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FROM STENIUS' CONSISTENCY PROOF TO SCHÜTTE'S CUT ELIMINATION FOR ω-ARITHMETIC

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Abstract. The book *Das Interpretationsproblem der Formalisierten Zahlentheorie und ihre Formale Widerspruchsfreiheit* by Erik Stenius published in 1952 contains a consistency proof for infinite ω-arithmetic based on a semantical interpretation. Despite the proof's reference to semantics the truth definition is in fact equivalent to a syntactical derivability or reduction condition. Based on this reduction condition Stenius proves that the complexity of formulas in a derivation can be limited by the complexity of the conclusion. This independent result can also be proved by cut elimination for ω-arithmetic which was done by Schütte in 1951.

In this paper we interpret the syntactic reduction in Stenius' work as a method for cut elimination based on invertibility of the logical rules. Through this interpretation the constructivity of Stenius' proof becomes apparent. This improvement was explicitly requested from Stenius by Paul Bernays in private correspondence (In a letter from Bernays begun on the 19th of September 1952 (Stenius & Bernays, 1951–75)). Bernays, who took a deep interest in Stenius' manuscript, applied the described method in a proof Herbrand's theorem. In this paper we prove Herbrand's theorem, as an application of Stenius' work, based on lecture notes of Bernays (Bernays, 1961). The main result completely resolves Bernays' suggestions for improvement by eliminating references to Stenius' semantics and by showing the constructive nature of the proof. A comparison with Schütte's cut elimination proof shows how Stenius' simplification of the reduction of universal cut formulas, which in Schütte's proof requires duplication and repositioning of the cuts, shifts the problematic case of reduction to implications.

§1. Introduction. The Finnish professor of logic, Erik Stenius (1911–1990), published both in the field of mathematics and in philosophy. After his dissertation in philosophy, which he worked on in Zurich under the guidance of Paul Finsler and Paul Bernays, he held a position as professor of philosophy at Åbo Akademi in Turku from about 1950 until 1963, when he became professor of philosophy at the University of Helsinki.

In 1952 he published a book which gives a constructive consistency proof for Peano Arithmetic through an interpretation of formulas in ω-arithmetic. The book entitled *Das Interpretationsproblem der Formalisierten Zahlentheorie und ihre Formale Widerspruchsfreiheit* (The interpretation problem for formal number theory and its formal consistency) received moderate international attention. For example, Georg Kreisel reviewed the work in the Journal of Symbolic Logic (Kreisel, 1953). However, its foremost international proponent was Bernays who gave lectures about the proof and applied its method in sketches of a proof of Herbrand's theorem. Bernays discussed the proof with Stenius in person and by correspondence during the 1950's and 60's.

He explained the result at lectures held at a Princeton seminar in 1952 (Bernays, 1952), mentioned the application of the result to Herbrand's theorem at the Logic Colloquium

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in Amsterdam (Bernays, 1954) and gave an extensive account of this application during lectures at the University of Pennsylvania in 1961 (Bernays, 1961).

The main result in Stenius' monograph is a constructive consistency proof for Peano Arithmetic, but the focus is on a heuristic argument that explains a method developed by Stenius. Stenius' motivates his outset by noting that because of the nonconstructive nature of Tarski's semantics the constructivist approach to mathematics requires another rigorous philosophical foundation. To fill in this gap in the foundational work he builds a new constructive formal semantics, which is sound and complete, based on an expansion of the notion of truth tables from propositional logic to predicate logic and arithmetic. The main problem with an expansion of truth tables is that the confirmation of the truth-value for a universally quantified formula intuitively requires checking each element of an infinite domain. Thus, in order to avoid all infinite features Stenius' presents a solution to the problem by restriction of the acceptable truth-values to values that do not produce contradictory results with respect to derivability in the arithmetical system. This restriction implies that the intended formal semantics is equivalent to a syntactic derivability or reduction condition. Stenius proves this equivalence formally after which he leaves the heuristic extension of truth tables behind and instead adopts the syntactic condition as the basic definition of truth.

Correspondence between Bernays and Stenius can be found among Stenius' manuscripts in the archive of the National Library of Finland (Stenius & Bernays, 1951–75). In Bernays' letters to Stenius[1] he expresses appreciation of Stenius' manuscript and confirms the correctness of the result. Bernays informs Stenius of Schütte's equivalent result of cut elimination for ω-arithmetic (Schütte, 1951), but he notes the merits of Stenius' independent work. Bernays' main criticism of the manuscript is Stenius' reference to the truth-interpretation. He comments that the proof would benefit from an elimination of the heuristic argument and could be reworked to clearly show the constructive manipulation of derivations that is only implicitly presented by Stenius. The desired improvement is a concrete reduction procedure for derivations, which shows that the complexity of all formulas in a derivation can be restricted to the complexity of the conclusion of the derivation.

Stenius continued to work in the direction proposed by Bernays. Among Stenius' manuscripts in the archive of the National Library of Finland there is a notebook with several reworked versions of the proof (*Stenius*, 1956). However, these versions never resulted in a new published proof. In the reworked versions Stenius replaces his axiomatic Hilbert-Bernays calculus with various systems of rules in order to simplify the application of his reduction condition.

The system used in Stenius' published work is the axiomatic Hilbert–Bernays calculus for Peano Arithmetic where the finite induction rule has been replaced with an infinite ω-induction rule. The inclusion of the ω-induction rule might be questioned because of the desideratum to restrict methods to constructively acceptable ones (see (Kreisel, 1953)). However, Stenius makes this choice because of his ambition to create not only a sound semantics, but also a complete semantics and indeed he does satisfy the requirements for completeness through ω-completeness. To prove soundness for the defined semantics Stenius converts applications of the rule of modus ponens into applications of weakening and cut. The weakening and cut rules are admissible in the system and Stenius shows that they also are truth preserving, which implies that the modus ponens rule is truth preserving. Since the structural rules, weakening and cut, are characteristic of sequent

[1] In a letter from Bernays begun on the 19th of September 1952.

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calculus systems it suggests that a full sequent calculus system might be a convenient setting for Stenius' proof.

1.1. The aim of this paper. The aim of this paper is to explain Stenius' proof and to carry out the improvements suggested by Bernays. The idea of Stenius proof can be interpreted as a method for cut elimination based on invertibility of the logical rules. We will call this the Stenius–Bernays method. The choice of sequent calculus as the basic system brings to light a new meaning of the Stenius–Bernays method in a sequent calculus terminology. The new setting also makes possible the comparison to Schütte's equivalent method for cut elimination in ω-arithmetic.

Despite Stenius' own thoughts that the reductions for specific derivations might be complicated[2] this paper will show that the Stenius–Bernays method can be transformed into a general reduction procedure in a sequent calculus that inherits properties from the Hilbert–Bernays calculus. The direct application of the method to a cut elimination proof completely resolves the suggestions for improvement made by Bernays and makes the constructiveness of Stenius' proof apparent. In addition we will give a complete formal proof of Herbrand's theorem following the suggested implementation of the Stenius–Bernays method made by Bernays.

§2. Stenius' proof.

In this section a summary of Stenius' proof will be presented.

2.1. The axiomatic system for ω-arithmetic. The axiomatic system for ω-arithmetic that Stenius uses is the Hilbert-Bernays system for Peano Arithmetic, where the induction rule has been replaced with an infinite ω-rule. The system includes rules for implication and universal quantification.

DEFINITION 2.1. *Let the system for ω-arithmetic called Z_{ax} include the following:*

- *Propositional and arithmetical axioms for each propositionally true formula and each true atomic formula.*
- *A quantifier axiom $\forall x A(x) \supset A(y)$.*
- *A substitution rule, which substitutes a term for the substitution variable, x:*

$$\frac{A(x)}{A(t)}\ Subst.$$

- *The modus ponens rule:*

$$\frac{A \supset B \quad A}{B}\ \supset E$$

- *A quantifier rule, where the eigenvariable y is not free in A:*

$$\frac{A \supset B(y)}{A \supset \forall x B(x)}\ \forall I$$

- *A rule for ω-induction, where there is a premise, $A(k)$, for each closed term k:*

$$\frac{\begin{array}{c}\vdots\\ \ldots \quad A(k) \quad \ldots\end{array}}{A(x)}\ SInd$$

[2] (Stenius, 1952), p. 98.

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The substitution rule creates a distinction between two kinds of variables. Some variables, so called parametrized variables, will be globally replaceable in a derivation with arbitrary terms while others will be restricted by their use in the derivation. The replacement of the latter kind of variables is restricted because an arbitrary replacement would not result in a valid derivation. In particular, the eigenvariables in instances of the quantifier rule and the substitution variables in instances of the substitution rule cannot be arbitrarily replaced.

DEFINITION 2.2. *Variables in a derivation can be* parametrized *or* nonparametrized. *A variable is nonparametrized if it occurs as a substitution variable in the premise of an instance of the substitution rule or if it occurs as an eigenvariable in the premise of an instance of the quantifier rule. In all other cases a variable is parametrized.*

The deduction theorem for an axiomatic system, which corresponds to an implication introduction rule in a natural deduction, can be proven for Z_{ax} under certain variable restrictions. The deduction theorem 2.3 is necessary in order to prove completeness for Z_{ax}.

THEOREM 2.3 (Deduction theorem). *If B is derivable in Z_{ax} from the open assumption A and all variables of A are parametrized in the derivation, then $A \supset B$ is derivable in Z_{ax}.*

The variable restriction that all variables of A should be parametrized in the derivation of B is necessary to prevent misuse of substitution variables. If a term has been substituted for a variable in a derivation of B and this variable was allowed to occur in the antecedent of a derivable implication, then by substituting another term for the same variable the derivable implication could become false. One example is the derivation

$$\frac{x = 0}{1 = 0}\ \mathit{Subst.}$$

If the deduction theorem 2.3 could be applied, then we could derive the implication $x = 0 \supset 1 = 0$. Thus, the following derivation of a false implication would be valid.

$$\frac{x = 0 \supset 1 = 0}{0 = 0 \supset 1 = 0}\ \mathit{Subst.}$$

2.2. Stenius' semantic definition of truth. The title of Stenius' monograph announces that he will discuss the interpretation problem for formal number theory. In a letter to Kreisel Stenius summarises the problem statement:

> The best short formulation of this problem as it is treated [in the published work] is probably the following: Find an interpretation that is useful for an 'interpretative' consistency proof by a modification of truth tables for universal and existential formulas. (Stenius' letter to Kreisel 2.7.1953)[3]

Stenius explains in the letter that he uses the word 'interpretation' for a semantic concept, which assigns meaning to predicates by a model. The main consequence of the

[3] Die beste kurze Formulierung dieses Problems wie es dort behandelt wird währe wohl die folgende: eine Modifikation der durch die Wahrheitstafeln der All- und Existenzoperatoren gegebenen Interpretation zu finden, die führ einen 'interpretativen' Wiederspruchsfreiheitsbeweis brauchbar sei.

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interpretation is a constructive consistency proof for Peano Arithmetic. Though Stenius approaches the question of consistency by stating a wider interpretation problem.

> The problem of finding a finite interpretation for number theory, I call the *interpretation problem* for formalised number theory. This problem is of course of interest in itself, which extends beyond the consistency proof. (Stenius, 1952)[p. 11.][4]

To interpret formal number theory Stenius builds a new constructive formal semantics based on an expansion of the notion of truth tables from propositional logic to predicate logic and arithmetic. The focus of the presentation is on a heuristic argument that explains the method he is developing.

The main problem with an expansion of truth tables is that the confirmation of the truth value for a universally quantified formula intuitively requires checking the truth value for each element of an infinite domain. In order to avoid all infinite features Stenius presents a solution to the problem, which restricts the acceptable truth values to values that do not produce contradictory results with respect to derivability in the arithmetical system.

The argument of how to create a semantics can be summarised thus: If all universally quantified sub formulas in a formula of predicate logic are treated as atomic formulas, then we can make a truth table by assigning truth values to both the atomic and quantified formulas. The set of *fundamental formulas* comprises atomic and universal formulas. If truth values are assigned to fundamental formulas, then truth tables can be generated. However, if a specific atomic formula is true, then the rows of the truth table that assigns the value 'false' to the formula should be disregarded. In the case of Peano Arithmetic the truth value of atomic formulas is decidable, so the problem of which rows to disregard is decidable. A similar argument should also be made for universal formulas, because some truth values should be excluded. If these truth values were assumed then it would lead to contradictions with the intuitive meaning of the sub formula, which is obtained by removing the outermost quantification. Therefore, Stenius argued that only the truth values that are consistent with the internal structure of the quantified formula should be considered as valid assignments of truth values.

> A formula is 'identically true' in the sense of the propositional calculus, if in any such distribution of truth values to the components that gives the formula the truth value 'false' in a truth table, at least one fundamental formula receives both truth values, so that any such distribution is contradictory. Stenius (1952)[p. 32.][5]

This restriction implies that the intended formal semantics is equivalent to a syntactic derivability or reduction condition. Stenius gives a formal proof of the equivalence of the truth definitions after which he leaves the heuristic extension of truth tables behind and instead adopts the syntactic condition as the basic definition of truth.

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[4] Das Problem, eine finite Interpretation der formalisierten Zahlentheorie zu finden, nenne ich das *Interpretationsproblem* für die formalisierte Zahelntheorie. Dieses Problem besitzt natürlich ein Interesse an sich, welches sich über das der Widerspruchsfreiste hinaus erstreckt.

[5] Eine Formel ist 'identisch wahr' im Sinne des Aussagenkalküls, wenn bei jeder solchen Verteilung der Wahrheitswerte der Komponenten, die ihr nach den Wahrheitstafeln der Wert falsch gibt, wenigstens eine Fundamentalformel beide Wahrheitswerte erhält, so dass jede solche Verteilung widerspruchsvoll ist.

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2.3. Stenius' syntactic definition of truth. Let the complexity of formulas be defined based on the quantification structure. We will follow Stenius' terminology for the complexity measure for formulas and denote the formally defined notion 'order' (from the German ordnung). The complexity measure should not be confused with the set-theoretic quantification over sets. The order-measure adopted here is a measure for the quantifier depth of first-order formulas.

DEFINITION 2.4. *The composite formulas are disjunctions, conjunctions, implications and universally quantified formulas. Whereas negation and existential quantification are defined notions.*

The order *of a formula A,* $order(A)$*, is inductively defined:*

- $order(P) = 0$ *if* P *is an atomic formula.*
- $order(A \vee B) = order(A\&B) = order(A \supset B) = max(order(A), order(B))$.
- $order(\forall x A) = order(A) + 1$.

Bernays pointed out to Stenius[6] that his proof gives an implicit reduction, which produces a derivation where the complexity of all formulas is limited by the complexity of the conclusion. The explicit reductions are hidden behind Stenius' references to a semantical interpretation.

The reduction of the formulas of a high order in a derivation begins by considering an equivalent formula of a particular form; a conjunction of fundamental implications.

DEFINITION 2.5. *A* fundamental formula *is an atomic formula or a universal formula.*

For each formula, A, there is an equivalent formula consisting of a conjunction of implications of the form: $M_1\& \ldots \&M_r \supset N_1 \vee \cdots \vee N_{r'}$*, where each* M_i *and* N_j *are fundamental formulas. These implications are the* fundamental implications *of the formula A.*

If a formula is not an atomic formula, then each of its fundamental implications that contains a fundamental formula of high order can be converted into a reduced implication by replacing the high-order quantification with free reduction variables that are fresh and independent.

Reduction of formulas: Let A be a conjunction of fundamental implications of order $n + 1$. The *reduced implication* of each fundamental implication is obtained by replacing the universally quantified subformulas of order $n + 1$, say $\forall x B(x)$, in the antecedent and consequent with the formula $B(y)$. The *reduction variable*, y, should be free and independent, which means that we choose a fresh variable for each reduced universal formula. The conjunction of the reduced implications is the reduced formula of A.

If a fundamental implication of A is of the form $M_1\& \ldots \&M_r \supset N_1 \vee \cdots \vee N_{r'}$ as in definition 2.5, then its reduced implication is denoted by $M_1'\& \ldots \&M_r' \supset N_1' \vee \cdots \vee N_{r'}'$.

The reduction of formulas can be applied to reduce the complexity of a given derivation.

DEFINITION 2.6. *The system* Z_{ax} *can be restricted by the order n producing a subsystem,* Z_{ax}^n*, that only allows formulas of order n or less in the derivations.*

The goal of the reduction of a derivation is to eliminate every formula that has a higher complexity than the conclusion from the derivation and thus derive the formula in a restricted system. The form of the sought derivation is described in the syntactical truth definition.

[6] Letter from Bernays to Stenius begun on the 19th of September 1952.

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DEFINITION 2.7 (Syntactical definition of truth). *The* z-true *of propositional combinations of atomic formulas is determined by truth tables based on the truth values of atomic formulas.*

A formula A of order $n+1$ *is* z-true *if and only if for each fundamental implication,* $M_1\&\ldots\&M_r \supset N_1 \vee \cdots \vee N_{r'}$*, of the formula the consequent of the reduced implication,* $N'_1 \vee \cdots \vee N'_{r'}$*, is derivable from the antecedent of the reduced implication,* $M'_1\&\ldots\&M'_r$*, in the system,* Z^n_{ax}*, which is restricted by the order of the reduced implication.*

A variable condition applies to the derivation in the restricted system, namely, that all variables in M'_i *and* N'_j*, except the reduction variables, should be parametrized (see definition 2.2).*

Stenius' syntactical truth definition defines a formula as z-true if the formula is derivable without introducing formulas of a higher complexity. Thus, completeness of the system Z_{ax} is easy to prove. A constructive proof of the soundness theorem, on the other hand, shows how to eliminate formulas of a high order by transforming a given derivation.

2.4. Stenius' proof of soundness and completeness. Using the truth definition 2.7 Stenius shows completeness and soundness for predicate logic (in chapter 4 of (Stenius, 1952)) and extends the result to the system Z_{ax} for ω-arithmetic (in chapter 5).

THEOREM 2.8 (The completeness theorem). *If a formula is z-true, then it is derivable in* Z_{ax}.

Proof. By the reduction condition 2.7 the consequent of the reduced implication, $N'_1 \vee \cdots \vee N'_{r'}$, is derivable from the antecedent of the reduced implication, $M'_1\&\ldots\&M'_r$, in the system, Z^n_{ax}. With the rules of Z^{n+1}_{ax} the reduced conjunction $M'_1\&\ldots\&M'_r$ is derivable from the unreduced conjunction $M_1\&\ldots\&M_r$. The variable condition put on the reduction condition 2.7 shows that all variables in $M_1\&\ldots\&M_r$ are parametrized. Thus, the deduction theorem 2.3 applies and the implication $M_1\&\ldots\&M_r \supset N'_1 \vee \cdots \vee N'_{r'}$ is derivable. Using the quantifier rule quantification can be reintroduced in the consequent of the implication, thereby, deriving the fundamental implication from the derivation of the reduction condition 2.7. □

THEOREM 2.9 (The soundness theorem). *If a formula is derivable in* Z_{ax}*, then it is z-true.*

Proof sketch. Soundness is proved by induction on the length of the derivation. First Stenius shows that the axioms are true and then by working his way down in the derivation he shows that from the reduction condition of the premises of a rule he can construct a derivation that satisfies the reduction condition for the conclusion.

The proof is straightforward for all axioms and most rules. However, the proof that modus ponens is truth preserving requires special attention. If the order of the antecedent of the implication is higher than the consequent, then the conclusion of the modus ponens rule is of lower order than the premises. Stenius treats this case by replacing the modus ponens rule with an application of the weakening rule followed by an application of the cut rule. In Stenius' system weakening and cut rules are clearly admissible rules. The modus ponens premise $A \supset B$ can be replaced with the propositionally equivalent formula $\neg A \vee B$ and by applying weakening to the other premise A we can derive $A \vee B$. By applying cut on the formula A we derive the formula B, which is the conclusion of the modus ponens rule. That the modus ponens rule is truth preserving can then be shown by proving the same property for the weakening rule and the cut rule. □

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§3. A sequent calculus system. The details of Stenius' proof can be clarified by following the advice of Bernays. If a sequent calculus system for ω-arithmetic is used and the soundness proof is converted into a cut-elimination theorem, then the reduction procedure shows the constructive transformations hidden behind Stenius' proof idea.

The inference rules of the sequent calculus system can be chosen to closely correspond to the axiomatic system. A classical multi-succedent calculus for predicate logic, P_{sc}, can be extended with a substitution rule, arithmetical rules and a rule for ω-induction to obtain a system, Z_{sc}, for ω-arithmetic. The logical rules of the system are based on a classical Gentzen calculus improved by Ketonen to make the propositional rules invertible.

Sequents are of the form $\Gamma \to \Delta$ where both the *antecedent* Γ and the *succedent* Δ are (possibly empty) multisets. This means that the structural exchange rule is superfluous.

The existential quantifier and negation will be defined notions, that is, $\exists x A(x) \equiv_{df} \neg\forall x \neg A(x)$ and $\neg A \equiv_{df} A \supset \bot$.

Initial sequent:

$$A, \Gamma \to \Delta, A$$

Initial sequent, $L\bot$:

$$\bot, \Gamma \to \Delta$$

Substitution rule:

$$\frac{A(t), \Gamma \to \Delta}{A(x), \Gamma \to \Delta}\ LSub.,\text{VR1}$$

Logical rules:

$$\frac{A, B, \Gamma \to \Delta}{A\&B, \Gamma \to \Delta}\ L\& \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A\&B}\ R\&$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ L\vee,\text{VR2} \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\ R\vee$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ L\supset \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\ R\supset,\text{VR3}$$

$$\frac{A(t/x), \Gamma \to \Delta}{\forall x A, \Gamma \to \Delta}\ L\forall \qquad \frac{\Gamma \to \Delta, A(y/x)}{\Gamma \to \Delta, \forall x A}\ R\forall,\text{VR4}$$

Structural rules:

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\ LC \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}\ RC$$

$$\frac{\Gamma \to \Delta, A \quad A, \Gamma \to \Delta}{\Gamma \to \Delta}\ Cut$$

The calculus P_{sc}.

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The formula introduced in the conclusion of the logical rules is called the *principal formula* of the rule. The subformulas of the principal formula in the premises of the rule are the *active formulas* of the rule. For the substitution rule and the structural rules, as well as the arithmetical rules the induction rule defined below, a similar definition applies; that the active formula occurs in the premise and the principal formula in the conclusion. The formula that occurs both in the antecedent and succedent is the active formula of the initial sequent and $\bot$ is active in the $L\bot$ initial sequent.

3.1. Variables in the sequent calculus. The addition of the substitution rule changes the treatment of variables from a standard sequent calculus to a system that has variable restrictions induced from the axiomatic system. We have four variable restrictions in the rules that are indicated in the calculus P_{sc} by labels, VR. We recall from definition 2.2 that parametrized variables can be substituted with an arbitrary term and the derivation remains valid. The nonparametrized variables are substitution variables and eigenvariables.

DEFINITION 3.1. *We assume that the two sets of variables, substitution variables and eigenvariables, are distinct.*

Variable restriction 1: *In the premise of the substitution rule the term t is arbitrary and the substitution variable x should not occur in Δ or in $A(t)$. However, x may appear in Γ.*

Variable restriction 2: *The variable restriction in the rule $L\vee$ is that all nonparametrized variables in A and B should be distinct.*

Variable restriction 3: *The variable restriction in the rule $R\supset$ is that no nonparametrized variables should occur in the antecedent, A, of the implication.*

Variable restriction 4: *The standard variable restriction of rule $R\forall$ requires that the eigenvariable, y, should not occur free in the conclusion of the rule. We assume that all eigenvariables in a derivation are distinct and only used in the derivation above the quantifier rule of which it is the eigenvariable.*

As eigenvariables hides a universality of statements in the succedent of sequents, the substitution variables will hide a universality of statements in the antecedent of sequents. The formulas that contain these nonparametrized variables can be treated as if there was a universal quantifier in front of the formula that quantifies the nonparametrized variable. The hidden universal nature implies that restrictions have to be put on the inference rules in order to prevent derivability of falsehoods.

The only rule that moves a formula from the antecedent to the succedent of a sequent is the right implication rule. This rule corresponds to the deduction theorem 2.3 in the axiomatic system. As is required by the deduction theorem all variables that occur in the antecedent of the implication should be parametrized and in particular not be substitution variables. The corresponding variable restriction, $VR3$, prevents substitution variables from occurring in the formula that moves to the succedent. Thus, substitution variables will only occur in the antecedents of sequents in the calculus Z_{sc}. This restriction prevents substitution variables from being treated as parametrized variables in the derivation.

Variable restrictions also apply to the substitution rule and the left disjunction rule in order to preserve a consequent use of the nonparametrized hidden universal variables. The restriction, $VR1$, is needed in the former case to prevent that we from the initial sequent $y = x \to y = x$ derive the inconsistent sequent $x = x \to y = x$. In the latter case we need the restriction, $VR2$, because the implication $\forall x(A(x) \vee B(x)) \supset \forall x A(x) \vee \forall x B(x)$ is not valid.

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3.2. Extension of the calculus to an arithmetical system. We will extend the calculus with Peano's arithmetical axioms and an infinite induction rule.

By converting axioms into rules of inference it is possible to preserve the structural properties of a sequent calculus system (Negri&von Plato, 1998). In particular full cut elimination holds for the extended system. The system of rules for the arithmetical part of Peano Arithmetic is found in (von Plato, 2006). The added arithmetical rules of our system Z_{sc} are rules with or without premises that derive atomic formulas in the succedent of the sequent. In the rules with premises we do not need to add the principal formula of the rule in the succedent of the premises because we do not aim to construct a system with contraction as an admissible rule.

Arithmetical rules:

Arithmetical rules based on Peano's axioms.

Infinite induction:

$$\frac{\ldots \quad \Gamma(n), \Gamma' \to \Delta', \Delta(n) \quad \ldots}{\Gamma(x), \Gamma' \to \Delta', \Delta(x)} \; Ind$$

The calculus Z_{sc}.

The infinite induction rule has a countably infinite number of premises, one for each numeral n. The rule can have multiple induction formulas both in the antecedent and the succedent. We denote the multisets of induction formulas with $\Gamma(n)$ and $\Delta(n)$ and also allow arbitrary contexts Γ' and Δ'. In the conclusion of the rule the numerals of the induction formulas have been replaced with an *induction variable*.

Substitution of a term for an induction variable in a derivation requires a modification of the instance of the induction rule for the derivation to remain valid (Lemma 3.6). The modification is necessary because in the conclusion of the induction rule we have a variable and not an arbitrary term in the induction formulas. The modification makes it possible to treat induction variables as parametrized variables.

We allow induction formulas in the antecedent of the premises in order to get invertibility of the right implication rule, in lemma 5.1, without using the substitution rule.

3.3. Equivalence of the sequent calculus system and the axiomatic system. The axiomatic system Z_{ax} and the sequent calculus system Z_{sc} are equivalent.

THEOREM 3.2. *Let the disjunction of all the formulas in the multiset Δ be D. If a sequent $\Gamma \to \Delta$ is derivable in Z_{sc}, then D is derivable from the open assumptions Γ in Z_{ax}.*

THEOREM 3.3. *If a formula D is derivable from the open assumptions Γ in Z_{ax}, then the sequent $\Gamma \to D$ is derivable in Z_{sc}.*

3.4. Admissibility of rules. Admissibility of rules is an appealing property for rules that extend deductive systems. If a rule is shown to be admissible, then it is possible to use it as if it was included in the system, while knowing that it does not introduce new derivable formulas in the system.

DEFINITION 3.4. *A rule is* admissible *in a system if the conclusion of the rule is derivable whenever the premises of the rule are derivable.*

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Weakening rules are admissible in the sequent calculus.

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\ LW \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}\ RW$$

In the right weakening rule we assume that the weakening formula does not contain variables that are nonparametrized in the derivation of the premise.

Because each sequent in the calculus has an arbitrary context we can add the weakening formula in the context of each sequent in the derivation of the premise of an instance of weakening. Thus, the weakening rules are admissible.

LEMMA 3.5. *The weakening rule is admissible in* Z_{sc}.

The induction rule of Z_{sc} can be replaced with a general the term-induction rule that allows the conclusion of induction formulas with variables, but also terms.

$$\frac{\ldots \quad \Gamma(n), \Gamma' \to \Delta', \Delta(n) \quad \ldots}{\Gamma(t), \Gamma' \to \Delta', \Delta(t)}\ TInd$$

LEMMA 3.6. *The term-induction rule is admissible in* Z_{sc}.

Proof. We choose a term and show that we can substitute it for the induction variable and modify the derivation to make the induction rule valid in the system Z_{sc}.

If the chosen term, t, contains variables, then we can state the variables explicitly. For simplicity we assume that there is one variable, y, in t. Thus, the induction term is $t(y)$. In the desired conclusion of the term-induction, $\Gamma(t(y)), \Gamma' \to \Delta', \Delta(t(y))$, we can replace the variable y with each numeral n. By each of these replacements we obtain a closed term. Since each closed term, $t(n)$, is equal to a numeral, m, we can derive $\Gamma(t(n)), \Gamma' \to \Delta', \Delta(t(n))$ by altering the arithmetical parts of the derivation of the premise $\Gamma(m), \Gamma' \to \Delta', \Delta(m)$. Thus, by applying induction we can exchange the numeral in $t(n)$ with the induction variable y. The argument can be generalized if the induction-term contains several variables. □

Since the described method of admissibility does not add any cuts in the derivation we can from now on assume that substitution is allowed for induction variables. Therefore, we treat induction variables as parametrized variables.

§4. Two strategies for proving cut elimination. Two known basic strategies for proving cut elimination can be identified. Gentzen proved cut elimination for classical predicate logic by introducing a multicut rule that eliminates multiple copies of the cut formula (Gentzen, 1934). The introduction of a multicut rule, which eliminates multiple copies of the cut formula, makes it possible to disregard contractions on the cut formula. The multicut rule can be permuted up in a derivation until leafs are reached and the multicut can be trivially eliminated.

Another strategy that avoids the introduction of multicut is a cut elimination strategy based on invertibility of rules. Similar proofs that give direct proofs of cut elimination for predicate logic without the use of multicut are found in (Buss, 1998) and (von Plato, 2001).

The crucial property of the calculus that is used in these proofs is the invertibility of the propositional rules. Gentzen's original sequent calculus was improved in 1944 by Oiva Ketonen (Ketonen, 1944) who discovered a classical invertible propositional sequent calculus that made a root-first approach to proof search possible. After a root-first decomposition the derivability of the sequent is equivalent to the derivability of the leaves.

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Ketonen's improvements include replacing Gentzen's two $L\&$-rules with a single rule that has both conjuncts in the premise and a similar change to the $R\vee$-rule. He also found the classical $L\supset$-rule with shared contexts.

However, because invertibility in a standard sequent calculus does not apply to the quantifier rules an elimination of a cut on a quantified formula is problematic. In particular the left universal and the right existential rules are not invertible. The proof of (Buss, 1998) presents a solution by duplication and repositioning of the cuts on the quantified formulas. Whereas the proof of Buss presents the solution for the existential quantifier, we will focus on the universal quantifier. A cut on a universal formula can be transformed into cuts on instances that replace the left universal rules that derived copies of the universal cut formula.

As an example we consider the following derivation:

$$\dfrac{\Gamma \to \Delta, \forall x B(x), \forall x A(x) \quad \dfrac{\dfrac{A(y), \Gamma \to \Delta, B(y)}{\forall x A(x), \Gamma \to \Delta, B(y)}\, L\forall}{\forall x A(x), \Gamma \to \Delta, \forall x B(x)}\, R\forall}{\Gamma \to \Delta, \forall x B(x)}\, cut$$

Because of invertibility of the right universal rule the left cut premise can be inverted. Thus, if one attempted to simply eliminate the left universal rule as an attempt to covert the cut into a cut on the shorter formula $A(y)$, then the resulting derivation

$$\dfrac{\Gamma \to \Delta, \forall x B(x), A(y) \quad \dfrac{A(y), \Gamma \to \Delta, B(y)}{A(y), \Gamma \to \Delta, \forall x B(x)}\, R\forall}{\Gamma \to \Delta, \forall x B(x)}\, cut$$

would not be a valid derivation, because the eigenvariable condition of the right universal rule is violated. However, the problem can be solved, with the method of (Buss, 1998), by moving the cut above the right universal rule. If the cut occurs above the rule with the eigenvariable, then the additional occurrence of the eigenvariable, that previously was removed by quantification, is now removed by a cut.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to \Delta, \forall x B(x), A(y)}{\Gamma \to \Delta, \forall x B(x), B(y), A(y)}\, RW \quad \dfrac{A(y), \Gamma \to \Delta, B(y)}{A(y), \Gamma \to \Delta, \forall x B(x), B(y)}\, RW}{\Gamma \to \Delta, \forall x B(x), B(y)}\, cut}{\Gamma \to \Delta, \forall x B(x), \forall x B(x)}\, R\forall}{\Gamma \to \Delta, \forall x B(x)}\, RC$$

The additional weakening rules gives the shared contexts for the application of the cut rule and the contraction gives us the same conclusion as the original derivation.

For an arbitrary derivation one has to take into account that there can be contractions on a universal cut formula. In this case a cut would replace each rule that introduced a copy of the cut formula.

The novelty of Stenius' proof is apparent when the proof method is converted into a cut elimination procedure. Namely, the inclusion of a substitution rule in the sequent calculus Z_{sc} makes the left universal rule invertible. However, the variable restriction on the right implication rule will imply that invertibility of this propositional rule inherits the restriction. Thus, cuts on formulas that do not fulfil this restriction will have to be converted into cuts on formulas that do.

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4.1. Schütte's cut elimination procedure. Proofs of invertibility of rules[7] can be extended from predicate logic to ω-arithmetic. Schütte's proof of cut elimination for ω-arithmetic (Schütte, 1951) uses a sequent calculus which is equivalent to Stenius' calculus and the strategy is a version of the cut elimination by invertibility.

Schütte's calculus comprises rules for negation, disjunction and the universal quantifier. The sequents of Schütte's system are disjunctions of atomic formulas, universal formulas and their negations. The disjunctions can be converted to sequents if the negated formulas are collected into the antecedent and the other formulas are collected into the succedent. The assumptions are true atomic formulas or negations of false atomic formulas. The rules of the system correspond to a left and right disjunction rule, a left negation rule and a left and right universal rule. The structural rules of the system are exchange, contraction and cut. The right negation rule is missing because it is trivial and the right disjunction rule is a weakening rule. The right universal rule is a combination of ω-induction and a quantifier rule, because it directly introduces a quantified formula in the conclusion.

Schütte's cut elimination theorem uses a strategy based on invertibility of the rules. He proves that the left disjunction and the left negation rules are invertible. Due to the invertibility of these rules cuts on propositional formulas can be converted into cuts on shorter formulas. The right universal rule, that is, the infinite induction rule is gown to be invertible and cuts on instances of the quantified formula are inserted where quantified cut formula was introduced by the left universal rule.

§5. Invertibility of the sequent calculus rules. The cut elimination strategy for Z_{sc} is also based on invertibility. The novelty of a cut elimination procedure for the sequent calculus with a substitution rule is that the left universal rule is invertible. However, the invertibility of the system is not total as the left implication rule inherits variable conditions on the inversion. The need for duplication and repositioning of cuts is shifted from quantified formulas, as in Schütte's proof, to implications.

The proofs of invertibility are justified by a principle of structural induction on infinite derivations or a form of bar induction, which is discussed in section 7.

LEMMA 5.1 (Invertibility of the propositional rules). *If the conclusion of one of the propositional rules is derivable in Z^n_{sc}, then also the premises of the rule are derivable. For the $L\supset$-rule with the principal formula $A \supset B$ we require that all the variables in A are parametrized to obtain invertibility. For the rule $L\vee$ invertibility applies even if the nonparametrized variables of the disjuncts are not distinct.*

Proof. The proof is based on the method of tracing up in the derivation of the sequent and replacing the sequents that introduce the invertible formula in the derivation. Thus, we trace up from the end sequent following every occurrence of the invertible formula until we reach an initial sequent or a rule where the formula is principal. If the rule is a contraction rule, then we continue the trace from each contraction formula in the premise. If the rule we reach is an induction where the trace-formula is principal, then we continue the trace from the active formula in each premise. If the rule we reach is a substitution rule where the trace-formula is principal, then we continue the trace from the active formula in the premise. Thus, an initial sequent, arithmetical rule or a logical rule where the trace formula is principal is reached. In each case the sequent can be replaced with an inverted

[7] See (Buss, 1998) and (von Plato, 2001)

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sequent and the derivation below it recomposed. In the recomposition of the derivation we use some properties of the substitution and induction rule to extend the result from predicate logic to ω-arithmetic.

In the proof of invertibility of the right universal rule we can derive $A, \Gamma \to \Delta, B$ instead of $\Gamma \to \Delta, A \supset B$ because the induction rule allows induction formulas in the antecedent of the sequents. In the proof of invertibility of the left conjunction and right disjunction rule we use the fact that the induction rule allows multiple induction formulas in the antecedent and the succedent respectively. For the invertibility of the left conjunction rule we the variable restriction of the substitution rule allows us to use the same substitution variable multiple times, which is necessary when the conjunction $A\&B$ is replaced with two formulas A and B. □

LEMMA 5.2 (Invertibility of the right universal rule). *If the sequent* $\Gamma \to \Delta, \forall x A(x)$ *is derivable in* Z^n_{sc}*, then there is a derivation of the sequent* $\Gamma \to \Delta, A(y)$ *for a fresh parametrized variable* y.

In Z_{sc} it is possible to prove a limited invertibility for the left universal rule. The limiting condition of the invertibility is that the term of the derivable instance of the universal formula will be a fresh nonparametrized variable.

LEMMA 5.3 (Invertibility of the left universal rule). *If the sequent* $\forall x A(x), \Gamma \to \Delta$ *is derivable in* Z^n_{sc}*, then there is a derivation of the sequent* $A(y), \Gamma \to \Delta$ *for some fresh nonparametrized variable* y.

Proof. The derivation of the inverted sequent is obtained by replacing left universal rules that derive copies of the universal formula, $\forall x A(x)$, by instances of substitution with a fresh substitution variable y. □

Due to the invertibility lemma 5.1 a root-first decomposition of the propositional structure is possible for each formula.

LEMMA 5.4 (Root-first decomposition lemma). *Let the conjunction of the fundamental implications of a formula* A *be* $\bigwedge(M_1\& \ldots \& M_r \supset N_1 \vee \cdots \vee N_{r'})$*. Then the sequent* $\Gamma \to \Delta, A$ *is derivable in* Z^n_{sc} *if and only if the sequent* $M_1, \ldots, M_r, \Gamma \to \Delta, N_1, \ldots, N_{r'}$ *is derivable in* Z^n_{sc} *for each fundamental implication of* A.

By the invertibility lemmas 5.2 and 5.3 for the quantifier rules the quantification of the fundamental formulas M_i and N_j in lemma 5.4 can be reduced to instances with fresh and independent reduction variables. In these reduced sequents that correspond to the reduced implications the reduction variables in the antecedent are nonparametrized and the variables in the succedent are parametrized.

LEMMA 5.5 (Reduction lemma). *If a sequent* $\Gamma \to \Delta, A$ *is derivable in* Z^n_{sc}*, then for each fundamental implication of* A *a sequent* $M'_1, \ldots, M'_r, \Gamma \to \Delta, N'_1, \ldots, N'_{r'}$ *corresponding to the reduced implication is derivable in* Z^n_{sc}.

The converse lemma also holds, but the reintroduction of the quantifiers may increase the maximum order of the formulas. The lemma is a sequent calculus version of Stenius' completeness theorem 2.8.

LEMMA 5.6 (Completeness lemma). *If each sequent* $M'_1, \ldots, M'_r, \Gamma \to \Delta, N'_1, \ldots, N'_{r'}$ *corresponding to the reduced implications of the formula* A *is derivable in* Z^n_{sc} *and all variables except the reduction variables in each* M'_i *are parametrized, then the sequent,* $\Gamma \to \Delta, A$*, is derivable in* Z^{n+1}_{sc}.

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Proof. We apply left universal rules on the reduced formulas in the antecedent of each sequent and derive $M_1, \ldots, M_r, \Gamma \to \Delta, N'_1, \ldots, N'_{r'}$. Then we apply right universal rules on the reduced formulas in the succedent of each sequent and derive $M_1, \ldots, M_r, \Gamma \to \Delta, N_1, \ldots, N_{r'}$. The eigenvariable condition holds because the reduction variables of the formulas are fresh and independent. By lemma 5.4 the derivability of A is equivalent to the derivability of its fundamental implications. Thus, we can by propositional rules derive the sequent $\Gamma \to \Delta, A$. □

5.1. The soundness theorem in sequent calculus. Lemma 5.6 proves that the sequent calculus Z_{sc} is complete. What remains to be proved is the sequent calculus version of the soundness theorem. The soundness theorem is a strengthening of lemma 5.5 above that puts a stricter limit on the order of all formulas occurring in the derivations of the reduced implications.

LEMMA 5.7 (Soundness). *If a sequent, $\Gamma \to \Delta, A$, is derivable in Z^{n+1}_{sc} and the formulas of Γ and Δ are of order $m < n + 1$, then for each fundamental implication of A a sequent $M'_1, \ldots, M'_r, \Gamma \to \Delta, N'_1, \ldots, N'_{r'}$ corresponding to the reduced implication is derivable in Z^n_{sc}.*

By examining the rules of the calculus Z_{sc} it can be observed that the only rule, where the maximum order of the formulas in a sequent can be higher in the premises than in the conclusion, is the cut rule. Thus, if potential cuts on formulas of $n + 1$ can be eliminated from the derivation of the sequent $M'_1, \ldots, M'_r, \Gamma \to \Delta, N'_1, \ldots, N'_{r'}$, then this is a derivation in Z^n_{sc}. In fact, full cut elimination is provable for Z^n_{sc} by theorem 6.2.

§6. A cut-elimination procedure based on invertibility. As was mentioned in section 4 a cut elimination strategy for predicate logic based on invertibility requires duplication and repositioning of cuts if the cut formula is a universally quantified. A similar cut elimination strategy can be applied to the system Z^n_{sc}, but the complications are transferred from the universal case to the rule that does not have full invertibility, that is, the left implication rule. However, if an implication contains nonparametrized variables, then a cut on the formula can be converted into cuts that contain fewer nonparametrized variables.

LEMMA 6.1. *Consider a derivation in Z^n_{sc}. If the antecedent, A, of the cut formula $A \supset B$ in a right premise of a cut, $A \supset B, \Gamma \to \Delta$, contains nonparametrized variables, then the cut can be converted into a valid derivation in Z^n_{sc} with cuts on implications that contain fewer nonparametrized variables.*

Proof. Let the cut on the formula $A(y) \supset B$ in a given derivation be

$$\dfrac{\Gamma \to \Delta, A(y) \supset B \qquad \begin{array}{c}\Pi\\ \vdots\\ A(y) \supset B, \Gamma \to \Delta\end{array}}{\Gamma \to \Delta}\ cut$$

where y is a nonparametrized variable. By tracing up in the derivation Π we locate substitution rules on the implication. If the formula occurs in the context of a rule during the trace, then the trace continues from each premise of the rule. There are two cases in which the trace continues if the formula is the principal formula of the rule; contraction and induction. Because induction variables are parametrized the nonparametrized variables cannot be induction variables.

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If the trace terminates in an initial sequent, and the implication is not the active formula, then it can be deleted. If the implication is the active formula, then we can replace the initial sequent by the left cut premise. If the trace terminates in an initial sequent $L\bot$ or an arithmetical rule without premises, then the implication is in the context of the sequent and can be deleted.

If the trace terminates in a substitution rule in which the formula is principal,

$$\frac{A(t) \supset B, \Gamma' \to \Delta'}{A(y) \supset B, \Gamma' \to \Delta'}\ LSub.$$

Then the rule substitution rule can be replaced with a cut on the premise of the substitution, which contains fewer nonparametrized variables.

$$\dfrac{\dfrac{\dfrac{\Gamma \to \Delta, A(t) \supset B}{\Gamma, \Gamma' \to \Delta, \Delta', A(t) \supset B}\ RWandLW \quad \dfrac{A(t) \supset B, \Gamma' \to \Delta'}{A(t) \supset B, \Gamma, \Gamma' \to \Delta, \Delta'}\ RWandLW}{\Gamma, \Gamma' \to \Delta, \Delta'}\ cut}{\begin{array}{c}\vdots\ \Pi \\ \dfrac{\Gamma, \Gamma \to \Delta, \Delta}{\Gamma \to \Delta}\ RCandLC\end{array}}$$

Because the variable y in the derivation of the left cut premise of the original cut is parametrized it can be substituted with the substitution term t to obtain the left cut premise of the new cut. We note that the variable does not occur in Δ because it is nonparametrized in the antecedent of the right cut premise. Potential occurrences of the substitution variable in the multi set Γ are also replaced with the term, but can be reintroduced by applications of substitution. Thus, we have a derivation of the sequent $\Gamma \to \Delta, A(t) \supset B$.

If the trace terminates in an instance of the left implication rule in which the formula is principal

$$\frac{\Gamma' \to \Delta', A(y) \quad B, \Gamma' \to \Delta'}{A(y) \supset B, \Gamma' \to \Delta'}\ L\supset$$

then the nonparametrized variables would occur in the succedent of the left premise and is not parametrized in the subderivation of the premise. These occurrences of the implication can be left unaltered.

The derivation below can be recomposed by the rules that we have traced past. □

6.1. Full cut elimination. We can now state the theorem for full cut elimination in the system Z^n_{sc}.

THEOREM 6.2 (Cut-elimination theorem). *If the sequent* $\Gamma \to \Delta$ *is derivable in* Z^{n+1}_{sc}, *then it is derivable without cut.*

Proof. By structural induction on infinite derivations, or bar induction, we prove that cuts can be eliminated from the derivation. We assume that cut-elimination holds for the premises of a cut.

$$\dfrac{\begin{array}{c}\Pi \\ \vdots \\ \Gamma \to \Delta, A \end{array} \quad \begin{array}{c} A, \Gamma \to \Delta \\ \vdots \\ \Pi \end{array}}{\Gamma \to \Delta}\ cut$$

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By induction on the order of the length of the cut formula and a subinduction on number of nonparametrized variables in the cut formula we prove that the conclusion of the rule can be derived without cut.

By the lemma 5.1 we have total invertibility for all propositional rules except for implication. If the cut formula is an implication that has nonparametrized variables in the antecedent, then by lemma 6.1 the cut can be converted into cuts on implications with fewer nonparametrized variables. By invertibility of implications that contain no nonparametrized variables we can eliminate these cuts.

Assume that we have a cut on a universal formula, $\forall x A(x)$. By lemma 5.3 we have partial invertibility of the left universal rule and can derive the sequent $A(x), \Gamma \to \Delta$ by replacing the cut formula $\forall x A(x)$ in the right cut premise with $A(x)$ for a nonparametrized variable x. By lemma 5.2 we have complete invertibility of the right universal rule and can derive the sequent $\Gamma \to \Delta, A(y)$ where y is a fresh parametrized variable. Because y is parametrized and only occurs in A we can substitute x for y and derive $\Gamma \to \Delta, A(x)$. By a cut on the shorter formula we derive the conclusion of the original cut, $\Gamma \to \Delta$.

Thus, to complete the proof we need to eliminate cuts on atomic formulas. This is done in the proof of the following lemma 6.3. □

6.2. Cut-elimination on atomic formulas.

LEMMA 6.3 (Cut-elimination on atomic formulas). *Cuts on atomic formulas in derivations in Z_{sc}^{n+1} can be eliminated.*

Proof. Cut elimination for atomic formulas is proven by structural induction on derivations. We consider a cut on an atomic formula, A, in a given derivation.

We trace up from the right cut premise as in lemma 6.1 past rules where the formula is not principal. If we reach a rule where the formula is principal, then this rule must be a contraction, an induction or a substitution rule. If the rule is a contraction on the trace formula, then we continue the trace from each contraction formula. If the rule is an induction rule where the cut formula is one of the principal formulas, then the trace continues from the active formula of each premise. If the rule is a left substitution where the cut formula is the principal formula, then we continue the trace from the active formula in the premise.

Termination of the trace: Thus, we reach a leaf of the derivation with the atomic trace formula in the antecedent. If the leaf is an initial sequent, where the cut formula is active, then we replace the formula by the left cut premise and introduce the context of the replaced sequent by weakening. If the cut formula is not active, then it is deleted. If the leaf is an arithmetical rule without premises, then the cut formula is not active and can be deleted. The derivation below the sequent is then recomposed by deleting all descendants of the cut formula. □

§7. Principle of termination. Cut-elimination is proved by structural transformations of derivations by which cuts are simplified until they take on a form that can be eliminated. By Schütte (Schütte, 1951) the termination of the cut-elimination procedure for an infinite arithmetical system can be proved with transfinite induction on ordinals up to ε_0.

Gentzen (Gentzen, 1938) also used transfinite induction restricted to elementary computable predicates to prove consistency for a standard system of Peano Arithmetic. However, the earlier version of Gentzen's proof that was published posthumously in 1974 (Gentzen, 1974) implicitly used bar induction as the principle of termination.

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The proof is known for the controversy in its publication process. It was submitted to Mathematische Annalen in August 1935 but withdrawn before publication due to criticism that sparked a debate on constructive termination principles. Bernays, who apparently had critisised the proof for making nonconstructive assumptions, later clarified why the argument of the proof, in fact, is constructive (Bernays, 1970). The issue at hand is that for ω-arithmetic a bottom-up construction of the reduction tree of a sequent does not necessarily terminate, but if a top-down deduction tree is inductively given for a sequent, then the cut-elimination procedure can be proved by structural induction on the deduction tree (Tait, 2015). The principle for structural induction on derivation trees with a countably infinite number of premises follows from the principle of bar induction. Formally bar induction can be defined by:

DEFINITION 7.1 (Bar induction). *Let R and S be predicates for finite lists of natural numbers. Given the following premises:*

1. *R is decidable.*
2. *Every choice sequence has a finite prefix satisfying the predicate R.*
3. *If for a finite list α, $R(\alpha)$ holds, then $S(\alpha)$ also holds.*
4. *Let α be a finite list and α^*a a list extended by an element a. If $S(\alpha^*a)$ holds for all a, then $S(\alpha)$ also holds.*

We conclude that the predicate S holds for the empty list.

Assume that a derivation in ω-arithmetic is given. A branch in a derivation is defined as a list of sequents that begins with the conclusion of the derivation and is extended with one premise of the rule that concludes the previous sequent in the list. For any rule in the calculus we let numbers n code the different premises of the rule. Thus, a branch in the given derivation is coded by a list of natural numbers. Let the predicate R be satisfied by lists that code branches in the derivation tree that ends with a leaf. Let the predicate S be satisfied by a list if the list codes a partial branch in the derivation tree and the last number of the list codes a sequent in the derivation that has some property P.

The predicate R is decidable by comparing the list with the derivation tree. Every choice sequence has a finite prefix satisfying the predicate R because all branches are finite. The base case of the structural induction on derivations states that the leafs of the derivation have property P. Thus, if $R(\alpha)$ holds, then $S(\alpha)$ also holds. The inductive step of the structural induction on derivations states that if the premises of a rule have the property P, then we can prove that the conclusion of the rule also has property P. Thus, if $S(\alpha^*a)$ holds for all a, this means that P holds for all elements a that code extensions of a partial branch and $S(\alpha)$ states that the property P holds for the conclusion of the rule. By bar induction 7.1 it is concluded that P holds for the conclusion of the derivation.

The principle of termination used by Stenius is induction of the third kind (Induktion dritter Art).[8]

DEFINITION 7.2 (Induction of the third kind). *If it has been established that all members of the infinite recursive class T belong to the recursive class K and that the object α is related to all members of T by a binary relation, then α belongs to K.*

[8] (Stenius, 1952), p.23.

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Let T be the set of premises of an instance of the infinite induction rule, let α be the conclusion of the induction rule and let K be the class of sequents that satisfies the property P. If $P(\beta)$ holds for each premise of the induction rule, then by the principle of induction of the third kind one can conclude that $P(\alpha)$. Thus, induction of the third kind is a principle that validates the inductive step of a structural induction in the case of a rule with an infinite number of premises.

Definition 7.2 together with a principle for structural induction on finite rules is enough for Stenius to justify structural induction on derivations in ω-arithmetic. In section 7 of (Stenius, 1952) the relationship between induction of the third kind and transfinite induction is clarified. Stenius proves that the termination principle required for ω-arithmetic can be replaced with a finite structural induction (so called induction of second kind) combined with transfinite induction with a modification of the system.

> The deduction rule (a) [the rule for infinite induction] clearly operates with an infinite proof figure, and the induction on the system Z_a is therefore an induction of the third kind. But the use of an infinite proof figure can be eliminated by replacing the deduction rule (a) with a rule of inference defined by transfinite recursion. The rule operates on proof figures, that admittedly contain an arbitrary number of formulas, but they are always finite, and therefore the principle of induction of the third kind can be replaced with the principle of induction of the second kind combined with transfinite induction. (Stenius, 1952)[p. 92][9]

§8. Consistency of ω-arithmetic. As a corollary of the cut-elimination theorem we can prove a consistency theorem for ω-arithmetic.

DEFINITION 8.1. *A sequent system is inconsistent if the empty sequent $\rightarrow$ is derivable in the system. If a system is not inconsistent, then it is consistent.*

The consistency proof states that ω-arithmetic is consistent if and only if the quantifier-free system is consistent. That is Z_{sc} is consistent if and only if Z^0_{sc} is consistent. With our theorem of full cut elimination we even prove that ω-arithmetic is consistent if and only if the system restricted to atomic formulas is consistent.

COROLLARY 8.2. *Z_{sc} is consistent if and only if the system restricted to atomic formulas is consistent.*

Proof. A system restricted to atomic formulas is a subsystem of Z_{sc} and therefore a derivation of the empty sequent in the former system is a valid derivation in the latter.

If Z_{sc} is inconsistent, then the empty sequent $\rightarrow$ is derivable. If any formula with logical structure occurs in the derivation, then this formula must disappear with cut. By the cut-elimination theorem 6.2 we have a cut-free derivation of the empty sequent and therefore a derivation that only contains atomic formulas. □

[9] Die Schlussregel (a) [the rule for infinite induction] operiert - anschaulich gesprochen - mit einer >unendlichen< Schlussfigur, und die Induktion über das System Z_a ist daher eine Induktion >dritter< Art. Man kann aber die Verwendung einer unendlichen Schlussfigur dadurch eliminieren, dass man die Schlussregel (a) durch eine mittels transfiniter Rekursion definierte Schlussregel ersetzt, die mit Schlussfiguren operiert, die zwar beliebig viele Formeln erhalten, die aber stets endlich sind, wobei die Induktion >dritter< Art durch eine Induktion >zweiter< Art kombiniert mit einer transfiniten Induktion ersetzt wird.

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§9. Herbrand's theorem. Bernays gave a short lecture in 1954 at the International Congress of Mathematicians with the title *On the connection between Herbrand's theorem and the new results of Schütte and Stenius* (Über den Zusammenhang des Herbrandschen Satses mit den neueren Ergebnissen von Schütte and Stenius) (Bernays, 1954). Bernays had noticed that Herbrand's theorem follows as a corollary from Stenius' result if we restrict the calculus to a predicate calculus without ω-induction. In the abstract for the conference he writes that

> Recently Gentzen's proof has been simplified by Schütte and alternatively Stenius' proof theoretical investigation of number theory also contains a proof of Herbrand's theorem. In both of the two works, however, this result is not explicit. ... A proof of Herbrand's theorem can be extracted from the reflections of Stenius by applying his result to the following theorem: If an axiomatic first-order system that does not contain bound variables can derive a formula without bound variables with the help of predicate logic, then the bound variables can be eliminated from the derivation too. (Bernays, 1954)[10]

Herbrand's theorem states that if we can derive a formula in prenex form, then we can derive a disjunction of instances of the formula without the quantifiers. In general, Herbrand's theorem can be proved for a system of sequent calculus extended with mathematical rules as in (Negri&von Plato, 2001), p.142. The following formulation of the theorem is taken from the mentioned book.

Let the system PT be our sequent calculus system for predicate calculus (excluding the substitution rule) extended with purely universal axioms of a theory T converted into rules of proof. Thus, these rules have atomic formulas active in the succedent of the rule.

THEOREM 9.1 (Herbrand's theorem). *If the sequent* $\to \forall x \exists y_1 \ldots \exists y_k A$, *with* A *quantifier-free, is derivable in the system* PT, *then there are terms* t_{i_j} *with* $i \leqslant n$ *and* $j \leqslant k$, *such that*

$$\to \bigvee_{i=1}^{n} A(t_{i_1}/y_1, \ldots, t_{i_k}/y_k)$$

is derivable in PT.

Proof. If the sequent is derivable in PT we can add the left substitution rule to the system and get a system PT*. We have cut elimination for these systems as in ω-arithmetic. The invertibility of the $R\forall$, $L\forall$, $R\supset$ and $L\supset$-rules can be proved in the same way for the system PT* as for ω-arithmetic.

Let the formula $\forall x \exists y_1 \ldots \exists y_k A$ be in prenex form. We convert the formula into the classically equivalent $\forall x \neg \forall y_1 \ldots \forall y_k \neg A$.

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[10] Neuerdings ist der Gentzen'sche Beweis durch Schütte vereinfacht worden, und andererseits enthält auch die beweistheoretische Untersuchung von Stenius über die Zahlentheorie einen Beweis des Herbrand'schen Satzes. In den beiden letztgenannten Arbeiten tritt dies Resultat jedoch nicht ganz explicit hervor. ... Aus den Überlegungen von Stenius lässt sich ein Beweis des Herbrand'schen Satzes entnehmen, indem man den folgenden in den Stenius'schen Ergebnissen enthaltenen Satz verwendet: Wenn aus einem Axiomensystem der ersten Stufe, dessen Axiome keine gebundenen Variablen enthalten, sich eine Formel ohne gebundene Variable mit Hilfe des Prädikatenkalkuls herleiten lässt, dann können die gebundenen Variablen auch aus der Herleitung entfernt werden.

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Thus, we get that if the sequent $\rightarrow \forall x \exists y_1 \ldots \exists y_k A$ is derivable in PT*, then we can by invertibility derive the sequent $A(x, y_1, \ldots, y_k) \supset \bot \rightarrow$ for nonparametrized variables y_i and a parametrized variable x. Because the antecedent of the implication contains nonparametrized variables we do not have full invertibility of the $L \supset$-rule.

To eliminate the substitution rule from the derivation, we can trace up in the derivation. When we reach a contraction on the implication we delete the contraction rule and continue the trace from each contraction formula. When we reach a left substitution on the trace implication we delete the rule and continue the trace with the premise formula instead.

By continuing the trace we can only get other formulas, Γ, in the antecedent if the $R \supset$-rule has been used and this requires that no formula in Γ contains nonparametrized variables. Thus, all instances of the $LSub$-rule are on the implications that are traced.

When a $L \supset$-rule on an implication that we are tracing is reached we continue the trace from the left premise. The right premise will be a $L\bot$-initial sequent and the left premise of the rule will be a sequent $[A(x, t_1^i, \ldots, t_k^i) \supset \bot]_{i=1}^{p-1}, \Gamma \rightarrow \Delta, A(x, t_1^n, \ldots, t_k^n)$ with several copies of the implication in the antecedent. Eventually we will reach a leaf, which is an initial sequent or $L\bot$-initial sequent or mathematical rule without premises. In all cases we can delete eventual trace formulas from the context of the antecedent. By a recomposition of the derivation we get a derivation without $LSub$, by deleting the $LSub$-rules, contractions on the trace formula and the $L \supset$-rules. Instead we will have several instances of the formula A in the succedent that are bounded by the finite number of contractions in the derivation. Since the formula A is quantifier-free there are no $R\forall$-rules in the derivation that have eigenvariable restrictions that could be violated by keeping the terms t in the succedent.

Thus, we can derive the sequent $\rightarrow [A(t_{i_1}/y_1, \ldots, t_{i_k}/y_k)]_{i=1}^{n}$ in PT and by $R\vee$ we get the derivation of the sought disjunction. □

§10. Conclusion. Stenius' major work in proof theory presents a sound and complete semantic interpretation for infinite ω-arithmetic. The proof can be interpreted as a method for cut elimination that simplifies the treatment of quantification. When the proof is interpreted in sequent calculus the similarities between Schütte's and Stenius' work is apparent. Both proofs prove cut elimination by invertibility of the rules. The main difficulty of Schütte's standard calculus arises in the case of universal cut formulas. Due to lack of invertibility the proof transformation require a duplication and repositioning of the cuts, in order to convert a cut to cuts on sub formulas. By introducing a substitution rule invertibility can be proved for universal formulas, but as a consequence the implication reductions inherit the duplication and repositioning of the cuts. The simplification of the universal case makes the Stenius-Bernays method relevant for proving Herbrand's theorem, which was noticed by Bernays.

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