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# Finite approximations to infinite non-negative matrices

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1. *Introduction.* In applying the theory of infinite Markov chains to practical examples, it is important to know how the ergodic properties defined by the infinite stochastic or substochastic matrix under consideration are related to those of the $n \times n$ $(n = 1, 2, 3, \ldots)$ truncated corner sub-matrices. In particular, it is of interest whether the relevant eigenvalues and eigenvectors of the truncated matrices in some sense approximate to corresponding quantities for the infinite matrix as $n \to \infty$.

The technique of considering truncated matrices to obtain properties of the corresponding infinite matrix is well known in slightly different but sometimes related situations. For instance, in the theory of continuous-time Markov chains with a denumerable infinity of states, Reuter and Ledermann (6) have used truncated $n \times n$ matrices of the transition intensity matrix to construct a solution to the corresponding Kolmogorov differential equations, and discuss its uniqueness. These authors use essentially the same technique in a later paper (3) to give a detailed discussion of the spectral theory for birth-and-death processes. Mandl (4) also utilizes similar truncations to give a simple discussion of the ergodic properties of these processes, and to derive a simple sufficient condition for uniqueness of solution in the determining equations. However, it is apparent that the approach of these authors centres on differential equation properties and/or the special form of the intensity matrix, rather than eigenvector properties, and, in any case, will not in general carry over to the discrete-time case.

In the case of present interest, viz. discrete space and time, the method of truncations is rare. The most general study appears to have been made by Sarymsakov ((7), Sections 22–24) who, by considering the matrix of a chain consisting of a single non-cyclic essential class of states, and imposing the further restriction that all $n \times n$ truncated matrices except for a finite number, be also irreducible, gives several theorems. Two of these are proved below in clarified and modified form. His general approach relies on considering the determinant-cofactor properties of the truncated matrices as $n \to \infty$, which, however, leads to several difficulties as regards existence and uniqueness. The other references to the problem are only of a peripheral nature. Thus Beckmann, McGuire and Winsten ((1), page 42) state in reference to a special type of queueing matrix, that 'The matrix is infinite, so it has to be truncated. It can be proved that this type of truncation can still give approximations to the ergodic probabilities of the infinite matrix, however.' Finally, it is relevant to mention the well-known paper of Kac (2) who has used a finite approximation approach to certain random-walk processes.

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The present note is concerned with several aspects of this general problem. We shall not at first restrict ourselves to a stochastic or substochastic matrix, but shall work under the assumption that the infinite non-negative matrix is irreducible, that all its positive integral powers exist, that its common convergence parameter, $R$, is positive, and that all but a finite number of its corner truncations are also irreducible. This last condition is a non-vacuous one in the sense that simple infinite irreducible matrices may be found for which it does not hold. It is also relevant that the above assumptions do not exclude the possibility of cyclicity, and that they remain true with rows and columns interchanged (*row-column duality*). Under the stated conditions, we find that the extension of the Perron–Frobenius theory developed by Vere-Jones (**11**, **12**), Pruitt (**5**) *et al.* to infinite non-negative matrices is readily applicable, and yields several results of interest.

2. *Preliminary results.* Let $T = \{t_{ij}\}$ $(i, j = 1, 2, 3, \ldots)$ be a matrix with non-negative entries such that $T^m = \{t_{ij}^{(m)}\}$ exists for all positive integral $m$. We shall call $T$ irreducible if for every $i, j$ there exists a finite sequence of integers $k_1, k_2, \ldots, k_n$ such that

$$t_{ik_1} t_{k_1 k_2} \ldots t_{k_n j} > 0.$$

The period of the index $i$, $d_i$, is the greatest common divisor of those integers $m$ for which $t_{ii}^{(m)} > 0$. For an irreducible $T$, it is well known that $d_i \equiv d$ for all $i$, and $d$ is then called the period of the matrix.

Let us denote, further, by ${}_{(n)}T = \{{}_{(n)}t_{ij}\}$ $(i, j = 1, 2, \ldots, n)$ the $n \times n$ 'north-west' corner truncation of $T$. In place of assuming ${}_{(n)}T$ is irreducible for each $n$ apart from a finite number, we shall henceforth assume in fact that ${}_{(n)}T$ is irreducible for all $n = 1, 2, 3, \ldots$ for simplicity. We do this without loss of generality, since all our results are concerned essentially with sufficiently large $n$, and are very easily modified if necessary to include the previous case. For convenience, we adopt the convention that ${}_{(n)}t_{ij} = 0$ if $i$ or $j > n$. We then have the following preliminary lemmas, partly attributable to Sarymsakov ((**7**), Section 22)

LEMMA 2·1. *For* $m = 1, 2, \ldots$ $\lim_{n\to\infty} {}_{(n)}t_{ij}^{(m)} = t_{ij}^{(m)}$.

*Proof.* Since ${}_{(n)}t_{ij}^{(m)} \leqslant {}_{(n+1)}t_{ij}^{(m)} \leqslant t_{ij}^{(m)}$, $\lim_{n\to\infty} {}_{(n)}t_{ij}^{(m)}$ exists for $m = 1, 2, \ldots$; moreover, obviously $\lim_{n\to\infty} {}_{(n)}t_{ij}^{(1)} = t_{ij}$. Then, since

$${}_{(n)}t_{ij}^{(m+1)} = \sum_{k=1}^{\infty} {}_{(n)}t_{ik}^{(1)} \, {}_{(n)}t_{kj}^{(m)}$$

by dominated convergence and induction

$$\lim_{n\to\infty} {}_{(n)}t_{ij}^{(m+1)} = \sum_{k=1}^{\infty} t_{ik} t_{kj}^{(m)} = t_{ij}^{(m+1)}.$$

LEMMA 2·2. *The period* $d$ *of* $T$ *is also the period of* ${}_{(n)}T$, *for all sufficiently large* $n$.

*Proof.* For an irreducible matrix $T$ of period $d$, it is well known that $t_{ii}^{(md)} > 0$ for $m \geqslant M(i)$; hence $t_{11}^{(md)} > 0$ for $m \geqslant M(1)$. As a consequence of Lemma 2·1 therefore, ${}_{(n)}t_{11}^{(md)} > 0$, ${}_{(n)}t_{11}^{([m+1]d)} > 0$ for $n \geqslant N(m)$ and fixed $m \geqslant M(1)$. Hence if $d(n)$ is period of

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${}_{(n)}T$, $d(n) \leqslant d$, $n \geqslant N(m)$. Now since ${}_{(n)}t_{11}^{(md(n))} \leqslant t_{11}^{(md(n))}$, $d(n) \geqslant d$. Thus for $n \geqslant N(m)$, $d(n) = d$.

We now introduce the generating functions $T_{ij}(z) = \sum_{m=0}^{\infty} t_{ij}^{(m)} z^m$ ($t_{ij}^{(0)} = \delta_{ij}$ – Kronecker delta) $i, j = 1, 2, \ldots$ and the corresponding functions for the finite truncations. Then if we denote by $R_{ij}$ the radius of convergence of $T_{ij}(z)$, it is readily shown (Vere-Jones (11)) that $\infty > R_{ij} = R \geqslant 0$ for all $i, j$. For the irreducible finite matrices ${}_{(n)}T$ it is readily seen that $1/R_n$ is just the positive maximum-modulus eigenvalue guaranteed by the Perron–Frobenius theory. We shall assume, as previously mentioned, $R > 0$, which is always the case for a stochastic or substochastic matrix. The following theorem is then evident. *Note:* this theorem will be sharpened later; see Theorem 3·1.

THEOREM 2·1 (Sarymsakov). $R_n \downarrow R_\infty$, *where* $R \leqslant R_\infty < \infty$.

*Proof.* From the inequalities ${}_{(n)}t_{ij}^{(m)} \leqslant {}_{(n+1)}t_{ij}^{(m)} \leqslant t_{ij}^{(m)}$ we have that $\infty > R_n \geqslant R_{n+1} \geqslant R$. Hence the theorem follows by putting $\lim_{n\to\infty} R_n = R_\infty$.

To proceed we need to introduce some further quantities $f_{ij}^{(n)}$, $l_{ij}^{(n)}$ and their generating functions, following Vere-Jones (11, 12). Define $f_{ij}^{(0)} = l_{ij}^{(0)} = 0$, $f_{ij}^{(1)} = l_{ij}^{(1)} = t_{ij}$ and thereafter put inductively

$$f_{ij}^{(m+1)} = \sum_{k \neq j} t_{ik} f_{kj}^{(m)}, \qquad l_{ij}^{(m+1)} = \sum_{k \neq i} l_{ik}^{(m)} t_{kj},$$

$m = 1, 2, \ldots$. The corresponding quantities for the truncated matrices are as usual preceded by a subscript. It is easily seen that $L_{ij}(z) = \sum_{n=0}^{\infty} l_{ij}^{(n)} z^n$, $F_{ij}(z) = \sum_{n=0}^{\infty} f_{ij}^{(n)} z^n$ are convergent for $|z| < R$, and in particular that ($i \neq j$)

$$T_{jj}(z) F_{ij}(z) = T_{ii}(z) L_{ij}(z) = T_{ij}(z); \quad L_{ii}(z) = F_{ii}(z); \quad T_{ii}(z) = (1 - L_{ii}(z))^{-1} \qquad (2{\cdot}1)$$

for $|z| < R$. The following stated results due to Vere-Jones (12) will be required in the sequel: they apply in fact without the restriction of irreducibility on the finite truncations which we assume, but do require $R > 0$ and $T^m$ to exist for all positive integral $m$.

THEOREM 2·2. (A) *If for* $r > 0$, $\{x_i\}$ *is a non-zero solution to the left 'subinvariant' equation*

$$r \sum_{k=1}^{\infty} x_k t_{kj} \leqslant x_j$$

*and* $x_j \geqslant 0$, $j = 1, 2, 3, \ldots$, *then* $x_j > 0$ *for all* $j$ *and* $x_j/x_i \geqslant L_{ij}(r)$.

(B) $L_{ii}(r) \leqslant 1$ *for all* $i$, *and* $r \leqslant R$.

(C) For $r \leqslant R$, $L_{ij}(r) < \infty$ *for all* $i, j$ *and for fixed* $i$, $\{L_{ij}(r)\}$ *constitutes a left-subinvariant vector corresponding to* $r$, *remaining subinvariant if* $L_{ii}(r)$ *is replaced by unity.*

(D) *No left subinvariant vectors can exist for* $r > R$.

Similar statements follow for a right subinvariant vector $\{y_i\}$ with $L_{ij}$ replaced by $F_{ij}$.

From (B) it is clear that $L_{ii}(R) \leqslant 1$ for every $i = 1, 2, \ldots$. If $L_{ii}(R) < 1$ for some $i$, it can be shown that $L_{ii}(R) < 1$ for all $i$. Thus, following Vere-Jones, we may introduce the following classification of the matrices $T$: if $L_{ii}(R) < 1$ then $T$ is *R-transient*, whereas if $L_{ii}(R) = 1$, $T$ is *R-recurrent*. The $R$-recurrent class can be further subdivided

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by use of a generalization of the Erdos–Feller–Pollard theorem for Markov chains which states that if

$$L'_{ii}(R) = \infty,\ t_{ii}^{(n)}R^n \to 0 \text{ as } n \to \infty, \quad \text{and if} \quad L'_{ii}(R) < \infty,\ t_{ii}^{(nd)}R^{nd} \to d/RL'_{ii}(R),$$

(with corresponding statements for off diagonal elements $t_{ij}^{(n)}$). In the first case the $R$-recurrent matrix is called *R-null* and in the second case *R-positive* (note if $T$ is $R$-transient $t_{ii}^{(n)}R^n \to 0$). It is clear that for any finite irreducible truncation ${}_{(n)}T$,

$${}_{(n)}L_{ii}(R_n) = 1 \tag{2·2}$$

from the Perron–Frobenius theory, irrespective of the nature of $T$ (for ${}_{(n)}T$ is clearly $R$-positive).

It is relevant at this point to remark on the relation between the above classification, and that of Markov chains consisting of a single essential class of states, i.e. possessing an irreducible stochastic matrix $P$. The chain is said to be transient if $L_{ii}(1) < 1$ for some $i$ (in which case $p_{ij}^{(n)} \to 0$ for all $i$ and $j = 1, 2, \ldots$) and recurrent if, for some $i$, $L_{11}(1) = 1$. In the latter case, if also $L'_{ii}(1) = \infty$ for some $i$, the chain is null-recurrent, and $p_{ij}^{(n)} \to 0$ for all $i$ and $j$, but if $L'_{ii}(1) < \infty$ the chain is positive-recurrent and $p_{ii}^{(nd)} \to d/L'_{ii}(1) > 0$ for all $i$, with an analogous statement for all $p_{ij}^{(n)}$ where $d$ is the period of the transition matrix. Since $\mathbf{e} = (1, 1, \ldots)$ is a right subinvariant vector corresponding to $r = 1$, it follows from Theorem 2·2D that $R \geqslant 1$, so that our condition that $R > 0$ is always satisfied. Moreover, it is immediately clear that a null-recurrent chain has an $R$-null matrix and a positive-recurrent chain has an $R$-positive matrix with $R = 1$, and hence $R > 1$ may only occur if the chain is transient: in fact the matrix may sometimes be $R$-positive (Vere-Jones (11)). Thus it is clear that our theorems will apply to irreducible stochastic matrices $P$, with only the additional condition of only a finite number, at most, of reducible truncations.

To conclude this section we state another theorem (Vere-Jones (12), Pruitt (5)) which we shall require.

THEOREM 2·3. *Suppose $T$ is $R$-recurrent and $\{x_i\}$ is a non-trivial non-negative solution of the left subinvariant equations for $r = R > 0$, viz. $R\sum_i x_i t_{ij} \leqslant x_j$ $(j = 1, 2, \ldots)$. Then $\{x_i\}$ is strictly positive and invariant, and is unique up to a constant multiplier.*

3. *General limit theorems.* The following theorems answer several problems on the convergence of largest eigenvalues and their corresponding left eigenvectors. Analogous results for right eigenvectors can be proved similarly.

THEOREM 3·1. *In Theorem 2·1, in fact $R_\infty = R$.*

*Proof.* From their defining relations, in the same way as in the proof of Lemma 2·1, ${}_{(n)}l_{ij}^{(m)} \uparrow l_{ij}^{(m)}$ as $n \to \infty$. From Theorem 2·1, for all $n$, $R_n \downarrow R_\infty$, so that from Theorem 2·2 (B), ${}_{(n)}L_{ii}(R_\infty) \leqslant 1$. From Fatou's lemma, therefore, $L_{ii}(R_\infty) \leqslant 1$. Suppose $R_\infty > R$; then take $R^*$ such that $R_\infty > R^* > R$. Clearly $L_{ii}(R^*) < 1$, so that

$$T_{ii}(R^*) = (1 - L_{ii}(R^*))^{-1} < \infty,$$

which contradicts $R$ as the radius of convergence of $T_{ii}(z)$. Hence $R_\infty = R$. (A less direct proof could be obtained by proceeding initially as in Theorem 3·2, and invoking Theorem 2·2 (D).)

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Sarymsakov ((7), Section 23) was not able to identify $R_\infty$. Dealing with the stochastic case he concluded essentially that $R_\infty \geqslant 1$ and if $R_\infty > 1$, $t_{ii}^{(m)} \to 0$ as $m \to \infty$. This result follows immediately from the above considerations. Moreover, from the above theorem and Lemma 2·2 we have the immediate corollary that if $T$ has period $d$, then the $d$ maximum-modulus eigenvalues of the matrices ${}_{(n)}T$ for sufficiently large $n$ each converge: viz. $1/R_n\, e^{2\pi ik/d} \to 1/R\, e^{2\pi ik/d}$, $k = 0, 1, 2, \ldots, d-1$ as $n \to \infty$. The latter quantities have been studied by Šidak (9) and Vere-Jones ((12), Section 7). If $1/R$ is an eigenvalue (in the sense that there exists a left $R$-invariant (eigen) vector, then they are obviously eigenvalues of $T$.

Let us write $\{{}_{(n)}v_k\}$ $(k = 1, 2, \ldots, n)$ for the unique positive left eigenvector corresponding to the eigenvalue $1/R_n$ for the irreducible matrix ${}_{(n)}T$, and put ${}_{(n)}v_1 = 1$ for all $n$.

THEOREM 3·2. *The vector* $\{v_k\}$, $k = 1, 2, \ldots$ *where* $v_k = \liminf_{n\to\infty} {}_{(n)}v_k$ $(v_1 = 1)$ *is a positive left subinvariant vector of* $T$ *corresponding to* $R$. *If* $T$ *is* $R$*-recurrent, then*

$$\lim_{n\to\infty} {}_{(n)}v_k = v_k, \quad k = 1, 2, 3, \ldots,$$

*exists, and* $\{v_k\}$ *constitutes the unique positive invariant vector of* $T$ *with* $v_1 = 1$ *corresponding to* $R$.

*Proof.*

$$R_n \sum_{k=1}^{\infty} {}_{(n)}v_k \cdot {}_{(n)}t_{kj} = {}_{(n)}v_j.$$

Since $R_n \downarrow R > 0$ we have from Fatou's lemma and the above that

$$R \sum_{k=1}^{\infty} v_k . t_{kj} \leqslant v_j$$

(since $v_1 = 1$ and $T$ is irreducible with $R > 0$, necessarily $0 < v_k < \infty$).

If $T$ is $R$-recurrent, it follows from Theorem 2·3 that $\{v_k\}$ is the unique positive strictly invariant left eigenvector with $v_1 = 1$ corresponding to the eigenvalue $1/R$. To prove that in this case $v_k$ is the limit of ${}_{(n)}v_k$, suppose $s > 1$ is the first number for which

$$\limsup_{n\to\infty} {}_{(n)}v_s > \liminf_{n\to\infty} {}_{(n)}v_s.$$

Then choosing a sequence $\{n_{k(s)}\}$ for which

$$\lim_{n_{k(s)}\to\infty} {}_{(n)}v_s = \limsup_{n\to\infty} {}_{(n)}v_s = v_s^*$$

we could construct as above a left subinvariant vector corresponding to $R$ which was non-negative, with first element unity, but with $v_s^* \neq v_s$. This is impossible by Theorem 2·3, which completes the proof.

It is relevant to note that the first part of this theorem confirms the existence of a left-subinvariant vector for $r = R$, even for an $R$-transient matrix, a fact which is well known. The general existence of a left-subinvariant vector $\{v_k\}$ corresponding to $R$ under our special assumption of irreducible truncations implies that for every finite $n$, $R_n > R$, a result which we shall need. It may be proved by realizing that the matrix $\{t_{ij}^*\}$ (where $t_{ij}^* = Rv_j t_{ji}/v_i$) is substochastic, with $R^* = 1$, while its finite truncations, being strictly substochastic†, must have their convergence radii $R_n^* = R_n/R > 1$.

† Stochasticity of a finite truncation would render $\{t_{ij}^*\}$, and so $T$, reducible, which is in contradiction to our assumptions.



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It is also of interest to consider whether an essentially different normalization of the finite eigenvectors in the above theorem will produce the same result. In particular, suppose we know that $T$ is $R$-positive and has convergent left eigenvector corresponding to $1/R$ so that without loss of generality $\Sigma v_k = 1$(cf. Markov chains). Then it can be shown that the ${}_{(n)}v_k$, normalized so that $\sum_{k=1}^{n} {}_{(n)}v_k = 1$, approach $v_k$ as $n \to \infty$, if the matrix $T$ is, in addition, aperiodic and *geometrically ergodic* (see Vere-Jones (11)). The result follows essentially from the fact that geometric ergodicity implies each of the generating functions $L_{ij}(z)$ is analytic within a disc $|z| < R'$ where $R' > R$ for all $i, j$, and as a consequence $L_{ij}(R_n) < \infty$ for sufficiently large $n$. However, if the behaviour of the matrix is sufficiently well known to yield geometric ergodicity, then, generally speaking, this result is of little practical importance, and we omit a detailed proof.

The present section is concluded by the following theorem, derived from an idea of Sarymsakov ((7), Section 23). By ${}_{(n)}c_{ji}$ we denote the cofactor of the $(j, i)$ element of the $n \times n$ matrix $[{}_{(n)}I - R \cdot {}_{(n)}T]$, and by $\Delta(n)$ the determinant of this matrix, where ${}_{(n)}I$ is the $n \times n$ unit matrix ($\Delta(n) > 0$ since $R_n > R$)†.

THEOREM 3·3. (A) *As* $n \to \infty$ ${}_{(n)}c_{ji}/{}_{(n)}c_{ii} \uparrow v_{ij} > 0$ $(i, j = 1, 2, \ldots)$, *and for fixed* $i$ $v_{ij}$ *forms a positive left subinvariant vector of* $T$ *corresponding to* $R$. $(v_{ij} = L_{ij}(R), i \neq j.)$

(B) $\lim_{n\to\infty} \Delta(n) \geqslant 0$ *exists; if* $\lim \Delta(n) > 0$, $T$ *is* $R$*-transient. For an* $R$*-transient matrix* $\Delta(n)$ *and* ${}_{(n)}c_{ij}$ *have positive or zero limit together.*

*Proof.* Since $R_n > R$, ${}_{(n)}c_{ji} = \Delta(n) \times {}_{(n)}T_{ij}(R) > 0$ for $i, j = 1, 2, \ldots, n$. From (2·1) $(i \neq j)$

$$\left.\begin{aligned} {}_{(n)}L_{ij}(R) \cdot \Delta(n) &= {}_{(n)}c_{ji}[1 - {}_{(n)}L_{ii}(R)], \\ \Delta(n) &= {}_{(n)}c_{ii}[1 - {}_{(n)}L_{ii}(R)]. \end{aligned}\right\} \tag{3·1}$$

Hence ${}_{(n)}c_{ji}/{}_{(n)}c_{ii} = {}_{(n)}L_{ij}(R) \uparrow L_{ij}(R) < \infty$, $n \to \infty$. This inequality and the rest of assertion (A) follow from Theorem 2·2(C).

By cofactor expansion along the $(n+1)$th column for $j \leqslant n$

$${}_{(n+1)}c_{jj} = {}_{(n)}c_{jj} - R \sum_{\substack{k=1 \\ k\neq j}}^{n+1} t_{k,\,n+1} \cdot {}_{(n+1)}c_{jj|k,\,n+1},$$

where ${}_{(n+1)}c_{jj|k,\,n+1}$ is the $(k, n+1)$ cofactor of the matrix formed from

$$[{}_{(n+1)}I - R \cdot {}_{(n+1)}T]$$

by crossing out its $j$th row and column. The corresponding matrix formed from ${}_{(n+1)}T$ by crossing out its $j$th row and column is not obviously irreducible, but clearly its eigenvalues cannot exceed $1/R_{n+1}$ in modulus (consider its resolvent): hence

$${}_{(n+1)}c_{jj|k\ n+1,} \geqslant 0$$

as in the first step above. Thus $0 < {}_{(n+1)}c_{jj} \leqslant {}_{(n)}c_{jj}$, so that ${}_{(n)}c_{jj} \downarrow c_{jj} \geqslant 0$ (this is the definition of $c_{jj}$). Since ${}_{(n)}L_{ii}(R) \uparrow L_{ii}(R) \leqslant 1$ as $n \to \infty$, from (3·1)

$$\Delta(n) \downarrow c_{ii}[1 - L_{ii}(R)] \geqslant 0.$$

Hence $\lim \Delta(n) > 0 \Rightarrow L_{ii}(R) < 1$, $c_{ii} > 0$, so that $T$ is $R$-transient. This completes assertion (B).

† *The characteristic function* $|\lambda I - H|$ *(for any real square matrix* $H$*) is positive for real values of* $\lambda$ *exceeding its largest real root, if a real root exists.*

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It is also clear from the proof of the theorem that for fixed $i$, all $\lim_{n\to\infty} {}_{(n)}c_{ji} = c_{ji}$ exist and are positive or zero together, $j = 1, 2, \ldots$. In fact it now follows immediately by row-column duality that $\lim_{n\to\infty} {}_{(n)}c_{ji}$ for all $i, j$ are positive or zero together; and if all are positive then for fixed $i$ it is clear that the limits form a left subinvariant vector themselves. The following corollaries are important for irreducible Markov chains with stochastic transition matrices, $P$.

COROLLARY. (1) *$c_{ji}$ is independent of $i$ if the chain is recurrent.*

(2) $\lim_{n\to\infty} \det [{}_{(n)}I - {}_{(n)}P]$ *exists and is non-negative. A non-zero limit implies that the chain is transient, and $c'_{ji} > 0$ for all $i, j$, where the $c'_{ji}$ are corresponding cofactors.*

For the first part, we need to prove the statement only if all $c_{ji}$ are positive. It can be shown as in the theorem that $c_{ji} = c_{jj}F_{ij}(R)$ (here $R = 1$). From Theorems 2·2–2·3 therefore, the $c_{ji}$ for fixed $j$ form the unique positive right invariant vector corresponding to $R = 1$. But this must be a positive multiple of $\mathbf{e} = (1, 1, \ldots)$, and hence $c_{ji}$ is independent of $i$. We shall show with examples that both $c_{ji} > 0$ for all $i, j$ and $c_{ji} = 0$ for all $i, j$ can occur for both positive and null recurrent chains. The second part of the corollary follows immediately from the theorem, if the chain has $R = 1$; the only other possibility is for the chain to be transient and $R > 1$. In this case $R_n > R > 1$, hence it is clear from the body of the proof that the limit exists (and if positive, implies $L_{ii}(1) < 1$ as required). An example where the limit is positive will be exhibited.

It is clear that this theorem is most useful for Markov chains when we know beforehand that our chain is recurrent, since then $R = 1$, and the stationary distribution or measure is immediately computable†. The condition $\lim_{n\to\infty} \det [{}_{(n)}I - {}_{(n)}P] > 0$ is sufficient, but by no means necessary for transience. It will be seen to be a rather 'too strong' condition and it would certainly be of interest to investigate in more detail more intuitively obvious conditions under which it holds.

4. *Examples and applications to Markov chains.* (1) *Random walk with absorbing barrier at origin.* The random walk is assumed to take place on the non-negative integers $0, 1, 2, \ldots$ of which zero is an absorbing barrier. The submatrix corresponding to the transient states $1, 2, 3, \ldots$ is

$$T = \begin{pmatrix} 0 & b & 0 & 0 & . & . & . \\ a & 0 & b & 0 & . & . & . \\ 0 & a & 0 & b & & & \\ & . & . & . & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & . & . & . \end{pmatrix},$$

† Moreover, Prof. D. G. Kendall has pointed out to the author that then, if the invariant vector is $\{m_j\}$,

$$\frac{{}_{(n)}c_{ji}}{{}_{(n)}c_{ii}} \uparrow \frac{m_j}{m_i} \downarrow \left[\frac{{}_{(n)}c_{ji}}{{}_{(n)}c_{jj}}\right]^{-1} \quad \text{as} \quad n \to \infty,$$

from Theorem 3·3, which can thus be used to give *bounds* for $m_j : m_i$.



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where $a, b > 0$, $a+b = 1$, and is irreducible and periodic with period 2. Its $n \times n$ corner truncations for $n \geqslant 2$ are irreducible, and clearly have period 2 (Lemma 2·2). It can be shown $T$ is $R$-transient with $R = (4ab)^{-\frac{1}{2}}$, $a, b > 0$ and has, in this particular case, left and right eigenvectors $\{v_i\}, \{w_i\}$ corresponding to the eigenvalue $1/R$ where

$$v_j = v_1 j(b/a)^{\frac{1}{2}(j-1)}, \quad w_j = w_1 j(a/b)^{\frac{1}{2}(j-1)} \quad (j = 1, 2, \ldots)$$

(Seneta and Vere-Jones (8)).

The $n \times n$ truncation of this matrix for $n \geqslant 2$ has maximum-modulus eigenvalues $\pm 1/R_n$ and eigenvectors $\{{}_{(n)}v_j\}$, $\{{}_{(n)}w_j\}$ corresponding to $1/R_n$ given by

$$1/R_n = 2(ab)^{\frac{1}{2}} \cos(\pi/n+1),$$

$${}_{(n)}v_j = A_n(b/a)^{\frac{1}{2}(j-1)} \sin(\pi j/n+1),$$

$${}_{(n)}w_j = B_n(a/b)^{\frac{1}{2}(j-1)} \sin(\pi j/n+1)$$

for $j = 1, 2, \ldots, n$ (Takács (10), pages 68–69). Thus as $n \to \infty$

$$R_n - R \sim \pi^2/4(ab)^{\frac{1}{2}} n^2. \tag{4·1}$$

Renormalizing the vectors $\{{}_{(n)}v_j\}$, $\{{}_{(n)}w_j\}$, $\{v_j\}$, $\{w_j\}$ so that ${}_{(n)}v_1 = {}_{(n)}w_1 = v_1 = w_1 = 1$ (which will specify $A_n$, $B_n$, $w_1$, $v_1$ above) we have e.g.

$${}_{(n)}v_j - v_j \sim -j(j^2-1)(b/a)^{\frac{1}{2}(j-1)}\pi^2/6n^2 \quad \text{as} \quad n \to \infty, \quad \text{for} \quad j = 1, 2, 3, \ldots. \tag{4·2}$$

From (4·1) and (4·2) we obtain results predicted by Theorem 3·1 and the first part of Theorem 3·2, the rate of convergence being quite slow.

The general problems of rates of convergence are clearly very important in respect to the application of the preceding theorems; however, we do not pursue this topic further at present.

(2) *Random walk with elastic barrier at origin.* This Markov chain has an irreducible aperiodic transition matrix (and all its truncations irreducible and aperiodic) given by

$$P = \begin{pmatrix} a & b & 0 & 0 & . & . & . \\ a & 0 & b & 0 & . & . & . \\ 0 & a & 0 & b & & & \\ & . & . & . & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & . & . & . \end{pmatrix},$$

where $a$, $b > 0$, $a+b = 1$. It is well known that corresponding to the three cases $b < a$, $b = a$, $b < a$ the chain is positive-recurrent, null-recurrent, and transient respectively. In the latter case, consideration of $P_{11}(z)$ reveals that when $b > a$, $R = (4ab)^{-\frac{1}{2}} > 1$ and the matrix is $R$-transient. It can be shown that *in this case*

$$\Delta(n) = \det[{}_{(n)}I - R.{}_{(n)}P] = (1-Ra)\, n(\tfrac{1}{2})^{n-1} - abR^2(n-1)(\tfrac{1}{2})^{n-2}$$

so that $\Delta(n) \to 0$ as $n \to \infty$. Thus $\Delta(n)$ may have a zero limit when a matrix is $R$-transient. (It may also be shown that $\lim_{n\to\infty} [{}_{(n)}I - \theta.{}_{(n)}P] = 0$, for $0 < \theta < R$, in this

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example—which includes the case $\theta = 1$.) Moreover, it now follows from the theorem that $c_{(n)ij} \to 0$ for all $i, j$ in the transient case, whereas if the chain is recurrent

$$\begin{aligned} {}_{(n)}c_{jj} &= b^{j-1}.a^{n-j}.[(b/a)^{n-j+1}-1]/[b/a-1] \quad \text{if} \quad b < a \\ &= (n-j+1)\,(\tfrac{1}{2})^{n-1} \qquad\qquad\qquad\qquad\quad \text{if} \quad b = a \end{aligned}$$

so that as $n \to \infty$, ${}_{(n)}c_{jj} \to 0$ for all $j$, and hence ${}_{(n)}c_{ij} \to 0$ for all $i, j$.

(3) *Recurrent events.* In the Markov chain described by the transition matrix

$$P = \begin{pmatrix} f_1 & 1-f_1 & 0 & 0\ldots \\ \dfrac{f_2}{1-f_1} & 0 & \dfrac{1-f_1-f_2}{1-f_1} & 0\ldots \\ \dfrac{f_3}{1-f_1-f_2} & 0 & 0 & \dfrac{1-f_1-f_2-f_3}{1-f_1-f_2} \\ . & . & . & . \end{pmatrix}$$

the state 1 has a prescribed first-recurrence-time distribution,

$$\{f_n\}, \quad n = 1, 2, \ldots, \quad \sum_{n=1}^{\infty} f_n \leqslant 1.$$

If $\sum_{n=1}^{\infty} f_n = 1$ the state 1 is a recurrent event, and if $\sum_{n=1}^{\infty} f_n < 1$ it is transient. Let us suppose that $f_j > 0$ for all $j$, in which case $P$ is irreducible and acyclic (as are all its finite truncations); then the positive or null-recurrence, or transience of state 1 implies the corresponding property for the chain. It is easy to check that

$$\det\,[{}_{(n)}I - {}_{(n)}P] = 1-f_1-f_2-\ldots-f_n$$

$n = 1, 2, \ldots$, so that in this case $\det\,[{}_{(n)}I - {}_{(n)}P] \to 0$ *if and only if the chain is recurrent.* Moreover, for $j \leqslant n$, if $R = 1$,

$${}_{(n)}c_{jj} = 1-f_1-f_2-\ldots-f_{j-1}$$

($f_0 = 0$), whence it follows that when the chain is recurrent, $\lim_{n\to\infty} {}_{(n)}c_{jj}$ is positive. In fact, it is clear that in this case using the Corollary to Theorem 3·3, the unique stationary distribution or invariant measure, $\{v_j\}$ depending on whether we have positive or null recurrence, is given to a constant multiplier by

$$v_j = 1-f_1-f_2\ldots-f_{j-1} \quad (j = 1, 2, \ldots).$$

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