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A MODULAR ANDRÉ–OORT STATEMENT WITH DERIVATIVES

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Abstract In unpublished notes, Pila discussed some theory surrounding the modular function j and its derivatives. A focal point of these notes was the statement of two conjectures regarding j, j' and j'': a Zilber–Pink-type statement incorporating j, j' and j'', which was an extension of an apparently weaker conjecture of André–Oort type. In this paper, I first cover some background regarding j, j' and j'', mostly covering the work already done by Pila. Then I use a seemingly novel adaptation of the o-minimal Pila–Zannier strategy to prove a weakened version of Pila's 'Modular André–Oort with Derivatives' conjecture. Under the assumption of a certain number-theoretic conjecture, the central theorem of the paper implies Pila's conjecture in full generality, as well as a more precise statement along the same lines.

Keywords: definable set; rational point; André–Oort conjecture; derivatives of j; Pila–Wilkie theorem

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1. Introduction

The modular André–Oort conjecture can be viewed as a statement about the arithmetic and algebraic properties of the classical modular function

$$j : \mathbb{H} \to \mathbb{C}.$$

The statement is the following.

Theorem 1.1 (Pila, Modular André–Oort). *Let $V \subseteq \mathbb{C}^n$ be an algebraic variety. Then V contains only finitely many maximal special subvarieties.*

While various partial [1] and conditional [7] results were known, this was first proven unconditionally and in full generality by Pila, in his 2011 paper [14], using what is now a standard strategy employing ideas of o-minimality and point-counting. The connection with the j function is obscured behind the definition of 'special subvariety', which is as follows. It is well known that there are modular polynomials $\Phi_N \in \mathbb{Z}[X, Y]$ with the property that

$$\Phi_N(j(g\tau), j(\tau)) = 0$$

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whenever $g \in \mathrm{GL}_2^+(\mathbb{Q})$ is a primitive integer matrix of determinant N. So, although j is a transcendental function, it is very well behaved under the action of $\mathrm{GL}_2^+(\mathbb{Q})$. A fairly direct consequence of the existence of the polynomials Φ_N is the (also well-known) fact that $j(\tau)$ is an algebraic integer whenever $\tau \in \mathbb{H}$ is quadratic.

Loosely, a special subvariety of $\mathbb{C}^n$ is a variety induced by these relations. To be more precise, we have the following definition.

Definition 1.2. Let $n \in \mathbb{N}$.

Let $S_0 \cup S_1 \cup \cdots \cup S_k$ be a partition of $\{1, \ldots, n\}$, where $k \geq 0$ and $S_i \neq \emptyset$ for $i > 0$. For each $s \in S_0$, choose any point $q_s \in \mathbb{H}$. For each $i > 0$, let s_i be the least element of S_i and for each $s \in S_i \setminus \{s_i\}$ choose a matrix $g_{i,s} \in \mathrm{GL}_2^+(\mathbb{Q})$. A *weakly* $\mathbb{H}$*-special* subvariety of $\mathbb{H}^n$ is a set of the form

$$\{(\tau_1, \ldots, \tau_n) \in \mathbb{H}^n : \tau_s = q_s \text{ for } s \in S_0, \tau_s = g_{i,s}\tau_{s_i} \text{ for } s \in S_i, s \neq s_i, i = 1, \ldots, k\},$$

for some given data S_i, q_s, $g_{i,s}$.

A weakly $\mathbb{H}$-special subvariety is $\mathbb{H}$*-special* if $S_0 = \emptyset$ or if the constant factors q_s are imaginary quadratic numbers for all $s \in S_0$.

One can define special (henceforth called j-special) subvarieties in a similar way, as varieties in $\mathbb{C}^n$ cut out by the modular polynomials Φ_N. For our purposes, however, there is a simpler definition that suits us better. By abuse of notation, we will write j for the function

$$(\tau_1, \ldots, \tau_n) \mapsto (j(\tau_1), \ldots, j(\tau_n)).$$

A j-special variety in $\mathbb{C}^n$ is then the image under j of an $\mathbb{H}$-special variety; similarly, a weakly j-special variety is the image under j of a weakly $\mathbb{H}$-special variety.

In this paper, I investigate what happens when we consider not only j and its corresponding special varieties, but also the derivatives of j. In this setup, the situation is rather more complicated. To begin with, recall that j satisfies a certain third-order differential equation over $\mathbb{Q}$. Hence, it is enough to consider only j and its first two derivatives. With this in mind, let us define a function that will be our central object of study in this paper:

$$J = (j, j', j'') : \mathbb{H} \to \mathbb{C}^3,$$
$$J(\tau) = (j(\tau), j'(\tau), j''(\tau)).$$

Again, we will abuse notation and use J also to refer to the obvious map from $\mathbb{H}^n$ to $\mathbb{C}^{3n}$, defined as J on each coordinate.

Much of the setup here is due to Pila; in unpublished notes, he compiled various properties of J and gave definitions of what a J-special variety should be. Pila also made a conjecture analogous to 1.1, with respect to J. For the rest of this section, I will give some of the setup covered by Pila in his notes. Towards the end of the section, I state Pila's conjecture and a weakened version of the conjecture, which is the central theorem of the paper.

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1.1. Properties of J

The functions j' and j'' are not fully modular functions. Instead, they satisfy

$$j'(\gamma\tau) = (c\tau + d)^2 j'(\tau)$$

and

$$j''(\gamma\tau) = (c\tau + d)^4 j''(\tau) + 2c(c\tau + d)^3 j'(\tau),$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

This states that j' is a meromorphic modular form of weight 2, while j'' is a so-called *quasimodular* form of weight 4 and depth 1. The behaviour of j' and j'' at quadratic points is also worse than that of j: while $j(\tau)$ is algebraic at quadratic τ, $j'(\tau)$ and $j''(\tau)$ almost never are. In fact, such points are always transcendental unless τ is in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of i or $\rho = \mathrm{e}^{(2\pi i/3)}$ (see [4]). The number $j''(\tau)$, however, is algebraic over $j'(\tau)$, whenever τ is quadratic. The proof of this is as follows.

First, we define an almost holomorphic modular (AHM) function by

$$\chi^* = 1728 \cdot \frac{E_2^* E_4 E_6}{E_4^3 - E_6^2},$$

where the E_k are standard Eisenstein series and E_2^* is the weight 2 AHM form defined by

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \operatorname{Im} \tau}.$$

We can decompose χ^* as

$$\chi^* = \chi - \frac{3}{\pi \operatorname{Im} \tau} \cdot f,$$

where χ and f are, here and throughout the paper, holomorphic functions defined by

$$\chi = 1728 \cdot \frac{E_2 E_4 E_6}{E_4^3 - E_6^2}, \qquad f = 1728 \cdot \frac{E_4 E_6}{E_4^3 - E_6^2}. \tag{1.1}$$

The function χ^* and related AHM functions have been studied in several places, perhaps first by Masser in [8, Appendix 1], as well as by Mertens and Rolen [9] and Zagier [23, § 5.3], among others. I have studied χ^* in the context of an André–Oort result [20]; we will be making use of some results from that paper occasionally.

It follows from the basic theory of AHM functions (see [20, § 2]) that any AHM function ϕ satisfies

$$\phi(\tau) = q(\operatorname{Im} \tau, j(\tau), j'(\tau), j''(\tau))$$

for some fixed rational function q. For χ^* in particular, this equation can be rearranged, yielding a rational function p_c such that

$$p_{\operatorname{Im} \tau}(j(\tau), \chi^*(\tau), j'(\tau)) = j''(\tau)$$

for all τ. Although the precise form of p_c will not be very relevant, we include it for the reader's interest; a calculation involving the Eisenstein series that make up j, j', j'' and

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χ^* shows that

$$p_c(W,X,Z) = \frac{1}{6}\frac{Z^2(X-7W+6912)}{W(W-1728)} + \frac{iZ}{c}.$$

In [8, Appendix 1, Theorem A1], Masser shows that a certain AHM function, which he calls ψ (the definition of which can be found at the top of page 114 of [8]), satisfies $\psi(\tau) \in \overline{\mathbb{Q}}$ whenever τ is quadratic. Our AHM function χ^* satisfies $\chi^* = (j-1728)\psi$, whence $\chi^*(\tau)$ is also algebraic for quadratic τ. As a consequence, we see that $p_{\mathrm{Im}\,\tau}$ has coefficients in $\overline{\mathbb{Q}}$ whenever τ is quadratic. It follows, as claimed, that

$$\mathrm{tr.deg.}_{\mathbb{Q}}(j(\tau), j'(\tau), j''(\tau)) = 1$$

for quadratic $\tau \notin \mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$.

By differentiating the modular polynomials, we see that J also has nice behaviour with respect to $g \in \mathrm{GL}_2^+(\mathbb{Q})$. We will briefly work out the details of this in the case of j'. Let $g \in \mathrm{GL}_2^+(\mathbb{Q})$ be a primitive integer matrix of determinant N. Since

$$\Phi_N(j(\tau), j(g\tau)) = 0,$$

we see that

$$\partial_X(\Phi_N)(j(\tau), j(g\tau))j'(\tau) + \partial_Y(\Phi_N)(j(\tau), j(g\tau))j'(g\tau)\frac{\mathrm{d}(g\tau)}{\mathrm{d}\tau} = 0$$
$$\implies j'(g\tau) = -j'(\tau)m_g(\tau)\lambda_N(j(\tau), j(g\tau)),$$

where

$$\lambda_N = \frac{\partial_X(\Phi_N)}{\partial_Y(\Phi_N)}$$

and

$$m_g(\tau) = \frac{(c\tau+d)^2}{\det g} = \left(\frac{\mathrm{d}(g\tau)}{\mathrm{d}\tau}\right)^{-1}.$$

For j'', we have a similar relation, though a bit more complex. In both cases, the nature of the relation differs depending on whether g is upper triangular. If g fails to be upper triangular (i.e. $c \neq 0$), then the relation between the functions $j'(g\tau)$ and $j'(\tau)$ (respectively for j'') only exists over $\mathbb{C}(\tau)$. Otherwise the relationship is over $\mathbb{C}$. Hence

$$\mathrm{tr.deg.}_{\mathbb{C}}(J, J \circ g) = 3$$

(as functions on $\mathbb{H}$) if g is upper triangular and

$$\mathrm{tr.deg.}_{\mathbb{C}}(J, J \circ g) = 4$$

otherwise.

Since the behaviour of J is affected by whether or not a matrix is upper triangular, it makes sense to make the following definition.

Definition 1.3. A weakly $\mathbb{H}$-special variety G is called a geodesic upper-triangular (GUT) variety if all of the $g_{i,s} \in \mathrm{GL}_2^+(\mathbb{Q})$ arising in its definition (see Definition 1.2) are upper-triangular matrices.

We will be making use of GUT varieties often, later on.

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1.2. Special sets for J

In the classical case, the special sets were just the images, under j, of $\mathbb{H}$-special sets. An important feature is that they are bi-algebraic; for $\mathbb{H}$-special sets G, both G and $j(G)$ are algebraic sets defined over $\overline{\mathbb{Q}}$. In the J case, as noted earlier, $J(G)$ is not necessarily an algebraic set, let alone defined over $\overline{\mathbb{Q}}$. The solution to this is fairly simple; just take Zariski closures over $\overline{\mathbb{Q}}$. We are still following Pila, and will use the following notation, also of his design.

Notation 1.4. For any subset $S \subseteq \mathbb{H}^n$, define $\langle\langle S\rangle\rangle$ to be the $\overline{\mathbb{Q}}$-Zariski closure of $J(S)$. That is, $\langle\langle S\rangle\rangle$ is the smallest algebraic variety, defined over $\overline{\mathbb{Q}}$, which contains $J(S)$.

Abusing notation slightly, we also use this terminology to apply to individual points in $\mathbb{H}$, equating points $\tau \in \mathbb{H}$ with singleton sets $\{\tau\}$ in the obvious way.

Definition 1.5. A J-special subvariety of $\mathbb{C}^{3n}$ is an irreducible component of any set of the form $\langle\langle G\rangle\rangle$, where G is an $\mathbb{H}$-special set.

With this definition made, we might conjecture a direct analogue of 1.1, with j-special varieties replaced by J-special ones. This fails for a fairly obvious reason. Consider the variety $V \subseteq \mathbb{C}^3$ defined just by

$$X_1 = j(\tau),$$

for some fixed quadratic $\tau \in \mathbb{H} \setminus \mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$. Then by modularity of j, V contains all the points $J(\gamma\tau)$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. In particular, taking Zariski closures, we have

$$\langle\langle \gamma\tau\rangle\rangle \subseteq V$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Since

$$\langle\langle \gamma\tau\rangle\rangle = \{(j(\tau), w, p_{\mathrm{Im}\,\gamma\tau}(j(\tau), \chi^*(\tau), w)) : w \in \mathbb{C}\},$$

one sees that there are infinitely many distinct such $\langle\langle \gamma\tau\rangle\rangle$. So V contains infinitely many J-special sets. They are maximal, since $\langle\langle \mathbb{H}\rangle\rangle = \mathbb{C}^3 \not\subseteq V$.

With this in mind, we need a version of 1.1 that takes into account the action of $\mathrm{SL}_2(\mathbb{Z})$. The abovementioned conjecture of Pila is one such version. To state it, we will need a definition.

Definition 1.6. Let $\mathcal{S}$ be a collection of subsets of $\mathbb{H}^n$. We say $\mathcal{S}$ is $\mathrm{SL}_2(\mathbb{Z})$-finite if there is some finite subcollection $\mathcal{T} \subseteq \mathcal{S}$, such that every $S \in \mathcal{S}$ takes the form

$$S = \gamma \cdot T,$$

for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$, $T \in \mathcal{T}$. Otherwise, $\mathcal{S}$ is $\mathrm{SL}_2(\mathbb{Z})$-infinite.

As above, we abuse notation and apply this terminology also to collections of points, equating points τ with singleton sets $\{\tau\}$.

Now we can state Pila's conjecture.

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Conjecture 1.7 (Pila, 'Modular André–Oort with Derivatives'). *Let $V \subseteq \mathbb{C}^{3n}$ be a proper algebraic variety defined over $\overline{\mathbb{Q}}$. There exists an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma(V)$, consisting of* proper *$\mathbb{H}$-special varieties of $\mathbb{H}^n$, with the following property. Every J-special subvariety of V is contained in $\langle\langle G\rangle\rangle$ for some $G \in \sigma(V)$.*

We will be approaching this conjecture using a variant of the Pila–Zannier strategy and o-minimality. With a seemingly novel adaptation of the usual strategy, we are able to make some good progress. The o-minimal methods are sufficient to give us good control over quadratic points of the form

$$(g_1\sigma, \dots, g_n\sigma).$$

In this case, the methods yield a bound on the size of the determinants of the $g_i \in \mathrm{GL}_2^+(\mathbb{Q})$ and on the discriminant of σ. This is a good step towards Conjecture 1.7.

There is significant difficulty, however, in dealing with points having a more complex $\mathrm{GL}_2^+(\mathbb{Q})$-structure. The difficulty lies in the possibility that, for two quadratic points $\tau, \sigma \in \mathbb{H}$, it might happen that $j'(\tau)$ and $j'(\sigma)$ are algebraically dependent even when τ and σ lie in distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits. As we will discuss shortly, we do not expect this to happen, but it does not seem to be feasible to exclude the possibility using o-minimal methods. So we need the following definition.

Definition 1.8. Let $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{H}^n$ be a quadratic point, and suppose that $\tau_1, \dots, \tau_n$ lie in precisely k distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits. We say τ is j'-generic if

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(j'(\tau_1), \dots, j'(\tau_n)) = k.$$

Now we can state the central theorem of the paper.

Theorem 1.9. *Let $V \subseteq \mathbb{C}^{3n}$ be a proper algebraic variety defined over $\overline{\mathbb{Q}}$. There exists an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma(V)$, consisting of* proper *$\mathbb{H}$-special varieties of $\mathbb{H}^n$, with the following property. Every j'-generic $\mathbb{H}$-special point in $J^{-1}(V)$ is contained in some $G \in \sigma(V)$.*

Note in particular that any special point of the form $(g_1\sigma, \dots, g_n\sigma)$, with $g_i \in \mathrm{GL}_2^+(\mathbb{Q})$, is automatically j'-generic, unless some $g_i\sigma$ is in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of i or ρ. Hence we have the following easy corollaries.

Corollary 1.10. *Let $V \subseteq \mathbb{C}^{3n}$ be a proper algebraic variety defined over $\overline{\mathbb{Q}}$. There exists an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma(V)$, consisting of* proper *$\mathbb{H}$-special varieties of $\mathbb{H}^n$, with the following property. Every $\mathbb{H}$-special point of the form $(g_1\sigma, \dots, g_n\sigma) \in J^{-1}(V)$, $g_i \in \mathrm{GL}_2^+(\mathbb{Q})$, is contained in some $G \in \sigma(V)$.*

Proof. This follows directly from 1.9, except for the possible existence of points with a coordinate lying in $\mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$. But any such points are automatically contained within $\mathrm{SL}_2(\mathbb{Z})$-finitely many proper $\mathbb{H}$-special varieties. □

Corollary 1.11. *Let $V \subseteq \mathbb{C}^3$ be a proper algebraic variety defined over $\overline{\mathbb{Q}}$. Then $J^{-1}(V)$ contains only $\mathrm{SL}_2(\mathbb{Z})$-finitely many $\mathbb{H}$-special points.*

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This last, of course, is simply Conjecture 1.7 for $n = 1$. To get 1.7 in full generality, we would need something like the following.

Conjecture 1.12. *Let $\tau_1, \dots, \tau_n \in \mathbb{H}$ be quadratic points, lying in distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits, none of which lies in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of i or ρ. Then $j'(\tau_1), \dots, j'(\tau_n)$ are algebraically independent over $\overline{\mathbb{Q}}$.*

In other words: 'All quadratic points in $\mathbb{H}^n$ are j'-generic, except those with a coordinate in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of i or ρ.' In this form, it is easy to see that the conjecture, together with Theorem 1.9, implies 1.7.

Theorem 1.13. *Assume Conjecture 1.12. Then Conjecture 1.7 holds.*

Proof. Immediate from Theorem 1.9. □

Should we believe Conjecture 1.12? It is strong, having a similar flavour to existing modular Schanuel statements; but it does fit into the existing body of conjectures. See, for instance, Bertolin's elliptico-toric conjecture (CET) [**2**], from which Conjecture 1.12 follows immediately. In turn, CET is a special case of the Grothendieck–André period conjecture. So 1.12 fits well with what we might expect. It is also much stronger than we need; to get Conjecture 1.7, various weaker transcendence statements would suffice. The weaker statements are much less clean to state and fit less obviously into the existing literature, so we stick with Conjecture 1.12 for this paper.

The paper is broken down as follows. Section 2 is dedicated to proving some Ax–Lindemann-type results. In this section lies the primary novelty of the paper; the Ax–Lindemann results we prove here are necessarily of a very new and unusual shape, in order to account for some problems that arise with point-counting. This section is completely independent from Conjecture 1.12 and does not involve considerations of j'-genericity. It is in § 3, where we discuss the point-counting aspects, that the issue of j'-genericity arises. Section 4 brings everything together to conclude the proof of 1.9. The penultimate section of the paper, § 5, is dedicated to proving a more precise version of Conjecture 1.7, under the assumption of Conjecture 1.12. In the final section, we apply this more precise result, together with a slight adaptation of the work of Scanlon [**17**], to produce versions of our results that are uniform in algebraic families.

2. Ax–Lindemann

2.1. Technicalities

We begin with a few crucial definitions.

Definition 2.1. A subset of $\mathbb{H}^n$ is a *Möbius variety* if, up to reordering of coordinates, it takes the form

$$G = \{(g_{1,1}\tau_1, \dots g_{1,r_1}\tau_1, \dots, g_{m,1}\tau_m, \dots, g_{m,r_m}\tau_m, t_{0,1}, \dots, t_{0,r_0}) : \tau_i \in \mathbb{H}\}, \tag{2.1}$$

where $t_{0,j} \in \mathbb{H}$ and $g_{i,j} \in \mathrm{SL}_2(\mathbb{R})$. We allow $m = 0$, so that in particular any point $\tau \in \mathbb{H}^n$ is a Möbius variety.

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A Möbius variety is called *basic* if $r_0 = 0$. So a basic Möbius variety is defined by finitely many matrices $g_{i,j} \in \mathrm{SL}_2(\mathbb{R})$, together with a permutation of the coordinates. We will often suppress the permutation of coordinates, assuming for simplicity that the variety takes exactly the form in (2.1).

Any positive-dimensional Möbius variety $G \subseteq \mathbb{H}^n$ has an underlying basic variety attached to it, namely the $B \subseteq \mathbb{H}^k$ attained by ignoring the constant coordinates $t_{0,j}$. We say that G is a *translation* of B (by the $t_{0,j}$), or that B is the basic variety underlying G.

Note. Any weakly $\mathbb{H}$-special set is itself a Möbius variety, since matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ can be rescaled to become elements of $\mathrm{SL}_2(\mathbb{R})$ without affecting their action on $\mathbb{H}$.

As we will see in later sections, the usual counting methods do not work out perfectly when applied to the derivatives problem. One would like to be able to count Möbius subvarieties contained in $J^{-1}(V)$, but the methods are slightly too coarse for this. One can, however, count those varieties that are, in some sense, *approximately* in $J^{-1}(V)$. This motivates the following.

Definition 2.2. Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety and $B \subseteq \mathbb{H}^k$ a basic Möbius variety, given by data $g_{i,j}$ as in (2.1). For each i, j, take a complex number $z_{i,j}$ and a real $c_{i,j} > 0$. Also take $n - k$ triples of complex numbers (w_l, x_l, y_l).

We say B is *adjacent to* V *via* $z_{i,j}, c_{i,j}, w_l, x_l, y_l$ if for all τ_i, we have, up to permutation of coordinates,

$$\left[\dots, j(g_{i,j}\tau_i), \frac{j'(g_{i,j}\tau_i)z_{i,j}}{m_{g_{i,j}}(\tau_i)}, p_{c_{i,j}}\left(j(g_{i,j}\tau_i), \chi^*(g_{i,j}\tau_i), \frac{j'(g_{i,j}\tau_i)z_{i,j}}{m_{g_{i,j}}(\tau_i)}\right), \dots, w_l, x_l, y_l, \dots\right] \in V.$$

If this holds for some choice of the possible data $z_{i,j}$, $c_{i,j}$, w_l, x_l, y_l, we simply say B is adjacent to V, and write

$$B \hookrightarrow V.$$

If this is the case and $G \subseteq \mathbb{H}^n$ is a translation of B, we will also say that G is adjacent to V (via the same data), and denote this by

$$G \hookrightarrow V.$$

This rather intricate definition turns out to be crucial in carrying out a suitable variant of the usual o-minimal strategy used for André–Oort problems. This is the main new idea of the paper: we follow the typical Pila–Zannier strategy for Diophantine problems of this type, but rather than counting those $\mathbb{H}$-special varieties that are contained in $J^{-1}(V)$ directly, we instead count the $\mathbb{H}$-special varieties that are adjacent to V. The notion of adjacency is constructed so as to be definable, invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ (on the $g_{i,j}$) and invariant under Galois action (on j and χ^*); those three conditions are precisely what we need to make the strategy work.

The first step of the strategy and the primary goal of this section is to prove an 'Ax–Lindemann-style' result pertaining to the notion of adjacency. The idea is that the only

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Möbius varieties (or more generally, real algebraic arcs) that can be adjacent to some variety V should be accounted for by weakly $\mathbb{H}$-special varieties. Given our overall goal—to count those Möbius varieties that are adjacent to V—this problem is directly analogous to the usual Ax–Lindemann theorems needed in the classical case. Unsurprisingly, we will need to use some existing Ax–Lindemann results.

Theorem 2.3. *Let $S \subseteq \mathbb{H}^n$ be an arc of a real algebraic curve and let G be the smallest weakly $\mathbb{H}$-special variety containing S. Suppose that G is a GUT variety. (Recall: this simply means that the matrices defining G are upper triangular.)*
Let $V \subseteq \mathbb{C}^{3n+1}$ be an algebraic variety such that

$$(\tau_1, J(\tau_1), \dots, J(\tau_n)) \in V$$

for all $(\tau_1, \dots, \tau_n) \in S$. Then in fact this holds for all $(\tau_1, \dots, \tau_n) \in G$.

Theorem 2.4. *Let $S \subseteq \mathbb{H}^n$ be an arc of a real algebraic curve and let G be the smallest weakly $\mathbb{H}$-special variety containing S. Let $V \subseteq \mathbb{C}^{2n}$ be an algebraic variety such that*

$$(\dots, j(\tau_j), \chi^*(\tau_j), \dots) \in V$$

for all $(\tau_1, \dots, \tau_n) \in S$. Then in fact this holds for all $(\tau_1, \dots, \tau_n) \in G$.

Both of the above were proven in [**20**], though the majority of the work towards 2.3 was done by Pila in [**15**]. Before we can apply these to prove our central Ax–Lindemann result, we will need some technical lemmas, the first of which is simply a strengthening of 2.3. For the statement of this theorem, recall that we defined in (1.1) two holomorphic functions χ and f, which together make up χ^*.

Theorem 2.5. *Let $S \subseteq \mathbb{H}^n$ be an arc of a real algebraic curve and let G be the smallest weakly $\mathbb{H}$-special variety containing S. Suppose that G is a GUT variety.*
Let $V \subseteq \mathbb{C}^{4n+1}$ be an algebraic variety such that

$$(\tau_1, j(\tau_1), j'(\tau_1), \chi(\tau_1), f(\tau_1), \dots, j(\tau_n), j'(\tau_n), \chi(\tau_n), f(\tau_n)) \in V$$

for all $(\tau_1, \dots, \tau_n) \in S$. Then in fact this holds for all $(\tau_1, \dots, \tau_n) \in G$.

Proof. Let F be a defining polynomial of V. We will represent the various coordinates as follows.

- The τ_1 coordinate will be represented by a variable T.
- The j-coordinates (i.e. the 2nd, 6th, and so on, coordinates) will be represented by variables $J_1, \dots, J_n$.
- The j'-coordinates will be represented by variables $K_1, \dots, K_n$.
- The χ-coordinates will be represented by variables $X_1, \dots, X_n$.
- The f-coordinates will be represented by variables $F_1, \dots, F_n$.

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Since j, j', χ and f are algebraically dependent, there is an irreducible polynomial p with the property that

$$p(j(\tau), j'(\tau), \chi(\tau), f(\tau)) = 0$$

for all τ. Consider the variety $W \subseteq \mathbb{C}^{4n+1}$ defined by

$$p(J_i, K_i, X_i, F_i) = 0$$

for each $1 \leq i \leq n$. Clearly $\dim W = 3n + 1$.

If we further impose the condition

$$F(T, J_1, K_1, X_1, F_1, \dots, J_n, K_n, X_n, F_n) = 0,$$

there are two possibilities. Either the resulting variety W_F still has dimension $3n + 1$ or it has dimension $3n$.

If $\dim W_F = 3n + 1$, it is automatically the case that

$$F(\tau_1, \dots, j(\tau_k), j'(\tau_k), \chi(\tau_k), f(\tau_k), \dots) = 0$$

for all $(\tau_1, \dots, \tau_n) \in \mathbb{H}^n$. On the other hand, if $\dim W_F = 3n$, then W_F amounts to the imposition of a relation between three of the four functions. That is, there are distinct $A, B, C \in \{J, K, X, F\}$ and a polynomial H, in $3n + 1$ variables, such that W_F is defined by

$$(T, J_1, \dots F_n) \in W$$

and

$$H(T, A_1, B_1, C_1, \dots, A_n, B_n, C_n) = 0.$$

We then have, for the corresponding $f_A, f_B, f_C \in \{j, j', \chi, f\}$ that

$$H(\tau_1, \dots, f_A(\tau_i), f_B(\tau_i), f_C(\tau_i), \dots) = 0$$

for all $(\tau_1, \dots, \tau_n) \in S$. By Theorem 2.3, this must then hold for all $(\tau_1, \dots, \tau_n) \in G$. In particular,

$$F(\tau_1, \dots, j(\tau_i), j'(\tau_i), \chi(\tau_i), f(\tau_i), \dots) = 0$$

for all $(\tau_1, \dots, \tau_n) \in G$.

This holds for each defining polynomial F of V, so we are done. □

Corollary 2.6. *Let $S \subseteq \mathbb{H}^n$ be an arc of a real algebraic curve and let ϕ be an algebraic function in $4n + 1$ variables. Let G be the smallest weakly $\mathbb{H}$-special variety containing S, and suppose that G is a GUT variety. Writing*

$$\widetilde{\pi}(\tau_1, \dots, \tau_n) = (j(\tau_1), j'(\tau_1), \chi(\tau_1), f(\tau_1), \dots, j(\tau_n), j'(\tau_n), \chi(\tau_n), f(\tau_n)),$$

we suppose that, on some branch of ϕ,

$$\phi(\tau_1, \widetilde{\pi}(\tau)) = 0$$

for all $\tau = (\tau_1, \dots, \tau_n) \in S$. Then this holds for all $\tau \in G$, excluding perhaps some exceptional set corresponding to branch points of ϕ.

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Proof. There exists an irreducible polynomial p such that

$$p(\phi(\mathbf{X}), \mathbf{X}) = 0$$

for all $\mathbf{X}$. Then in particular we have

$$p(0, \tau_1, \widetilde{\pi}(\tau)) = 0$$

for all $\tau = (\tau_1, \dots, \tau_n) \in S$. By Theorem 2.5, we have this relation for all $\tau \in G$.

We can pick a point $\mathbf{q} = (\tau_1, \dots, \tau_n) \in S$, a G-open neighbourhood U of $\mathbf{q}$ and a $\mathbb{C}$-open neighbourhood V of zero with the following property. Whenever $\tau \in U$, the only root of

$$p(X, \tau_1, \widetilde{\pi}(\tau)) \tag{2.2}$$

lying in V is zero (which is a root by the first comment). Now, for all $\tau \in \mathbb{H}^n$, we have

$$p(\phi(\tau_1, \widetilde{\pi}(\tau)), \tau_1, \widetilde{\pi}(\tau)) = 0$$

by definition of p. In other words, $\phi(\tau_1, \widetilde{\pi}(\tau))$ is a root of (2.2). However, as $\tau \in U$ gets arbitrarily close to $\mathbf{q}$, the value of $\phi(\tau_1, \widetilde{\pi}(\tau))$ gets arbitrarily close to zero. Hence it eventually lies in V. The only root of (2.2) within V is zero, whence for some G-open neighbourhood of $\mathbf{q}$, we have

$$\phi(\tau_1, \widetilde{\pi}(\tau)) = 0.$$

By analytic continuation, this holds for all $\tau \in G$, excluding some exceptional set corresponding to the branch points and branch cuts of ϕ. □

We conclude this subsection with one more technical lemma. While it may appear entirely unmotivated, hopefully the need for such a lemma will become clear during the proof of Theorem 2.8.

Lemma 2.7. *Let $S \subseteq \mathbb{H}^n$ be an arc of a real algebraic curve and let ϕ be an algebraic function in $4n$ variables. Suppose that*

$$\operatorname{Im} \tau_1 = \phi(\dots, j(\tau_k), j'(\tau_k), \chi(\tau_k), f(\tau_k), \dots)$$

for all $(\tau_1, \dots, \tau_n) \in S$. Let G be the smallest weakly $\mathbb{H}$-special variety containing S, and suppose that G is a GUT variety. Then $\operatorname{Im} \tau_1$ is constant on S.

Proof. Write $y = \operatorname{Im} \tau_1$ and suppose that y is non-constant. Let us retain the abbreviation

$$\widetilde{\pi}(\tau) = (j(\tau_1), j'(\tau_1), \chi(\tau_1), f(\tau_1), \dots, j(\tau_n), j'(\tau_n), \chi(\tau_n), f(\tau_n)).$$

Then S can be parametrized as

$$S = \{(x_1(y) + iy, x_2(y) + iy_2(y), \dots, x_n(y) + iy_n(y)) : y \in U\}$$

for some interval $U \subseteq \mathbb{R}$ and algebraic functions x_i, y_i.

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Now, take one of the polynomials $p(x_1, y_1, \dots, x_n, y_n)$ defining S. We can write

$$p\big[\tau_1 - i\phi(\widetilde{\pi}(\tau)), \phi(\widetilde{\pi}(\tau)), x_2(\phi(\widetilde{\pi}(\tau))), y_2(\phi(\widetilde{\pi}(\tau))), \dots, x_n(\phi(\widetilde{\pi}(\tau))), y_n(\phi(\widetilde{\pi}(\tau)))\big] = 0 \tag{2.3}$$

for all $\tau = (\tau_1, \dots, \tau_n) \in S$. In other words, we have an algebraic function ψ such that

$$\psi(\tau_1, \widetilde{\pi}(\tau)) = 0$$

for all $\tau \in S$. By Corollary 2.6, this holds for all $\tau \in G$, whence (2.3) holds for all $\tau \in G$.

By assumption, G is a GUT variety. Since y is assumed to be non-constant on S, the variable τ_1 cannot be constant on G. So up to permutation of coordinates G looks like

$$\{(\tau_1, g_2\tau_1, \dots, g_k\tau_1) : \tau_1 \in \mathbb{H}\} \times H$$

for some upper-triangular matrices $g_i \in \mathrm{GL}_2^+(\mathbb{Q})$ and some GUT variety H.

For any $\tau_1 \in \mathbb{H}$, $\tau' \in H$ and any $t \in \mathbb{Z}$, we then have

$$\sigma(t) := (\tau_1 + t, g_2(\tau_1 + t), \dots, g_k(\tau_1 + t), \tau') \in G.$$

Since the g_i are upper triangular, we can find an integer N with the following property. For every $t \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ with

$$g_i(\tau_1 + tN) = k + g_i(\tau_1),$$

for all i. By the periodicity of j, j', χ and f, it follows that

$$\widetilde{\pi}(\sigma(tN)) = \widetilde{\pi}(\sigma(0))$$

for all $t \in \mathbb{Z}$. So for all $\tau = (\tau_1, \dots, \tau_n) \in G$ and all $t \in \mathbb{Z}$ we have

$$p(\tau_1 + tN - i\phi(\widetilde{\pi}(\tau)), \phi(\widetilde{\pi}(\tau)), x_2(\phi(\widetilde{\pi}(\tau))), y_2(\phi(\widetilde{\pi}(\tau))), \dots, x_n(\phi(\widetilde{\pi}(\tau))), y_n(\phi(\widetilde{\pi}(\tau)))) = 0.$$

In particular, whenever $\tau = (x_1(y) + iy, x_2(y) + iy_2(y), \dots, x_n(y) + iy_n(y)) \in S$, we have

$$p(x_1(y) + tN, y, x_2(y), y_2(y), \dots, x_n(y), y_n(y)) = 0.$$

This holds for *every* polynomial p defining S. Since S has only one real dimension, it must therefore be a horizontal line in the τ_1 coordinate. That is, y is constant. Contradiction! □

2.2. Ax–Lindemann for adjacency

We can now prove our main Ax–Lindemann theorem. The idea is to show that a real algebraic arc in $\mathbb{H}^n$ that is 'adjacent' to a variety V (in a suitable sense) must be contained in a weakly $\mathbb{H}$-special variety that is itself adjacent to V.

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Theorem 2.8. *Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety. Let S be an arc of a real algebraic curve lying in $(\mathbb{H} \times \mathrm{SL}_2(\mathbb{R}))^n$. Define*

$$\widehat{S} = \{(g_1\tau_1, \dots, g_n\tau_n) : (\tau_1, g_1, \dots, \tau_n, g_n) \in S\}.$$

Suppose that

$$\left(\dots, j(g_i\tau_i), \chi^*(g_i\tau_i), \frac{j'(g_i\tau_i)}{m_{g_i}(\tau_i)}, \dots \right) \in V$$

for all $(\tau_1, g_1, \dots, \tau_n, g_n) \in S$, and fix any $(\tau_1^0, g_1^0, \dots, \tau_n^0, g_n^0) \in S$.

Then there is a weakly $\mathbb{H}$-special variety G with $\widehat{S} \subseteq G$ and

$$\left(\dots, j(\sigma_i), \chi^*(\sigma_i), \frac{j'(\sigma_i)}{m_{g_i^0}(\tau_i^0)}, \dots \right) \in V$$

for all $(\sigma_1, \dots, \sigma_n) \in G$.

Idea of proof. First, we attempt to parametrize the relevant algebraic arcs in terms of the imaginary part y_j of one of the variables. With suitable manipulations, we reach one of two outcomes: either a particular complex analytic relation involving just $\widetilde{\pi}$ holds, or y_j is equal to an algebraic function in $\widetilde{\pi}$. In the first case, the result comes fairly easily. In the second case, we apply Lemma 2.7 to see that y_j is constant; this situation can be dealt with easily.

Note. We will throughout write G for the weakly $\mathbb{H}$-special closure of $\widehat{S}$: the smallest weakly $\mathbb{H}$-special variety containing $\widehat{S}$. The functions $j(g_j\tau_j)$ and $\chi^*(g_j\tau_j)$ are unaffected if we replace g_j by γg_j for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and it is easy to see, using the transformation law for j, that the same is true of $j'(g_j\tau_j)/m_{g_j}(\tau_j)$. In particular, by applying suitable elements of $\mathrm{SL}_2(\mathbb{Z})$ to S, we may assume that G is a GUT variety. This will be useful several times throughout the paper.

Proof of 2.8. Given $(\dots, \tau_j, g_j, \dots) \in S$, let us write

$$g_j\tau_j = \sigma_j = x_j + iy_j$$

and

$$m_{g_j}(\tau_j) = \rho_j = u_j + iv_j.$$

We wish to parametrize the real algebraic arc

$$\widetilde{S} = \{(\dots, g_j\tau_j, m_{g_j}(\tau_j), \dots) : (\dots, \tau_j, g_j, \dots) \in S\} \subseteq (\mathbb{H} \times \mathbb{C})^n$$

in terms of one of the y_j. Thus the content of the argument depends on whether or not the y_j are constant on $\widetilde{S}$. In fact there will be a number of possible cases, depending on whether the y_j and/or the ρ_j are constant on $\widetilde{S}$.

Case 1. The $y_j = \operatorname{Im} \sigma_j$ and ρ_j are all constant on $\widetilde{S}$.

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Let us write $a_j = y_j$ to make it a little clearer that these are constant. We have

$$\left[\dots, j(\sigma_j), \chi(\sigma_j) - \frac{3f(\sigma_j)}{\pi a_j}, \frac{j'(\sigma_j)}{\rho_j}, \dots\right] \in V \tag{2.4}$$

for some constants ρ_j and all $\sigma = (\sigma_1, \dots, \sigma_n) \in \widehat{S}$. So we can apply Theorem 2.5 to see that (2.4) holds for all $\sigma \in G$. Our goal is to show that it still holds if we replace each of the $\chi(\sigma_j) - (3f(\sigma_j)/\pi a_j)$ by $\chi^*(\sigma_j)$. For this purpose, we may assume without loss of generality that G takes the form

$$\{(\tau, g_2\tau, \dots, g_n\tau) : \tau \in \mathbb{H}\},$$

with each $g_i \in \mathrm{GL}_2^+(\mathbb{Q})$ an upper-triangular matrix. (Any higher-dimensional GUT variety is simply a Cartesian product of such varieties, along with some constant factors. We have $\operatorname{Im}\sigma_j = a_j$ for any constant factors σ_j, whence $\chi(\sigma_j) - (3f(\sigma_j)/\pi a_j) = \chi^*(\sigma_j)$ already, so by induction we can ignore the constant factors also.)

Taking Zariski closures (over $\mathbb{C}$) in (2.4), we get

$$\left[\dots, j(g_j\tau), \chi(g_j\tau) - \frac{3f(g_j\tau)}{\pi a_j}, w_j, \dots\right] \in V \tag{2.5}$$

for all $\sigma = (\tau, g_2\tau, \dots, g_n\tau) \in G$ and $(w_1, \dots, w_n) \in V_{j(\sigma)}$. Here, $V_{j(\sigma)}$ is a variety that depends only on the variables $j(\tau)$, $j(g_j\tau)$ and that contains the point

$$\left(\frac{j'(\tau)}{\rho_1}, \dots, \frac{j'(g_n\tau)}{\rho_n}\right).$$

Aside. Consider a general upper-triangular matrix

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}).$$

Let $A = \gcd(b, d)$, let $D = ad/A$ and let k, m be integers such that $mb + kd = A$. For all integers t, we have

$$\begin{pmatrix} b/A & -k + tb \\ d/A & m + td \end{pmatrix} \cdot \begin{pmatrix} A & -ma \\ 0 & D \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & tD \end{pmatrix}.$$

The leftmost matrix is in $\mathrm{SL}_2(\mathbb{Z})$, while the matrix

$$H(g) = \begin{pmatrix} A & -ma \\ 0 & D \end{pmatrix}$$

has the same determinant as g. Note (taking $t = 0$ above) that

$$\begin{pmatrix} b/A & -k \\ d/A & m \end{pmatrix} H(g) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = g. \tag{2.6}$$

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From these matrix equations and the transformation laws of j, χ and f, we can easily see that

$$j\left(g\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) = j(H(g)\tau), \qquad \text{for all } t.$$

$$\chi\left(g\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) \to \chi(H(g)\tau) \qquad \text{as } t \to \infty.$$

$$f\left(g\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) \to 0 \qquad \text{as } t \to \infty.$$

$$j\left(\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) = j(\tau).$$

$$\chi\left(\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) \to \chi(\tau) \qquad \text{as } t \to \infty.$$

$$f\left(\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) \to 0 \qquad \text{as } t \to \infty.$$

Now recall: Equation (2.5) holds for all $\sigma \in G$ and $(w_1, \dots, w_n) \in V_{j(\sigma)}$. In particular, it holds for all

$$\sigma_t = \left(\begin{pmatrix}0 & -1\\ 1 & t\end{pmatrix}\tau, g_1\begin{pmatrix}0 & -1\\ 1 & t\end{pmatrix}\tau, \dots, g_n\begin{pmatrix}0 & -1\\ 1 & t\end{pmatrix}\tau\right) \in G,$$

and all $(w_1, \dots, w_n) \in V_{j(\sigma_t)}$. By choosing t to be an increasingly large multiple of a suitable D, we see from the discussions in the aside above that

$$j(\sigma_t) \to j(\sigma'), \qquad \chi(\sigma_t) \to \chi(\sigma'), \qquad f(\sigma_t) \to 0$$

as $t \to \infty$, where

$$\sigma' = (\tau, H(g_2)\tau, \dots, H(g_n)\tau).$$

Thus, by replacing σ by σ_t in (2.5) and letting $t \to \infty$, we see by continuity that

$$[\dots, j(H(g_j)\tau), \chi(H(g_j)\tau), w_j, \dots] \in V$$

for all $\sigma' = (\tau, H(g_2)\tau, \dots, H(g_n)\tau)$ and all $(w_1, \dots, w_n) \in V_{j(\sigma')}$. This is an algebraic condition on the points $j(\sigma'), \chi(\sigma')$, holding for all $\sigma' = (\tau, H(g_2)\tau, \dots, H(g_n)\tau)$, so by [**20**, Lemma 3.2] we can replace the copies of χ by χ^*. That is

$$[\dots, j(H(g_j)\tau), \chi^*(H(g_j)\tau), w_j, \dots] \in V,$$

for all $\sigma' = (\tau, H(g_2)\tau, \dots, H(g_n)\tau)$ and all $(w_1, \dots, w_n) \in V_{j(\sigma')}$. Replacing τ by $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\tau$ and using the modularity of χ^* and j, we see from (2.6) that

$$[\dots, j(g_j\tau), \chi^*(g_j\tau), w_j, \dots] \in V$$

for all $\sigma = (\tau, g_2\tau, \dots, g_n\tau)$ and all $(w_1, \dots, w_n) \in V_{j(\sigma)}$.

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Recalling that $V_{j(\sigma)}$ contains $(j'(\tau)/\rho_1, \dots, j'(g_n\tau)/\rho_n)$, this yields

$$\left[\dots, j(\sigma_j), \chi^*(\sigma_j), \frac{j'(\sigma_j)}{\rho_j}, \dots\right] \in V,$$

for all $\sigma \in G$. Since the ρ_j are constant on $\widetilde{S}$, they necessarily match the $m_{g_j^0}(\tau_j^0)$, so we are done.

Case 2. At least one of the y_j is non-constant on $\widetilde{S}$.

Without loss of generality, we can suppose that y_1 is non-constant and write $y = y_1$. We can parametrize

$$\widetilde{S} = \{(\dots, x_j(y) + iy_j(y), u_j(y) + iv_j(y), \dots,) : y \in I\},$$

for some interval $I \subseteq \mathbb{R}$ and algebraic functions x_j, y_j, u_j, v_j (setting $y_1(y) = y$). Letting F be a defining polynomial of V, we have

$$F\left[\dots, j(\sigma_j), \chi(\sigma_j) - \frac{3f(\sigma_j)}{\pi y_j(y)}, \frac{j'(\sigma_j)}{u_j(y) + iv_j(y)}, \dots\right] = 0.$$

We can rewrite this; there is an algebraic function s such that the above holds if and only if

$$s(y, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma = (\sigma_1, \dots, \sigma_n) \in \widehat{S}$.

Since s is an algebraic function, there is a non-trivial irreducible polynomial p_s such that

$$p_s(s(\mathbf{X}), \mathbf{X}) = 0$$

for all $\mathbf{X}$. In particular,

$$p_s(0, y, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$ and $y = \operatorname{Im}\sigma_1$. Since p_s is irreducible and non-trivial, we get a non-trivial $q_s(\mathbf{X}) = p_s(0, \mathbf{X})$. (It is clear that $p_s(t, \mathbf{X}) \neq t$.)

Now we apply the following iterative procedure to q_s, starting with $q = q_s$.

(1) Inspect separately each coefficient r_k of T^k in $q(T, \dots)$. If

$$r_k(\widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$ and all k, then terminate. Otherwise, let q' be the polynomial produced by removing from q all terms $r_k T^k$ for which $r_k(\widetilde{\pi}(\sigma))$ vanishes on $\widehat{S}$.

(2) If q' is irreducible, terminate. Otherwise, there is a factor q'' of q' with

$$q''(y, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$ and $y = \operatorname{Im}\sigma_1$.

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(3) We have a polynomial q'', which retains the property that

$$q''(y, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$ and $y = \operatorname{Im} \sigma_1$. Repeat from step 1, with q'' instead of q.

This must eventually terminate, since step 2 will always reduce the degree of the polynomial in question. So we have two possibilities.

Subcase 2a: We terminated at step 1.

In this case, working backwards we see that every coefficient r_k of T^k in q_s has the property that

$$r_k(\widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$. Using the fact that G is a GUT variety, we can apply Theorem 2.5 to see that this holds for all $\sigma \in G$. In particular,

$$q_s(y, \widetilde{\pi}(\sigma)) = 0$$

for all $y \in \mathbb{C}$ and all $\sigma \in G$. In other words, zero is a root of

$$p_s(X, y, \widetilde{\pi}(\sigma)) \tag{2.7}$$

for all $y \in \mathbb{C}$ and all $\sigma \in G$.

Now we proceed much as we did in the proof of Corollary 2.6. Let

$$\mathbf{a} = (g_1^0 \tau_1^0, \ldots, g_n^0 \tau_n^0) \in \widehat{S}.$$

We can choose a G-open neighbourhood U of $\mathbf{a}$ and a $\mathbb{C}$-open neighbourhood W of zero with the following property. Whenever $\sigma = (\sigma_1, \ldots, \sigma_n) \in U$, the only root of (2.7) lying in W is zero itself.

Now recall that

$$s(y, \widetilde{\pi}(\sigma))$$

is a root of (2.7) for all y and all σ. This is just the definition of p_s. Since s vanishes at $\mathbf{a} \in \widehat{S}$, there is a G-open neighbourhood

$$\mathbf{a} \in U' \subseteq U$$

such that

$$s(\operatorname{Im} g_1^0 \tau_1^0, \widetilde{\pi}(\sigma)) \in W$$

for all $\sigma \in U'$. But the only root of (2.7) lying in W is 0. So it must be the case that

$$s(\operatorname{Im} g_1^0 \tau_1^0, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in U'$ and hence for all $\sigma \in G$.

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Recalling the definition of S, we see that

$$F\left[\dots, j(\sigma_j), \chi(\sigma_j) - \frac{3f(\sigma_j)}{\pi y_j(y)}, \frac{j'(\sigma_j)}{u_j(y)+iv_j(y)}, \dots\right] = 0,$$

for all $\sigma = (\sigma_1, \dots, \sigma_n) \in G$ and for *constants*

$$y_j = y_j(\operatorname{Im} g_1^0\tau_1^0) \text{ and } \rho_j = u_j(\operatorname{Im} g_1^0\tau_1^0) + iv_j(\operatorname{Im} g_1^0\tau_1^0).$$

By construction, we have $\rho_j = m_{g_j^0}(\tau_j^0)$.

If we repeat this whole procedure for each defining polynomial of V, we get

$$\left[\dots, j(\sigma_j), \chi(\sigma_j) - \frac{3f(\sigma_j)}{\pi y_j}, \frac{j'(\sigma_j)}{m_{g_j^0}(\tau_j^0)}, \dots\right] \in V$$

for all $\sigma \in G$ and *constants* $y_j = y_j(\operatorname{Im} g_1^0\tau_1^0)$. Now we are in exactly the same position as we were for (2.4), so we conclude as we did earlier.

Subcase 2b: We terminated at step 2.

Then we have an irreducible polynomial q such that

$$q(y, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma = (\sigma_1, \dots, \sigma_n) \in \widehat{S}$. Moreover, for every k, the coefficient r_k of T^k in $q(T, \dots)$ has the property that

$$r_k(\widetilde{\pi}(\sigma))$$

does *not* vanish identically for $\sigma \in \widehat{S}$. Thus we can extract an algebraic function ϕ such that

$$y = \phi(\widetilde{\pi}(\sigma))$$

for all $\sigma \in \widehat{S}$ and $y = \operatorname{Im}\sigma_1$. Now we may apply Lemma 2.7 (once again using the fact that the weakly $\mathbb{H}$-special closure of $\widehat{S}$ is a GUT variety) and see that y is constant on $\widehat{S}$, a contradiction.

Case 3. All the y_j are constant on $\widetilde{S}$, but the ρ_j may vary.

Write $y_j = a_j$. We may assume that at least one of the σ_j is non-constant on $\widetilde{S}$, since otherwise the conclusion is automatic. Without loss of generality, let us say it is σ_1.

We have the relation

$$\left[\dots, j(\sigma_j), \chi(\sigma_j) - \frac{3f(\sigma_j)}{\pi a_j}, \frac{j'(\sigma_j)}{\rho_j}, \dots\right] \in V, \tag{2.8}$$

holding for all $(\dots, \sigma_j, \rho_j, \dots) \in \widetilde{S}$. Now let A be the smallest complex algebraic component containing $\widetilde{S}$. A method of 'complexifying the parameter' due to Pila (see [**13**,

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Lemma 2.1]) then tells us that (2.8) holds for all

$$(\dots, \sigma_j, \rho_j, \dots) \in A.$$

Since the weakly $\mathbb{H}$-special closure of $\widehat{S}$ is a GUT variety G, the projection of A onto the σ_j coordinates also has G as its weakly $\mathbb{H}$-special closure. Hence we can find a real algebraic arc

$$T \subseteq A \subseteq (\mathbb{H} \times \mathbb{C})^n$$

with the following properties:

- On T, the imaginary part of σ_1 is not constant.
- Whenever $(\dots, \sigma_j, \rho_j, \dots) \in T$, we have

$$\left[\dots, j(\sigma_j), \chi(\sigma_j) - \frac{3f(\sigma_j)}{\pi a_j}, \frac{j'(\sigma_j)}{\rho_j}, \dots\right] \in V. \tag{2.9}$$

- The projection of T onto the σ_j coordinates has G as its weakly $\mathbb{H}$-special closure.

We can then parametrize T in terms of $y = \operatorname{Im} \sigma_1$ exactly as in Case 2 earlier. We rewrite each polynomial relation (2.9) as an algebraic function s, yielding

$$s(y, \widetilde{\pi}(\sigma_1, \dots, \sigma_n)) = 0$$

whenever $(\dots, \sigma_j, \rho_j, \dots) \in T$ and $y = \operatorname{Im} \sigma_1$. By the same analysis as in Case 2, we end up with

$$s(\operatorname{Im} g_1^0 \tau_1^0, \widetilde{\pi}(\sigma_1, \dots, \sigma_n)) = 0$$

for all $(\sigma_1, \dots, \sigma_n) \in G$. Substituting this back into (2.9) yields

$$\left[\dots, j(\sigma_j), \chi(\sigma_j) - \frac{3f(\sigma_j)}{\pi a_j}, \frac{j'(\sigma_j)}{m_{g_1^0}(\tau_1^0)}, \dots\right] \in V$$

for all $(\sigma_1, \dots, \sigma_n) \in G$. Since a_j is constant, we can then conclude exactly as we did for (2.4). □

The above theorem has obvious consequences regarding the notion of adjacency. We present two crucial corollaries, both on the topic of adjacency.

Corollary 2.9 (Ax–Lindemann for Adjacency). *Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety. Let S be an arc of a real algebraic curve lying in $(\mathbb{H} \times \mathrm{SL}_2(\mathbb{R}))^n$. Define*

$$\widehat{S} = \{(g_1\tau_1, \dots, g_n\tau_n) : (\tau_1, g_1, \dots, \tau_n, g_n) \in S\}.$$

Suppose that, for some $c_k \in \mathbb{R}$, we have

$$\left[\dots, j(g_k\tau_k), \frac{j'(g_k\tau_k)}{m_{g_k}(\tau_k)}, p_{c_k}\left(j(g_k\tau_k), \chi^*(g_k\tau_k), \frac{j'(g_k\tau_k)}{m_{g_k}(\tau_k)}\right), \dots\right] \in V$$

for all $(\tau_1, g_1, \dots, \tau_n, g_n) \in S$, and fix any $(\tau_1^0, g_1^0, \dots, \tau_n^0, g_n^0) \in S$.

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Then there exists a weakly $\mathbb{H}$-special variety G such that

$$\widehat{S} \subseteq G,$$

and such that $G \hookrightarrow V$ via the following data.

- *Some*[*] $z_{i,j} \in \mathbb{C}$.
- *Real numbers $c_{i,j}$ that are, up to the obvious coordinate-wise relabelling, equal to the given c_k.*
- *Triples*

$$(w_l, x_l, y_l) = \left(j(g_k\tau_k), \frac{j'(g_k\tau_k)}{m_{g_k^0}(\tau_k^0)}, p_{c_k}\left(j(g_k\tau_k), \chi^*(g_k\tau_k), \frac{j'(g_k\tau_k)}{m_{g_k^0}(\tau_k^0)}\right)\right),$$

corresponding to those coordinates k on which G is constant.

The above corollary will be used in our point-counting arguments in the next section, in order to take a real algebraic arc and produce from it a weakly $\mathbb{H}$-special variety with certain adjacency properties. We will also need the following corollary, which we will eventually use to ensure that there are only finitely many basic Möbius varieties adjacent to our fixed variety V.

Corollary 2.10. *Let $V \subseteq \mathbb{C}^{3n}$ be a variety and let $B \subseteq \mathbb{H}^k$ be a basic Möbius variety, adjacent to V via some data including real numbers $c_{i,j}$ and triples $(w_l, x_l, y_l) \in \mathbb{C}^3$. Suppose that B is maximal among the Möbius subvarieties of $\mathbb{H}^k$ with this property (adjacent via the same $c_{i,j}$, (w_l, x_l, y_l), though the rest of the data may vary). Then B is $\mathbb{H}$-special, i.e. all of the $g_{i,j}$ defining B lie in $\mathrm{GL}_2^+(\mathbb{Q})$.*

Proof. Immediate from Corollary 2.9. □

3. Point-counting

In this section, we will discuss the necessary o-minimality and point-counting considerations. For familiar readers, the results here should fit well with expectations, though they are necessarily rather less neat than their equivalents in more classical settings. We are assuming some basic familiarity with o-minimality and the Pila–Wilkie theorems; see [16, 21, 22].

The first crucial fact we need is that all of the relevant functions are definable in an appropriate sense. Namely, the restrictions of j, j', j'', χ^*, χ and f to any standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ are definable in (the o-minimal structure)

[*] These can be calculated in terms of G and the $m_{g_k^0}(\tau_k^0)$, but we will not need to know what they are.

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$\mathbb{R}_{\mathrm{an,exp}}$. We will use the 'most standard' fundamental domain

$$\mathbb{F} = \{\tau \in \mathbb{H} : -1/2 \leq \operatorname{Re}\tau \leq 1/2, |\tau| \geq 1\}.$$

The definability can be seen in various ways; either as a consequence of the theory of elliptic curves and a result of Peterzil and Starchenko on the definability of the Weierstrass $\wp$-function [12], or via q-expansions. Given this fact, the idea is a fairly standard one:

First assume that a variety in $\mathbb{C}^{3n}$ contains an infinite set of special points. Then the preimage of that variety (under J), intersected with $\mathbb{F}^n$, is a definable set in $\mathbb{R}_{\mathrm{an,exp}}$. The usual idea is then to take Galois conjugates of the special points, forcing this definable set to contain many quadratic points of bounded height, in the sense of the Pila–Wilkie theorem. Then it contains a real algebraic arc, allowing us to apply the Ax–Lindemann results of the previous section.

The missing ingredient so far is the Galois aspect. As we shall see, it is enough to have suitable Galois information about points of the form $j(\tau)$ or $\chi^*(\tau)$, with τ quadratic. For this, we have the following.

Proposition 3.1. *Let $\tau \in \mathbb{H}$ be a quadratic point and consider the algebraic numbers $j(\tau)$ and $\chi^*(\tau)$. Let σ be a Galois conjugation acting on $\mathbb{Q}(j(\tau)) \supseteq \mathbb{Q}(\chi^*(\tau))$. Let τ' be a quadratic point such that $j(\tau') = \sigma(j(\tau))$. Then $\chi^*(\tau') = \sigma(\chi^*(\tau))$.*

For a proof, see [20, Proposition 5.2]. This tells us, essentially, that to keep track of the Galois conjugates of $\chi^*(\tau)$, we need only keep track of the Galois conjugates of $j(\tau)$. We already have sufficiently good control of the Galois conjugates of $j(\tau)$; it is a consequence of the Siegel bound for class numbers of quadratic fields [18] that

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] \gg D^{1/4}. \tag{3.1}$$

Here $D = D(\tau)$, where throughout $D(\tau)$ will represent the absolute value $|\delta(\tau)|$ of the discriminant $\delta(\tau)$ of τ.

In fact, we can do much better than (3.1), but this is sufficient for our purposes. In particular, there will be $\gg D^{1/4}$ Galois conjugates of a point $(j(\tau), \chi^*(\tau))$, over any fixed number field. This fact is central to our main 'point-counting theorem', 3.5, for which we will also need to be able to compare discriminant of quadratic points with their *height*.

Definition 3.2. For a tuple of algebraic points $\alpha \in \overline{\mathbb{Q}}^n$, we will write $\mathrm{Ht}(\alpha)$ to be the *absolute multiplicative height* of α, as defined in [3, Definition 1.5.4].

This definition also extends to elements of $\mathrm{GL}_2^+(\mathbb{Q})$ by identifying an element of $\mathrm{GL}_2^+(\mathbb{Q})$ with a tuple in $\mathbb{Q}^4$. Equivalently, we have

$$\mathrm{Ht}\begin{pmatrix} p_1/q_1 & p_2/q_2 \\ p_3/q_3 & p_4/q_4 \end{pmatrix} = \max(|p_1|, |q_1|, \ldots, |p_4|, |q_4|),$$

with each p_i/q_i written in lowest terms.

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Lemma 3.3. *Let $\tau, \sigma \in \mathbb{F}$ be quadratic points, with*

$$D(\tau), D(\sigma) \leq D.$$

Then there are positive constants c, δ such that: if $\tau \in \mathrm{GL}_2^+(\mathbb{Q}) \cdot \sigma$, then there is some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $\tau = g\sigma$ and

$$\mathrm{Ht}(g) \leq cD^{\delta}.$$

Proof. There are points $\tau' \in \mathrm{SL}_2(\mathbb{Z}) \cdot \tau$ and $\sigma' \in \mathrm{SL}_2(\mathbb{Z}) \cdot \sigma$, corresponding to elliptic curves E_τ and E_σ with coefficients in $\mathbb{Q}(j(\tau))$ and $\mathbb{Q}(j(\sigma))$ respectively. The points τ' and σ' have the same discriminants as τ and σ respectively.

An isogeny estimate due to Pellarin [11, Theorem 2] tells us that in this situation there is an isogeny between E_τ and E_σ of degree at most

$$10^{78} d^4 \max(1, \log d)^2 \max(1, h_F(E_\tau))^2, \tag{3.2}$$

where d is the product of the degrees of $j(\tau)$ and $j(\sigma)$ over $\mathbb{Q}$ and $h_F(E_\tau)$ is the semi-stable Faltings height of E_τ. From [19, Proposition 2.1] (see also [5, § 3]), we also know that

$$h_F(E_\tau) \leq 12(h(j(\tau)) + c \log \max\{2, h(j(\tau))\}),$$

where h is the logarithmic Weil height. (The heights discussed in this lemma are not relevant anywhere else so we will not define them.) Further, Habegger and Pila show in [6, Lemma 4.3] that

$$h(j(\tau)) \leq c_\epsilon D(\tau)^\epsilon$$

for any $\epsilon > 0$.

Moreover, since τ, σ have $D(\tau), D(\sigma) \leq D$, there are integer matrices g, h, of determinant at most D with $g\tau = \tau$ and $h\sigma = \sigma$. Namely,

$$g = \begin{pmatrix} -B & -2C \\ 2A & B \end{pmatrix},$$

where

$$Ax^2 + Bx + C$$

is the minimal polynomial of τ, with $A, B, C \in \mathbb{Z}$ and $(A, B, C) = 1$, whence

$$D(\tau) = |B^2 - 4AC| = 4AC - B^2 = \det g,$$

and similarly for h and σ.

Now $\Phi_{\det g}(j(\tau), j(\tau)) = 0$, and it is well known that $\deg \Phi_N \ll N^2$, so the degrees of $j(\tau)$, $j(\sigma)$ over $\mathbb{Q}$ are $O(D^2)$. Putting all this together with (3.2) tells us that there is an isogeny between E_τ and E_σ of degree M, with

$$M \ll_\epsilon D^{16+\epsilon},$$

and in particular $M = O(D^{17})$.

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On the $\mathbb{H}$ side, this tells us that we have a primitive integer matrix α', of determinant M, with $\alpha'\tau' = \sigma'$. Consequently there is such a matrix α, still of determinant M, with $\alpha\tau = \sigma$. By the work either of Orr [**10**, Proposition 1.4] or Habegger–Pila [**6**, Lemma 5.2], in this situation there is a positive b and a $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$ with $\tau = \beta\sigma$, $\det\beta = M$ and

$$\mathrm{Ht}(\beta) \ll M^b,$$

whence

$$\mathrm{Ht}(\beta) \ll D^{17b},$$

as required. □

The final piece we need before we can state our main 'point-counting theorem' is the following definition.

Definition 3.4. Two weakly $\mathbb{H}$-special varieties G and H are called *internally* $\mathrm{SL}_2(\mathbb{Z})$-*equivalent* if H can be obtained from G by replacing some (or all) matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ arising in the definition of G by some $g\gamma$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ (which may vary for different g).

Now we can state Theorem 3.5, in which we use the assumption of j'-genericity for the first, and only, time in the paper.

Theorem 3.5. *Let $V \subseteq \mathbb{C}^{3n+r}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$. Consider it as a family of algebraic varieties V_a, with $a \in \mathbb{C}^r$. Let d be a positive integer. Then there is a number $D = D(V, d)$ with the following property.*

Whenever $a \in \mathbb{C}^r$ is algebraic of degree at most d and $\tau \in J^{-1}(V_a)$ is a j'-generic quadratic point with $D(\tau) > D$, and none of the coordinates of τ lie in $\mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$, there are two positive-dimensional $\mathbb{H}$-special varieties G and H, internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent, with

$$\tau \in G \text{ and } H \hookrightarrow V_a.$$

Moreover, if $G = \mathbb{H}^k \times \{(\tau_{k+1}, \ldots, \tau_n)\}$ for some k and some τ_i (so that $G = H$), then in fact $\tau \in G \subseteq J^{-1}(V_a)$.

Proof. Let K be a number field containing a field of definition for V.

Consider a partition of $\{1, \ldots, n\}$, written as

$$S_1 \cup \cdots \cup S_k,$$

with each $S_i \neq \emptyset$. For each i, let $s_i = \min S_i$ and $r_i = \#S_i - 1$. Given

$$\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{H}^k$$

and

$$g = (g_{1,1}, \ldots, g_{1,r_1}, \ldots, g_{k,1}, \ldots, g_{k,r_k}) \in \mathrm{SL}_2(\mathbb{R})^{n-k},$$

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define the following set:

$$\mathcal{Z}_{a,\sigma,g} = \Big\{(\tau,h)\in\mathbb{H}^k\times \mathrm{GL}_2^+(\mathbb{R})^{n-k} : \det h_{i,j} = \det g_{i,j},$$
$$\Big[\dots, j(\tau_i), j'(\tau_i), p_{\mathrm{Im}\,\sigma_i}(j(\tau_i),\chi^*(\tau_i),j'(\tau_i)),\dots$$
$$\dots, j(h_{i,j}\tau_i), j'(h_{i,j}\tau_i)\frac{m_{g_{i,j}}(\sigma_i)}{m_{h_{i,j}}(\tau_i)},$$
$$p_{\mathrm{Im}\,g_{i,j}\sigma_i}\left(j(h_{i,j}\tau_i),\chi^*(h_{i,j}\tau_i),j'(h_{i,j}\tau_i)\frac{m_{g_{i,j}}(\sigma_i)}{m_{h_{i,j}}(\tau_i)}\right),\dots\Big]\in V_a\Big\}.$$

Consider this as a family of sets, fibred over $\mathbb{C}^r\times\mathbb{H}^k\times \mathrm{SL}_2(\mathbb{R})^{n-k}$. There is one such family for each of the finitely many partitions of $\{1,\dots,n\}$, and we consider them all together. They are certainly *not* definable families. However, for a given partition, the family

$$Z_{a,\sigma,g} = \{(\tau,h)\in\mathcal{Z}_{a,\sigma,g} : \tau_i, h_{i,j}\tau_i\in\mathbb{F}, i\le k, j\le r_k\}$$

is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Given some algebraic $a\in\mathbb{C}^r$, let us consider a j'-generic special point $\tau\in J^{-1}(V_a)$. Assume no coordinate of τ lies in $\mathrm{SL}_2(\mathbb{Z})\cdot\{i,\rho\}$, and set $D = D(\tau)$.

Up to permutation of coordinates, τ looks like

$$\tau = (\sigma_1, g_{1,1}\sigma_1,\dots,g_{1,r_1}\sigma_1,\dots,\sigma_k,g_{k,1}\sigma_k,\dots,g_{k,r_k}\sigma_k), \tag{3.3}$$

with the σ_i lying in distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits, and $g_{i,j}\in\mathrm{GL}_2^+(\mathbb{Q})$. So τ corresponds to a partition of $\{1,\dots,n\}$ in the obvious way. Writing $g_{i,j}$ as primitive integer matrices, let $N_{i,j} = \det g_{i,j}$. The j'-genericity of τ means that

$$j'(\sigma_1),\dots,j'(\sigma_k)$$

are algebraically independent over $\overline{\mathbb{Q}}$. So we see that the $\overline{\mathbb{Q}}$-Zariski closure $\langle\langle\tau\rangle\rangle$ of $J(\tau)$ is the set of all points of the form

$$[\dots,\ j(\sigma_i), w_i, p_{\mathrm{Im}\,\sigma_i}(j(\sigma_i),\chi^*(\sigma_i),w_i),$$
$$\dots, j(g_{i,j}\sigma_i), -w_i\lambda_{N_{i,j}}\big(j(\sigma_i),j(g_{i,j}\sigma_i)\big)m_{g_{i,j}}(\sigma_i),$$
$$p_{\mathrm{Im}\,g_{i,j}\sigma_i}(j(g_{i,j}\sigma_i),\chi^*(g_{i,j}\sigma_i),-w_i\lambda_{N_{i,j}}\big(j(\sigma_i),j(g_{i,j}\sigma_i)\big)m_{g_{i,j}}(\sigma_i)),\dots],$$

for some $w_1,\dots,w_k\in\mathbb{C}$.

We will show that the existence of this τ implies that $Z_{a,\sigma,g}$ contains 'many' quadratic points of bounded height. As in the previous section, these 'many' points will arise from taking Galois conjugates of the point $(j(\tau),\chi^*(\tau))$. We will shortly define a variety that serves to translate the problem from the realm of J into the realm of j and χ^*, making this approach possible.

To define the abovementioned variety, we will write a general element of $\mathbb{C}^{2n}$ as

$$(\dots,X_i,Y_i,\dots,X_{i,j},Y_{i,j},\dots),$$

with $i\le k$ and $j\le r_i$, matching the structure of the underlying partition of $\{1,\dots,n\}$.

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With this notation, define a variety $V_{a,\sigma,g} \subseteq \mathbb{C}^{2n}$ by

$$\begin{aligned} V_{a,\sigma,g} = \{(\mathbf{X},\mathbf{Y}) \in \mathbb{C}^{2n} : \forall w_1,\dots,w_k \in \mathbb{C},\\ [\dots, X_i, w_i, p_{\operatorname{Im}\sigma_i}(X_i,Y_i,w_i),\dots, X_{i,j}, -w_i\lambda_{N_{i,j}}(X_i,X_{i,j})m_{g_{i,j}}(\sigma_i),\\ p_{\operatorname{Im} g_{i,j}\sigma_i}\big(X_{i,j},Y_{i,j},-w_i\lambda_{N_{i,j}}(X_i,X_{i,j})m_{g_{i,j}}(\sigma_i)\big),\dots] \in V_a\}. \end{aligned}$$

Then $V_{a,\sigma,g}$ is a subvariety of $\mathbb{C}^{2n}$, defined over $K(a,\sigma,i\operatorname{Im}\sigma)$. By construction of $V_{a,\sigma,g}$, which mirrors $\langle\langle\tau\rangle\rangle$, we see that $\langle\langle\tau\rangle\rangle \subseteq V_a$, whence $V_{a,\sigma,g}$ must contain the point $(j,\chi^*)(\tau)$.

Hence $V_{a,\sigma,g}$ also contains every Galois conjugate (over $K(a,\sigma,i\operatorname{Im}\sigma)$) of $\pi(\tau)$. By Proposition 3.1, such a Galois conjugate must take the form $(j,\chi^*)(\tau')$, for some quadratic τ' with $D(\tau') = D(\tau)$. Moreover, thanks to the modular polynomials Φ_N, τ' must have the same $\mathrm{GL}_2^+(\mathbb{Q})$-structure as τ. That is:

$$\tau' = (\sigma_1', g_{1,1}'\sigma_1', \dots, g_{1,r_1}'\sigma_1', \dots, \sigma_k', g_{k,1}'\sigma_k', \dots, g_{k,r_k}'\sigma_k'), \tag{3.4}$$

where the σ_i' are quadratic points and $g_{i,j}' = g_{i,j}\gamma_{i,j}$, for some $\gamma_{i,j} \in \mathrm{SL}_2(\mathbb{Z})$. Further, by the modularity of j and χ^*, we can ensure that $\sigma_i' \in \mathbb{F}$.

For each τ' arising this way, let us take $w_i = j'(\sigma_i')$ in the definition of $V_{a,\sigma,g}$. Noting that

$$-j'(\sigma_i')\lambda_{N_{i,j}}(j(\sigma_i'), j(g_{i,j}'\sigma_i))m_{g_{i,j}}(\sigma_i) = j'(g_{i,j}'\sigma_i')\frac{m_{g_{i,j}}(\sigma_i)}{m_{g_{i,j}'}(\sigma_i')},$$

we see that $(\sigma',g') \in \mathcal{Z}_{a,\sigma,g}$. Further, there is $\gamma_{i,j}' \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_{i,j}'g_{i,j}'\sigma_i' \in \mathbb{F}$. This yields $(\sigma',\gamma'g') \in Z_{a,\sigma,g}$.

By (3.1), we know that

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] \geq c_{1/4}D^{1/4},$$

for some absolute constant $c_{1/4}$. So if $a \in \overline{\mathbb{Q}}^r$ is of degree at most d, then

$$\begin{aligned} [\mathbb{Q}(j(\tau)) : K(a,\sigma,i\operatorname{Im}\sigma)] &= \frac{[\mathbb{Q}(j(\tau)) : \mathbb{Q}]}{[K(a,\sigma,i\operatorname{Im}\sigma) : \mathbb{Q}]}\\ &\geq \frac{[\mathbb{Q}(j(\tau)) : \mathbb{Q}]}{[K:\mathbb{Q}](rd+2n+2n)}\\ &\geq \frac{c_{1/4}D^{1/4}}{[K:\mathbb{Q}](rd+4n)}\\ &= c(V,d)D^{1/4}, \end{aligned}$$

where $c(V,d) = \dfrac{c_{1/4}}{[K:\mathbb{Q}](rd+4n)}$.

Each of the Galois conjugates of $(j,\chi^*)(\tau)$ over $K(a,\sigma,i\operatorname{Im}\sigma)$ corresponds to a different $(\sigma',\gamma'g') \in Z_{a,\sigma,g}$, so there are at least $c(V,d)D^{(1/4)}$ points of degree at most 2 in $Z_{a,\sigma,g}$.

By [**14**, Proposition 5.7], we know $\mathrm{Ht}(\sigma') \leq c_1 D$, for some $c_1 > 0$. Further, by Lemma 3.3, the corresponding matrices $\gamma_{i,j}'g_{i,j}\gamma_{i,j}$ can be chosen such that $\mathrm{Ht}(\gamma'g') \leq c_2 D^{c_3}$, for positive c_2 and c_3. (In fact we know $c_3 > 1$, so clearly we can ensure $c_1 D \leq c_2 D^{c_3}$.)

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Putting all this together tells us that

$$N(Z_{a,\sigma,g}, c_2 D^{c_3}) = \#\left\{(\tau,h) \in Z_{a,\sigma,g} : \begin{array}{c}\tau \text{ quadratic},\\ h \in \mathrm{GL}_2^+(\mathbb{Q}),\\ \mathrm{Ht}(\tau,h) \leq c_2 D^{c_3}\end{array}\right\} \geq c(V,d) D^{1/4}. \tag{3.5}$$

On the other hand, the uniform Pila–Wilkie Theorem [16] gives us, for each $\epsilon > 0$, a constant $b(V,\epsilon) > 0$ such that, for any a, σ and g:

$$N(Z_{a,\sigma,g} \setminus Z^{\mathrm{alg}}_{a,\sigma,g}, T) \leq b(V,\epsilon) T^{\epsilon}.$$

In this inequality, we can set $T = c_2 D^{c_3}$ and choose, for example, $\epsilon = 1/8c_3$, to get

$$N(Z_{a,\sigma,g} \setminus Z^{\mathrm{alg}}_{a,\sigma,g}, c_2 D^{c_3}) \leq b(V) D^{1/8}. \tag{3.6}$$

Now, if the chosen point $\tau \in J^{-1}(V)$ has

$$D(\tau) > \frac{b(V)^8}{c(V,d)^8},$$

then the two inequalities (3.5) and (3.6) are inconsistent unless the corresponding $Z^{\mathrm{alg}}_{a,\sigma,g} \neq \emptyset$. In fact, we still get the same inconsistency in these inequalities unless $Z^{\mathrm{alg}}_{a,\sigma,g}$ contains the $(\sigma', \gamma' g')$ corresponding to one of the τ' arising from the Galois conjugates of $(j, \chi^*)(\tau)$. In particular, we have a real semi-algebraic arc $T \subseteq (\mathbb{H} \times \mathrm{GL}_2^+(\mathbb{R}))^n$ with

$$(\sigma', \gamma' g') \in T \subseteq Z_{a,\sigma,g}.$$

We would like to apply Corollary 2.9 to some algebraic arc constructed from T. Indeed, let

$$S = \left\{\left(\ldots, \tau_i, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \ldots, \tau_i, \overline{h_{i,j}}, \ldots\right) : (\tau,h) \in T\right\},$$

where $\overline{h}$ is the element of $\mathrm{SL}_2(\mathbb{R})$ corresponding to the image of h as an element of $\mathrm{PGL}_2(\mathbb{R})$. Also let

$$\widehat{S} = \{(\ldots, \tau_i, \ldots, h_{i,j}\tau_i, \ldots) : (\tau,h) \in T\}.$$

Note that $\widehat{S}$ is indeed an arc, rather than just point. This is easy to see; if τ_i and $h_{i,j}\tau_i$ are all constant on $\widehat{S}$, then h must be constant, up to determinant, on T. Since the determinant of $h_{i,j}$ is fixed in the definition of $Z_{a,\sigma,g}$, it follows that τ_i and $h_{i,j}$ are both constant on T, whence T itself is just a point, which is a contradiction.

Corollary 2.9 yields an $\mathbb{H}$-special set H_0 with

$$\widehat{S} \subseteq H_0 \hookrightarrow V.$$

Since $\widehat{S}$ is not just a point, H_0 is certainly positive-dimensional. Further, since $(\sigma', \gamma' g') \in T$, we have $\gamma\tau' \in \widehat{S}$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$. Hence, some $\mathrm{SL}_2(\mathbb{Z})$-translate H_1 of H_0 contains τ', and remains adjacent to V.

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Let us write, without loss of generality,

$$H_1 = B \times \{\tau'_{k+1}, \dots, \tau'_n\} \hookrightarrow V,$$

for some basic $\mathbb{H}$-special variety B. Since the definition of adjacency does not depend on any constant coordinates, we also have

$$H = B \times \{\tau_{k+1}, \dots, \tau_n\} \hookrightarrow V,$$

where the τ_i now correspond to suitable coordinates of our original point τ. Since τ and τ' had the same $\mathrm{GL}_2^+(\mathbb{Q})$-structure (see (3.3) and (3.4)), it follows that there is an $\mathbb{H}$-special G, internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent to H, which contains τ, as required.

It remains to show that if $H = G = \mathbb{H}^k \times \{\tau_{k+1}, \dots, \tau_n\}$, then in fact $G \subseteq J^{-1}(V)$. For this we must refer to the finer detail of Corollary 2.9. In this situation it tells us that $\mathbb{H}^k$ is adjacent to V via

- some positive reals c_i;
- some $z_i \in \mathbb{C}$;
- triples of the form

$$(w, x, y) = \Bigg(j(g'_{i,j}\sigma'_j),\, j'(g'_{i,j}\sigma'_j)\frac{m_{g_{i,j}}(\sigma_j)}{m_{g'_{i,j}}(\sigma'_j)},\\ p_{\operatorname{Im} g_{i,j}\sigma_j}\left(j(g'_{i,j}\sigma'_j), \chi^*(g'_{i,j}\sigma'_j), j'(g'_{i,j}\sigma'_j)\frac{m_{g_{i,j}}(\sigma_j)}{m_{g'_{i,j}}(\sigma'_j)}\right)\Bigg).$$

Applying the definitions of adjacency and the polynomial p_{c_i}, it follows that

$$J(\sigma_1, \dots, \sigma_k) \times \{(\mathbf{w}, \mathbf{x}, \mathbf{y})\} \in V_a$$

for any σ_i with $\operatorname{Im}\sigma_i = c_i$. Here $(\mathbf{w}, \mathbf{x}, \mathbf{y})$ consists of triples as described above. Taking Zariski closures, it follows that

$$\mathbb{C}^{3k} \times \{(\mathbf{w}, \mathbf{x}, \mathbf{y})\} \subseteq V_a.$$

Finally, recall that we had a Galois automorphism that had the effect of sending

$$(j, \chi^*)(g_{i,j}\sigma_j) \text{ to } (j, \chi^*)(g'_{i,j}\sigma'_j)$$

and

$$j'(g_{i,j}\sigma_j) \text{ to } j'(g'_{i,j}\sigma'_j)\frac{m_{g_{i,j}}(\sigma_j)}{m_{g'_{i,j}}(\sigma'_j)}.$$

Applying the inverse Galois conjugation now yields

$$\begin{aligned}&\mathbb{C}^{3k} \times \{[\dots, j(g_{i,j}\sigma_j), j'(g_{i,j}\sigma_j), p_{\operatorname{Im} g_{i,j}\sigma_j}(j(g_{i,j}\sigma_j), \chi^*(g_{i,j}\sigma_j), j'(g_{i,j}\sigma_j)), \dots]\}\\ &\quad = \mathbb{C}^{3k} \times \{[\dots, J(g_{i,j}\sigma_j), \dots]\} \subseteq V_a,\end{aligned}$$

which, unpacking the notation, yields $J(H) \subseteq V_a$, as required. □

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The above result is one of two crucial pieces of 'counting' that we need in order to prove 1.9. The other half is the following proposition. The idea is that 3.5 will be used to count the number of isolated points that can arise in $J^{-1}(V)$ (the 'zero-dimensional pieces'), and this next proposition will count the 'positive-dimensional pieces', namely the basic $\mathbb{H}$-special varieties. Together, 3.5 and 3.6 will act as the engine driving the inductive argument at the heart of the proof of 1.9.

Proposition 3.6. *Let $V \subseteq \mathbb{C}^{3n+r}$ be a variety, considered as a family of varieties V_a, $a \in \mathbb{C}^r$. Let $\mathcal{B}$ be the collection of basic $\mathbb{H}$-special varieties $B \subseteq \mathbb{H}^k$ (for any k) having the following properties:*

- *For some $a \in \mathbb{C}^r$, B is adjacent to V_a via*
  - *some real numbers $c_{i,j}$;*
  - *some complex numbers $z_{i,j}$;*
  - *some triples (w_l, x_l, y_l).*
- *B is maximal with this property among $\mathbb{H}$-special subvarieties of $\mathbb{H}^k$, for the given a, $c_{i,j}$ and (w_l, x_l, y_l) (the $z_{i,j}$ may vary).*

Then $\mathcal{B}$ is $\mathrm{SL}_2(\mathbb{Z})$-finite.

Proof. By Corollary 2.10, the only basic Möbius B that are maximally adjacent to any V_a are necessarily $\mathbb{H}$-special. For such B, all the defining $g_{i,j}$ are in $\mathrm{GL}_2^+(\mathbb{Q})$, so the collection of such B is parametrized by a countable set. The conditions specified are definable (compare with, for instance, [**14**, Proposition 10.2]), so we have a countable definable set, which is therefore finite. □

To be more precise, we will in fact be applying the following easy corollary.

Corollary 3.7. *The collection $\mathcal{C}$ of basic $\mathbb{H}$-special varieties C that are internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent to some $B \in \mathcal{B}$ (as described in Proposition 3.6) is $\mathrm{SL}_2(\mathbb{Z})$-finite.*

Proof. Up to the action of $\mathrm{SL}_2(\mathbb{Z})$, there are only finitely many primitive integer matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ of any given determinant. Hence, given any $\mathbb{H}$-special G, defined by some matrices g_i, there are only $\mathrm{SL}_2(\mathbb{Z})$-finitely many possibilities for $g_i\gamma_i$, $\gamma_i \in \mathrm{SL}_2(\mathbb{Z})$, and therefore only finitely many $\mathbb{H}$-special varieties internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent to G.

With this in mind, the assertion is immediate from 3.6. □

4. Bringing the proof together

As we bring everything together to prove our central theorem, let us recall the statement.

Theorem 1.9. *Let $V \subseteq \mathbb{C}^{3n}$ be a proper algebraic variety defined over $\overline{\mathbb{Q}}$. There exists an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma(V)$, consisting of* proper *$\mathbb{H}$-special varieties of $\mathbb{H}^n$, with the following property. Every j'-generic $\mathbb{H}$-special point in $J^{-1}(V)$ is contained in some $G \in \sigma(V)$.*

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This will clearly follow from the following uniform version.

Theorem 4.1. *Let $V \subseteq \mathbb{C}^{3n+r}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$, considered as an algebraic family of varieties,*

$$V_a \subseteq \mathbb{C}^{3n}, \quad a \in \mathbb{C}^r.$$

For each positive integer d, there is an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma_d(V)$, consisting of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$, with the following property.

Whenever $a \in \overline{\mathbb{Q}}^r$ satisfies $\max[\mathbb{Q}(a_i) : \mathbb{Q}] \leq d$ and V_a is a proper *subvariety of $\mathbb{C}^{3n}$, every j'-generic quadratic point in $J^{-1}(V_a)$ is contained in some $G \in \sigma_d(V)$.*

Proof. First, let us note: we can safely ignore any τ that have a coordinate lying in $\mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$, since these all clearly lie in $\mathrm{SL}_2(\mathbb{Z})$-finitely many proper $\mathbb{H}$-special subvarieties.

We work by induction on n. When $n = 1$, the result follows immediately from Theorem 3.5.

For larger n, the first stage is to construct a variety V^* that is designed to account for all possible positive-dimensional special subvarieties of any V_a.

Let $\mathcal{B}$ be the collection of basic $\mathbb{H}$-special subvarieties afforded by applying Proposition 3.6 to V. Let $\mathcal{C}$ be the collection of $\mathbb{H}$-special subvarieties internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent to some $B \in \mathcal{B}$. If any of these are equal to $\mathbb{H}^k$, for some k, remove them from $\mathcal{C}$.

Now let

$$\mathcal{G}_1^* = \left\{ \omega(B \times \mathbb{H}^{n-k}) : \begin{array}{c} B \in \mathcal{C}, 1 \leq k \leq n, B \subseteq \mathbb{H}^k, \\ \omega \text{ a permutation of the coordinates} \end{array} \right\}.$$

Since, by 3.7, $\mathcal{C}$ is $\mathrm{SL}_2(\mathbb{Z})$-finite, $\mathcal{G}_1^*$ is too.

Next, consider the variety $V_k \subseteq \mathbb{C}^{3(n-k)+r}$, defined over $\overline{\mathbb{Q}}$, for $1 \leq k < n$, by

$$V_k = \{(\mathbf{X}, a) \in \mathbb{C}^{3(n-k)+r} : \text{The translation of } \mathbb{C}^{3k} \text{ by } \mathbf{X} \\ \text{(for some choice of ordering of coordinates) is contained in } V_a\}.$$

Clearly V_k is a proper subvariety of $\mathbb{C}^{3(n-k)+r}$. By our inductive assumption, there is some $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\mathcal{F}_k$ of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^{n-k}$. Every j'-generic $\mathbb{H}$-special point in $J^{-1}((V_k)_a)$ is contained in some $F \in \mathcal{F}_k$, provided that $(V_k)_a \neq \mathbb{C}^{3(n-k)}$.

Let

$$\mathcal{G}_2^* = \left\{ \omega(\mathbb{H}^k \times F) : \begin{array}{c} F \in \mathcal{F}_k, 1 \leq k < n, \\ \omega \text{ a permutation of coordinates} \end{array} \right\}.$$

Then $\mathcal{G}_2^*$ is $\mathrm{SL}_2(\mathbb{Z})$-finite.

Let

$$\mathcal{G}^* = \mathcal{G}_1^* \cup \mathcal{G}_2^*,$$

and let

$$V^* = \left\langle \left\langle \bigcup_{G \in \mathcal{G}^*} G \right\rangle \right\rangle.$$

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Since $\mathcal{G}^*$ consists of $\mathrm{SL}_2(\mathbb{Z})$-finitely many proper $\mathbb{H}$-special subvarieties, V^* is a proper subvariety of $\mathbb{C}^{3n}$.

Suppose that, for some a of degree at most d, the set $J^{-1}(V_a \setminus V^*)$ contains a j'-generic quadratic τ with $D(\tau) > D = D(V, d)$. By 3.5, there are positive-dimensional $\mathbb{H}$-special varieties G and H, internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent, with

$$\tau \in G, \qquad H \hookrightarrow V_a,$$

and maximal with these properties.

Assume first that H does not take the form $\mathbb{H}^k \times \{(\tau_{k+1}, \dots, \tau_n)\}$. Then since $H \hookrightarrow V_a$, we have $H \in \mathcal{B}$, whence $G \in \mathcal{C}$. It follows that $\tau \in \bigcup_{G \in \mathcal{G}_1^*} G$, whence $J(\tau) \in V^*$, which is a contradiction.

If $G = H = \mathbb{H}^k \times \{(\tau_{k+1}, \dots, \tau_n)\}$, for some k, then 3.5 tells us that $\tau \in G \subseteq J^{-1}(V_a)$. Taking Zariski closures, it follows that

$$\mathbb{C}^{3k} \times \{J(\tau_{k+1}, \dots, \tau_n)\} \subseteq V_a,$$

whence $(\tau_{k+1}, \dots, \tau_n)$ is a j'-generic quadratic point of $J^{-1}((V_k)_a)$. This implies $G \in \mathcal{G}_2^*$, whence $\tau \in V^*$, which is a contradiction.

Hence, there are no a (of degree at most d) for which there is $\tau \in J^{-1}(V_a \setminus V^*)$ with $D(\tau) > D$. In particular, the collection $\mathcal{H}^*$ of j'-generic τ that lie in *any* of the $V_a \setminus V^*$ is $\mathrm{SL}_2(\mathbb{Z})$-finite. So whenever a is of degree at most d and $V_a \neq \mathbb{C}^{3n}$, every j'-generic quadratic point in $J^{-1}(V_a)$ is contained either in an element of $\mathcal{G}^*$ or $\mathcal{H}^*$. Both $\mathcal{G}^*$ and $\mathcal{H}^*$ are $\mathrm{SL}_2(\mathbb{Z})$-finite, so we are done. □

In the next section, we will state a more precise version of Conjecture 1.7, and prove it under the assumption of 1.12. This is part of why we have gone to the trouble of proving the uniform version 4.1 of 1.9; we need it for our more precise result.

5. A more precise statement

Readers familiar with the normal shape of André–Oort statements may have noticed that 1.7 is rather weaker than one might expect. Even taking into account the action of $\mathrm{SL}_2(\mathbb{Z})$, a more natural analogue of the classical case might look like the following.

Statement 5.1. *Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$. Then the collection of maximal J-special subvarieties of V is $\mathrm{SL}_2(\mathbb{Z})$-finite.*

This turns out to be false. The reason for its failure is fairly simple; the modular relations that relate $j'(g\tau)$ (for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$) with $j'(\tau)$ include not just $j(\tau)$, $j'(\tau)$, $j(g\tau)$ and $j'(g\tau)$, but also instances of $m_g(\tau) = (c\tau + d)^2/N$. With the right polynomial, one can therefore enforce arbitrary relations between $m_g(\tau)$ and other $m_h(\sigma)$ arising in other coordinates.

Similarly, the modular relation for j'' introduces new variables to the equation. Indeed, by differentiating the modular polynomials one can find a rational function μ_N (in seven variables) such that

$$j''(g\tau) = \mu_N(j(\tau), j(g\tau), j'(\tau), j'(g\tau), j''(\tau), c, (c\tau + d)),$$

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where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a primitive integer matrix of determinant N. Moreover, μ_N is linear in the c-coordinate. Hence we are able to enforce relations on the c that can arise, as well as the $m_g(\tau)$.

Example 5.2. Let us see some examples to illustrate this issue.

(1) Let $W \subseteq \mathbb{C}^2$ be an algebraic variety defined over $\mathbb{Q}$. Suppose that W has at least one solution (x, y) where x and y are both squares of quadratic points in $\mathbb{H}$. Fix two positive integers M and N.
Writing a general element of $\mathbb{C}^{12}$ as $(X_1, Y_1, Z_1, \ldots, X_4, Y_4, Z_4)$, consider the variety $V \subseteq \mathbb{C}^{12}$ defined (over $\overline{\mathbb{Q}}$) by

$$\Phi_M(X_1, X_2) = 0, \quad \Phi_N(X_3, X_4) = 0,$$
$$\left(\frac{-Y_2}{Y_1 \lambda_M(X_1, X_2)}, \frac{-Y_4}{Y_3 \lambda_N(X_3, X_4)}\right) \in W.$$

Then the special points of $J^{-1}(V)$ are precisely the points $(\tau, g\tau, \sigma, h\sigma)$, where g and h are (arbitrary) primitive integer matrices with determinant M and N respectively, and τ, σ are quadratic points satisfying

$$(m_g(\tau), m_h(\sigma)) \in W.$$

Since W has at least one solution that is a square of a quadratic point, we can certainly find τ and σ to solve this equation. Indeed, by modifying τ and σ we can solve this equation for any g and h of the right determinant. The resulting collection of special points is certainly $\mathrm{SL}_2(\mathbb{Z})$-infinite, but no positive-dimensional $\mathbb{H}$-special variety is contained in $J^{-1}(V)$.
This example therefore serves as a counterexample to the hypothetical statement 5.1. Since the points all lie within the $\mathrm{SL}_2(\mathbb{Z})$-translates of the $\mathbb{H}$-special set

$$\{(\tau_1, g\tau_1, \tau_2, h\tau_2) : \tau_i \in \mathbb{H}\},$$

our main Theorem 1.9 is still fine! Let us see one more example.

(2) Fix a quadratic point $\sigma \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$(c\sigma + d)^2 = m_\gamma(\sigma) = \frac{j'(\gamma\sigma)}{j'(\sigma)}$$

and

$$c = \frac{j''(\gamma\sigma) - j''(\sigma)(c\sigma + d)^4}{2(c\sigma + d)^3 j'(\tau)},$$

whence

$$c^2 = \frac{(j''(\gamma\sigma) - j''(\sigma)(c\sigma + d)^4)^2}{4(c\sigma + d)^6 j'(\sigma)^2}$$
$$= \frac{(j''(\gamma\sigma) - j''(\sigma)((j'(\gamma\sigma))/(j'(\sigma)))^2)^2}{4((j'(\gamma\sigma))/(j'(\sigma)))^3 j'(\sigma)^2}.$$

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So for the appropriate rational function q we have

$$c^2 = q(j'(\sigma), j'(\gamma\sigma), j''(\sigma), j''(\gamma\sigma)).$$

Given a variety $W \subseteq \mathbb{C}^2$, defined over $\overline{\mathbb{Q}}$, we can then define $V \subseteq \mathbb{C}^9$ by

$$\Phi_N(X_1, X_2), \quad X_3 = j(\sigma),$$

$$\forall w \in \mathbb{C}, \left(\frac{-Y_2}{Y_1 \lambda_N(X_1, X_2)}, q[w, Y_3, p_{\mathrm{Im}\,\sigma}(j(\sigma), \chi^*(\sigma), w), Z_3] \right) \in W.$$

Then the $\mathbb{H}$-special points of $J^{-1}(V)$ are exactly those points

$$(\tau, g\tau, \gamma\sigma),$$

where

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is any primitive integer matrix of determinant N and

$$((c\tau + d)^2, C^2) \in W.$$

Once again, if W is suitable then this is an $\mathrm{SL}_2(\mathbb{Z})$-infinite collection, but no positive-dimensional $\mathbb{H}$-special set lies in $J^{-1}(V)$.

With variants of the examples above, one can produce varieties whose special points satisfy almost any arbitrary relation, provided the relation is written in terms of variables c, $(c\tau + d)$ corresponding to matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$, and C, D corresponding to some $\mathrm{SL}_2(\mathbb{Z})$-translate of a fixed σ.

The idea of our stronger result is that these relations should be the only obstruction to a result like 5.1. In order to state this precisely, we will need to go through some technicalities.

Given a proper $\mathbb{H}$-special set $G \subseteq \mathbb{H}^n$, there is an underlying partition of $\{1, \ldots, n\}$ that can be written as

$$S_0 \cup S_1 \cup \cdots \cup S_h \cup T_1 \cup \cdots \cup T_k,$$

with $\#S_i > 1$ for $i > 0$ and $\#T_j = 1$ for all j. (The condition that G is proper is equivalent to requiring that $k < n$.) For $i > 0$, let $r_i = \#S_i - 1$ and let s_i be the smallest element of s_i. Also associated with G are some matrices $g_{i,1}, \ldots, g_{i,r_i} \in \mathrm{GL}_2^+(\mathbb{Q})$, so that the τ corresponding to each element of S_i (except s_i) is defined by $\tau = g_{i,j}\tau_{s_i}$.

Given such a G and given a tuple of matrices

$$\gamma = (\gamma_1, \ldots, \gamma_{\#S_0}, \gamma_{1,1}, \ldots, \gamma_{1,r_1}, \ldots, \gamma_{h,1}, \ldots, \gamma_{h,r_h}) \in \mathrm{SL}_2(\mathbb{Z})^{\#S_0 + \sum r_i}, \tag{5.1}$$

let c_i, d_i be the bottom row of γ_i, and $c_{i,j}, d_{i,j}$ be the bottom row of the matrix $\gamma_{i,j}g_{i,j}$.

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This is all building towards the following definitions.

Definition 5.3. A variety $W \subseteq \mathbb{C}^{2\#S_0+2\sum r_i}$, defined over $\overline{\mathbb{Q}}$, is called a G-variety. For a G-variety W and a given $\gamma \in \mathrm{SL}_2(\mathbb{Z})^{\#S_0+\sum r_i}$ (as above), we define $W^\gamma \subseteq \mathbb{H}^{\{s_1,\dots,s_h\}}$ to be the set of $(\tau_{s_1},\dots,\tau_{s_h})$ such that

$$(\dots, c_i, d_i, \dots, c_{i,j}, c_{i,j}\tau_{s_i} + d_{i,j}, \dots) \in W.$$

If γ is an element of the full group $\mathrm{SL}_2(\mathbb{Z})^n$, then it consists of $\gamma' \in \mathrm{SL}_2(\mathbb{Z})^{\#S_0+\sum r_i}$, as above, together with some more matrices

$$\alpha_{s_1},\dots,\alpha_{s_h} \in \mathrm{SL}_2(\mathbb{Z}),$$

corresponding to the s_i-coordinates, and

$$\beta_1,\dots \beta_k \in \mathrm{SL}_2(\mathbb{Z})$$

corresponding to the singleton coordinates in the T_i. For such a $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$, we will abuse notation and write

$$W^\gamma = (\alpha_{s_1},\dots,\alpha_{s_h}) \cdot W^{\gamma'}.$$

(The β_i have no meaningful effect.)

Notation 5.4. We will write

$$\mathrm{Sp}(\gamma, W) = \{(\tau_1,\dots,\tau_n) \in \mathbb{H}^n : (\tau_{s_1},\dots,\tau_{s_h}) \in W^\gamma \text{ and every } \tau_i \text{ is quadratic}\}.$$

In the case where $h = 0$, so that we have no τ-coordinates to work with, the variety W^γ only enforces conditions on the γ corresponding to the S_0-coordinates. In this case, we will use the convention that

$$\mathrm{Sp}(\gamma, W) = \{\tau \in \mathbb{H}^n : \tau \text{ is quadratic.}\}$$

if $(\dots, c_i, d_i, \dots) \in W$, and

$$\mathrm{Sp}(\gamma, W) = \emptyset$$

otherwise.

Before we can state our more precise version of 1.7, we need one more definition.

Definition 5.5. A pair (G, W), with G a proper $\mathbb{H}$-special set and W a G-variety is said to be *geodesically minimal* if

$$\bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})^n} \mathrm{Sp}(\gamma, W)$$

is *not* contained in any $\mathrm{SL}_2(\mathbb{Z})$-finite collection of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$.

Theorem 5.6 (Precise Modular André–Oort with Derivatives). *Assume Conjecture 1.12. Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$. Then there is a finite collection $\sigma(V)$ of $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$, and for each $G \in \sigma(V)$ an associated G-variety W_G, with the following properties.*

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- *For every $G \in \sigma(V)$, (G, W_G) is geodesically minimal.*
- *The set of quadratic points in $J^{-1}(V)$ is precisely*

$$\bigcup_{\substack{G\in\sigma(V)\\ \gamma\in \mathrm{SL}_2(\mathbb{Z})^n}} \gamma\cdot G\cap \mathrm{Sp}(\gamma, W_G).$$

Proof. Under the assumption of Conjecture 1.12, Theorem 1.9 yields a finite collection $\sigma(V)$ of proper $\mathbb{H}$-special subvarieties, such that the special points of $J^{-1}(V)$ are contained in

$$\bigcup_{\substack{G\in\sigma(V)\\ \gamma\in \mathrm{SL}_2(\mathbb{Z})^n}} \gamma\cdot G.$$

Let us look first at a single $G \in \sigma(V)$, and associate some data with it, as in the definitions above. Associated with G is a partition of $\{1, \dots, n\}$,

$$S_0 \cup S_1 \cup \cdots \cup S_h \cup T_1 \cup \cdots \cup T_k,$$

with T_i singletons, $\#S_i > 1$. As above, we have some associated data

$$s_i = \min S_i, \qquad r_i = \#S_i - 1 \text{ for } i > 0,$$

$$\sigma_1, \dots, \sigma_{\#S_0} \in \mathbb{H} \text{ quadratic}$$

and, for $1 \le i \le h$, $1 \le j \le r_i$,

$$g_{i,j} \in \mathrm{GL}_2^+(\mathbb{Q}), \text{ a primitive integer matrix with determinant } N_{i,j}.$$

For ease of notation, we will assume that the coordinates are ordered nicely, with the first few coordinates in S_0, the next few in S_1, and so on.

Recall: for each N, there is a rational function μ_N with the property that

$$j''(g\tau) = \mu_N(j(\tau), j(g\tau), j'(\tau), j'(g\tau), j''(\tau), c, (c\tau + d))$$

whenever

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a primitive integer matrix of determinant N. Moreover, it will be useful later to know that

$$\begin{aligned} &\mu_N\left(j(\tau), j(g\tau), j'(\tau), j'(g\tau)\frac{(c_0\tau_0 + d_0)^2}{(c\tau + d)^2}, j''(\tau), c_0, (c_0\tau_0 + d_0)\right) \\ &\qquad = j''(g\tau)\frac{(c_0\tau_0 + d_0)^4}{(c\tau + d)^4} + j'(g\tau)\frac{2(c_0\tau_0 + d_0)^3(c_0 d - c d_0)}{(c\tau + d)^3}. \end{aligned} \tag{5.2}$$

This follows by a straightforward, if tedious, calculation.

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Given $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$, and a point $\tau = (\tau_1, \dots, \tau_h) \in \mathbb{H}^h$, define a variety

$$V_{\gamma,\tau} \subseteq \mathbb{C}^{3(h+k)}$$

as follows. First, write γ as in (5.1). That is, γ consists of

$$\gamma' = (\gamma_1, \dots, \gamma_{\#S_0}, \gamma_{1,1}, \dots, \gamma_{1,s_1}, \dots, \gamma_{h,1}, \dots, \gamma_{h,s_h}) \in \mathrm{SL}_2(\mathbb{Z})^{\#S_0 + \sum r_i},$$

$$\alpha_{s_1}, \dots, \alpha_{s_h} \in \mathrm{SL}_2(\mathbb{Z})$$

corresponding to the s_i coordinates, and

$$\beta_1, \dots, \beta_k \in \mathrm{SL}_2(\mathbb{Z})$$

corresponding to the singleton coordinates in the T_i. Let c_i, d_i be the bottom row of γ_i and $c_{i,j}, d_{i,j}$ the bottom row of $\gamma_{i,j} g_{i,j}$.

Define $V'_{\gamma,\tau}$ by

$$(X_1, Y_1, Z_1, \dots, X_{h+k}, Y_{h+k}, Z_{h+k}) \in V'_{\gamma,\tau}$$

if and only if

$$\begin{aligned} \forall w_{i,j} \in \mathbb{C} \quad \text{with } & \Phi_{N_{i,j}}(X_i, w_{i,j}), i \le h, j \le r_i, \\ & [\dots, j(\sigma_i), j'(\gamma_i\sigma_i), j''(\gamma_i\sigma_i), \dots, X_i, Y_i, Z_i, \\ & \dots, w_{i,j}, -Y_i(c_{i,j}\tau_{s_i} + d_{i,j})^2 \lambda_{N_{i,j}}(X_i, w_{i,j}), \\ \mu_{N_{i,j}}(X_i, w_{i,j}, Y_i, -Y_i(c_{i,j}\tau_{s_i} + d_{i,j})^2 \lambda_{N_{i,j}}(X_i, w_{i,j}), & \quad Z_i, c_{i,j}, c_{i,j}\tau_{s_i} + d_{i,j}), \\ & \dots, X_{h+i}, Y_{h+i}, Z_{h+i}, \dots] \in V. \end{aligned}$$

Taking $\overline{\mathbb{Q}}$-Zariski closures replaces the $j'(\gamma_i\sigma_i)$ and $j''(\gamma_i\sigma_i)$ by suitable rational functions involving σ_i, $\operatorname{Im}\sigma_i$, $\chi^*(\sigma_i)$, c_i, d_i, and some complex numbers w that are allowed to be arbitrary. Making these replacements, we get a variety $V_{\gamma,\tau}$, defined over $\overline{\mathbb{Q}}$, depending polynomially on $c_i, d_i, c_{i,j}, (c_{i,j}\tau_i + d_{i,j})$ (and nothing else). Thus, each $V_{\gamma,\tau}$ is a fibre of an algebraic family of varieties $\widehat{V}$, defined over $\overline{\mathbb{Q}}$.

We now apply Theorem 4.1 to $\widehat{V}$. We get an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma_2(\widehat{V})$, consisting of $\mathbb{H}$-special subvarieties of $\mathbb{H}^{(h+k)}$, such that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$ and all quadratic $\tau \in \mathbb{H}^h$, either

$$V_{\gamma,\tau} = \mathbb{C}^{3(h+k)}$$

or the J-special subvarieties of $V_{\gamma,\tau}$ are accounted for by $\sigma_2(\widehat{V})$. The $H \in \sigma_2(\widehat{V})$ correspond in the obvious way to an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\mathcal{G}$ of proper $\mathbb{H}$-special subvarieties of G. We add all these $H \in \mathcal{G}$ to the overarching collection $\sigma(V)$.

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Now, a quadratic point lying in

$$\bigcup_{\gamma\in \mathrm{SL}_2(\mathbb{Z})^n} \gamma\cdot G\cap J^{-1}(V)$$

corresponds to a quadratic point $\tau=(\tau_1,\dots,\tau_n)\in G$ together with $\gamma\in \mathrm{SL}_2(\mathbb{Z})^n$ such that $\gamma\tau\in J^{-1}(V)$. Such a pair (τ,γ) necessarily satisfies

$$J(\tau')\in V_{\gamma,\tau},$$

where

$$\tau'=(\alpha_{s_1}\tau_{s_1},\dots,\alpha_{s_h}\tau_{s_h},\beta_1\tau_{n-k+1},\dots,\beta_k\tau_n).$$

By the properties of $\sigma_2\big(\widehat{V}\big)$, either $\tau'\in H$, for some $H\in\sigma_2\big(\widehat{V}\big)$, or $V_{\gamma,\tau}=\mathbb{C}^{3(h+k)}$. In the first case, we have $\tau\in H$, for some $H\in\mathcal{G}$.

Define a set

$$R=\left\{(\tau,\gamma)\in\mathbb{H}^n\times \mathrm{SL}_2(\mathbb{Z})^n : \begin{array}{c}\tau \text{ is quadratic},\tau\in G, J(\gamma\tau)\in V,\\ \forall H\in\mathcal{G},\tau\notin H\end{array}\right\}.$$

If R is empty, we can stop here, removing G from $\sigma(V)$ entirely; it contributes no special points other than those already accounted for by $\mathcal{G}$.

If $R\neq\emptyset$, we continue. By the properties of $\mathcal{G}$, every $(\tau,\gamma)\in R$ must satisfy

$$V_{\gamma,\tau}=\mathbb{C}^{3(h+k)}.$$

Hence, in the definition of $V_{\gamma,\tau}$, we can replace $(X_1,\dots,Z_{h+k})$ with $J(z_1,\dots,z_{h+k})$, for *arbitrary* $z_i\in\mathbb{H}$. By (5.2) and an easy calculation involving λ_N, we see that

$$\Bigg[\dots,J(\gamma_i\sigma_i),\dots,J(z_i),$$
$$\dots,j(h_{i,j}z_i),j'(h_{i,j}z_i)\frac{(c_{i,j}\tau_{s_i}+d_{i,j})^2}{\delta_{i,j}^2},$$
$$j''(h_{i,j}z_i)\frac{(c_{i,j}\tau_{s_i}+d_{i,j})^4}{\delta_{i,j}^4}+2j'(h_{i,j}z_i)\frac{c_{i,j}(c_{i,j}\tau_{s_i}+d_{i,j})^3}{\delta_{i,j}^2},$$
$$\dots,J(z_{h+i}),\dots\Bigg]\in V$$

for all $(z_1,\dots,z_{h+k})\in\mathbb{H}^{h+k}$. Here $h_{i,j}$ is an upper-triangular matrix in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of $g_{i,j}$ and $\delta_{i,j}$ is its lower-right entry.

As before, by taking $\overline{\mathbb{Q}}$-Zariski closures we can replace the $J(\gamma_i\sigma_i)$ by some $\overline{\mathbb{Q}}$-rational functions of c_i and d_i. Thus the above equation, for each choice of $\mathbf{z}=(z_1,\dots,z_{h+k})$, defines a G-variety that we will call $W_{\mathbf{z}}$. The intersection

$$W=\bigcap_{\mathbf{z}\in\mathbb{H}^{h+k}} W_{\mathbf{z}}$$

is still a G-variety. (It is non-empty since R is non-empty.)

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We have seen that every $(\tau, \gamma) \in R$ satisfies

$$(\tau_{s_1}, \dots, \tau_{s_h}) \in W^\gamma.$$

Conversely, any quadratic $\mathbf{z} = (z_1, \dots, z_h)$, no matter its height, which is a solution of some W^γ, must come from a member of $\gamma \cdot G \cap J^{-1}(V)$. Indeed, in this situation we have

$$\Bigg[\dots, J(\gamma_i\sigma_i), \dots, J(z_i), \dots, j(h_{i,j}z_i), j'(h_{i,j}z_i)\frac{(c_{i,j}z_i + d_{i,j})^2}{\delta_{i,j}^2}, j''(h_{i,j}z_i)\frac{(c_{i,j}z_i + d_{i,j})^4}{\delta_{i,j}^4} + 2j'(h_{i,j}z_i)\frac{c_{i,j}(c_{i,j}z_i + d_{i,j})^3}{\delta_{i,j}^2}, \dots, J(z_{h+i}), \dots\Bigg] \in V,$$

for all $(z_{h+1}, \dots, z_{h+k}) \in \mathbb{H}^k$.

Brief calculations involving the transformation laws for j' and j'' show that

$$j'(h_{i,j}z_i)\frac{(c_{i,j}z_i + d_{i,j})^2}{\delta_{i,j}^2} = j'(\gamma_{i,j}g_{i,j}z_i)\frac{(c_{i,j}z_i + d_{i,j})^2}{(c_{i,j}z_i + d_{i,j})^2} = j'(\gamma_{i,j}g_{i,j}z_i)$$

and that

$$j''(h_{i,j}z_i)\frac{(c_{i,j}z_i + d_{i,j})^4}{\delta_{i,j}^4} + 2j'(h_{i,j}z_i)\frac{c_{i,j}(c_{i,j}z_i + d_{i,j})^3}{\delta_{i,j}^2} = j''(\gamma_{i,j}g_{i,j}z_i)\frac{(c_{i,j}z_i + d_{i,j})^4}{(c_{i,j}z_i + d_{i,j})^4} = j''(\gamma_{i,j}g_{i,j}z_i).$$

Thus we get

$$[\dots, J(\gamma_i\sigma_i), \dots, J(z_i), \dots, j(\gamma_{i,j}g_{i,j}z_i), j'(\gamma_{i,j}g_{i,j}z_i), j''(\gamma_{i,j}g_{i,j}z_i), \dots, J(z_{h+i}), \dots] \in V,$$

for every $(z_{h+1}, \dots, z_{h+k}) \in \mathbb{H}^k$. Hence $\mathbf{z}$ corresponds to the point

$$(\dots, \gamma_i\sigma_i, \dots, z_i, \dots, \gamma_i g_{i,j} z_i, \dots, z_{h+i}, \dots) \in \gamma \cdot G \cap J^{-1}(V),$$

for any choice of z_{h+i}.

To sum up, we have seen that the quadratic points in $\gamma \cdot G \cap J^{-1}(V)$ consist precisely of

- those quadratic points that lie in some $H \in \mathcal{G}$;
- those points that correspond to quadratic solutions of W^γ, that is,

$$\gamma \cdot G \cap \mathrm{Sp}(\gamma, W).$$

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We are not claiming that these possibilities are mutually exclusive!

Before we are done with G, we must check whether (G, W) is geodesically minimal. If it does happen to be geodesically minimal, we are done. Otherwise, by definition,

$$\bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})^n} \mathrm{Sp}(\gamma, W)$$

is contained in some $\mathrm{SL}_2(\mathbb{Z})$-finite collection of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^{h+k}$. This is not a problem; we simply remove G from $\sigma(V)$ entirely and replace it by the appropriate finite collection of proper subvarieties of G.

This is as much as we can do with a given $G \in \sigma(V)$. It may be helpful to have a brief summary here. For the given G we have done two things:

(1) Either removed G from $\sigma(V)$ entirely or associated with G a geodesically minimal G-variety W_G.

(2) Added to $\sigma(V)$ some finite collection $\mathcal{G}$ of proper subvarieties of G.

Moreover, the union of the $\mathbb{H}$-special subvarieties of $\gamma \cdot G \cap J^{-1}(V)$ is precisely

$$\gamma \cdot G \cap \mathrm{Sp}(\gamma, W_G),$$

together with the $\mathbb{H}$-special subvarieties of

$$\bigcup_{H \in \mathcal{G}} \gamma \cdot H \cap J^{-1}(V).$$

This is enough to conclude the theorem. Simply perform the above process to each $G \in \sigma(V)$ in turn, taking the G in *descending order of dimension*. Since each G can add to $\sigma(V)$ only finitely many varieties of strictly smaller dimension, the process will eventually terminate. □

6. Uniformity

In this final section, I will discuss uniform versions of the two main results of the document, Theorems 1.9 and 5.6. For this, we are closely following the work of Scanlon, who gives in [**17**] a very general approach to uniformizing results of this type. Unfortunately, our setting does not fit perfectly into Scanlon's framework; the full strength of his result is therefore not available to us. Fortunately, the central ideas of Scanlon's work, at least, are applicable, and give us some fairly nice results.

There are two main lemmas of Scanlon that we will use. The first is Lemma 3.1 from [**17**], which applies directly. We write it out here for completeness.

Lemma 6.1. *Let k be a field and K an algebraically closed field extension of k. Let X be a variety over k and X_K its base change to K. Let $A \subseteq X(K)$. Suppose that $Y \subseteq X_K$ is constructible. Then there is a natural number n, some constructible set $Z \subseteq X \times X^n$, defined over k, and some $a \in A^n$ such that $Z_a(K) \cap A = Y(K) \cap A$.*

Proof. See Lemma 3.1 from [**17**]. □

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The next lemma is the analogue of Lemma 3.2 from [17]. The statement needs some very slight modification before it applies in our setting, but the proof of the modified lemma is essentially identical to the original proof.

Lemma 6.2. *Let $k \subseteq K$ be algebraically closed fields, $X = \mathbb{A}^s$, $B = \mathbb{A}^t$ and $Y \subseteq X \times B$ a constructible subset. Let $A \subseteq X(K)$. Suppose that there exist $\alpha, \beta \in A$ that are k-Zariski separated, as defined below.*

Then there is a natural number n and a k-constructible set $Z \subseteq X \times X^n$ such that for any $b \in B(K)$, there is $a \in A^n$ for which $Y_b(K) \cap A = Z_a \cap A$.

Zariski separation property. *Let k be a field. For $m \in \mathbb{N}$ and $i \leq m$, let $p_i^{(m)}$ be the m-tuple*

$$(\alpha, \dots, \alpha, \beta, \alpha, \dots, \alpha),$$

with the β arising in the ith place. Let $P_i^{(m)}$ be the k-Zariski closure of $\{p_i^{(m)}\}$. Then α and β are k-Zariski separated if

$$p_i^{(m)} \notin P_j^{(m)}$$

for any $i \neq j$.

Proof. Throughout this proof we will suppress notation, writing $W = W(K)$ whenever W is a constructible set over K.

Consider K as a structure in the language $\mathcal{L} = (+, \cdot, P_A, \{a\}_{a \in K})$, where each constant symbol a is to be interpreted as the corresponding element $a \in K$ and P_A is an s-ary predicate to be interpreted as the set A. In this language, we can express the condition '$x \in A \cap W$' for any affine variety W over K.

Let T be the $\mathcal{L}$-theory of K, and then let $\mathcal{L}'$ be $\mathcal{L}$ together with some new constant symbols $b_1, \dots, b_t$. Write $b = (b_1, \dots, b_t)$.

Let $\mathcal{C}(k)$ be the set of k-constructible subsets of $X \times X^N$ (for some N) and consider the set

$$\Gamma = T \cup \{\forall c \in A^N \quad \exists x \in A \quad (x \in Y_b \setminus Z_c \vee x \in Z_c \setminus Y_b) : Z \in \mathcal{C}(k)\}.$$

Suppose that Γ is *not* finitely satisfiable. Let Γ_0 be a finite subset witnessing this. Since T is the theory of K, Γ_0 cannot be contained in T, so it includes references to some finitely many k-constructible sets $Z_1, \dots, Z_l$, with $Z_i \subseteq X \times X^{N_i}$. Since Γ_0 is not satisfiable, we have

$$\forall b \in B \quad \exists i \leq l \quad \exists c \in A^{N_i} \qquad \forall x \in A \quad (x \notin Y_b \setminus Z_c \wedge x \notin (Z_i)_c \setminus Y_b).$$

In other words, for every $b \in B$, there is some Z_i and some $c \in A^{N_i}$ such that

$$A \cap (Z_i)_c = A \cap Y_b.$$

Now let $N = \max\{N_i : i \leq l\}$ and let $n = N + l$. Define

$$Z = \bigcup_{i=1}^{l} Z_i \times X^{N - N_i} \times P_i^{(l)},$$

with $P_i^{(l)}$ corresponding to the k-Zariski closure of $(\alpha, \dots, \beta, \dots, \alpha)$ as in the statement of k-Zariski separation of α and β. This Z is a constructible set defined over k, and satisfies

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the conclusion of the lemma: if $b \in B$, then for some i, c, we have $A \cap (Z_i)_c = A \cap Y_b$. Letting $c' = \left(c, \alpha, \ldots, \alpha, p_i^{(l)}\right) \in A^{N+l}$, we get

$$A \cap Z_{c'} = A \cap (Z_i)_c = A \cap Y_b.$$

So suppose on the other hand that Γ is finitely satisfiable. By the compactness theorem, it is satisfiable. So we have an algebraically closed extension $L \supseteq K$ of K, a point $b \in B(L)$ and a set $A^* \subseteq X(L)$ such that, for every k-constructible $Z \subseteq X \times X^N$ and $c \in (A^*)^N$, we have

$$Y_b(L) \cap A^* \neq Z_c(L) \cap A^*.$$

This contradicts Lemma 6.1 (applied to k, L, A^*, X and Y_b). □

Using this, we would like to get a uniform version of 5.6. This can indeed be done, though perhaps not in exactly the manner we might like.

Given an algebraic family $V \subseteq \mathbb{C}^{3n+k}$, we can apply 6.2, with

$$A = \{J(\tau) : \tau \in \mathbb{H}^n \text{ quadratic}\}.$$

For the resulting constructible set Z, we can write

$$Z = \bigcup_{i=1}^{r} X_i \setminus Y_i$$

for some varieties X_i, Y_i defined over $\overline{\mathbb{Q}}$. If we apply 5.6 (under the assumption of Conjecture 1.12) to each of the X_i and Y_i, we get a finite collection $\sigma(V)$, consisting of *pairs* (G, H) of $\mathbb{H}$-special varieties, with corresponding G-varieties (W_G, W_H), such that the union of the $\mathbb{H}$-special subvarieties of $J^{-1}(V_b)$, for any fibre b, is precisely the fibre of the set

$$\bigcup_{(G,H)\in\sigma(V)} \left[\bigcup_{\gamma\in \mathrm{SL}_2(\mathbb{Z})} \gamma G \cap \mathrm{Sp}(\gamma, W_G) \setminus \bigcup_{\gamma\in \mathrm{SL}_2(\mathbb{Z})} \gamma H \cap \mathrm{Sp}(\gamma, W_H) \right]$$

at some quadratic $\tau = \tau(b)$.

This is not as good as we might like; one would prefer not to have the $\mathbb{H}$-special sets H in the picture. These arise thanks to the fact that Z is only constructible, rather than Zariski closed. It does not seem possible to get around this; the reason is essentially the same as the reason why the full strategy outlined in Scanlon's paper [**17**] does not work.

Let G and H be $\mathbb{H}$-special varieties and take a corresponding G-variety W_G and H-variety W_H. If it were the case that $(G \cap H, W_G \cap W_H)$ was geodesically minimal whenever (G, W_G) and (H, W_H) were, then we would be able to apply the rest of Scanlon's work. This is not necessarily the case, so the rest of Scanlon's work cannot be applied. Hence the above seems likely to be the best possible uniform version of 5.6.

Theorem 1.9, however, can be uniformized more cleanly, since it is less precise.

Proposition 6.3. *Assume Conjecture 1.12. Let $V \subseteq \mathbb{C}^{3n+k}$ be an algebraic variety (with arbitrary field of definition), considered as an algebraic family of fibres $V_b \subseteq \mathbb{C}^{3n}$. There is a natural number N, and an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma(V)$ of $\mathbb{H}$-special subvarieties of $\mathbb{H}^{n+N}$, with the following property.*

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For every $b \in \mathbb{C}^k$ with $V_b \neq \mathbb{C}^{3n}$, the $\mathbb{H}$-special points of $J^{-1}(V_b)$ are contained in

$$\bigcup_{G \in \sigma(V)} G_\tau,$$

where G_τ is the fibre of G at some fixed quadratic $\tau = \tau(b) \in \mathbb{H}^N$. Moreover, all of the G_τ are proper *subvarieties of $\mathbb{H}^n$.*

Proof. Let $V \subseteq \mathbb{C}^{3n+k}$ be a variety, considered as an algebraic family of fibres $V_b \subseteq \mathbb{C}^{3n}$. We will apply Lemma 6.2, with $X = \mathbb{C}^{3n}$, $Y = V$, $K = \mathbb{C}$, $k = \overline{\mathbb{Q}}$ and

$$A = \{J(\tau) : \tau \in \mathbb{H}^n \text{ quadratic}\}.$$

To apply the lemma, we need to find two $\overline{\mathbb{Q}}$-Zariski separated points $\alpha, \beta \in A$. This is easy; we only need the j-coordinates of α and β to be distinct. So Lemma 6.2 does apply; we get a $\overline{\mathbb{Q}}$-constructible set $Z \subseteq \mathbb{C}^{3(n+dn)}$ such that, for every $b \in \mathbb{C}^k$, there is some $a \in A^d$ such that

$$Z_a \cap A = V_b \cap A.$$

Write

$$Z = \bigcup_{i=1}^{r} X_i \setminus Y_i,$$

for some $\overline{\mathbb{Q}}$-varieties X_i and Y_i. We will apply Theorem 5.6 to each X_i and Y_i separately. We get some finite sets $\sigma(X_i)$ and $\sigma(Y_i)$ of $\mathbb{H}$-special varieties, with associated G-varieties, exactly describing the special subvarieties of the Z_i in the manner described in the statement of Theorem 5.6.

Now, given $b \in \mathbb{C}^k$, let $a \in A^d$ be such that $Z_a \cap A = V_b \cap A$. Let τ be a preimage of a under J.

First suppose that no $\sigma(X_i)$ contains any G such that, for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^N$, $G_{\gamma\tau} = \mathbb{H}^n$. Then we are done; the special points of $J^{-1}(V_b)$ are contained in the $\mathrm{SL}_2(\mathbb{Z})$-finite collection of proper $\mathbb{H}$-special varieties

$$\{\gamma' \cdot G_{\gamma\tau} : G \in \sigma(X_i), i \leq r, \gamma \in \mathrm{SL}_2(\mathbb{Z})^N, \gamma' \in \mathrm{SL}_2(\mathbb{Z})^n\}.$$

On the other hand, suppose that some $\sigma(X_i)$ contains G such that, for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^N$, $G_{\gamma\tau} = \mathbb{H}^n$. Then the G-variety associated with G cannot impose any condition on the coordinates corresponding to $\mathbb{H}^n$. Hence by the properties laid out in 5.6, we must have $(X_i)_a = \mathbb{C}^{3n}$.

Now apply the same argument to Y_i. There are two possibilities. Either

- the special points of $(Y_i)_a$ are contained in an $\mathrm{SL}_2(\mathbb{Z})$-finite collection

  $$\{\gamma' \cdot G_{\gamma\tau} : G \in \sigma(Y_i), i \leq r, \gamma \in \mathrm{SL}_2(\mathbb{Z})^N, \gamma' \in \mathrm{SL}_2(\mathbb{Z})^n\},$$

  with each $G_{\gamma\tau}$ being a *proper* $\mathbb{H}$-special subvariety of $\mathbb{H}^n$; or
- $(Y_i)_a = \mathbb{C}^{3n}$.

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In the first case, since the special points of $(Y_i)_a$ are contained in a lower-dimensional set, it follows that the special points of Z_a are Zariski dense, whence $V_b = \mathbb{C}^{3n}$. In the second case, $(X_i)_a \setminus (Y_i)_a$ contributes no new special points, so we can ignore this i and move on.

Finally, note that if $G_\tau = \mathbb{H}^n$ for some τ, then it must be the case for every τ' that either $G_{\tau'} = \mathbb{H}^n$ or $G_{\tau'} = \emptyset$. Thus, if some $G \in \sigma(X_i)$ has the property that (for some τ), $G_\tau = \mathbb{H}^n$, we can safely remove it. By the previous arguments, all the special points will still be covered by the rest of the $G \in \bigcup \sigma(X_i)$, except in the case where $V_b = \mathbb{C}^{3n}$.

Hence the ($\mathrm{SL}_2(\mathbb{Z})$-finite) collection

$$\sigma(V) = \left\{ \gamma \cdot G : \begin{array}{c} \gamma \in \mathrm{SL}_2(\mathbb{Z})^n \\ G \in \sigma(X_i), \text{ for some } i \\ \text{For every } \tau \in \mathbb{H}^N, G_\tau \neq \mathbb{H}^n \end{array} \right\}$$

satisfies the conclusion of the proposition. □

To conclude, we will state one final corollary of the above result, which simply says that Theorem 1.9 holds for arbitrary varieties, rather than just those defined over $\overline{\mathbb{Q}}$.

Corollary 6.4. *Assume Conjecture 1.12. Let $V \subseteq \mathbb{C}^{3n}$ be a proper algebraic variety (with arbitrary field of definition). There exists an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma(V)$, consisting of* proper $\mathbb{H}$*-special varieties of $\mathbb{H}^n$, such that every $\mathbb{H}$-special point in $J^{-1}(V)$ is contained in some $G \in \sigma(V)$.*

Proof. Immediate. □

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