ON FRULLANI INTEGRALS

By K. S. K. IYENGAR

Communicated by G. H. Hardy

Received 14 November 1940

1. The object of this note is to obtain necessary and sufficient conditions for the existence of

$$I(a,b) = \int_0^\infty \frac{\phi(at)-\phi(bt)}{t}\,dt \quad (0<a<b),$$

on the assumption that ϕ is integrable in any finite positive interval not including zero ($0<\alpha\leqslant x\leqslant\beta<\infty$). On reference to the literature on the topic*, I find no exhaustive analysis of this problem anywhere.

If $\rho = a/b$, it is easy to show that

$$I(a,b) = I(\rho) = I_1(\rho) - I_2(\rho),$$

where

$$I_1(\rho) = \lim_{\epsilon\to 0}\int_{\epsilon\rho}^{\epsilon}\frac{\phi(t)}{t}\,dt, \quad I_2(\rho) = \lim_{\epsilon\to 0}\int_{\epsilon\rho}^{\epsilon}\frac{\phi(1/t)}{t}\,dt, \tag{1·1}$$

so that the existence of I is equivalent to the existence of two independent limits I_1 and I_2. The main theorem of this note is as follows.

A necessary and sufficient condition for the existence of $I_1(\rho)$, for all $\rho>0$, given that $\phi(t)$ is integrable in any finite positive interval not including zero, is the existence of the two limits

$$\text{(i)}\quad \lim_{\epsilon\to 0}\int_{\epsilon}^{1}\phi(t)\,dt, \qquad \text{(ii)}\quad \lim_{u\to 0}\frac{1}{u}\int_0^u\phi(t)\,dt. \tag{1·2}$$

2. The following are the results of my analysis.

(i) If $I_1(\rho)$ exists for all ρ in any finite interval $0<\rho_1\leqslant\rho\leqslant\rho_2$, then it exists for all $\rho>0$. (2·1)

(ii) If $|\phi(t)|\leqslant K$ near $t=0$, and $I_1(\rho)$ exists for two values ρ_1 and ρ_2 such that $\log\rho_1/\log\rho_2$ is irrational, then $I_1(\rho)$ exists for all $\rho>0$. (2·2)

(iii) In cases (i) and (ii), $I_1(\rho) = A\log\rho$, where A is a constant. (2·3)

(iv) If $I_1(\rho)$ exists for one value of ρ, say $\rho=\rho_1$, then

$$(a)\ \lim_{\epsilon\to 0}\int_{\epsilon}^{1}\phi(t)\,dt \text{ exists;} \tag{2·4a}$$

$$(b)\ \text{if } \psi(u) = \frac{1}{u}\int_0^u\phi(t)\,dt, \text{ then } \psi(u)-\psi(u\rho_1)\to 0 \text{ as } u\to 0; \tag{2·4b}$$

and

$$(c)\ I_1(\rho) = \lim_{\epsilon\to 0}\int_{\epsilon\rho}^{\epsilon}\frac{\psi(t)}{t}\,dt. \tag{2·4c}$$

* See, for example, Hardy, *Messenger of Math.* 34 (1905), 11–18; Bromwich, *Infinite series*, 2nd ed. (1926), 479; de la Vallée Poussin, *Cours d'analyse*, 6th ed. (1926), ii, 34.

https://doi.org/10.1017/S0305004100021484 Published online by Cambridge University Press

(v) The main theorem as stated in § 1.

Obviously similar theorems may be proved about $I_2(\rho)$ and $\phi(1/t)$.

3. *Proof of* (2·1). It is plain that, if $I_1(\rho)$ exists for $\rho = \rho_1$, and $\rho = \rho_2$, then

$$I(\rho_1\rho_2) = \lim_{\epsilon\to 0}\int_{\epsilon\rho_1}^{\epsilon} + \lim_{\epsilon\to 0}\int_{\epsilon\rho_1\rho_2}^{\epsilon\rho_1} = I(\rho_1)+I(\rho_2);$$

and, more generally, that

$$I_1(\rho_1^m\rho_2^n) = mI_1(\rho_1)+nI_2(\rho_2). \tag{3·1}$$

Suppose now that $I_1(\rho)$ exists for $0<\rho_1\leqslant\rho\leqslant\rho_2$. Since $I_1(1/\rho) = -I_1(\rho)$, by (3·1), we may without loss of generality take $0\leqslant\rho_1\leqslant\rho\leqslant\rho_2\leqslant 1$. By (3·1), $I_1(\rho)$ exists when $\rho = (\rho_2-x)/\rho_2$, where $0\leqslant x\leqslant\rho_2-\rho_1$, i.e. when $1\geqslant\rho\geqslant\rho_3$, where $\rho_3 = \rho_1/\rho_2$. Take $y = \rho\rho_3^n$, $n = 0, \pm 1, \pm 2, \ldots$; then y takes all values in $0<y<\infty$ and, by (3·1), $I_1(y)$ exists for all $y>0$.

4. *Proof of* (2·2). Since by hypothesis $\log\rho_2/\log\rho_1$ is irrational, the set of points $m+n(\log\rho_2/\log\rho_1)$, where m and n are integers, is dense everywhere, and the set $\{\rho_1^m\rho_2^n\}$ is dense everywhere in $(0,\infty)$. Let $\{r_n\}$ denote this latter set, and let ρ be any positive number. Then

$$F_\epsilon(\rho) = \int_{\epsilon\rho}^{\epsilon}\frac{\phi(t)}{t}dt = \int_{\epsilon\rho}^{\epsilon r_n} + \int_{\epsilon r_n}^{\epsilon} = P+Q,$$

say. Since $|\phi|\leqslant K$ near the origin, $|P|\leqslant K\log(r_n/\rho)$, and by hypothesis $Q\to I(r_n)$ as $\epsilon\to 0$. Hence the oscillation of $F_\epsilon(\rho)$ at $\epsilon = 0$ is less than $2K|\log(r_n/\rho)|$. Since $\{r_n\}$ is everywhere dense in $(0,\infty)$, r_n can be taken as near as we like to ρ, so that $F_\epsilon(\rho)$ converges as $\epsilon\to 0$, that is $I_1(\rho)$ exists, for all $\rho>0$.

5 *Proof of* (2·3). Let $I_1(\rho)$ exist for all $\rho>0$. By (3·1),

$$I_1(\rho+h)-I_1(\rho) = I_1(1+h/\rho)-I_1(1),$$

so that $I_1(\rho)$ is either totally discontinuous or continuous, according as it is discontinuous or continuous at $\rho = 1$. Now

$$I_1(\rho) = \lim_{\epsilon\to 0}\int_{\epsilon\rho}^{\epsilon}\frac{\phi(t)}{t}dt = \lim_{\epsilon\to 0}F_\epsilon(\rho),$$

say, and $F_\epsilon(\rho)$ is continuous in ρ in $0<\rho<\infty$ for all $\epsilon>0$. Hence $I_1(\rho)$ is at worst pointwise discontinuous, and therefore continuous everywhere.

From (3·1), $$I_1(e^{p/q}) = \frac{p}{q}I_1(e),$$

and therefore $$I_1(e^{p/q})\to I_1(e^x)$$

as $p/q\to x$. Hence $$I_1(e^x) = xI_1(e)$$

or $$I_1(\rho) = A\log\rho. \tag{5·1}$$

https://doi.org/10.1017/S0305004100021484 Published online by Cambridge University Press

6. *Proof of* (2·4 *a*). Let $I_1(\rho)$ exist for $\rho = \rho_1$; we may take $\rho_1 < 1$. Then

$$\int_{\epsilon\rho_1}^{\epsilon} \frac{\phi(t)}{t}\,dt = \int_{\epsilon\rho_1}^{1} - \int_{\epsilon}^{1} = \int_{\epsilon}^{1} \frac{\phi(t\rho_1)-\phi(t)}{t}\,dt + \int_{1}^{1/\rho_1} \frac{\phi(t\rho_1)}{t}\,dt,$$

which implies the existence of $\lim_{\epsilon\to 0} \int_{\epsilon}^{1} \{\phi(t\rho_1)-\phi(t)\}\,dt$. Let $F(u) = \int_{u}^{1} \phi(t)\,dt$; then this implies that

$$F(u) - \frac{F(u\rho_1)}{\rho_1}$$

tends to a limit L as $u \to 0$.

Let $u > 0$ and $u_0 \geqslant u \geqslant u_0\rho_1$. Then

$$\left.\begin{aligned} F(u) - \frac{1}{\rho_1} F(u\rho_1) &= L + \eta_1, \\ F(u\rho_1) - \frac{1}{\rho_1} F(u\rho_1^2) &= L + \eta_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ F(u\rho_1^{m-1}) - \frac{1}{\rho_1} F(u\rho_1^m) &= L + \eta_m, \end{aligned}\right\} \tag{6·1}$$

and

$$F(u) - \frac{1}{\rho_1^m} F(u\rho_1^m) = \sum_{r=1}^{m} \frac{L+\eta_r}{\rho_1^{r-1}} = (L+\bar{\eta}_m)\frac{\rho_1^{-m}-1}{\rho_1^{-1}-1}, \tag{6·2}$$

where η_m and $\bar{\eta}_m$ tend to 0 when $m \to \infty$. Let $y = u\rho_1^m$, $m = 0, 1, 2, \ldots$, and $u_0 \geqslant u \geqslant u_0\rho_1$; then y takes all values in $u_0 \geqslant y > 0$, and (6·2) gives

$$F(y) = y\frac{F(u)}{u} - \frac{L+\bar{\eta}_m}{\rho_1^{-1}-1}\left(1-\frac{y}{u}\right). \tag{6·3}$$

Hence

$$\lim_{y\to 0} F(y) = -\frac{L\rho_1}{1-\rho_1}, \tag{6·4}$$

thus proving (2·4 *a*).

7. *Proof of* (2·4 *b*) *and* (2·4 *c*). Let

$$\int_0^u \phi(t)\,dt = F(u), \quad \frac{F(u)}{u} = \psi(u). \tag{7·1}$$

Then

$$\int_{\epsilon\rho_1}^{\epsilon} \frac{\phi(t)}{t}\,dt = \psi(\epsilon) - \psi(\epsilon\rho_1) + \int_{\epsilon\rho_1}^{\epsilon} \frac{\psi(t)}{t}\,dt. \tag{7·2}$$

If $y = \int_{\epsilon\rho_1}^{\epsilon} \frac{\psi(t)}{t}\,dt$, then

$$\psi(\epsilon) - \psi(\epsilon\rho_1) = \epsilon\frac{dy}{d\epsilon}; \tag{7·3}$$

and by hypothesis

$$y + \epsilon\frac{dy}{d\epsilon} \to I_1(\rho_1)$$

as $\epsilon \to 0$, i.e.

$$\frac{d(y\epsilon)}{d\epsilon} \to I_1(\rho_1).$$

https://doi.org/10.1017/S0305004100021484 Published online by Cambridge University Press

Hence
$$y\epsilon - y_0\epsilon_0 = \{I_1(\rho_1)+\eta\}(\epsilon-\epsilon_0), \tag{7·4}$$
where $\eta \to 0$. Since $F(u)$ is continuous near the origin,
$$y = \int_{\epsilon\rho_1}^{\epsilon} \frac{\psi(t)}{t}\,dt = \int_{\epsilon\rho_1}^{\epsilon} \frac{F(t)}{t^2}\,dt = \frac{F(t_1)}{\epsilon}\left(\frac{1}{\rho_1}-1\right) \quad (\epsilon \geqslant t_1 \geqslant \epsilon\rho_1),$$
and so $y\epsilon \to 0$ when $\epsilon \to 0$. Hence, making $\epsilon_0 \to 0$ in (7·4), we obtain
$$y\epsilon = \{I_1(\rho_1)+\eta\}\epsilon,$$
where again $\eta \to 0$. Hence
$$y \to I, \tag{7·5}$$
and, by (7·2),
$$\psi(\epsilon) - \psi(\epsilon\rho_1) \to 0, \tag{7·6}$$
thus proving (2·4 *b*) and (2·4 *c*).

8. *Proof of the main theorem.* Let $\rho > 0$.

(i) *Sufficiency.* As in (7·2),
$$\int_{\epsilon\rho}^{\epsilon} \frac{\phi(t)}{t}\,dt = \psi(\epsilon) - \psi(\epsilon\rho) + \int_{\epsilon\rho}^{\epsilon} \frac{\psi(t)}{t}\,dt.$$
Since by hypothesis $\psi(t) \to L$ (say) as $t \to 0$,
$$\psi(\epsilon) - \psi(\epsilon\rho) \to 0$$
as $\epsilon \to 0$, and
$$\int_{\epsilon\rho}^{\epsilon} \frac{\psi(t)}{t}\,dt \to L\log\frac{1}{\rho}. \tag{8·1}$$

(ii) *Necessity.* Given that $I_1(\rho)$ exists for all $\rho > 0$, we have to prove that $\psi(t)$ converges as $t \to 0$. Let $G_\epsilon(\rho) = \psi(\epsilon) - \psi(\epsilon\rho)$. By (2·4 *a*), $\psi(t)$ is continuous in $t > 0$, so that $G_\epsilon(\rho)$ is continuous for all $\rho > 0$, for a fixed $\epsilon > 0$; and, by (2·4 *b*), $G_\epsilon(\rho) \to 0$ when $\epsilon \to 0$. By a theorem due to Arzela, there exists, in every interval $0 < \rho_1 \leqslant \rho \leqslant \rho$, at least one point of uniform convergence of $G_\epsilon(\rho)$. That is to say, given any $\eta > 0$, there exist a ρ_0 in $\rho_1 \leqslant \rho_0 \leqslant \rho_2$, an interval $(\rho_0-\delta, \rho_0+\delta)$, and an $\epsilon_0 > 0$, such that
$$|\,G_\epsilon(\rho)\,| \leqq \eta \quad (\rho_0-\delta \leqslant \rho \leqslant \rho_0+\delta,\ \epsilon \leqslant \epsilon_0). \tag{8·2}$$

By (2·3),
$$I_1(\rho) = A\log\rho = B\log\frac{1}{\rho},$$
say. Suppose now that $\psi(t)$ oscillates and that its upper limit as $t \to 0$ is M. Then it follows from (2·4 *c*) that $M \geqslant B$. Suppose that $M > B$. Then there is a sequence $\{\epsilon_n\}$ tending to zero such that $\psi(\epsilon_n) = M_n \to M$ as $n \to \infty$. Then, by (8·2),
$$|\,G_{\epsilon_n}(\rho)\,| = |\,\psi(\epsilon_n) - \psi(\rho\epsilon_n)\,| \leqq \eta \quad (\rho_0-\delta \leqslant \rho \leqslant \rho_0+\delta)$$
for sufficiently large n. Hence
$$\int_{\epsilon_n(\rho_0-\delta)}^{\epsilon_n(\rho_0+\delta)} \frac{\psi(t)}{t}\,dt \geqslant (M-\eta)\log\frac{\rho_0+\delta}{\rho_0-\delta} \tag{8·3}$$

https://doi.org/10.1017/S0305004100021484 Published online by Cambridge University Press

for large n. But, by hypothesis,

$$\int_{\epsilon_n(\rho_0-\delta)}^{\epsilon_n(\rho_0+\delta)} \frac{\psi(t)}{t}\,dt \to B \log \frac{\rho_0+\delta}{\rho_0-\delta}.$$

Since η is arbitrary, we have $B \geqslant M$. Hence $M = B$.

Similarly we prove that the lower limit of $\psi(t)$ as $t \to 0$ is B. Hence $\psi(t) \to B$, thus proving the necessity.

9. It may make the point of my results clearer if I observe that, when

$$\rho(x) = \cos(\log x + C),$$

then $I(a, b)$ is convergent, and has the value 0, when

$$\rho = e^{2k\pi}$$

and k is an integer, but for no other values of ρ.

CENTRAL COLLEGE
BANGALORE
INDIA

https://doi.org/10.1017/S0305004100021484 Published online by Cambridge University Press