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Cycles Are Strongly Ramsey-Unsaturated

J. SKOKAN[1] and M. STEIN[2†‡]

[1]Department of Mathematics, London School of Economics and Political Science,
Houghton Street, London WC2A 2AE, UK
and
Department of Mathematics, University of Illinois at Urbana-Champaign,
1409 W. Green Street, Urbana, IL 61801, USA
(e-mail: jozef@member.ams.org)

[2]Centro de Modelamiento Matemático, Universidad de Chile, Santiago, Chile
(e-mail: mstein@dim.uchile.cl)

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We call a graph H *Ramsey-unsaturated* if there is an edge in the complement of H such that the Ramsey number $r(H)$ of H does not change upon adding it to H. This notion was introduced by Balister, Lehel and Schelp in [*J. Graph Theory* **51** (2006), pp. 22–32], where it is shown that cycles (except for C_4) are Ramsey-unsaturated, and conjectured that, moreover, one may add *any* chord without changing the Ramsey number of the cycle C_n, unless n is even and adding the chord creates an odd cycle.

We prove this conjecture for large cycles by showing a stronger statement. If a graph H is obtained by adding a linear number of chords to a cycle C_n, then $r(H) = r(C_n)$, as long as the maximum degree of H is bounded, H is either bipartite (for even n) or almost bipartite (for odd n), and n is large.

This motivates us to call cycles *strongly* Ramsey-unsaturated. Our proof uses the regularity method.

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1. Introduction

The Ramsey number $r(H)$ of a graph H is the smallest integer N such that every 2-colouring of the edges of the complete graph K_N on N vertices contains a monochromatic copy of H.

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‡ Corresponding author.

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It has been known [4, 10, 16] for a long time that the cycles C_n on n vertices have Ramsey numbers linear in n, while the Ramsey numbers of complete graphs K_n are exponential. So from the beginnings of Ramsey theory one important question has been hovering in the background: If we keep adding chords to C_n until we reach K_n, at what point will the Ramsey number jump up? Very little is known. When we are close to the complete graph, then, for $3 \leqslant n \leqslant 6$, it holds that $r(K_n) > r(K_n - e)$, and it is conjectured (see [2]) that this is also true for every $n > 6$.

At the other end of the spectrum, that is, when we are close to the cycle, all that is known is the result by Balister, Lehel and Schelp [2] mentioned in the abstract. Using their terms, we call a graph H *Ramsey-saturated* if $r(H + e) > r(H)$ for every edge e in the complement of H, and *Ramsey-unsaturated* otherwise. Balister, Lehel and Schelp [2] prove that the cycle C_n is Ramsey-unsaturated for every $n > 4$, and conjecture that one may add any edge e to C_n without increasing the Ramsey number, unless n is even and adding e destroys the bipartiteness of C_n (Conjecture 3 in [2]). Balister, Lehel and Schelp [2] also remark: 'One would expect to be able to add more than one edge to the cycle without increasing the Ramsey number.'

We show that this is indeed the case, at least asymptotically. Our main result is that one may add a linear number cn of chords to C_n without changing the Ramsey number, as long as the obtained graph H has bounded maximum degree $\Delta(H)$, and as long as H is bipartite if n is even, and almost bipartite if n is odd. A non-bipartite graph is called *almost bipartite* if the removal of one vertex results in a bipartite graph. Since our proof relies on the regularity method, our result holds only for large graphs, with the constant c depending only on $\Delta(H)$.

Theorem 1.1. *For every $\Delta \in \mathbb{N}$ there are $n_0 \in \mathbb{N}$ and $c > 0$ such that, for every $n \geqslant n_0$ and every collection D of chords of C_n with*

(1) $|D| \leqslant cn$ *and*

(2) $\Delta(C_n \cup D) \leqslant \Delta$,

the following two conditions are equivalent:

(a)

$$r(C_n \cup D) = r(C_n) = \begin{cases} 3n/2 - 1, & n > 4 \text{ is even}, \\ 2n - 1, & n > 4 \text{ is odd}, \end{cases}$$

(b) *either n is even and $C_n \cup D$ is bipartite, or n is odd and $C_n \cup D$ is almost bipartite.*

Observe that Theorem 1.1 confirms Conjecture 3 of [2], since an odd cycle plus one chord is almost bipartite. Also notice that the value of $r(C_n)$ in (a) comes from [10, 16].

When $n > 4$ is even, $r(C_n) = 3n/2 - 1 \leqslant 2(n-1)$ and the implication (a) $\Rightarrow$ (b) of Theorem 1.1 can easily be seen by considering the following construction. Colour all edges of a maximal cut in $K_{2(n-1)}$ blue, and the remaining edges red. By (a), there is a monochromatic copy of $C_n \cup D$. Since the components of the red subgraph of $K_{2(n-1)}$ are too small to contain $C_n \cup D$, there is a blue copy of $C_n \cup D$ in $K_{2(n-1)}$. Thus $C_n \cup D$ is bipartite.

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For odd $n > 4$, we know that $r(C_n) = 2n - 1$. Take the above colouring of $K_{2(n-1)}$ and add one new vertex to $K_{2(n-1)}$ all whose incident edges are coloured blue. Then, as above, we find a blue copy of $C_n \cup D$. This copy cannot be bipartite since it contains the odd cycle C_n, hence it must be almost bipartite.

The odd case of Theorem 1.1 follows from a slightly more general result. As a generalization of almost-bipartiteness, let us call a graph *k-almost bipartite* (for $k \in \mathbb{N}$) if it contains an independent set of k vertices such that the removal of these vertices leaves a bipartite graph, but the removal of any independent set of less than k vertices does not. Then a graph is 1-almost bipartite if and only if it is almost bipartite.

Theorem 1.2. *For every Δ and $k \in \mathbb{N}$ there are $n_0 \in \mathbb{N}$ and $c > 0$ such that, for every odd $n \geqslant n_0$ and every collection D of chords of C_n with*

(1) $|D| \leqslant cn$ *and*

(2) $\Delta(C_n \cup D) \leqslant \Delta$,

the following two conditions are equivalent:

(a) $r(C_n \cup D) = r(C_n) + k - 1$,

(b) $C_n \cup D$ *is k-almost bipartite.*

Again, the implication (a) $\Rightarrow$ (b) of Theorem 1.2 is straightforward. Suppose that $r(C_n \cup D) = r(C_n) + k - 1 = 2(n-1) + k$ and n is large (in particular, $n > k$). We partition the vertices of $K_{r(C_n \cup D)}$ into sets X, Y, and Z such that $|X| = |Y| = n - 1$ and $|Z| = k$ and colour all edges within X or Y or Z red and the rest blue. Clearly, the red subgraph has no sufficiently large component to contain $C_n \cup D$, so we find a blue copy of $C_n \cup D$ in $K_{2(n-1)+k}$. Since the blue subgraph is k-almost bipartite, we find that $C_n \cup D$ is k'-almost bipartite for some $k' \leqslant k$. Now, if $k' < k$, then we can use the implication (b) $\Rightarrow$ (a) for k' to obtain a contradiction. Hence, $C_n \cup D$ is k-almost bipartite.

Both our results are, apart from the value of c, best possible for every sufficiently large Δ. To see this, we first recall the following result of Graham, Rödl and Ruciński.

Theorem 1.3 (Graham, Rödl and Ruciński [13]). *There exists a constant $c_1 > 1$ such that, for every $\Delta \geqslant 1$ and for every $n \geqslant \Delta + 1$ (except for $\Delta = 1$ and $n = 2, 3, 5$), there exists a bipartite graph H with n vertices and maximum degree at most Δ which satisfies $r(H) > c_1^{\Delta} n$.* □

Let $\Delta \geqslant 3$ be such that $c_1^{\Delta-2} \geqslant 4$, and let n be even and larger than $2(\Delta + 2)$. Then Theorem 1.3 ensures that there is a bipartite graph H on $n/2 > 5$ vertices and of maximum degree at most $\Delta - 2$ such that $r(H) > c_1^{\Delta-2}(n/2) \geqslant 2n$. Clearly, we can add H into C_n respecting the natural bipartition of C_n. Denote the set of chords by D and note that

$$\Delta(C_n \cup D) \leqslant \Delta \quad \text{and} \quad |D| \leqslant e(H) \leqslant \frac{1}{2}(\Delta - 2)\frac{n}{2} < (\Delta/4)n.$$

On the other hand, we have

$$r(C_n) = 3n/2 - 1 < 2n \leqslant r(H) \leqslant r(C_n \cup D).$$

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Hence, this construction shows that the function from condition (1) of Theorem 1.1 cannot be improved to anything above $\Delta n/4$.

A similar construction works for large odd n. In this case, let $\Delta \geqslant 3$ be such that $c_1^{\Delta-2} \geqslant 6$, $n > \max\{2\Delta, 6k+1\}$, and let H be a bipartite graph on $(n+1)/2$ vertices with

$$\Delta(H) \leqslant \Delta - 2 \quad \text{and} \quad r(H) > c_1^{\Delta-2}(n+1)/2 \geqslant 3(n+1).$$

Let $v_1, v_2, \ldots, v_n$ be the (consecutive) vertices of C_n. We add k chords $v_1v_3, v_4v_6, \ldots, v_{3k-2}v_{3k}$ and add H into the path $v_{3k+1}, v_{3k+2}, \ldots, v_n$, respecting its natural bipartition. Denote the set of thus created chords by D. Then $C_n \cup D$ had maximum degree Δ, it is k-almost bipartite because it contains k vertex-disjoint triangles $v_1v_2v_3, v_4v_5v_6, \ldots, v_{3k-2}v_{3k-1}v_{3k}$ and removing vertices $v_1, v_4, \ldots, v_{3k-2}$ from $C_n \cup D$ yields a bipartite graph, and

$$|D| \leqslant e(H) + k \leqslant \frac{1}{2}(\Delta - 2)\frac{n}{2} + k < \frac{\Delta}{4}n.$$

On the other hand, we have

$$r(C_n) + k - 1 = 2n + k - 2 < 3(n+1) \leqslant r(H) \leqslant r(C_n \cup D).$$

Furthermore, Theorem 1.3 also implies that Theorems 1.1 and 1.2 cannot be true for non-constant Δ. Suppose that $\Delta = \Delta(n) \to \infty$ as $n \to \infty$ arbitrarily slowly. Note that we may assume that $2\log n > \Delta$, because for faster growing Δ, or more precisely, for all $\Delta \geqslant 2\log n$, we can add $K_{\log n, \log n}$ to C_n so that the resulting graph has maximum degree at most Δ and it is bipartite, but its Ramsey number is at least $r(K_{\log n, \log n}) > 2n$ for large n (see Chung and Graham [6]).

Now, let n_0 be such that $c_1^{\Delta}/\Delta^2 > 2$ for every $n > n_0$. Let H be a bipartite graph on $n/\Delta^2 > \Delta + 1$ vertices and of maximum degree at most Δ such that $r(H) > c_1^{\Delta}(n/\Delta^2) > 2n$. Again, we can add H into C_n respecting the natural bipartition of C_n. Denote the set of chords by D and note that

$$|D| \leqslant e(H) \leqslant \frac{1}{2}\left(\Delta\frac{n}{\Delta^2}\right) = o(n).$$

On the other hand, we have

$$r(C_n) = 3n/2 - 1 \leqslant 2n < r(H) \leqslant r(C_n \cup D).$$

2. Preliminaries

2.1. Notation and two facts

We use standard graph-theoretic notation. For a graph G, let $V(G)$ denote the set of its vertices and $E(G)$ the set of its edges. We denote by $|G|$ and $e(G)$ the number of vertices and edges of G, respectively. Given a set $A \subset V(G)$ of vertices and a set $E \subset E(G)$ of edges, $G[A]$ stands for the subgraph of G induced by the vertices of A, and $G - E$ is the subgraph of G with vertex set $V(G)$ and edge set $E(G) \setminus E$.

Given two disjoint sets of vertices, A and B, we write $E(A, B)$ for the set of edges with one endpoint in A and the other in B, we set $e(A, B) := |E(A, B)|$, and call $G[A, B]$ the bipartite subgraph of G with bipartition A, B and edge set $E(A, B)$. The quantity

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$d(A,B) := d(A,B)/|A||B|$ is the *density* of the pair (A,B). For a vertex a, we denote by $N(a)$ the set of all vertices adjacent to a and set $\deg(a) := |N(a)|$.

Next, let us clarify some notation on paths. The *length* of a path is the number of edges it contains. An *odd path* is a path of odd length, an *even path* is a path of even length. A path is *trivial* if it consists of one vertex only. An *x–y-path* is a path starting at vertex x and ending at vertex y. If x and y are vertices of some path P, then we denote by xPy the subpath of P starting at x and ending at y. If P is a path with one endvertex in a set A and the other one in a set B and does not meet $A \cup B$ otherwise, then we say that P is an *A–B-path*. For a set A, a non-trivial A–A-path is also called an *A-path*.

We finish this subsection by quickly proving two easy facts, which will be needed in Section 3.3.

Fact 2.1. *Let G be a bipartite graph, and let $H \subseteq G$ be such that all $V(H)$-paths of $G - E(H)$ are odd. Then H has a bipartition U_1, U_2 of its vertex set so that each $V(H)$-path in $G - E(H)$ is a U_1–U_2-path.*

Proof. Take any bipartition of G and consider its restriction U_1, U_2 to the vertices of H. Since the endpoints of every odd path cannot be in the same partite set, the statement of the fact follows. □

Fact 2.2. *Let H', G be a tripartite graphs such that G is obtained from H' by adding at most $|H'|$ pairwise internally vertex-disjoint $V(H')$-paths, each of odd length at least m, $m \geqslant 3$.*

Then there is a graph H with $H' \subseteq H \subseteq G$ and $|H| \leqslant 3|H'|$ that has a tripartition U_1, U_2, U_3 of its vertex set such that each $V(H)$-path in $G - E(H)$ is an odd U_1–U_2-path of length at least $m - 2$.

Proof. Let H' have colour classes U_1', U_2' and U_3'. For each $V(H')$-path $P = abc \cdots xyz$ in $G - E(H')$ that has at least one endvertex (a or z) in U_3', we add both b and y to H'. More precisely, if $a, z \in U_3'$, then we add b to U_1' and y to U_2', and if only one of a, z lies in U_3', say $a \in U_3'$, and $z \in U_i'$, $i \in \{1,2\}$, then we add b to U_i' and y to U_{3-i}'. Call the obtained tripartite graph H, with partition classes U_1, U_2 and U_3. Clearly, the $V(H)$-paths in $G - E(H)$ all have odd length $\geqslant m - 2$, and since H is obtained from H' by adding two vertices from each $V(H')$-path, we also have $|H| \leqslant 3|H'|$. □

2.2. Structural tools

We now present some structural results we will use, and which come from [1, 9, 14]. We also prove a corollary of one of these theorems, which will be tailor-made for our purposes.

The first two are well-known theorems of Erdős and Gallai.

Theorem 2.3 (Erdős and Gallai [9]). *Every graph G with at least $(n-1)(|G|-1)/2 + 1$ edges contains a cycle of length at least $n \geqslant 3$.* □

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Theorem 2.4 (Erdős and Gallai [9]). *Let G be a graph with $|G| \geqslant 3$ and minimum degree at least $(|G|+1)/2$. Then, for every two vertices u and v, there exists a u–v-path containing all vertices of G.* □

The next result is a recent one on embedding bounded degree graphs.

Lemma 2.5 (Allen, Brightwell and Skokan [1]). *Given a natural number $\Delta \geqslant 1$ and any $0 < \varepsilon < 1/(\Delta^2+4)$, let F be an n-vertex graph in which every vertex has degree at least $3\Delta\varepsilon n$, and all but at most εn vertices have degree at least $(1-2\varepsilon)n$. Let J be any n-vertex graph with $\Delta(J) \leqslant \Delta$. Then $J \subset F$.* □

We also need a well-known result, originally conjectured by Burr and Erdős [5] and proved by Chvátal, Rödl, Szemerédi and Trotter [7], that the graphs with bounded maximum degree have linear Ramsey numbers. Later improvements (on the value of the multiplicative constant) include results of Graham, Rödl and Ruciński [12] and Conlon, Fox and Sudakov [8].

Theorem 2.6 (Graham, Rödl and Ruciński [12]). *There exists a constant $c_2 > 1$ such that, for all $\Delta \geqslant 2$ and all $n \geqslant \Delta+1$, every graph H with n vertices and maximum degree at most Δ satisfies $r(H) \leqslant 2^{c_2\Delta(\log\Delta)^2} n$.* □

Finally, we will need stability results on monochromatic cycles in graphs whose edges are multicoloured. (A multi-colouring of the edges of some graphs is an assignment of colours to the edges in which an edge can have more than one colour.) It will come as no surprise that there are two cases, according to the parity of the cycle. We start with the odd case.

Theorem 2.7 (Kohayakawa, Simonovits and Skokan [14]). *For every $\gamma \in (0, 10^{-2})$, there is an $n_{\text{odd}} \in \mathbb{N}$ such that the following holds for all integers $\bar{m}, \bar{M}$ with $\bar{m} > n_{\text{odd}}$ and $\bar{M} \geqslant (3/2+80\gamma)\bar{m}$.*

Suppose that G is a graph on $\bar{M}$ vertices with $\delta(G) \geqslant \bar{M} - \gamma\bar{m}$. Then every multi-colouring of the edges of G with two colours contains either a monochromatic odd cycle C_ℓ for some $\ell \geqslant (1+\gamma)\bar{m}$ or there is a vertex $x \in V(G)$, a partition of $V(G)\setminus\{v\}$ into two sets V_1 and V_2, and an $i \in \{1,2\}$ such that:

(a) *all edges of $G[V_1] \cup G[V_2]$ have only colour i,*

(b) *all edges of $G[V_1, V_2]$ have only colour $3-i$, and*

(c) *$|V_1|, |V_2| < (1+\gamma)\bar{m}$.* □

A similar result holds for even cycles.

Theorem 2.8 (Kohayakawa, Simonovits and Skokan [14]). *For every $\beta \in (0, 10^{-6})$, there is an $n_{\text{even}} \in \mathbb{N}$ such that the following holds for all integers m, M with $m > n_{\text{even}}$ and $M \geqslant (3/2-\beta)m$.*

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If the edges of a graph G on M vertices with $\delta(G) \geqslant M - \beta m$ are multi-coloured with two colours, then G either contains a monochromatic even cycle C_ℓ for some $\ell \geqslant (1+\beta)m$ or there is a partition of $V(G)$ into sets V_1, V_2 and W, and an $i \in \{1,2\}$ such that:

(a) *all but at most $\beta^{1/5}|V_1|^2$ edges of $G[V_1]$ have only colour i,*

(b) *all but at most $\beta^{1/5}|V_1||V_2|$ edges of $G[V_1, V_2]$ have only colour $3-i$, and*

(c) $|V_1| \geqslant (1-\beta)m$, $|V_2| \geqslant (1-\beta)m/2$, $|W| < \beta^{1/5}m$. □

In addition to Theorem 2.8, we will use the following corollary (which one might consider a 'degree form' of Theorem 2.8).

Corollary 2.9. *The sets V_1 and V_2 from Theorem 2.8 contain sets A and B, respectively, such that:*

(a′) *the minimum degree in colour i in $G[A]$ is at least $(1-\beta^{1/20})|A|$, the maximum degree in colour $3-i$ in $G[A]$ is at most $\beta^{1/20}|A|$,*

(b′) *in colour $3-i$ in $G[A,B]$, every vertex of A has degree at least $(1-\beta^{1/20})|B|$ and every vertex of B has degree at least $(1-\beta^{1/20})|A|$, and*

(c′)
$$|A| \geqslant \left(\frac{2}{3} - \beta^{1/20}\right)\frac{3}{2}m, \quad |B| \geqslant \left(\frac{1}{3} - \beta^{1/20}\right)\frac{3}{2}m.$$

Proof. Observe that, firstly, condition (a) implies that, in V_1, there are at most $2\beta^{1/10}|V_1|$ vertices that, in colour $3-i$, have degree at least $\beta^{1/10}|V_1|$ inside V_1. Secondly, (b) implies that V_1 contains at most $\beta^{1/10}|V_1|$ vertices sending at least $\beta^{1/10}|V_2|$ edges of colour i to V_2. Finally, (b) implies that, in V_2, there are at most $\beta^{1/10}|V_2|$ vertices that send at least $\beta^{1/10}|V_1|$ edges of colour i to V_1.

Remove these in total at most $3\beta^{1/10}|V_1|$ vertices from V_1 and at most $\beta^{1/10}|V_2|$ vertices from V_2 to obtain sets A and B as required. In fact, by construction, each vertex of A is incident with at most $\beta^{1/10}|V_1| \leqslant \beta^{1/20}|A|$ edges of colour $3-i$ that go to other vertices of A. Thus (a′) holds. Similarly we can see that (b′) holds.

For (c′), we have to see that
$$|A| \geqslant \left(1 - \frac{3}{2}\beta^{1/20}\right)m \quad \text{and} \quad |B| \geqslant (1-3\beta^{1/20})\frac{m}{2}.$$
This is true since $|A| \geqslant (1-3\beta^{1/10})|V_1|$ and $|B| \geqslant (1-\beta^{1/10})|V_2|$ by construction. Using (c), we thus obtain (c′). □

2.3. Regularity

In this section we introduce some very well-known concepts; the reader familiar with regularity is invited to skip this section.

We start with giving the standard definition of regularity.

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Definition. Given $\varepsilon > 0$ and disjoint subsets A, B of the vertex set of a graph G, we say that the pair (A, B) is *ε-regular in G* if, for every pair (A', B') with $A' \subseteq A$, $|A'| \geqslant \varepsilon|A|$, $B' \subseteq B$, $|B'| \geqslant \varepsilon|B|$, we have

$$|d(A', B') - d(A, B)| < \varepsilon.$$

When there is no danger of confusion, we simply say that the pair (A, B) is ε-regular.

It follows from the definition that any pair with density close to 1 must be regular.

Fact 2.10. *Let $0 < \varepsilon < 1$ and let (A, B) be a pair of density at least $1 - \varepsilon^3$ in some graph G. Then the pair (A, B) is ε-regular.* □

We now turn to the regularity lemma.

Theorem 2.11 (regularity lemma: Szemerédi [17]). *For every $\varepsilon > 0$ and $m_{\mathrm{reg}} > 0$, there are M_{reg} and n_{reg} such that, for each graph G on at least n_{reg} vertices, there exists a partition $V_0, V_1, \ldots, V_t$ of $V(G)$ such that:*

(i) $m_{\mathrm{reg}} \leqslant t \leqslant M_{\mathrm{reg}}$,

(ii) $|V_1| = |V_2| = \cdots = |V_t|$ *and* $|V_0| \leqslant \varepsilon|V(G)|$,

(iii) *for each i, $1 \leqslant i \leqslant t$, all but at most εt of the pairs (V_i, V_j), $1 \leqslant j \leqslant t$, are ε-regular.*

We shall refer to a partition of $V(G)$ satisfying conditions (i)–(iii) *as ε-regular with respect to G.* □

We continue by stating a definition and three facts on ε-regular pairs, all very well known, which can be found for example in [15].

Definition. Let (V_1, V_2) be an ε-regular pair of density d in some graph G. A vertex $v_1 \in V_1$ is called ε-typical in G with respect to V_2 if v_1 has at least $(d - \varepsilon)|V_2|$ neighbours in V_2.

Fact 2.12. *Let (V_1, V_2) be an ε-regular pair of density d in some graph G. Then all but at most $\varepsilon|V_1|$ vertices in V_1 are ε-typical in G with respect to V_2.* □

Fact 2.13. *Let (V_1, V_2) be an ε-regular pair of density d in some graph G. If $\ell \in \mathbb{N}$ is such that $(d - \varepsilon)^{\ell - 1} \geqslant \varepsilon$, then all but at most $\ell\varepsilon|V_1|^{\ell}$ ℓ-tuples of vertices in V_1 have at least $(d - \varepsilon)^{\ell}|V_2|$ common neighbours in V_2.* □

Fact 2.14. *Let $0 < \varepsilon \leqslant 1/4$ and let (V_1, V_2) be an ε-regular pair of density d. Then, for every $V_1' \subseteq V_1$, $V_2' \subseteq V_2$ such that $|V_1'| \geqslant \varepsilon^{1/2}|V_1|$ and $|V_2'| \geqslant \varepsilon^{1/2}|V_2|$, the pair (V_1', V_2') is $\varepsilon^{1/2}$-regular with density at least $d - \varepsilon$.* □

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We now state the so-called embedding lemma. Given a graph F and a positive integer s, let $F(s)$ be the graph obtained from F by replacing each vertex v of F with an independent set V_v of size s, and each edge vw of F with the complete bipartite graph on V_v and V_w.

Lemma 2.15 (embedding lemma [15]). *For all $\Delta \in \mathbb{N}$ and $p \in (0,1]$, if $\varepsilon' > 0$ is such that $(p-\varepsilon')^\Delta - \Delta\varepsilon' > p^\Delta/2$, then the following holds for every graph H of maximum degree Δ.*

Let F be a graph on vertices $1,\dots,r$, and let G be a graph on the union of the sets $V_1,\dots,V_r$, where each V_i has the same size λ, and for each edge $ij \in E(F)$ the pair (V_i,V_j) is ε'-regular and of density at least p. Suppose $s \leqslant p^\Delta\lambda/2$. If $H \subseteq F(s)$, then also $H \subseteq G$. □

In particular, we note that the conditions of Lemma 2.15 are satisfied if, for $\varepsilon' > 0$ and $0 < p < 1$, we have

$$\varepsilon' \leqslant \frac{p^\Delta}{4\Delta} \quad \text{and} \quad s \leqslant 2\Delta\varepsilon'\lambda. \tag{2.1}$$

Remark. The proof of Lemma 2.15 yields a stronger result. If $\sqrt{\varepsilon'}(p-\varepsilon')^\Delta\lambda - \Delta\varepsilon'\lambda > s$ and $H \subseteq F(s)$, then not only $H \subseteq G$ but also, for every vertex v of H (which is to be embedded to V_i in G), one can specify a 'target' set $C_v \subset V_i$ of size at least $\sqrt{\varepsilon'}\lambda$ into which v will be embedded.

Moreover, for $\varepsilon' > 0$ and $0 < p < 1$, the condition $\sqrt{\varepsilon'}(p-\varepsilon')^\Delta\lambda - \Delta\varepsilon'\lambda > s$ holds for

$$\varepsilon' \leqslant \left(\frac{p^\Delta}{4\Delta}\right)^2 \quad \text{and} \quad s \leqslant 2\Delta\varepsilon'\lambda. \tag{2.2}$$

We shall use this strengthening together with the next lemma to prove an embedding result suitable for a situation in which we have to deal with irregular pairs (see Lemma 2.17).

Lemma 2.16. *Let G be a graph, let $\ell \in \mathbb{N}$, $c > 0$ and $\rho > 0$, and suppose A and B are disjoint sets of vertices in G such that $d(A,B) > c$.*

Then there exists a subset $U \subset A$ of size at least $c|A|/2$ such that all but at most $\rho|U|^\ell$ ℓ-tuples of vertices in U have at least $\rho c^\ell|B|/2$ common neighbours in B. □

Lemma 2.16 can be viewed as a version of Fact 2.13 for pairs that are non-regular. The case $\ell = 2$ of this lemma is proved in [11, Lemma 5.1]. The proof of this Lemma for $\ell > 2$ is a straightforward generalization of the proof for $\ell = 2$ and we shall omit it here.

The next lemma will be useful in Section 3.5.1, where we need to deal with pairs that are potentially irregular.

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Lemma 2.17 (embedding lemma with a non-regular pair). *Let $\Delta \in \mathbb{N}$, let $p \in (0,1]$, and suppose $\varepsilon' > 0$ is such that*

$$\varepsilon' \leqslant \frac{p^{2\Delta}}{(2\Delta)^{4\Delta+1}}.$$

Let G be a graph on the union of the sets X_1, X_2, X_3, where each X_i has size at least λ, pairs (X_1, X_3) and (X_2, X_3) are ε'-regular and pairs (X_1, X_2), (X_2, X_3), (X_1, X_3) all have density greater than p. If H is a tripartite graph of maximum degree at most Δ, satisfying $|H| \leqslant 2\Delta\varepsilon'\lambda$, then $H \subseteq G$.

Proof. Let H be as in the lemma, with tripartition U_1, U_2, U_3. We will find an embedding of H to G that maps vertices from U_i into X_i.

Our idea is as follows. We first find an embedding f of U_1 to X_1' such that, for every $v \in U_i$, $i = 2, 3$, the set $f(N_H(v) \cap U_1)$ will have at least $\sqrt{\varepsilon'}|X_i|$ common neighbours in X_i. Then, using the remark after Lemma 2.15, we extend f to an embedding of H in G: for every $v \in U_i$, $i = 2, 3$, we define the target set C_v for v by $C_v = N_G(f(N_H(v) \cap U_1)) \cap X_i$, observe that $|C_v| \geqslant \sqrt{\varepsilon'}|X_i|$, and apply the remark after Lemma 2.15 with $F = K_2$.

So, we only need to find f. Set $\rho' = (2\Delta)^{-\Delta}$. We first apply Lemma 2.16 with $A = X_1$, $B = X_2$, $\ell = \Delta$, $c = p$ and $\rho = (\rho')^2$ and obtain a set $X_1' \subset X_1$ such that $|X_1'| \geqslant p|X_1|/2$ and all but at most $\rho|X_1'|^\Delta \leqslant \rho'\binom{|X_1'|}{\Delta}/2$ Δ-tuples of vertices in X_1' have at least $\rho p^\Delta |X_2|/2 \geqslant \sqrt{\varepsilon'}|X_2|$ common neighbours in X_2.

Next, by Fact 2.13, all but at most

$$\varepsilon'\Delta|X_1|^\Delta \leqslant \varepsilon'\Delta(2/p)^\Delta|X_1'|^\Delta \leqslant \rho'\binom{|X_1'|}{\Delta}/2$$

Δ-tuples of vertices in X_1' have at least $(p - \varepsilon')^\Delta|X_3| \geqslant \sqrt{\varepsilon'}|X_3|$ common neighbours in X_3.

We call a subset T of X_1' of size Δ *good* if T has at least $\sqrt{\varepsilon'}|X_i|$ common neighbours in X_i for $i = 2, 3$. We also say that a set $S \subset X_1'$ of size less than Δ is *good* if S is contained in more than

$$(1 - (2\Delta)^{|S|-\Delta})\binom{|X_1'|}{\Delta - |S|}$$

good sets T of size Δ. For example, it follows from the previous two paragraphs that the empty set is good.

For a good set S, we define B_S to be the set of all vertices $w \in X_1' \setminus S$ such that $S \cup \{w\}$ is not good. It can be easily shown (see the proof of Lemma 6.4 in [11]) that $|B_S| \leqslant |X_1'|/2\Delta$ for every good S.

Fix any ordering $v_1, v_2, \ldots, v_{|U_1|}$ of the vertices in U_1 and set $L_0 = \emptyset$, $L_i = \{v_1, \ldots, v_i\}$. Following the proof of Lemma 6.4 in [11], we construct f inductively on i so that for every i and every $v \in U_i$, $i = 2, 3$, the set $f(N_H(v) \cap L_i)$ is good.

Since the empty set is good, the statement is true for $i = 0$. Suppose that the statement is true for some $i \geqslant 0$. First of all, note that if $v \in U_i$, $i = 2, 3$, is not adjacent to v_{i+1}, then

$$f(N_H(v) \cap L_{i+1}) = f(N_H(v) \cap L_i)$$

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is good for any choice of $f(v_{i+1})$. Since the maximum degree of H is at most Δ, there are at most Δ sets $T \subseteq L_{i+1}$ such that $T = N(v) \cap L_{i+1}$ and $v \in U_i$ is a neighbour of v_{i+1}. By the induction assumption, $f(T \setminus \{v_{i+1}\})$ is good, so there are at most $|X_1'|/2\Delta$ bad vertices in X_1' corresponding to this set. Summing over all sets T, there are at most $|X_1'|/2$ vertices in X_1' to which we cannot map v_{i+1}. Hence, as long as $|H| \leqslant |X_1'|/2$, we can find a suitable image for v_{i+1} and, consequently, we can define f as desired. □

2.4. Embedding long paths

In this section we prove an important lemma, Lemma 2.19, which will be used later in the proof of our main theorems. This lemma is about embedding a family of long paths into a sequence of ε-regular pairs. As a starting point, we use a lemma from [3] that gives useful conditions under which we can embed a path into one regular pair.

Lemma 2.18 (Benevides [3]). *For every $\delta' \in (0,1)$ and for every $\varepsilon' \in (0, \delta'/20)$, there is an $n_{2.18} = n_{2.18}(\delta', \varepsilon')$ with the following property. Let $n \geqslant n_{2.18}$ and let (X_1, X_2) be an ε'-regular pair (in some graph) with density at least δ' and with $|X_1| = |X_2| = n$. Then, for every ℓ, $1 \leqslant \ell \leqslant n - 2\varepsilon' n/\delta'$, and for every two vertices $v_1 \in X_1, v_2 \in X_2$ such that $\deg(v_1), \deg(v_2) \geqslant \delta' n/2$, there exists a v_1–v_2-path of length $2\ell + 1$.* □

We now show how to embed more paths into a longer sequence of regular pairs.

Lemma 2.19. *Let G be a graph, let $\varepsilon \in (0, 1/625)$, and let $P_1, \dots, P_{k'}$ be a collection of pairwise internally vertex-disjoint paths of odd lengths. Suppose that $\ell \in \mathbb{N}$ is even and $V_1, \dots, V_\ell \subseteq V(G)$ are such that*

(i) $|V_1| = |V_2| = \cdots = |V_\ell| \geqslant n_{2.18}(\varepsilon^{1/4}, \varepsilon^{1/2})/4\varepsilon^{1/2}$,

(ii) *$V_1, \dots, V_\ell$ are pairwise disjoint, except (possibly) for V_1 and V_ℓ,*

(iii) *for all $i \in [\ell - 1]$, the pair (V_i, V_{i+1}) is ε-regular of density at least $2\varepsilon^{1/4}$,*

(iv) $k' \leqslant \varepsilon^{1/2}|V_1|$,

(v) *for all $i \in [k']$, $p^i := |E(P_i)| \geqslant 3\ell$,*

(vi) $\sum_{i=1}^{k'} p^i \leqslant \big(1 - 2\varepsilon^{1/4}\big)(\ell - 2)|V_1|$,

(vii) *for all $i \in [k']$, the first vertex of P_i is embedded in a vertex $a_i \in V_1$ and the last vertex of P_i is embedded in a vertex $b_i \in V_\ell$, and*

(viii) *for all $i \in [k']$, a_i is ε-typical in G with respect to V_2 and b_i is ε-typical in G with respect to $V_{\ell-1}$.*

Then we can extend $A := \bigcup_{i=1}^{k'} \{a_i, b_i\}$ to an embedding of all of $\bigcup_{i=1}^{k'} P_i$ in G such that the only vertices from $V_1 \cup V_\ell$ used are those from A.

Notice that the sets V_1 and V_ℓ can have a non-empty intersection. Consequently, in such a case, it is possible to choose $a_i = b_i$ and obtain a cycle of length p^i instead of path of length p^i. However, in this paper, we will always make sure that $a_i \neq b_i$.

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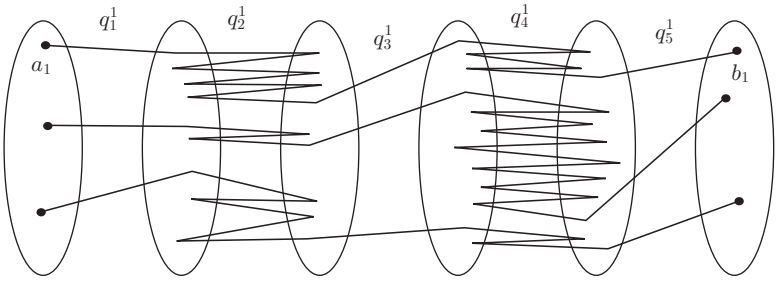


Figure 1. The numbers q_j^i stand for the number of edges of P_i we wish to embed in the pair (V_j, V_{j+1}).

Proof. For every $j \in [\ell - 1]$ and for all $i \in [k']$, we choose an odd number q_j^i that will indicate how many edges of P_i we plan to embed into the pair (V_j, V_{j+1}). We choose the odd numbers q_j^i so that they satisfy the following three conditions:

(a) $q_j^i = 1$ if j is odd, and $q_j^i \geqslant 3$ if j is even,

(b) $\sum_{j=1}^{\ell-1} q_j^i = p^i$ for each $i \in [k']$, and

(c) $\sum_{i=1}^{k'} (q_j^i + 1) \leqslant 2\big(1 - 2\varepsilon^{1/4}\big)|V_j|$ for each even $j \in [\ell - 1]$.

Observe that such a choice is possible because of assumptions (i), (v) and (vi).

Now, for $j \in [\ell]$ and $i \in [k']$, set

$$Q_j^i := \sum_{h=1}^{j-1} q_h^i + 1$$

and let v_j^i denote the Q_j^ith vertex on P_i. In particular, the pre-image of $w_1^i := a_i$ is v_1^i and the pre-image of $w_\ell^i := b_i$ is v_ℓ^i.

Next, successively, for each $i \in [k']$ we embed P_i in the following way. For each $j = 2, \ldots, \ell - 1$, we will consider certain sets $W_j^i \subset V_j$. The idea is that, for even j, the sets W_j^i and W_{j+1}^i contain possible images for $V(v_j^i P_i v_{j+1}^i) \cap V_j$. The sets W_j^i will satisfy the following conditions:

(d) $W_2^i \subset N(w_1^i) \cap V_2$, $W_{\ell-1}^i \subset N(w_\ell^i) \cap V_{\ell-1}$, and $W_j^i \subset V_j$ for each $j = 3, \ldots, \ell - 2$,

(e) W_j^i contains no vertex of $P_1, \ldots, P_{i-1}$, for any $j = 2, \ldots, \ell - 1$,

(f) $\big(1 - 2\varepsilon^{1/4}\big)|W_j^i| > (q^i_{2\lfloor \frac{j}{2} \rfloor} + 1)/2$ for each $j = 2, \ldots, \ell - 1$,

(g) $|W_j^i| > 4\varepsilon^{1/2}|V_j|$ for $j = 2, \ldots, \ell - 1$, and

(h) $|W_j^i| = |W_{j+1}^i|$ for each even $j = 2, 4, \ldots, \ell - 2$.

Observe that (g), together with Fact 2.14, ensures that, for $j = 2, \ldots, \ell - 2$,

$$(W_j^i, W_{j+1}^i) \text{ is } \varepsilon^{1/2}\text{-regular with density at least } 2\varepsilon^{1/4} - \varepsilon > \varepsilon^{1/4}. \tag{2.3}$$

First, assuming the existence of the sets W_j^i, let us embed all the P_i. We will show later that the sets $W_j^i \subset V_j$ satisfying (d)–(h) do exist. We shall start with the vertices v_j^i, which will be placed in the following way.

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For $j = 2, \dots, \ell - 1$, the vertex v_j^i is embedded in a vertex w_j^i of W_j^i such that

$$w_j^i \text{ is } \varepsilon^{1/2}\text{-typical in } G \text{ with respect to } W_{j-1}^i \text{ and } W_{j+1}^i, \tag{2.4a}$$

and

$$w_j^i w_{j+1}^i \text{ is an edge of } G \text{ if } q_j^i = 1. \tag{2.4b}$$

By Fact 2.12, all but at most $2\varepsilon^{1/2}|W_j^i|$ vertices in W_j^i satisfy (2.4a). In order to see (2.4b), observe that the sets of vertices satisfying (2.4a) for j and for $j+1$ are large enough to apply $\varepsilon^{1/2}$-regularity of (W_j^i, W_{j+1}^i), hence there must be an edge $w_j^i w_{j+1}^i$ between them. (Note that condition (2.4b) holds also for $j = 1, \ell - 1$, because of (d).)

The last step of our embedding is the embedding of the subpaths $v_j^i P_i v_{j+1}^i$ for each even $j \in [\ell]$. By (a) we know that $v_j^i P_i v_{j+1}^i$ has odd length $q_j^i \geqslant 3$. We wish to embed the edges of $v_j^i P_i v_{j+1}^i$ into the pair (W_j^i, W_{j+1}^i). For this, we plan to use Lemma 2.18 with $\delta' = \varepsilon^{1/4}$ and $\varepsilon' = \varepsilon^{1/2}$. The assumptions of Lemma 2.18 are verified in (2.3), (f), (h) and (2.4a), and the only condition left to check is that $|W_j^i| \geqslant n_{2.18}$. But this follows from

$$|W_j^i| \overset{\text{(g)}}{\geqslant} 4\varepsilon^{1/2}|V_j| \overset{\text{(i)}}{\geqslant} n_{2.18}(\varepsilon^{1/4}, \varepsilon^{1/2}) = n_{2.18}(\delta', \varepsilon').$$

This completes the embedding of the paths P_i. It only remains to show that the sets $W_j^i \subset V_j$ satisfying (d)–(h) do exist. Let $i \leqslant k'$ be given and assume we have already embedded all paths P_h with $h < i$. Let U be the set of all vertices used by our embedding of $\bigcup_{h<i} P_h$.

Now, for $j = 2, \ell - 1$, choose $W_j^i \subseteq \big(N(w_j^i) \cap V_j\big) - U$ and, for $j = 3, \dots, \ell - 2$, $W_j^i \subseteq V_j - U$ as large as possible, but such that $|W_j^i| = |W_{j+1}^i|$ for each even $j = 2, 4, \dots, \ell - 2$. This choice clearly guarantees conditions (d), (e) and (h).

To show (g), observe that, for $j = 2, \ell - 1$, we have

$$\begin{aligned} |N(w_j^i) \cap V_j - U| &\overset{\text{(viii)}}{\geqslant} \lfloor (2\varepsilon^{1/4} - \varepsilon)|V_j| \rfloor - k' \\ &\overset{\text{(iv)}}{>} \varepsilon^{1/4}|V_j| - \varepsilon^{1/2}|V_j| \\ &> 4\varepsilon^{1/2}|V_j|. \end{aligned}$$

Furthermore, for $j = 3, \dots, \ell - 2$, we know by (c) that

$$|V_j - U| \geqslant |V_j| - (1 - 2\varepsilon^{1/4})|V_j| \geqslant 4\varepsilon^{1/2}|V_j|,$$

and thus (g) is established.

In a similar way we see (f). For all even $j \in [\ell - 1]$, we have

$$\begin{aligned} |V_j - U| &= |V_j| - \sum_{h=1}^{i-1} \frac{q_j^h + 1}{2} \\ &\overset{\text{(c)}}{\geqslant} \frac{1}{1 - 2\varepsilon^{1/4}} \sum_{h=1}^{k'} \frac{q_j^h + 1}{2} - \sum_{h \in [k'], h \neq i} \frac{q_j^h + 1}{2} \\ &\geqslant \frac{1}{1 - 2\varepsilon^{1/4}} \frac{q_j^i + 1}{2}. \end{aligned}$$

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For odd $j \in [\ell - 1]$ it suffices to notice that $|W_j^i| = |W_{j-1}^i|$ by (h). □

3. The proof of the main theorems

3.1. Fixing the parameters

We shall prove at the same time, and with the same parameters c and n_0, Theorem 1.1 for even $n \geqslant n_0$ and Theorem 1.2 for odd $n \geqslant n_0$. Recall that we only need to worry about the implication (b) $\Rightarrow$ (a).

Suppose that $\Delta > 2$ and, for odd n, also a fixed $k \in \mathbb{N}$ are given. Define

$$\varepsilon := \frac{1}{(k \cdot \Delta)^{8000 c_2 \Delta^2 (\log \Delta)^2}}, \tag{3.1}$$

where the constant c_2 comes from Theorem 2.6, and set

$$d := 4 \max\{\varepsilon^{1/(100\Delta)}, (4\Delta \cdot 2\varepsilon)^{1/\Delta}\} < \frac{1}{(k \cdot \Delta)^{80\Delta}} \tag{3.2}$$

and

$$m_{\mathrm{reg}} := \frac{\max\{n_{\mathrm{even}}, n_{\mathrm{odd}}\}}{d^{1/4}}, \tag{3.3}$$

where $c_2 > 1$ comes from Theorem 2.6, n_{even} is given by Theorem 2.8 for the input

$$\beta := \varepsilon^{1/5},$$

and n_{odd} is given by Theorem 2.7 for the input

$$\gamma := d^{1/16}.$$

For the parameters ε and m_{reg}, the regularity lemma (Theorem 2.11) yields numbers M_{reg} and n_{reg}. We now set

$$c := \frac{\varepsilon^2}{16 M_{\mathrm{reg}}^2} \tag{3.4}$$

and

$$n_0 := \frac{M_{\mathrm{reg}}}{(1-\varepsilon)} \max\left\{ \frac{n_{2.18}(\varepsilon^{1/4}, \varepsilon^{1/2})}{4\varepsilon^{1/2}}, \frac{n_{2.18}(d^{1/12}, d^{1/6})}{4d^{1/6}} \right\}, \tag{3.5}$$

where function $n_{2.18}$ comes from Lemma 2.18.

Now suppose $n \geqslant n_0$ and let

$$N := \begin{cases} r(C_n) & \text{if } n \text{ is even,} \\ r(C_n) + k - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Let a 2-colouring of the edges of K_N be given, with colours red and blue, say, which induce the (spanning) subgraphs G^r and G^b.

Further, let D be a set of chords of C_n satisfying assumptions (1) and (2) of Theorems 1.1 and 1.2, and such that $C_n \cup D$ is bipartite if n is even and k-almost bipartite if n is odd. Our aim is to find a monochromatic copy of $C_n \cup D$ in K_N.

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3.2. Applying regularity to G^r and G^b

The regularity lemma (Theorem 2.11) applied to G^r yields a partition $V_0, V_1, \dots, V_t$ of $V(G^r)$ that is ε-regular with respect to G^r. It is well known that this partition is also ε-regular with respect to G^b.

Substituting each V_i, $i > 0$, with a vertex i and each ε-regular pair (V_i, V_j) with an edge ij, we obtain the so-called reduced graph R with vertex set $[t] = \{1, 2, \dots, t\}$ such that

(I) $M_{\text{reg}} \geqslant t \geqslant m_{\text{reg}}$,

(II) $N/t \geqslant |V_i| \geqslant (1-\varepsilon)N/t$,

(III) $\delta(R) \geqslant t - 1 - \varepsilon t$.

We define an edge-(multi)colouring of R as follows. Colour the edge $ij \in E(R)$ red (blue) if the density of (V_i, V_j) in G^r (in G^b) is at least d, where d comes from (3.2). Notice that every edge of R receives a colour because $d < 1/2$.

Let us fix some more notation: if xy is a red/blue edge in R (or in G^r or in G^b), then we call x a *red/blue neighbour* of y.

3.3. Preparing $C_n \cup D$

Set

$$z := \frac{\varepsilon^2 n}{M_{\text{reg}}|D|}$$

and, since $|D| \leqslant cn$, notice that

$$z \geqslant \frac{\varepsilon^2}{cM_{\text{reg}}} \overset{(3.4)}{\geqslant} 16M_{\text{reg}}. \tag{3.6}$$

Let H'' be the subgraph of $C_n \cup D$ induced by D and all $V(D)$-paths in C_n of length at most z. Then

$$|H''| \leqslant 2z|D| = 2\frac{\varepsilon^2 n}{M_{\text{reg}}}. \tag{3.7}$$

Now, $C_n \cup D - E(H'')$ is the union $\mathcal{P}''$ of several $V(D)$-paths, each of them longer than z. We wish to have them all of the same parity. For this reason, for each such even path $P = v_1, v_2, \dots, v_{2i+1}$, we remove vertex v_2 and edge v_1v_2 from P, add them to H'', and call the obtained graph H'. Observe that we have added at most $|H''|$ new vertices. Thus,

$$|H'| \leqslant 2|H''|, \tag{3.8}$$

and

$$\text{all } V(D)\text{-paths of } C_n \cup D - E(H') \text{ are odd.} \tag{3.9}$$

If n is even, then $C_n \cup D$, and thus also H', is bipartite. By Fact 2.1 and by (3.9), we can arrange the colour classes U_1, U_2 of H' so that $C_n \cup D - E(H')$ is the edge-disjoint union of U_1–U_2-paths of length at least $z - 1$. Set $H := H'$.

If n is odd, then, since $C_n \cup D \supseteq H'$ is k-almost-bipartite, the graph H' is 3-colourable. By Fact 2.2, there is a supergraph H of H' contained in $C_n \cup D$, with tripartition U_1, U_2, U_3

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of its vertex set such that all $V(H)$-paths in $C_n \cup D - E(H)$ are odd U_1–U_2-paths of length at least $z - 3$.

So, in both cases we have obtained a graph H with partition classes U_1, U_2 and (if n is odd) U_3 such that, by (3.7) and by (3.8),

$$|H| \leqslant 3|H'| \leqslant 12\frac{\varepsilon^2 n}{M_{\mathrm{reg}}}. \tag{3.10}$$

Furthermore, for each $V(H)$-path P of $C_n - E(H)$, we have that

$$P \text{ is an odd } U_1\text{–}U_2\text{-path}, \tag{3.11}$$

and, since $z \geqslant 16M_{\mathrm{reg}} \geqslant 6$ by (3.6), also

$$|E(P)| \geqslant z - 3 \geqslant \frac{z}{2} \geqslant 8M_{\mathrm{reg}}. \tag{3.12}$$

From now on, we split our proof into two cases, depending on the parity of n. We shall deal with even n in Section 3.4, and with odd n in Section 3.5.

3.4. Even *n*

We wish to apply Theorem 2.8 to $G := R$ with $\beta = \varepsilon^{1/5}$ (as fixed above), $m := \lceil 2t/3 \rceil$, and $M := t$. Clearly, by (I) and by the choice of m_{reg}, we have that

$$m \geqslant \left\lceil \frac{2}{3} m_{\mathrm{reg}} \right\rceil > n_{\mathrm{even}}$$

and

$$t \geqslant m_{\mathrm{reg}} > 1/\varepsilon. \tag{3.13}$$

Thus, as

$$m \leqslant \frac{2}{3}t + 1 = \frac{2}{3}M + 1,$$

we obtain that

$$M \geqslant \frac{3}{2}m - \frac{3}{2} \overset{(3.13)}{\geqslant} \frac{3}{2}m - \frac{3}{2}\varepsilon M \geqslant \frac{3}{2}m - \frac{9}{4}\varepsilon m \geqslant \left(\frac{3}{2} - \beta\right)m.$$

Finally, since $4\varepsilon \leqslant \beta$, we know that

$$\begin{aligned} \delta(R) &\overset{(III)}{\geqslant} M - 1 - \varepsilon t \\ &\overset{(3.13)}{>} M - 2\varepsilon t \\ &\geqslant M - 4\varepsilon\frac{t}{2} \\ &\geqslant M - 4\varepsilon m \\ &\geqslant M - \beta m. \end{aligned} \tag{3.14}$$

So, Theorem 2.8 and Corollary 2.9 yield either

(A) R has a monochromatic even cycle C_ℓ with $\ell \geqslant \frac{2}{3}(1 + \varepsilon^{1/5})t$, or

(B) there is a colour $i \in \{1, 2\}$, and sets $A, B \subseteq V(R)$ such that, for $\xi = \varepsilon^{1/100} < 1/16^{10}$,

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(B1) the minimum degree in colour i in $R[A]$ is at least $(1-\xi)|A|$ and the maximum degree in colour $3-i$ in $R[A]$ is at most $\xi|A|$,

(B2) in colour $3-i$ in $R[A,B]$, every vertex in A has degree at least $(1-\xi)|B|$ and every vertex in B has degree at least $(1-\xi)|A|$,

(B3) $|A| \geqslant (\frac{2}{3}-\xi)t$ and $|B| \geqslant (\frac{1}{3}-\xi)t$.

Depending on which case we obtained, we split our proof further into two cases.

3.4.1. Case (A). In this case, R contains an even monochromatic (say, red) cycle C_ℓ with vertices $V_1, V_2, \dots, V_\ell, V_1$ (in this order). Let $V_1' \subseteq V_1$, $V_\ell' \subseteq V_\ell$ contain all those vertices that are ε-typical in G^r with respect to V_2 or $V_{\ell-1}$, respectively. Thus, because of Fact 2.12, we have

$$\lambda := \min\{|V_1'|, |V_\ell'|\} \geqslant (1-\varepsilon)|V_1|.$$

It follows from the definition of ε-regularity that the pair (V_1', V_ℓ') is 2ε-regular with density at least $d-\varepsilon > d/2 \geqslant (4\Delta \cdot 2\varepsilon)^{1/\Delta}$. Moreover, by condition (2) of Theorems 1.1 and 1.2, we know that $\Delta(H) \leqslant \Delta$.

Thus we have verified almost all conditions of Lemma 2.15, which we would like to use with parameters Δ, $p := d-\varepsilon$, $\varepsilon' := 2\varepsilon$, λ, $s := |H|$ and graphs $F := K_2$, H, $G := G^r[V_1' \cup V_\ell']$ in order to embed the bipartite graph H into $G^r[V_1' \cup V_\ell']$. By (2.1), it is only left to check that

$$|H| \leqslant 2\Delta \cdot 2\varepsilon \cdot \lambda.$$

This inequality holds because we have

$$\begin{aligned} |H| &\overset{(3.10)}{\leqslant} 12\frac{\varepsilon^2}{M_{\mathrm{reg}}} \cdot n \leqslant 12\frac{\varepsilon^2}{M_{\mathrm{reg}}} \cdot N \\ &\overset{(3.1)}{\leqslant} 4\Delta(1-2\varepsilon) \cdot 2\varepsilon \cdot \frac{N}{M_{\mathrm{reg}}} \overset{(\mathrm{I})}{\leqslant} 4\Delta(1-\varepsilon)^2 \cdot 2\varepsilon \cdot \frac{N}{t} \\ &\overset{(\mathrm{II})}{\leqslant} 4\Delta \cdot 2\varepsilon \cdot (1-\varepsilon)\min\{|V_1|, |V_\ell|\} \leqslant 4\Delta \cdot 2\varepsilon \cdot \min\{|V_1'|, |V_\ell'|\} \\ &= 2\Delta \cdot 2\varepsilon \cdot \lambda. \end{aligned} \tag{3.15}$$

So Lemma 2.15 guarantees that we can embed H as planned. We will now embed the rest of $C_n \cup D$, that is, the long connections between the vertices of H.

By (3.11), we know that the edges of $C_n - E(H)$ form internally vertex-disjoint paths $P_1, P_2, \dots, P_{k'}$ such that for all $i \in [k']$ the first vertex of P_i is mapped to $a_i \in V_1$, while the last vertex of P_i is embedded in $b_i \in V_\ell$. By the choice of V_1' and V_ℓ', we know that, for each $i \in [k'] = \{1, \dots, k'\}$,

$$\begin{aligned} &a_i \text{ is } \varepsilon\text{-typical with respect to } V_2, \\ &\text{and } b_i \text{ is } \varepsilon\text{-typical with respect to } V_{\ell-1}. \end{aligned} \tag{3.16}$$

We now wish to employ Lemma 2.19 to embed the remaining part of H into G^r. We have just verified that (vii) and (viii) of Lemma 2.19 hold. Also, (ii) and (iii) are clear by construction.

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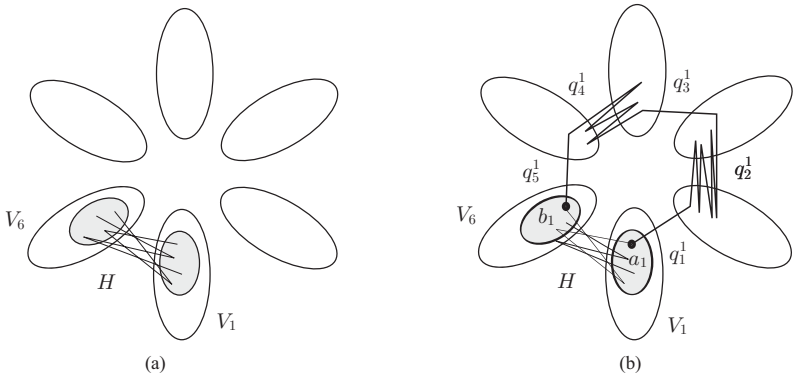


Figure 2. Our embedding of $C_n \cup D = H \cup \bigcup P_i$ in G^r. (a) First, the graph H is embedded in the pair (V_1, V_ℓ) via Lemma 2.15. (b) Then we use Lemma 2.19 to embed the long connections.

For (iv), note that by construction,

$$k' \leqslant |H| \overset{(3.10)}{<} \varepsilon^{1/2}|V_j|.$$

For (v), it suffices to observe that for each $i \in [k']$, inequality (3.12) implies that

$$p^i \geqslant z - 3 \geqslant 8M_{\mathrm{reg}} \geqslant 3\ell.$$

Let us next establish (vi). For this, it is enough to see that, as $M \geqslant n \geqslant 1/\varepsilon$, we have

$$\begin{aligned}
\sum_{i=1}^{k'} p^i &\leqslant n \\
&= \frac{2}{3}(N+1) \\
&\leqslant \frac{2}{3} \cdot \frac{N}{1-\varepsilon} \\
&\overset{(II)}{\leqslant} \frac{2}{3} \cdot \frac{|V_1|t}{(1-\varepsilon)^2} \\
&\leqslant \frac{2}{3} \cdot \frac{t}{1-3\varepsilon^{1/4}}(1-2\varepsilon^{1/4})|V_1| \\
&\leqslant \left(\frac{2}{3}(1+\varepsilon^{1/5})t - 2\right)(1-2\varepsilon^{1/4})|V_1| \\
&\overset{(A)}{\leqslant} (\ell-2)(1-2\varepsilon^{1/4})|V_1|.
\end{aligned}$$

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Finally, for (i) note that

$$\begin{aligned} 4\varepsilon^{1/2}|V_j| &\overset{(II)}{\geqslant} 4\varepsilon^{1/2}(1-\varepsilon)\frac{N}{t} \\ &\geqslant 4\varepsilon^{1/2}(1-\varepsilon)\frac{n_0}{t} \\ &\overset{(3.4)}{\geqslant} \frac{M_{\mathrm{reg}} n_{2.18}}{t} \\ &\overset{(I)}{\geqslant} n_{2.18}. \end{aligned}$$

This means we can use Lemma 2.19 as planned and thus finish our embedding in this case.

3.4.2. Case (B). We assume the colour i is red and colour $3-i$ to be blue. Let A_G denote the union of all vertices that belong to clusters in A. We estimate the number of blue edges in A_G as follows. There are

- at most $|A| \cdot \binom{n/t}{2}$ blue edges within clusters in A,
- at most $|A| \cdot \varepsilon t \cdot (n/t)^2$ blue edges in irregular pairs in A,
- at most $\xi t|A| \cdot (n/t)^2$ blue edges in pairs corresponding to blue edges in A,
- at most $\binom{|A|}{2} \cdot d(n/t)^2$ blue edges in the remaining pairs in A (those which are coloured only by red).

Using the facts that $|A| \leqslant t$, that $1/t < \varepsilon$, and that $\xi = \varepsilon^{1/100} \leqslant d/4$, we find that A_G contains in total at most

$$\left(\frac{\varepsilon}{2} + \varepsilon + \xi + \frac{d}{2}\right) n^2 \leqslant dn^2$$

blue edges. Since the size of A_G is bounded by

$$|A_G| \geqslant |A| \cdot \frac{(1-\varepsilon)N}{t} \overset{(B3)}{\geqslant} (1-\xi)(1-\varepsilon)n,$$

we obtain that A_G contains a subset A' of size at least $|A_G| - 2\sqrt{d}n \geqslant (1-3d^{1/2})n$ such that, in G^r, each vertex of A' has degree at least $|A'| - d^{1/2}n$.

Let C be the set of all vertices in $V(G) \setminus A'$ that send at least $6\Delta d^{1/2}n$ red edges to A' and suppose that $|A'| + |C| \geqslant n$. Since

$$2d^{1/2} \overset{(3.2)}{<} \frac{2}{\Delta^{20}} < \frac{1}{\Delta^2 + 4},$$

we may use Lemma 2.5 and conclude that the red graph $G^r[A' \cup C]$ contains $C_n \cup D$.

So let us from now on assume that $|A'| + |C| < n$. Setting $B' := V(G) \setminus (A' \cup C)$, this means that $|B'| \geqslant n/2$. Moreover, by definition of B', each vertex from B' sends at least $|A'| - 6\Delta d^{1/2}n$ blue edges to A'. We embed $C_n \cup D$ in $G^b[A', B']$ as follows.

Say $C_n \cup D$ has the bipartition classes X and Y. We know that $|X| = |Y| = |C_n|/2$. Now, embed all of X into B' arbitrarily. Then, we embed the vertices in Y successively in the following way. Say we wish to embed some vertex $y \in Y$. Vertex y has at most Δ

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neighbours in X. The images of these neighbours in B' have at least $|A'| - \Delta \cdot 6\Delta d^{1/2}n > n/2$ common blue neighbours in A'. Since we have used at most $|Y| - 1 < n/2$ vertices of A' in earlier steps, there is a vertex in A' we may use to embed y. Thus we manage to embed all of Y as desired in A', which finishes the embedding of $C_n \cup D$.

3.5. Odd n

In this case, we wish to apply Theorem 2.7 with parameters $\gamma := d^{1/16}$, $\bar{m} := t/2$, and $\bar{M} := t$ to the graph $G := R$. From (I) and (3.3), we get that $t > n_{\text{odd}}$, and, clearly, we have $\bar{M} \geqslant (3/2 + 80\gamma)\bar{m}$. Finally, the fact that $\delta(G) \geqslant M - \beta m \geqslant \bar{M} - \gamma\bar{m}$ follows from (3.14).

Thus Theorem 2.7 yields one of the following substructures:

(C) R has a monochromatic odd cycle C_ℓ with $\ell \geqslant (1 + d^{1/16})t/2$, or

(D) there is a colour $i \in \{1, 2\}$, and sets $A, B \subseteq V(R)$ so that

(D1) all edges in $R[A]$ and in $R[B]$ have only colour i,

(D2) all edges between A and B have only colour $3 - i$, and

(D3) $|A|, |B| \geqslant (1 - d^{1/16})t/2$.

Depending on which case we obtained, we split our proof further into two cases.

3.5.1. Case (C). Suppose that C_ℓ is red and contains the vertices $V_1, V_2, \ldots, V_\ell, V_1$ (in this order). Now, we split our proof further depending on whether or not C_ℓ contains a pair (V_i, V_{i+1}) with large density in G^r.

3.5.1.1. *C_ℓ has a pair (V_i, V_{i+2}) of red density at least d.*

Say (V_ℓ, V_2) is a pair of density at least d in G^r. Our plan is to embed H into the 3-partite subgraph of G^r induced by V_ℓ, V_1, V_2 such that all vertices of H that have neighbours in $G - H$ are mapped to vertices of V_ℓ or V_2 that are typical in red with respect to $V_{\ell-1}$ or to V_3, respectively.

However, the pair (V_ℓ, V_2) may not be ε-regular.[1] Hence, we need to employ Lemma 2.17 instead of Lemma 2.15. For this, let $V'_\ell \subseteq V_\ell$ and $V'_2 \subseteq V_2$ consist of all vertices that are ε-typical in G^r with respect to $V_{\ell-1}$ or to V_3, respectively. Then, by Fact 2.12, we have

$$\lambda := \min\{|V'_2|, |V'_\ell|\} \geqslant (1 - \varepsilon)|V_2|,$$

further, from the definition of ε-regularity we have that the pairs (V'_ℓ, V_1) and (V_1, V'_2) are 2ε-regular. In G^r, the pair (V_ℓ, V_2) has density at least d, therefore, the pair (V'_ℓ, V'_2) has density at least

$$\frac{d|V_\ell||V_2| - 2\varepsilon|V_\ell||V_2|}{|V'_\ell||V'_2|} \geqslant d - 2\varepsilon.$$

It follows from (3.1) and (3.2) that

$$2\varepsilon \leqslant \frac{(d - 2\varepsilon)^{2\Delta}}{(2\Delta)^{4\Delta+1}},$$

and inequality (3.15) from the even case implies that $|H| \leqslant 2\Delta \cdot 2\varepsilon \cdot \lambda$.

[1] As a matter of fact, it is possible that none of the pairs $(V_1, V_3), (V_4, V_6), \ldots, (V_\ell, V_2)$ is ε-regular.

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Recall that the tripartition classes of H were U_1, U_2 and U_3 such that (see (3.11)) $C_n - E(H)$ is a collection of internally vertex-disjoint odd U_1–U_2-paths. Applying Lemma 2.17 with parameters Δ, $p := d - 2\varepsilon$, $\varepsilon' := 2\varepsilon$, λ, sets $X_1 := V'_\ell$, $X_2 := V'_2$ and $X_3 := V_1$, and graphs $G := G^r$, H yields an embedding of H into G that maps U_1 to $X_1 = V'_\ell$, U_2 to $X_2 = V'_2$ and U_3 to $X_3 = V_1$.

We will now embed the rest of $C_n \cup D$, that is, the long connections between the vertices of H. We plan to use Lemma 2.19 on G^r with $\ell^{\text{Lem. 2.19}} := \ell - 1$ and $V_j^{\text{Lem. 2.19}} := V_{j+1}$.

By (3.11), we know that the edges of $C_n - E(H)$ form internally vertex-disjoint odd paths $P_1, P_2, \dots, P_{k'}$ such that, for all $i \in [k']$, the first vertex of P_i is mapped to some $a_i \in V_2$, while the last vertex of P_i is embedded to some $b_i \in V_\ell$. By the choice of V'_2 and V'_ℓ, we know that, for each $i \in [k']$, we have that

$$a_i \text{ is } \varepsilon\text{-typical to } V_3 \text{ and } b_i \text{ is } \varepsilon\text{-typical to } V_{\ell-1}. \tag{3.17}$$

This means that conditions (ii), (iii), (vii) and (viii) of Lemma 2.19 are satisfied. Conditions (i), (iv) and (v) follow as in the even case.

For (vi), we calculate similarly as in the even case:

$$\begin{aligned}\sum_{i=1}^{k'} p^i &\leqslant n \\ &\leqslant \frac{1}{2}(N+1) \\ &\leqslant \left(\frac{1}{2}(1+d^{1/16})t - 2\right)(1-2\varepsilon^{1/4})|V_1| \\ &\overset{(C)}{\leqslant} (\ell-2)(1-2\varepsilon^{1/4})|V_1|.\end{aligned}$$

Thus, with the help of Lemma 2.19 we may complete the embedding of H to an embedding of $C_n \cup D$.

3.5.1.2. All pairs (V_i, V_{i+2}) have red density less than d.
In this case, all pairs (V_i, V_{i+2}) have blue density at least $1 - d$. By Fact 2.10, all these pairs are $d^{1/3}$-regular, and $V_1, V_3, \dots, V_{\ell-1}, V_1$ form a blue C'_ℓ. Let $V'_1 \subseteq V_1$ be the set of all vertices in V_1 that are typical in red with respect to V_ℓ and V_2, and typical in blue with respect to $V_{\ell-1}$ and V_3. Since $|V'_1| \geqslant |V_1| - 2\varepsilon|V_1| - 2d^{1/3}|V_1|$ by Fact 2.12, by (3.10), and by Theorem 2.6, we have

$$\begin{aligned}|V'_1| &\geqslant (1 - 2\varepsilon - 2d^{1/3})|V_1| \geqslant (1 - 3d^{1/3})\frac{N}{t} \geqslant \frac{n}{M_{\text{reg}}} \\ &\geqslant 2^{c_2\Delta(\log\Delta)^2} \cdot 12\frac{\varepsilon^2 n}{M_{\text{reg}}} \geqslant 2^{c_2\Delta(\log\Delta)^2}|H| \geqslant r(H).\end{aligned}$$

Hence, by Theorem 2.6, we find a monochromatic copy of H inside V'_1.

Now only have to embed the long connections (in G^r if our copy of H is red, and in G^b if our copy of H is blue). This will be done as before, either in C_ℓ or in C'_ℓ, with the help of Lemma 2.19.

If the monochromatic copy of H inside V'_1 is red, then the input for Lemma 2.19 is $\ell^{\text{Lem. 2.19}} := \ell + 1$ and $V_j^{\text{Lem. 2.19}} := V_j$ for $j \in [\ell + 1]$, where we let $V_{\ell+j} := V_j$ for every j.

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Conditions (i)–(viii) of Lemma 2.19 are seen to hold as above, so we can finish our embedding as planned.

If the monochromatic copy of H inside V_1' is blue, then the input for Lemma 2.19 is $\ell^{\text{Lem. 2.19}} := \ell + 1$ and $V_j^{\text{Lem. 2.19}} := V_{1+2(j-1)}$ for $j \in [\ell+1]$, where we let $V_{\ell+j} := V_j$ for every j.

Conditions (i)–(viii) of Lemma 2.19 (with ε replaced by $d^{1/3}$) are seen to hold as before, with a bit of extra care for (vi):

$$\begin{aligned}\sum_{i=1}^{k'} p^i &\leqslant n \\ &\leqslant \frac{1}{2}(N+1) \\ &\leqslant \left(\frac{1}{2}(1+d^{1/16})t - 2\right)(1-2d^{1/12})|V_1| \\ &\overset{\text{(C)}}{\leqslant} (\ell-2)(1-2d^{1/12})|V_1|.\end{aligned}$$

3.5.2. Case (D). Assume again that colour i is red and $3-i$ is blue.

Again, let A_G, B_G denote the union of all vertices that belong to clusters in A, or in B, respectively. We have that

$$|A_G| \geqslant |A| \cdot \frac{(1-\varepsilon)N}{t} \geqslant (1-\varepsilon)(1-d^{1/16})\frac{2n-1+k}{2} \geqslant (1-2d^{1/16})n.$$

Similarly, $|B_G| \geqslant (1-2d^{1/16})n$.

Now we shall estimate the number of blue edges in $G[A_G]$ and $G[B_G]$ and the number of red edges in $G[A_G, B_G]$. There are

- at most $|A| \cdot \binom{n/t}{2}$ blue edges within the clusters in A,
- at most $|A| \cdot \varepsilon t \cdot (n/t)^2$ blue edges in irregular pairs in A (*i.e.*, non-edges of $R[A]$, here we use (III)),
- at most $\binom{|A|}{2} \cdot d(n/t)^2$ blue edges in the remaining pairs in A (here we use (D1)).

Using the facts that $|A| \leqslant t$, that $1/t < \varepsilon$, and that $\varepsilon < d/4$, we find that A_G contains in total at most

$$\left(\frac{\varepsilon}{2} + \varepsilon + \frac{d}{2}\right)n^2 \leqslant dn^2$$

blue edges. In the same way we obtain that B_G contains at most dn^2 blue edges.

As for the red edges in $G[A_G, B_G]$, there are

- at most $|A| \cdot \varepsilon t \cdot (n/t)^2$ red edges in irregular pairs corresponding to non-edges in $R[A,B]$, and
- at most $|A| \cdot |B| \cdot d(n/t)^2$ red edges in the pairs corresponding to edges in $R[A,B]$ (here we use (D2)).

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Using the facts that $|A|, |B| \leqslant t$, that $1/\varepsilon < t$, and that $\varepsilon < d/4$, we find that $G[A_G, B_G]$ contains at most

$$(\varepsilon + d)n^2 \leqslant 2dn^2$$

red edges. We remove at most $3\sqrt{d}n$ vertices with degree at least $\sqrt{d}n$ in one of $G^b[A]$, $G^b[B]$, and $G^r[A, B]$, and obtain sets $A' \subset A_G$ and $B' \subset B_G$ such that

$$|A'|, |B'| \geqslant (1 - 2d^{1/16} - 3\sqrt{d})n \geqslant (1 - 3d^{1/16})n$$

and such that the graphs $G^b[A']$, $G^b[B']$, and $G^r[A', B']$ have maximum degree at most $\sqrt{d}n$.

Set $\varepsilon' = 3d^{1/16}$, and let C_A and C_B be the sets of all vertices in $V(G) \setminus (A' \cup B')$ that send at least $3\Delta\varepsilon' n$ red edges to A' or to B', respectively. If $|C_A| + |A'| \geqslant n$, then we use Lemma 2.5 (note that $\varepsilon' < 1/(\Delta^2 + 4)$ by (3.2)) and conclude that the red graph $G^r[A' \cup C_A]$ contains $C_n \cup D$.

So let us from now on assume that $|C_A| + |A'|, |C_B| + |B'| < n$. This means that there is a set S of

$$2n - 1 + k - 1 - \big(|C_A| + |A'| + |C_B| + |B'|\big) \geqslant k$$

vertices which each send at least $|A'| - 3\Delta\varepsilon' n$ blue edges to A' and at least $|B'| - 3\Delta\varepsilon' n$ blue edges to B'. Let A^S and B^S be the set of their common blue neighbours in A', or in B', respectively. Note that

$$|A^S|, |B^S| \geqslant (1 - \varepsilon')n - k \cdot 3\Delta\varepsilon' n = \big(1 - (3k\Delta + 1)\varepsilon'\big)n \tag{3.18}$$

$$\geqslant n/2, \tag{3.19}$$

where the last inequality follows from (3.1).

We embed $C_n \cup D$ as follows: let Z be the set of k independent vertices such that $(C_n \cup D) - Z$ is a bipartite graph with bipartition X and Y. Clearly, the sets X and Y differ in size by at most k, and therefore we have that $|X|, |Y| \leqslant n/2$.

Embed all of Z into S and all of X into B^S arbitrarily ((3.19) ensures that X fits into B^S). Then embed the vertices in Y successively, as in the even case. For each vertex $y \in Y$ that we wish to embed in A^S, we know that the images of its at most Δ neighbours have at least

$$|A^S| - \Delta \cdot \sqrt{d}n \overset{(3.18)}{\geqslant} \big(1 - 4k\Delta\varepsilon' - \sqrt{d}\Delta\big)n > 0$$

common neighbours in A^S (the last inequality follows from (3.1) and from (3.2)), and thus there is space for embedding y properly.

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