LINKING SPHERES

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Received 22 May 1959

Introduction. Andrews and Curtis have shown (1) that one can embed two S^n's in E^{n+2} for $n = 2$, in such a way that one sphere cannot be shrunk to a point in the residue space of the other. In this paper the result is shown to be true for any $n \geqslant 1$. (The result is obvious for $n = 1$.) The method is to calculate the appropriate homotopy group of the residue space of one sphere, and to show that the embedding of the other sphere represents a non-zero element of the group. The two spheres can both be embedded analytically.

Spun knots. In this section, we describe a method of embedding knotted S^n's in E^{n+2}, $n \geqslant 2$. Let E^n be n-dimensional Euclidean space with coordinates $x_1, \ldots, x_n$. Let E^{n-1} be embedded in E^n by putting $x_n = 0$. Let $H = \{x; x \in E^3, x_3 \geqslant 0\}$.

Let K be a polygonal knot with a small linear segment removed, lying in H and such that the intersection of K with E^2 is its end-points. Let $Y = H - K$ and let Y_0 be the intersection of Y with E^2. Let B^n be the n-dimensional ball, with boundary S^{n-1}. Let $X = (Y \times S^{n-1}) \cup (Y_0 \times B^n)$, these two subspaces being identified on $Y_0 \times S^{n-1}$. E^{n+2} is homeomorphic to $(H \times S^{n-1}) \cup (E^2 \times B^n)$. Therefore X is homeomorphic to E^{n+2} with an S^n removed. This S^n is called a *spun knot*. When $n = 2$, the process is equivalent to Artin's method of 'spinning' H about the axis E^2 to obtain knotted 2-spheres (2).

Lemma 1. *Y is a $K(\pi, 1)$.*

We embed Y in $H' = \{x; x \in E^3, x_3 > -1\}$ and add to K two linear segments of unit length, perpendicular to E^2. We then compactify H' and obtain a knot embedded in S^3. Then Y is a deformation retract of the residue space of this knot. By (26·3) of (3), Y is a $K(\pi, 1)$.

Let $p\colon \tilde{Y} \to Y$ be the universal covering of Y. We emphasize that *$\tilde{Y}$ is contractible*, since all its homotopy groups vanish. Let $\tilde{Y}_0 = p^{-1}Y_0$. Let $\tilde{X} = (\tilde{Y} \times S^{n-1}) \cup (\tilde{Y}_0 \times B^n)$, identified on $\tilde{Y}_0 \times S^{n-1}$. We then have the commutative diagram

$$\begin{array}{ccc} \tilde{Y} \times S^{n-1} & \to & \tilde{X} \\ \downarrow & & \downarrow \\ Y \times S^{n-1} & \to & X \end{array}$$

where the vertical maps are induced by p and the horizontal maps are inclusions. $\tilde{X}$ is obviously a covering space of X, with the covering map as in the diagram. Our aim is to calculate $\Pi_n(X)$, which is isomorphic to $\Pi_n(\tilde{X})$ for $n \geqslant 2$.

Let Z denote the additive group of integers and let S be a non-empty set. Then we denote by $Z(S)$ the free Abelian group generated by the elements of S.

https://doi.org/10.1017/S0305004100034496 Published online by Cambridge University Press

LEMMA 2. $H_0(\tilde{Y}_0) \cong Z(\Pi_1(Y, Y_0))$.

The set of components of $\tilde{Y}_0$ is in (1-1) correspondence with the set of homotopy classes of paths in Y beginning at a base-point and ending in Y_0. The correspondence is obtained by lifting each path representing an element of $\Pi_1(Y, Y_0)$ to a path with base-point in $\tilde{Y}$ and noting in which component of $\tilde{Y}_0$ the other end-point lies.

We have

$$H_r(\tilde{X}, \tilde{Y} \times S^{n-1}) \cong H_r(\tilde{Y}_0 \times B^n, \tilde{Y}_0 \times S^{n-1}) = \begin{cases} 0 & \text{if } 0 \leqslant r < n, \\ H_{r-n}(\tilde{Y}_0) & \text{if } r \geqslant n. \end{cases}$$

The first isomorphism is obtained by excision, and the second from the Künneth relations.

For $1 \leqslant r < n$, the r-dimensional skeleton of $\tilde{X}$ is contained in $\tilde{Y} \times S^{n-1}$. Therefore $\Pi_r(\tilde{X}, \tilde{Y} \times S^{n-1}) = 0$. For $1 \leqslant r < n$, we have the exact sequence

$$\Pi_{r+1}(\tilde{X}, \tilde{Y} \times S^{n-1}) \to \Pi_r(\tilde{Y} \times S^{n-1}) \to \Pi_r(\tilde{X}) \to 0.$$

The homomorphism $\Pi_{r+1}(\tilde{X}, \tilde{Y} \times S^{n-1}) \to \Pi_r(\tilde{Y} \times S^{n-1})$ is onto. This can be seen from the commutative diagram

$$\begin{array}{ccc} \Pi_{r+1}(\tilde{X}, \tilde{Y} \times S^{n-1}) & \to & \Pi_r(\tilde{Y} \times S^{n-1}) \\ \downarrow & & \downarrow \\ \Pi_{r+1}(B^n, S^{n-1}) & \to & \Pi_r(S^{n-1}), \end{array}$$

where the vertical maps are induced by the projection $\tilde{X} \to B^n$, resulting from the product structure of $\tilde{X}$, and the horizontal maps are boundary homomorphisms. The vertical map on the right is an isomorphism, since $\tilde{Y}$ is contractible. The lower horizontal and the left vertical maps are obviously onto. Therefore $\Pi_r(\tilde{X}) = 0$ for $1 \leqslant r < n$.

We have the diagram

$$\begin{array}{ccccccccc} H_n(\tilde{Y} \times S^{n-1}) & \to & H_n(\tilde{X}) & \to & H_n(\tilde{X}, \tilde{Y} \times S^{n-1}) & \to & H_{n-1}(\tilde{Y} \times S^{n-1}) & \to & H_{n-1}(\tilde{X}) \\ \| & & \| & & \| & & \| & & \| \\ 0 & & \Pi_n(\tilde{X}) & & H_0(\tilde{Y}_0) & & Z & & 0 \\ & & \| & & \| & & & & \\ & & \Pi_n(X) & & Z(\Pi_1(Y, Y_0)), & & & & \end{array}$$

where the horizontal line is exact. The vertical isomorphisms first and fourth from the left are a result of the contractibility of $\tilde{Y}$. The second and fifth isomorphisms are Hurewicz isomorphisms. The third vertical isomorphism has already been dealt with.

THEOREM. *$\Pi_n(X)$ is the free Abelian group generated by the non-trivial homotopy classes of paths in Y, beginning at a base-point in Y_0, and ending in Y_0.*

We have the exact sequence

$$0 \to \Pi_n(X) \to Z(\Pi_1(Y, Y_0)) \to Z \to 0.$$

It is only necessary to prove that any generator of $Z(\Pi_1(Y, Y_0))$ corresponding to a path, maps onto a generator of Z. We have

$$H_n(\tilde{X}, \tilde{Y} \times S^{n-1}) \cong H_n(\tilde{Y}_0 \times B^n, \tilde{Y}_0 \times S^{n-1}) \cong H_0(\tilde{Y}_0).$$

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This isomorphism can be induced as follows: with every singular 0-cube in $\tilde{Y}_0$ we associate that singular n-cube in $\tilde{Y}_0 \times B^n$ which is the product of the degenerate map into the 0-cube in $\tilde{Y}_0$ and a definite homeomorphism h of an n-cube onto B^n. (The isomorphism is the same as that produced by the Künneth relation, but this is not relevant, since any isomorphism will do.) The theorem now follows.

Realizing elements. We now wish to show that the isomorphism proved in the theorem is induced by spinning a knot. Explicitly, let a be an arc in $Y = H - K$ whose intersection with E^2 is its end-points. Let a represent a non-trivial element in $\Pi_1(Y, Y_0)$. *Then the corresponding generator of* $\Pi_n(X)$ *is represented by the spun knot associated with* a.

Let $\tilde{a}$ be a lifting of a from Y to $\tilde{Y}$. Let the end-points of a and $\tilde{a}$ be denoted by a_0, a_1 and $\tilde{a}_0, \tilde{a}_1$ where a_0 and $\tilde{a}_0$ are base-points in Y_0 and $\tilde{Y}_0$, respectively. Let Sa denote the spun knot

$$(a \times S^{n-1}) \cup (a_0 \times B^n) \cup (a_1 \times B^n)$$

and let $S\tilde{a}$ denote $\quad (\tilde{a} \times S^{n-1}) \cup (\tilde{a}_0 \times B^n) \cup (\tilde{a}_1 \times B^n).$

Then the map $S\tilde{a} \to Sa$ induced by $p\colon \tilde{Y} \to Y$ is a homeomorphism. Let us follow the image of the element in $\Pi_n(X)$ represented by Sa, in the sequence of groups

$$\Pi_n(X) \cong \Pi_n(\tilde{X}) \cong H_n(\tilde{X}) \to H_n(\tilde{X}, \tilde{Y} \times S^{n-1}) \cong H_n(\tilde{Y}_0 \times B^n, \tilde{Y}_0 \times S^{n-1})$$
$$\cong H_0(\tilde{Y}_0) \cong Z(\Pi_1(Y, Y_0)).$$

Then $\{Sa\}$ becomes, in $\Pi_n(\tilde{X})$, $\{S\tilde{a}\}$; in $H_n(\tilde{X})$, $\langle S\tilde{a}\rangle$, the homology class of the singular cycle represented by the homeomorphism of the boundary of an $(n+1)$-dimensional cube onto $S\tilde{a}$. In $H_n(\tilde{X}, \tilde{Y} \times S^{n-1})$, the image of $\{Sa\}$ becomes $\langle \tilde{a}_1 \times B^n\rangle - \langle \tilde{a}_0 \times B^n\rangle$, where $\langle \tilde{a}_i \times B^n\rangle$ is the homology class of that singular n-cube which is the product of a degenerate cube into $\tilde{a}_i$ with the homeomorphism h. The image of $\{Sa\}$ in $H_0(\tilde{Y}_0)$ is therefore $\langle \tilde{a}_1\rangle - \langle \tilde{a}_0\rangle$, where $\langle \tilde{a}_i\rangle$ is the obvious 0-dimensional homology class. The image of $\{Sa\}$ in $Z(\Pi_1(Y, Y_0))$ is just the homotopy class of a, by applying the isomorphism of Lemma 2 and neglecting the constant path.

Applications.

COROLLARY 1. *We can embed two* S^n*'s in* $E^{n+2}(n \geqslant 1)$ *in such a way that either* (i) *neither can be shrunk to a point in the residue space of the other:*
or (ii) *one can and one cannot be shrunk to a point in the residue space of the other.*

For $n = 1$, case (i), take the S^1's to be linked circles. For $n = 1$, case (ii), let one S^1 be a circle and the other a trefoil knot. Let the trefoil knot link the circle twice, once in each direction. This can be done so as to satisfy (ii).

For $n > 1$, let K first be a semicircle with end-points in E^2. Then the spun knot obtained from K is equally well obtained by the embeddings $S^n \subset E^{n+1} \subset E^{n+2}$, where the embeddings are the standard ones. If we compactify E^{n+2}, then we have $S^n \subset S^{n+1} \subset S^{n+2}$, where the embeddings are equatorial. $S^{n+2} - S^n$ is of the same homotopy type as S^1. Therefore it is aspherical and $\Pi_n(S^{n+2} - S^n) = 0$. This implies that $\Pi_n(E^{n+2} - S^n) = 0$, where $S^n \subset E^{n+2}$ is the standard embedding.

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We now take K to be a trefoil knot with a small linear segment removed. Then we can choose a semicircular path a which represents a non-trivial element of $\Pi_1(Y, Y_0)$. By the preceding paragraph, the spun knot obtained from K can be shrunk to a point in the residue space of the spun knot obtained from a. This gives case (ii).

To obtain case (i), we choose a to be a trefoil knot with a small linear segment removed, such that a represents a non-trivial element of $\Pi_1(Y, Y_0)$. (K is the same as in the preceding paragraph.) We also choose a so that K represents a non-trivial element in $\Pi_1(H-a, E^2-a)$. This can be done by taking a to be the reflexion of K through a suitably chosen line perpendicular to E^2.

In Corollary 2, we compactify E^{n+2}, the space in which the spun knots are embedded. This does not affect homotopy groups up to and including dimension n.

COROLLARY 2. *The following conditions are equivalent*:

(i) *the residue space of a spun knot is a* $K(\Pi, 1)$;

(ii) *the spun knot is unknotted*;

(iii) *the residue space is a* $K(Z, 1)$.

If the residue space is a $K(\Pi, 1)$ then $\Pi_1(Y, Y_0) = 0$ from our theorem. Therefore the homomorphism $\Pi_1(Y_0) \to \Pi_1(Y)$ induced by inclusion is onto. The group $\Pi_1(Y_0)$ is the free group on two generators, which become the same element in $\Pi_1(Y)$. Therefore $\Pi_1(Y) = Z$. (It cannot be finite since $H_1(Y) = Z$, as is well known.) Therefore by (28·1) of (3) and the proof of our Lemma 1, K is unknotted. Unknotting K automatically unknots the spun knot associated with it. Therefore K is embedded as a semicircle and we have the situation described in the proof of Corollary 1. Therefore the residue space is of the same homotopy type as an S^1 and the corollary is proved.

Analytic spun knots. X is embedded in E^{n+2} by the embedding of

$$(Y_0 \times B^n) \cup (Y \times S^{n-1}) \quad \text{in} \quad (E^2 \times B^n) \cup (H \times S^{n-1}).$$

If we shrink B^n to a point then the spaces are replaced by homeomorphic images, as is easily seen. So we now consider a spun knot to be obtained as before, but with B^n shrunk to a point.

Let $I = \{t;\ -1 \leqslant t \leqslant 1\}$. Let the embedding $a: I \to H$ such that $a(I) = K$ be given by $a(t) = (x_1(t), x_2(t), x_3(t))$. Let $q: H \times S^{n-1} \to E^{n+2}$ be given by

$$q(x_1, x_2, x_3, y_1, \ldots, y_n) = (x_1, x_2, x_3 y_1, \ldots, x_3 y_n),$$

where $\Sigma y_i^2 = 1$. Let $r: I \times S^{n-1} \to S^n$ be given by

$$r(t, y_1, \ldots, y_n) = (t, y_1(1-t^2)^{\frac{1}{2}}, \ldots, y_n(1-t^2)^{\frac{1}{2}}).$$

Then we have

$$\begin{array}{ccccc} & & H \times S^{n-1} & & \\ & \overset{a\times 1}{\nearrow} & & \overset{q}{\searrow} & \\ I \times S^{n-1} & & & & E^{n+2} \\ & \underset{r}{\searrow} & & \underset{g}{\nearrow} & \\ & & S^n & & \end{array}$$

where g is chosen so that the diagram is commutative.

An easy computation shows that g is analytic and regular if $x_3(t) = (1-t^2)^{\frac{1}{2}} f(t)$,

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where $f(t) \neq 0$ for $t \in I$ and x_1, x_2 and f are analytic functions of t. Obviously any knotted arc in H, whose intersection with E^2 is at its end-points only, can be obtained to within a suitable approximation by using polynomials for x_1, x_2 and f.

My sincere thanks are due to E. C. Zeeman for much help and advice.

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https://doi.org/10.1017/S0305004100034496 Published online by Cambridge University Press