Φ, T, μ) is an ergodic measure preserving transformation.

Definition 5.3. Let $2 \le p \in \mathbb{N}$ and $x = (x_1, \ldots x_d) \in \mathbb{R}^d$. Define‡

$$Q^{\boxdot}_{x,n} = \prod_{i=1}^{d} \left[x_i - \frac{1}{2} p^{-n}, x_i + \frac{1}{2} p^{-n} \right].$$

We continue to use the simplified symbol $Q^{\boxdot}_x$ for $Q^{\boxdot}_{x,0} = \prod_{i=1}^d [x_i - \frac{1}{2}, x_i + \frac{1}{2}]$.

Note that $Q^{\boxdot}_{x,n}$ is a cube of side length p^{-n}, that $Q^{\boxdot}_{x,n+1} \subset Q^{\boxdot}_{x,n}$ for all $n \in \mathbb{N}$, and that $x = \bigcap_{n\in\mathbb{N}} Q^{\boxdot}_{x,n}$. We associate with $\{Q^{\boxdot}_{x,n}\}_{n\in\mathbb{N}}$ the sequence of homotheties $\{\mathbf{H}\langle Q^{\boxdot}_{x,n}, Q^{\boxdot}_x\rangle\}_{n\in\mathbb{N}}$, where $\mathbf{H}\langle Q^{\boxdot}_{x,n}, Q^{\boxdot}_x\rangle : \mathbb{R}^d \to \mathbb{R}^d$ is the unique homothety that induces a bijection from $Q^{\boxdot}_{x,n}$ onto $Q^{\boxdot}_x$.

Definition 5.4. Let $\theta \in \mathcal{M}(\mathbb{R}^d)$ be a Radon measure on $\mathbb{R}^d$ and $x \in |\theta|$. The sequence $\{\theta^{\boxdot}_{x,n}\}_{n\in\mathbb{N}}$ of probability measures on $Q^{\boxdot}_x$, defined by

$$\theta^{\boxdot}_{x,n} = \left. \frac{\mathbf{H}\langle Q^{\boxdot}_{x,n}, Q^{\boxdot}_x\rangle_*(\theta)}{\theta(Q^{\boxdot}_{x,n})} \right|_{Q^{\boxdot}_x},$$

will be called the discrete scaling scenery of θ around x.

Recall that $T^n(\theta, x)$ is an ordered pair, whose first coordinate is an 'uncentered' extended measure. Let us now relate it to the measure $\theta^{\boxdot}_{x,n}$, which is 'centered at x'. It is easy to verify that they both are (restricted) blow-ups by p^n of θ, which differ only by translation of the origin, and by the cube on which they are normalized. This is formalized as follows.

Definition 5.5.

(1) For $y \in \mathbb{R}^d$, let $\tau_y : \mathbb{R}^d \to \mathbb{R}^d$ be the translation map $\tau_y : x \mapsto x + y$. This maps acts on $\mathcal{M}(\mathbb{R}^d)$ by the push-forward

$$\tau_{y*} : \theta \mapsto \tau_{y*}(\theta). \tag{5.3}$$

† Note that if an event occurs for μ-almost every (θ, x), then it occurs for ν-almost every $\theta \in \mathcal{M}_Q(\mathbb{R}^d)$, and for every such θ, it occurs for θ-almost every $x \in Q$.

‡ This notation agrees with that of Definition 2.1, up to a constant factor in timescale: here, the cube side length is $p^{-t} = e^{-t \cdot \log p}$ instead of e^{-t}.

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(2) Suppose that (Φ, T, μ) is a CP system. For $(\theta, x) \in \Phi$ and a point† $y \in |\theta|_0$, define $\psi_y(\theta, x) \in \mathcal{M}(\mathbb{R}^d)$ by

$$\psi_y(\theta, x) = \frac{\tau_{(y-x)*}(\theta)}{\theta(Q_y^{\boxdot})}\bigg|_{Q_y^{\boxdot}}. \tag{5.4}$$

In other words, ψ_y translates the extended measure θ by y (moving the reference point from x to y), restricts it to a measure on the cube $Q_y^{\boxdot}$ and normalizes it to obtain a probability measure. In fact, (5.4) defines a map

$$\psi : \{(y, (\theta, x)) \mid (\theta, x) \in \Phi \text{ and } y \in |\theta|_0\} \to \mathcal{M}(\mathbb{R}^d). \tag{5.5}$$

The next lemma shows that this ψ is key in our discussion.

LEMMA 5.6. *Let (Φ, T, μ) be a CP system and let $(\theta, x) \in \Phi$. Then:*
(1) $\theta_{x,n}^{\boxdot} = \psi_x(T^n(\theta, x))$*; and*
(2) $\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,n}^{\boxdot}) = \psi_0(T^n(\theta, x))$.

Proof. (1) It will be convenient to write$(\theta_n, x_n) \in \Phi$ for $T^n(\theta, x)$. We have

$$\psi_x(T^n(\theta, x)) = \psi_x((\theta_n, x_n)) = \frac{\tau_{(x-x_n)*}(\theta_n)}{\theta_n(Q_x^{\boxdot})}\bigg|_{\theta_x^{\boxdot}} = \theta_{x,n}^{\boxdot},$$

and the statement follows.

(2) We have

$$\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,n}^{\boxdot}) = \tau_{(-x)*}(\theta_{x,n}^{\boxdot}) = \tau_{(-x)*}(\psi_x(T^n(\theta, x))),$$

where $\tau_{(-x)*}$ acts on measures as defined in (5.3). But it is easy to verify that always $\tau_{(-x)*}(\psi_x) = \psi_0$. □

Note that the mapping $\theta \mapsto \theta(Q_y^{\boxdot})$, for some fixed y, is not necessarily weak-* continuous as a function of θ. Hence the mapping $\psi_y(\cdot)$ is not, in general, continuous. Fortunately, it is straightforward to show that ψ is a pointwise limit of measurable functions, and hence measurable‡. In conclusion, $\theta_{x,n}^{\boxdot} = \psi_x(T^n(\theta, x))$, where ψ is a measurable map. We have thus recovered the discrete scaling scenery of θ around a point x, as a function of the T-orbit of the point $(\theta, x) \in \Phi$.

6. *Uniform scaling scenery in ergodic CP processes*

We are now in position to prove Theorem 5.1, whereby almost every measure in an ergodic CP process has uniform scaling scenery.

† Recall Definition 5.2.

‡ Take a sequence $h_m : \mathbb{R}^d \to \mathbb{R}$ of continuous, compactly supported functions that converges pointwise to $\mathbf{1}_{Q_0}$, such that $0 \le \mathbf{1}_{Q_0} \le g_m \le 1$ for all m. Then the map

$$(y, (\theta, x)) \mapsto \psi_y^{(m)}(\theta, x) = \frac{\tau_{(y-x)*}(\theta)}{\int_{\mathbb{R}^d} h_m(x-y)\, d\theta(x)}$$

is measurable, indeed continuous, and $\psi_y^{(m)}(\theta, x) \to \psi_y(\theta, x)$ as $m \to \infty$. See [**G**] for details.

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Proof. Let (Φ, T, μ) be the ergodic CP system corresponding to the given CP process (ν, p). First note that if $(\theta, x) \in \Phi$ then $x \in |\theta|_0$ (see Definition 5.2), so the scaling scenery of θ around x is defined. Since $\mathcal{C}(\mathcal{P}(Q_0^{\boxdot}))$ is separable, we can find a countable, dense subalgebra $\mathcal{A}$. Now, for $\ell \in \mathbb{N}$, $\bar{g} = (g_0, \ldots, g_\ell) \in \mathcal{A}$ and $\bar{\tau} = (\tau_0, \ldots, \tau_\ell) \in \mathbb{Q}^{\ell+1}$, define a mapping $\Psi_{\bar{g};\bar{\tau}} : \Phi \to \mathbb{R}$ by†

$$\Psi_{\bar{g};\bar{\tau}}(\theta, x) = \int_{t=0}^{t=1} \prod_{i=0}^{\ell} g_i([\psi_0(\theta, x)]^{\boxdot}_{0,t+\tau_i})\, dt. \tag{6.1}$$

The following proposition shows that $\Psi_{\bar{g};\bar{\tau}}$ is well defined, integrable, and, most importantly, that the ergodic averages of $\Psi_{\bar{g};\bar{\tau}}$ are exactly the limits we need to calculate.

PROPOSITION 6.1. *For $\ell \in \mathbb{N}$, $\bar{g} = (g_0, \ldots, g_\ell) \in \mathcal{A}$ and $\bar{\tau} = (\tau_0, \ldots, \tau_\ell) \in \mathbb{Q}^{\ell+1}$, the function $\Psi_{\bar{g};\bar{\tau}} : \Phi \to \mathbb{R}$ satisfies the following.*

(1) *The integrand in (6.1) is defined for all $(\theta, x) \in \Phi$, and the integral exists.*

(2) *$\Psi_{\bar{g};\bar{\tau}}$ is Borel measurable and, moreover, $\Psi_{\bar{g};\bar{\tau}} \in \mathcal{L}^1(\Phi, \mu)$.*

(3) *For all $N \in \mathbb{N}$ we have*

$$\frac{1}{N} \sum_{n=0}^{N-1} \Psi_{\bar{g};\bar{\tau}}(T^n(\theta, x)) = \frac{1}{N} \int_{t=0}^{t=N} \prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta^{\boxdot}_{x,t+\tau_i}))\, dt. \tag{6.2}$$

(4) *If the limit $(1/N)\sum_{n=0}^{N-1} \Psi_{\bar{g};\bar{\tau}}(T^n(\theta, x))$ exists, so does the limit*

$$\lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{t=T} \prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta^{\boxdot}_{x,t+\tau_i}))\, dt$$

and the two are equal.

Let us show how the rest of the proof follows from this proposition. For a given ℓ, $\bar{g} \in \mathcal{A}$ and $\bar{\tau} \in \mathbb{Q}^{\ell+1}$, by the above proposition, $\Psi_{\bar{g};\bar{\tau}}$ is integrable on (Φ, μ). Appealing to Birkhoff's ergodic theorem, and taking the intersection over all ℓ, $\bar{g}$ and $\bar{\tau}$, we can find $\mathcal{G} \subset \Phi$ with $\mu\mathcal{G} = 1$ such that for every $(\theta, x) \in \mathcal{G}$, every $\ell \in \mathbb{N}$, and $\bar{g} = (g_0, \ldots, g_\ell) \in \mathcal{A}$ and every rational $\ell + 1$-tuple $\bar{\lambda} = (\lambda_0, \ldots, \lambda_\ell) \in \mathbb{Q}^{\ell+1}$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Psi_{\bar{g};\bar{\lambda}}(T^n(\theta, x))$$

exists, and in fact

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Psi_{\bar{g};\bar{\lambda}}(T^n(\theta, x)) = \int_{\Phi} \Psi_{\bar{g};\bar{\lambda}}\, d\mu. \tag{6.3}$$

Now, Proposition 6.1 (to be proved below) implies that for every $(\theta, x) \in \bar{\mathcal{G}}$, every $\bar{g} = (g_0, \ldots, g_\ell) \in \mathcal{A}$ and every rational $\ell + 1$-tuple $\bar{\lambda} = (\lambda_0, \ldots, \lambda_\ell) \in \mathbb{Q}^{\ell+1}$, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{t=T} \prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta^{\boxdot}_{x,t+\lambda_i}))\, dt = \int_{\bar{\Phi}} \Psi_{\bar{g};\bar{\lambda}}\, d\mu, \tag{6.4}$$

† We use square brackets, e.g $[\theta]^{\boxdot}_{x,t}$, when the measure θ is given by a complex expression.

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and therefore the limit on the left is independent of the choice of (θ, x). Since $\mu\mathcal{G} = 1$, by the remark to Definition 5.2, we can assume that if $(\theta, x) \in \mathcal{G}$, then also $(\theta, y) \in \Phi$ for θ-a.e. $y \in Q$. Fix some $(\theta, x) \in \mathcal{G}$. Since for θ-a.e. $y \in Q$ we have $(\theta, y) \in \mathcal{G}$, we find that for any $\ell \in \mathbb{N}$, $\bar{g} \in \mathcal{A}$ and $\bar{\lambda} \in \mathbb{Q}^{\ell+1}$, the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{t=T} \prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_y^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{y,t+\lambda_i}^{\boxdot}))\, dt$$

exists and is identical for θ-almost every $y \in Q$. We thus find ourselves in the conditions of Lemma 2.5, which implies that θ has uniform scaling scenery, and that the uniform scaling process $\eta[\theta]$ is uniquely determined by the values

$$\int_{\mathcal{P}(Q_0^{\boxdot})^{\mathbb{R}}} \left(\prod_{i=0}^{\ell} f_i(\mathbf{y}_{\lambda_i}) \right) d\eta(\mathbf{y}).$$

However, by the definition of $\eta[\theta]$,

$$\lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{t=T} \prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\lambda_i}^{\boxdot}))\, dt = \int_{\mathcal{P}(Q_0^{\boxdot})^{\mathbb{R}}} \left(\prod_{i=0}^{\ell} f_i(\mathbf{y}_{\lambda_i}) \right) d\eta(\mathbf{y}),$$

which was found in (6.4) to be independent of the choice of $(\theta, x) \in \mathcal{G}$. This means that the map $(\theta, x) \mapsto \eta[x, \theta]_\sim$ is constant on $\mathcal{G}$. □

We still have to prove Proposition 6.1. Let us first observe a few useful properties of the scaling scenery.

LEMMA 6.2. *Suppose that $(\theta, x) \in \Phi$. We have*

$$\theta_{x,s+t}^{\boxdot} = [\theta_{x,s}^{\boxdot}]_{x,t}^{\boxdot}.$$

LEMMA 6.3. *Let (Φ, T, μ) be a CP system. Suppose that $(\theta, x) \in \Phi$. Then*

$$[\psi_0(T^n(\theta, x))]_{0,t}^{\boxdot} = \mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,n+t}^{\boxdot}).$$

LEMMA 6.4. *Let $\theta \in \mathcal{M}(\mathbb{R}^d)$ and $0 \in |\theta|$. The mapping $\mathbb{R} \times \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ defined by $(t, \theta) \longmapsto \theta_{0,t}^{\boxdot}$ is measurable both in t and in θ.*

Proof of Proposition 6.1. By Lemma 6.4, the integrand is a composition of measurable and continuous functions, hence measurable as a function of t. Since it is also bounded, we find that the integral above exists and is finite. A similar argument shows that $\Psi_{\bar{g};\bar{\tau}} \in \mathcal{L}^1(\Phi, \mu)$.

Next, for $n \in \mathbb{N}$ we have

$$\begin{aligned}
\Psi_{\bar{g};\bar{\tau}}(T^n(\theta, x)) &= \int_{t=0}^{t=1} \prod_{i=0}^{\ell} g_i([\psi_0(T^n(\theta, x))]_{0,t+\tau_i}^{\boxdot})\, dt \\
&= \int_{t=0}^{t=1} \prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,n+t+\tau_i}^{\boxdot}))\, dt \\
&= \int_{t=n}^{t=n+1} \prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\tau_i}^{\boxdot}))\, dt,
\end{aligned}$$

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where we have used Lemma 6.3. This yields

$$\frac{1}{N}\sum_{n=0}^{n-1}\Psi_{\bar{g};\bar{\tau}}(T^n(\theta,x)) = \frac{1}{N}\int_{t=0}^{t=N}\prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\tau_i}^{\boxdot}))\,dt,$$

which justifies (6.2).

Next, assume that the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{n-1}\Psi_{\bar{g};\bar{\tau}}(T^n(\theta,x))$$

exists. We have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{n-1}\Psi_{\bar{g};\bar{\tau}}(T^n(\theta,x)) = \lim_{N\to\infty}\frac{1}{N}\int_{t=0}^{t=N}\prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\tau_i}^{\boxdot}))\,dt.$$

Now, denote $M = \max_{0\le i\leqslant \ell}\sup|g_i|$, so

$$\lim_{T\to\infty}\left|\frac{1}{T}\int_{T}^{\lceil T\rceil}\prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\tau_i}^{\boxdot}))\,dt\right| \le \lim_{T\to\infty}\frac{1}{T}\int_{T}^{\lceil T\rceil} M^{\ell+1} = 0,$$

where $\lceil T\rceil$ is the upper integer value of T. But (6.2) implies that

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T\prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\tau_i}^{\boxdot}))\,dt\\
&\quad= \lim_{T\to\infty}\frac{\lceil T\rceil}{T}\cdot\frac{1}{\lceil T\rceil}\int_0^{\lceil T\rceil}\prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\tau_i}^{\boxdot}))\,dt\\
&\quad= \lim_{N\to\infty}\frac{1}{N}\int_{t=0}^{t=N}\prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\tau_i}^{\boxdot}))\,dt\\
&\quad= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{n-1}\Psi_{\bar{g};\bar{\tau}}(T^n(\theta,x)).
\end{aligned}$$

We thus find that if the limit $\lim_{N\to\infty}(1/N)\sum_{n=0}^{n-1}\Psi_{\bar{g};\bar{\tau}}(T^n(\theta,x))$ exists, then it must equal

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\prod_{i=0}^{\ell} g_i(\mathbf{H}\langle Q_x^{\boxdot}, Q_0^{\boxdot}\rangle_*(\theta_{x,t+\tau_i}^{\boxdot}))\,dt. \qquad \square$$

COROLLARY 6.5. *The classical Cantor measure has uniform scaling scenery.*

Proof. Let γ be the classical (ternary) Cantor measure on $[0,1]$ and let δ_γ be the probability distribution on $\mathcal{P}(Q)$ that is supported on $\{\gamma\}$. Clearly, (δ_γ, p) is a CP process on $Q\subset\mathbb{R}^1$ for $p=3$. As a stationary process, it is isomorphic to the Bernoulli shift $(\frac{1}{2},\frac{1}{2})$, and is therefore a (non-degenerate) ergodic CP process†. Now apply Theorem 5.1. $\square$

† It is interesting to observe that the situation is far from being this simple for other choices of grid-size parameter, say $p=2$.

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Note that, as in the case of a typical Brownian occupation measure, the map $x \mapsto \eta[x, \gamma]_\sim$ is γ-a.e. constant, but not *everywhere* constant: if x is an endpoint of a ternary interval, then the empirical scaling process of γ at x differs from the uniform scaling process.

Remark 6.6. Theorem 5.1 implies that we can map any ergodic CP system (Φ, T, μ) to its corresponding unique uniform scaling process η. We raise the question of whether this mapping is *onto*. More precisely, consider an arbitrary measure-valued stationary process in continuous time, which arises as the uniform scaling process that is attached to a measure with uniform scaling scenery. Can this process be obtained as the unique uniform scaling process, corresponding to some ergodic CP process†?

7. *Uniform scaling scenery in self-similar measures*

In this section we prove Theorem 3.2. The proof exploits Theorem 5.1, and is an example of the general approach [**F2**] for applying the CP process notion to problems in fractal geometry. In this section, we only need to consider CP processes on Q, and not their extension to $\mathbb{R}^d$.

Let $X \subset \mathbb{R}^d$ be a closed set. Recall the following [**Fa**].

(1) An *iterated function system* (IFS) *of contracting similitudes* is a family of affine maps $\phi_1, \ldots, \phi_n : X \to X$ such that

$$\|\phi_i(x) - \phi_i(y)\| = c_i \|x - y\|$$

for some *contraction factors* $0 < c_i < 1$, $i = 1 \ldots N$.

(2) A compact set $A \subset X$ is called an *attractor* of the IFS $(\phi_1, \ldots, \phi_n)$ if

$$A = \biguplus_{i=1}^{n} \phi_i(A).$$

(3) A measure μ on an attractor A of the IFS $(\phi_1, \ldots, \phi_n)$ is called a *self-similar measure* associated with the probability vector $(p_1, \ldots, p_n)$ if

$$\mu = \sum_{i=1}^{n} p_i \cdot \phi_{i*}(\mu). \tag{7.1}$$

(4) The IFS $(\phi_1, \ldots, \phi_n)$ is said to satisfy the *open set condition* if there exists $V \subset X$ open, such that $\bigcup_{i=1}^{n} \phi_i(V) \subset V$ and $\phi_i(V) \cap \phi_j(V) = \emptyset$ for all $i \neq j$.

(5) An IFS on a closed set $X \subset \mathbb{R}^d$ has a unique non-empty attractor A.

(6) For the IFS of contracting similitudes $(\phi_1, \ldots, \phi_n)$ that satisfies the open set condition, the Hausdorff dimension of its attractor A is the unique solution s to the equation

$$\sum_{i=1}^{n} (c_i)^s = 1 \tag{7.2}$$

where r_i is the contraction factor of the map ϕ_i, for $i = 1, \ldots, n$.

† This question is due to Furstenberg [**FP**] and is addressed in a work in progress by Hochman [**H**].

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(7) For any probability vector $(p_1, \ldots, p_n)$ there is a unique associated self-similar measure μ on A. If $p_1, \ldots, p_n > 0$, then $|\mu| = A$.

The Cantor measure, or its analog on the Sierpińiski triangle, are popular examples of self-similar measures.

The general CP processes technique, which we now employ, is outlined in [**F2**]. There, the notions of 'mini-set', 'micro-set' and 'gallery' are central. We start with the analogous definitions in the world of probability measures on Q.

Definition. (Mini-measures and micro-measures.) Suppose that $\theta \in \mathcal{P}(Q)$.

(1) A measure $\nu \in \mathcal{P}(Q)$ is called a *mini-measure* of θ if $\nu = (f_*\theta)\big|_Q$ (namely, $f_*\theta$ restricted to Q) for some $f : \mathbb{R}^d \to \mathbb{R}^d$ of the form $f : x \mapsto \lambda x + u$, where $\mathbb{R} \ni \lambda \geq 1$ and $u \in \mathbb{R}^d$.

(2) A measure $\nu \in \mathcal{P}(Q)$ is called a *micro-measure* of θ if there exists a sequence $(\nu_n)_{n=1}^{\infty}$ of mini-measures of ν, such that $\nu_n \to \nu$ in the weak-* topology of $\mathcal{P}(Q)$.

(3) A measure $\theta \in \mathcal{P}(Q)$ is called *homogeneous* if any micro-measure of θ is actually a mini-measure (that is, if the set of mini-measures of θ is weak-* closed).

(4) A measure $\theta \in \mathcal{P}(Q)$ is called *recurrent homogeneous* if it is homogeneous, and is a mini-measure of each of its own mini-measures.

Consider a general probability measure θ_1 on Q. We now claim that for any grid-size parameter p, there exists a (non-degenerate) ergodic CP process that is supported on micro-measures of θ_1 (following [**F2**, §4]).

Informally, the idea is as follows. Regardless of our choice of dimension d and grid-size parameter p, the state-space $\mathcal{P}(Q)$ of the Markov chain, underlying the CP processes on $Q \subset \mathbb{R}^d$, is compact. Hence, the set of stationary distributions ν is non-empty, compact, convex, and spanned by its extremals, which produce ergodic CP processes. This means that an extremal stationary distribution $\nu \in \mathcal{P}(Q)$, for which (ν, p) is an ergodic CP process, exists. Formally, we appeal to the following result [**F2**, Theorem 5.1].

THEOREM. *Let $Q \subset \mathbb{R}^d$ be the unit cube, and choose $p \geq 2$. Suppose that $\mathcal{G} \subset \mathcal{P}(Q)$ is a (weak-*) closed subset, with the property that for each $\theta \in \mathcal{G}$, all the micro-measures of θ are also in $\mathcal{G}$. Then there exists an ergodic CP process (ν, p), where ν is a distribution on $\mathcal{P}(Q)$ supported on $\mathcal{G}$, such that for ν-almost every $\theta \in \mathcal{P}(Q)$, $\dim|\theta| = \sup_{\theta' \in \mathcal{G}}\{\dim|\theta'|\}$.*

Here and below, $\dim|\theta|$ stands for the Hausdorff dimension of the support of θ. Recall the technical condition of 'non-degeneracy' of a CP process (Definition 4.1).

COROLLARY 7.1. *Let $\theta_1 \in \mathcal{P}(Q)$ with $\dim|\theta_1| > 0$. Then there exists a (non-degenerate) ergodic CP process (ν, p), where ν is a distribution on $\mathcal{P}(Q)$ that is supported on the set of micro-measures of θ_1.*

Proof. Let $Q \subset \mathbb{R}^d$ and let $\mathcal{G} \subset \mathcal{P}(Q)$ be the smallest (weak-*) closed set in $\mathcal{P}(Q)$ that contains θ_1 and is closed under the maps $\theta \mapsto \theta^{\lambda}$ for any $\lambda \in \Lambda$. Then all elements of $\mathcal{G}$ are micro-measures of θ_1. Invoking the above theorem, we obtain an ergodic CP process (ν, p), where ν is a distribution on $\mathcal{P}(Q)$ that is supported on $\mathcal{G}$, such that for ν-almost every $\theta \in \mathcal{P}(Q)$,

$$\dim|\theta| = \sup_{\theta' \in \mathcal{G}}\{\dim|\theta'|\} \geq \dim|\theta_1| > 0. \tag{7.3}$$

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It follows from (7.3) that the ergodic CP process $(\nu,\ p)$ is not degenerate on the unit cube $Q \subset \mathbb{R}^k$ for some $1 \leq k \leqslant d$ [**G**]. □

Now we can harvest results. Let $\theta_1 \in \mathcal{P}(Q)$ be any probability measure with $\dim|\theta_1| > 0$. Using Corollary 7.1 and restricting Q to a subspace $\mathbb{R}^k$ $(1 \leq k \leq d\ )$ if necessary, construct a non-degenerate ergodic CP process $(\nu,\ p)$, supported on the micro-measures of θ_1. By Theorem 5.1, for ν-almost every $\theta \in \mathcal{P}(Q)$, θ has uniform scaling scenery. Let θ be such a measure. Since ν is supported on micro-measures of θ_1, θ is a micro-measure of θ_1. We thus obtain the following corollary.

COROLLARY. *Any measure $\theta \in \mathcal{P}(Q)$ with $\dim|\theta| > 0$ has a micro-measure with uniform scaling scenery.*

If, in addition, ν is homogeneous, then θ above must be a mini-measure of θ_1. Therefore we have the following corollary.

COROLLARY. *Any homogeneous measure $\theta \in \mathcal{P}(Q)$ with $\dim|\theta| > 0$ has a mini-measure with uniform scaling scenery.*

Finally, if θ_1 happens to be recurrent homogeneous, it is a mini-measure of each of its own mini-measures. In particular, it is a mini-measure of its measure θ_1 that has uniform scaling scenery. But the property of uniform scaling scenery is clearly inherited by mini-measures, as it is independent of scale. Since θ has uniform scaling scenery, and θ_1 is a mini-measure of θ, we conclude that θ_1 has uniform scaling scenery. Concisely, we have the following corollary.

COROLLARY 7.2. *Any recurrent homogeneous measure $\theta \in \mathcal{P}(Q)$ with $\dim|\theta| > 0$ has uniform scaling scenery.*

We are finally in position to prove Theorem 3.2.

Proof. For the sake of brevity, we discuss the special case where $(\phi_1, \ldots, \phi_n)$ are homotheties. The general case, where $(\phi_1, \ldots, \phi_n)$ are similitudes, requires the additional observation that the uniform scaling scenery property is preserved under orthogonal transformations. Assume that A is the attractor of the IFS of contracting homotheties $(\phi_1, \ldots, \phi_n)$, which satisfies the open set condition, and let θ be a self-similar probability measure on A corresponding to the strictly positive probability vector $(p_1, \ldots, p_n)$. Since the family of mini-measures of θ is a continuous image of a compact set, it is clearly closed in $\mathcal{P}(Q)$. The defining property of self-similar measures (7.1) implies that θ is a mini-measure of each of its own mini-measures. Thus θ is recurrent homogeneous. Since $n \geq 2$, namely since the IFS at hand consists of two maps or more, (7.2) implies that its attractor must have strictly positive Hausdorff dimension. Now apply Corollary 7.2. □

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REFERENCES

[Fa] K. Falconer. *Fractal Geometry—Mathematical Foundations and Applications*, 2nd edn. Wiley, Chichester, 2003.

[Fi1] A. M. Fisher. Convex-invariant means and a pathwise central limit theorem. *Adv. Math.* **63** (1987), 213–246.

[Fi2] A. M. Fisher. Small-scale structure via flows. *Progr. Probab.* **57** (2004), 59–78.

[FP] H. Furstenberg. Private communication.

[F1] H. Furstenberg. Intersections of Cantor sets and transversality of semigroups. *Problems in Analysis (Princeton Mathematical Series, 31)*. Ed. R. C. Gunning. 1970, pp. 41–59.

[F2] H. Furstenberg. Ergodic fractal measures and dimension conservation. *Ergod. Th. & Dynam. Sys.* **28** (2008), 405–422.

[FW] H. Furstenberg and B. Weiss. Markov processes and Ramsey theory for trees, Special issue on Ramsey theory. *Combin. Probab. Comput.* **12** (2003), 548–563.

[G] M. Gavish. Measures with uniform scaling scenery, *M.Sc. Thesis*, Einstein Institute of Mathematics, Hebrew University of Jerusalem, 2008. Available online at http://math.huji.ac.il/∼matang02/GavishUniformScalingSceneryThesis.pdf.

[H] M. Hochman. Dynamics on fractals and fractal distributions, in preparation.

[LPP1] R. Lyons, R. Pemantle and Y. Peres. Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* **23** (1995), 1125–1138.

[LPP2] R. Lyons, R. Pemantle and Y. Peres. Ergodic theory on Galton–Watson trees: speed of random walk and dimension of harmonic measure. *Ergod. Th. & Dynam. Sys.* **15** (1995), 593–619.

[LPP3] R. Lyons, R. Pemantle and Y. Peres. Biased random walks on Galton–Watson trees. *Probab. Theory Related Fields* **106** (1996), 249–264.

[LP] R. Lyons and Y. Peres. Probability on trees and networks. Cambridge University Press, 2009, in preparation. Current version available at http://my.page.iu.edu/∼rdlyons.

[PrMö] P. Mörters and D. Preiss. Tangent measure distributions of fractal measures. *Math. Ann.* **312** (1998), 53–93.

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