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Variational methods on periodic and quasi-periodic solutions for the N-body problem

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Abstract. The purpose of this article is two fold. First, we show how quasi-periodic solutions for the N-body problem can be constructed by variational methods. We illustrate this by constructing uncountably many quasi-periodic solutions for the four- and six-body problems with equal masses. Second, we show by examples that a system of N masses can possess infinitely many simple or multiple choreographic solutions. In particular, it is shown that the four-body problem with equal masses has infinitely many double choreographic solutions and the six-body problem with equal masses has infinitely many simple and double choreographic solutions. Our approach is based on the technique of binary decomposition and some variational properties of Keplerian orbits.

1. *Introduction*

This paper is concerned with variational methods for the Newtonian N-body problem. As a Hamiltonian system, it is natural to study the N-body problem by considering critical points of the action functional defined on a suitable path space. However, analytical results from this viewpoint are very limited. From a variational point of view it is often difficult to distinguish solutions with collisions from classical solutions, since generally the presence of collisions does not result in a significant increment in the value of the action functional. In the case of three equal masses, the most notable success is due to Chenciner and Montgomery [**5**], who proved the existence of the figure-8 orbit by comparing the action functional with the action for Keplerian orbits. When the number of masses increases, we immediately encounter technical difficulties with effective estimation for the value of the action functional.

The figure-8 orbit is the first non-trivial simple choreographic solution ever found. A *simple choreographic solution* is a periodic solution with the property that all masses chase one another along a single orbit. If the orbit of a periodic solution consists of two or more closed curves, each of which is the trajectory of at least two masses, then it is called a *multiple choreographic solution*. Many relative equilibria give rise to simple

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or multiple choreographic solutions; they will be referred to as *trivial* choreographic solutions. Relative to the vast number of numerical discoveries [**4, 12**], very few non-trivial choreographic solutions have rigorous existence proofs. Progress in this direction beyond the discovery of the figure-8 orbit includes [**1, 3, 6, 9, 10, 14**] and, most recently, [**2, 7**].

The major purpose of this paper is to demonstrate how quasi-periodic solutions for the N-body problem can be constructed by variational methods. This was first numerically investigated in [**4**], in which it was suggested that a choreographic solution can give rise to a family of quasi-periodic solutions and 'satellite' choreographic solutions. None of this type of quasi-periodic solutions have analytical proof for their existence. Addressing this question, in §§5 and 6 we construct uncountably many quasi-periodic solutions for the four- and six-body problems with equal masses. Another goal of this paper is to show that a system of N masses can possess infinitely many simple or multiple choreographic solutions. In particular, it is shown that the four-body problem with equal masses has infinitely many double choreographic solutions and the six-body problem with equal masses has infinitely many simple and double choreographic solutions. Our approach is based on several variational principles (§2), the variational properties of Keplerian orbits (§3) and the technique of binary decomposition (§4).

2. *Preliminaries*

Consider a system of N (≥ 2) masses $m_1, m_2, \ldots, m_N$ moving in $\mathbb{C}$ in accordance with Newton's law of gravitation:

$$m_k \ddot{x}_k = \frac{\partial}{\partial x_k} U(x), \quad k = 1, \ldots, N, \tag{1}$$

where $x_k \in \mathbb{C}$ is the position of m_k and

$$U(x) = U(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|x_i - x_j|}$$

is the potential energy. The kinetic energy is given by

$$K(\dot{x}) = \sum_{i=1}^{N} \frac{1}{2} m_i |\dot{x}_i|^2.$$

For any fixed positive constant T, equation (1) is the Euler–Lagrange equation for the action functional $\mathcal{A}_T : H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^N) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{A}_T(x) := \int_0^T K(\dot{x}) + U(x)\, dt.$$

A path x in $H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^N)$ is said to have *d-rotational symmetry*, $d \geq 1$, if there exists some $T > 0$ such that

$$x(t) = e^{(2\pi/d)i} x(t + T)$$

for any $t \in \mathbb{R}$. The number T is called a *relative period* of x. Let

$$H_{d,T} := \{x \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^N) : x(t) = e^{(2\pi/d)i} x(t + T)\}.$$

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The conventional definition of the inner product on the Sobolev space $H^1([0, T], \mathbb{C}^N)$ also defines an inner product on $H_{d,T}$:

$$\langle x, y\rangle_{d,T} = \int_0^T \langle x(t), y(t)\rangle + \langle \dot{x}(t), \dot{y}(t)\rangle \, dt.$$

Here $\langle \cdot, \cdot \rangle$ stands for the standard scalar product on $(\mathbb{R}^2)^N \cong \mathbb{C}^N$. Critical points of $\mathcal{A}_T$ on $H_{d,T}$ are critical points of $\mathcal{A}_T$ on $H^1([0, T], \mathbb{C}^N)$. We can easily verify that, for any $x \in H_{d,T}$ and $\tau \in \mathbb{R}$,

$$\mathcal{A}_T(x) = \int_\tau^{T+\tau} K(\dot{x}) + U(x) \, dt,$$
$$\langle x, y\rangle_{d,T} = \int_\tau^{T+\tau} \langle x(t), y(t)\rangle + \langle \dot{x}(t), \dot{y}(t)\rangle \, dt.$$

Following these observations, any critical point x of $\mathcal{A}_T$ on $H_{d,T}$ is a solution of (1), but possibly with collisions. If we can show that x has no collision on $[0, T)$, then there is no collision at all and x indeed solves (1) for any $t \in \mathbb{R}$. Moreover, x is periodic if d is rational and quasi-periodic if d is irrational.

Let G be a group of linear transformations on $H_{d,T}$. The space $H^G_{d,T}$ of G-invariant paths is defined by

$$H^G_{d,T} = \{x \in H_{d,T} : g \cdot x = x \text{ for any } g \in G\}.$$

The existence of solutions invariant under the action of G is equivalent to the existence of critical points for the action functional $\mathcal{A}_T$ in $H^G_{d,T}$. Critical points of $\mathcal{A}_T$ when restricted to $H^G_{d,T}$, called *G-critical points*, are not necessarily critical points of $\mathcal{A}_T$ on $H_{d,T}$. In this regard we quote a result of Palais [**11**].

PALAIS' PRINCIPLE OF SYMMETRIC CRITICALITY. *Suppose the group G is orthogonal and $\mathcal{A}_T$ is G-invariant. Then G-critical points of $\mathcal{A}_T$ are critical points of $\mathcal{A}_T$ on $H_{d,T}$.*

Given a group G of orthogonal transformations on $H_{d,T}$, the minimizing problem for the functional $\mathcal{A}_T$ on $H^G_{d,T}$ is not always solvable. By using Fatou's lemma and the fact that any norm is weakly sequentially lower semicontinuous, it is an easy exercise to show that $\mathcal{A}_T$ is weakly sequentially lower semicontinuous on $H_{d,T}$. Following a standard argument in calculus of variations, if we can show that $\mathcal{A}_T$ is coercive on $H^G_{d,T}$ (which is weakly closed), then $\mathcal{A}_T$ attains its infimum on $H^G_{d,T}$. Below we provide a simple criterion for coercivity.

A subset Y of $H_{d,T}$ is said to be *non-central* if there exists some $\nu \in (0, 2]$ such that, for any $x \in Y$, there corresponds a $t_x \in (0, T]$ satisfying

$$\langle x(0), x(t_x)\rangle \le (1 - \nu)|x(0)| \cdot |x(t_x)|. \tag{2}$$

Clearly (2) holds if $\nu \le 0$ and fails if $\nu > 2$. Each path in Y has to move away from its initial position by a certain angle (relative to the origin).

PROPOSITION 2.1. *Suppose that the subset Y of $H_{d,T}$ is non-central. Then $\mathcal{A}_T$ is coercive on Y.*

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Proof. Let $\delta : Y \to \mathbb{R}$ be defined by

$$\delta(x) := \max_{s_1, s_2 \in [0,T]} |x(s_1) - x(s_2)|.$$

We first consider the case $\nu \in (0, 2)$. Let $t_x \in (0, T]$ be chosen so that (2) is satisfied. If $x(0) \neq 0$ and $x(t_x) \neq 0$ and θ is the angle between $x(0)$ and $x(t_x)$, $0 < \theta \leq \pi$, then clearly

$$|x(0) - x(t_x)| \geq \sin(\theta)|x(0)|$$

and the equality holds only when $x(0) - x(t_x)$ is perpendicular to $x(t_x)$. By (2),

$$\cos(\theta) \leq 1 - \nu, \quad \sin(\theta) \geq C_\nu := \sqrt{\nu(2-\nu)}.$$

If $x(0) = 0$ or $x(t_x) = 0$, then $|x(0) - x(t_x)| \geq |x(0)| \geq C_\nu |x(0)|$. Therefore, we always have

$$|x(0) - x(t_x)| \geq C_\nu |x(0)|.$$

Note that $C_\nu > 0$ because $\nu \in (0, 2)$. For any $t \in [0, T]$,

$$|x(t)| \leq |x(0)| + \delta(x) \leq \frac{1}{C_\nu}|x(0) - x(t_x)| + \delta(x) \leq \left(\frac{1}{C_\nu} + 1\right)\delta(x),$$

so that

$$\int_0^T |x|^2\, dt \leq \left(\frac{1}{C_\nu} + 1\right)^2 \delta(x)^2 T.$$

By the Cauchy–Schwartz inequality,

$$\delta(x)^2 \leq \left(\int_0^T |\dot{x}|\, dt\right)^2 \leq T \int_0^T |\dot{x}|^2\, dt.$$

Therefore, the $H_{d,T}$-norm of x is controlled by the value of its action:

$$\begin{aligned} \|x\|^2_{H_{d,T}} &= \int_0^T |x|^2 + |\dot{x}|^2\, dt \\ &\leq \left(\left(\frac{1}{C_\nu} + 1\right)^2 T^2 + 1\right) \int_0^T |\dot{x}|^2\, dt \\ &< \left(\left(\frac{1}{C_\nu} + 1\right)^2 T^2 + 1\right)\left(\frac{2}{m}\right) \mathcal{A}_T(x). \end{aligned}$$

Here $m = \min_i\{m_i\}$. This implies that $\mathcal{A}_T$ restricted to Y is coercive.

The other case, $\nu = 2$, is similar. Let t_x be as before. It follows easily from (2) that

$$|x(0)| \leq |x(0) - x(t_x)| \leq \delta(x).$$

Thus

$$|x(t)| \leq |x(0)| + \delta(x) \leq 2\delta(x)$$

for any $t \in [0, T]$. The fact that $\mathcal{A}_T$ is coercive on Y follows by the same argument as above. □

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As far as the Newtonian N-body problem is concerned, the principal difficulty for the variational approach lies in the fact that many solutions with collisions have about the same action as many classical solutions. In order to obtain physically realizable and meaningful solutions, we ought to distinguish classical solutions from collision ones. Below we introduce a useful theorem by Marchal that excludes a large class of paths with collisions when we consider a minimizing problem with fixed ends.

Given any finite group G acting on $H^G_{d,T}$ and any closed time interval $[\tau_1, \tau_2]$, there is a canonical projection $H^G_{d,T} \to H^1([\tau_1, \tau_2], \mathbb{C}^N)$. We say $[\tau_1, \tau_2]$ is a *fundamental domain* of the action if the projection is injective and there is no proper closed subinterval with this property. Let $[\tau_1, \tau_2]$ be a fundamental domain of the group action. If $\mathcal{A}_T$ is coercive on $H^G_{d,T}$ and $x \in H^G_{d,T}$ is a minimizer, then clearly x is also a minimizer for the fixed-ends problem:

$$\inf\left\{\int_{\tau_1}^{\tau_2} K(\dot{y}) + U(y)\,dt : y \in H^1([\tau_1, \tau_2], \mathbb{C}^N),\ y(\tau_1) = \xi_1,\ y(\tau_2) = \xi_2\right\}, \quad (3)$$

where $\xi_1 = x(\tau_1)$ and $\xi_2 = x(\tau_2)$. A fundamental result by Marchal [**10**] (see also [**3**]) states the following.

MARCHAL'S THEOREM. *Let $\xi_1, \xi_2 \in \mathbb{C}^N$ be given. Then minimizers of the fixed-ends problem (3) are collision-free on the interval (τ_1, τ_2).*

Marchal's theorem does not exclude the possibility of having collisions on the boundary of a fundamental domain. Recently Ferrario and Terracini [**7**] obtained an algebraic criterion on the symmetry group for which Marchal's theorem can be extended to the boundary of a fundamental domain. This criterion is not applicable to the major results in this paper.

3. *The Kepler problem revisited*

The Newtonian two-body problem is known as the *Kepler problem* in honor of Johannes Kepler's discovery of three laws of planetary motion, based on which Newton deduced in 1687 the celebrated law of universal gravitation. Partly due to the fact that the Kepler problem is often considered completely understood, and partly due to some technical difficulties with variational methods for the N-body problem, very few attempts have been made and little attention has been paid to investigations of the variational nature of Keplerian orbits.

In 1977, Gordon [**8**] proved a minimizing property for elliptical Keplerian orbits, but it was not until very recently that his result found some interesting applications, one of which is the proof of existence of the figure-8 orbit in the three-body problem [**5**]. In this section we further exploit the variational nature of Keplerian orbits in a different function space. We see later that the estimates herein are quite useful in the search for some symmetric solutions.

The action functional $\mathcal{A}_T$ defined on $H^1([0, T], \mathbb{C}^2)$ takes the form

$$\begin{aligned}\mathcal{A}_T(x) &= \int_0^T \frac{1}{2}(m_1|\dot{x}_1|^2 + m_2|\dot{x}_2|^2) + \frac{m_1 m_2}{|x_1 - x_2|}\,dt \\ &= \mathcal{A}^0_T(\mathbf{r}) + \mathcal{A}^1_T(\hat{x}),\end{aligned}$$

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where $\mathbf{r} = x_2 - x_1$, $\hat{x}$ is the center of mass, and

$$\mathcal{A}_T^0(\mathbf{r}) = \int_0^T \frac{m_1 m_2}{2(m_1+m_2)}|\dot{\mathbf{r}}|^2 + \frac{m_1 m_2}{|\mathbf{r}|}\,dt,$$
$$\mathcal{A}_T^1(\hat{x}) = \int_0^T \frac{m_1+m_2}{2}|\dot{\hat{x}}|^2\,dt.$$

Granting that linear momentum is an integral of motion, it is customary to drop the integral $\mathcal{A}_T^1$ and consider critical points of $\mathcal{A}_T^0$ defined on $H^1([0,T],\mathbb{C})$. Let $\mu = m_1m_2/(m_1+m_2)$ be the reduced mass and $\alpha = m_1m_2$, and let $\mathbf{r} = re^{i\theta}$ be the polar form of $\mathbf{r}$. The functional $\mathcal{A}_T^0$ can be written

$$\mathcal{A}_T^0(\mathbf{r}) = \int_0^T \frac{\mu}{2}|\dot{\mathbf{r}}|^2 + \frac{\alpha}{|\mathbf{r}|}\,dt = \int_0^T \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\alpha}{r}\,dt.$$

Consider the space of loops

$$\Lambda_T = \{\mathbf{r} \in H^1([0,T],\mathbb{C}) : \mathbf{r}(0) = \mathbf{r}(T), \mathbf{r}(t) \neq 0 \text{ and } \mathrm{Deg}(\mathbf{r},0) \neq 0\}.$$

In [**8**], Gordon proved the following.

GORDON'S THEOREM. *The functional $\mathcal{A}_T^0$ attains its infimum over Λ_T at elliptical Keplerian orbits with prime period T and attains its infimum over $\partial\Lambda_T$, the boundary of Λ_T, at collision–ejection Keplerian orbits with prime period T. The values of $\mathcal{A}_T^0$ over these orbits are all equal to*

$$\inf_{\bar{\Lambda}_T} \mathcal{A}_T^0 = 3\left(\frac{\mu\alpha^2\pi^2}{2}\right)^{1/3} T^{1/3}. \tag{4}$$

Periodic collision–ejection Keplerian orbits can be viewed as degenerate elliptical orbits with eccentricity 1. It is easy to see that without the topological constraint the action functional has no minimum. Gordon's theorem can be extended to a larger space that includes some loops with zero winding number about the origin; see [**2**].

Given any $\phi \in (0,\pi]$, consider the following path space:

$$\Gamma_{T,\phi} = \{\mathbf{r} = re^{i\theta} \in H^1([0,T],\mathbb{C}) : r(0)r(T)\cos\phi = \langle \mathbf{r}(0), \mathbf{r}(T)\rangle\}.$$

The symbol $\langle\cdot,\cdot\rangle$ stands for the standard scalar product in $\mathbb{R}^2 \cong \mathbb{C}$.

THEOREM 3.1. *For any $\phi \in (0,\pi]$, the functional $\mathcal{A}_T^0$ attains its infimum on $\Gamma_{T,\phi}$ and*

$$\inf_{\Gamma_{T,\phi}} \mathcal{A}_T^0 = \tfrac{3}{2}(\mu\alpha^2\phi^2)^{1/3}T^{1/3}. \tag{5}$$

Let $\mathbf{r}_\phi$ be a minimizer of $\mathcal{A}_T^0$ on $\Gamma_{T,\phi}$.

(a) *If $\phi = \pi$, then $\mathbf{r}_\pi$ is half of an elliptical Keplerian orbit with prime period $2T$, or a collinear Keplerian orbit that starts (respectively ends) with zero velocity and ends (respectively starts) at origin.*

(b) *If $\phi \in (0,\pi)$, then $\mathbf{r}_\phi$ is a portion of a circular Keplerian orbit with prime period $2\pi T/\phi$.*

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Consider the following subset of $\Gamma_{T,\phi}$:

$$\Delta_{T,\phi} = \{\mathbf{r} = re^{i\theta} \in H^1([0,T],\mathbb{C}) : r(0), r(T) \neq 0, \theta(0) = \theta(T) - \phi = 0\}.$$

The space $\Delta_{T,\phi}$ consists of paths that start from the positive real axis and end on the half line consisting of complex numbers with argument ϕ. The weak and strong closure coincide since it is convex and, hence, we speak of the 'closure' of $\Delta_{T,\phi}$ with no ambiguity. The closure $\bar{\Delta}_{T,\phi}$ of $\Delta_{T,\phi}$ consists of paths that start from the closure of the positive real axis and end on the closure of the half line consisting of complex numbers with argument ϕ.

LEMMA 3.2. *Given any* $\mathbf{r} \in \Gamma_{T,\phi}$, *there is an* $A \in \mathsf{O}(2)$ *and* $\tilde{\mathbf{r}} \in \bar{\Delta}_{T,\phi}$ *such that* $\tilde{\mathbf{r}} = A\mathbf{r}$ *and* $\mathcal{A}_T^0(\mathbf{r}) = \mathcal{A}_T^0(\tilde{\mathbf{r}})$.

Proof. This is quite obvious because the space $\Gamma_{T,\phi}$ is actually the image of $\mathsf{O}(2)$ acting on $\bar{\Delta}_{T,\phi}$. To be more precise, the possibilities are:

- if $\mathbf{r}(0) = \mathbf{r}(T) = 0$, then let $\tilde{\mathbf{r}} = \mathbf{r}$;
- if $\mathbf{r}(0) = 0$, $\mathbf{r}(T) \neq 0$ and $\mathrm{Arg}(\mathbf{r}(T)) = \varphi$, then let $\tilde{\mathbf{r}} = e^{(\phi-\varphi)i}\mathbf{r}$;
- if $\mathbf{r}(0) \neq 0$, $\mathbf{r}(T) = 0$ and $\mathrm{Arg}(\mathbf{r}(0)) = \varphi$, then let $\tilde{\mathbf{r}} = e^{-\varphi i}\mathbf{r}$;
- if $\mathbf{r}(0), \mathbf{r}(T) \neq 0$, $\mathrm{Arg}(\mathbf{r}(0)) = \varphi$, $\mathrm{Arg}(\mathbf{r}(T)) = \varphi + \phi$, then let $\tilde{\mathbf{r}} = e^{-\varphi i}\mathbf{r}$;
- if $\mathbf{r}(0), \mathbf{r}(T) \neq 0$, $\mathrm{Arg}(\mathbf{r}(0)) = \varphi$, $\mathrm{Arg}(\mathbf{r}(T)) = \varphi - \phi$, then let $\tilde{\mathbf{r}} = e^{\varphi i}\bar{\mathbf{r}}$.

Clearly $\mathcal{A}_T^0(\mathbf{r}) = \mathcal{A}_T^0(\tilde{\mathbf{r}})$ in each case. □

LEMMA 3.3. *If* $\phi \in (0,\pi)$, *then* $\inf_{\Delta_{T,\phi}} \mathcal{A}_T^0 < \inf_{\partial\Delta_{T,\phi}} \mathcal{A}_T^0$.

Proof. Let $\mathbf{r}_\phi \in \bar{\Delta}_{T,\phi}$ be a minimizer of $\mathcal{A}_T^0$ on $\bar{\Delta}_{T,\phi}$. Let $\xi_1 = \mathbf{r}_\phi(0)$, $\xi_2 = \mathbf{r}_\phi(T)$. Clearly $\mathbf{r}_\phi$ minimizes the fixed-ends problem:

$$\inf\{\mathcal{A}_T^0(\mathbf{r}) : \mathbf{r} \in H^1([0,T],\mathbb{C}), \mathbf{r}(0) = \xi_1, \mathbf{r}(T) = \xi_2\}.$$

This implies that $\mathbf{r}_\phi$ has to be a Keplerian orbit. By Marchal's theorem, $\mathbf{r}_\phi(t) \neq 0$ on $(0,T)$.

If $\mathbf{r}_\phi \in \partial\Delta_{T,\phi}$, then either $\mathbf{r}_\phi(0) = 0$ or $\mathbf{r}_\phi(T) = 0$. By (4), the case with both ends at zero has a higher $\mathcal{A}_T^0$ value $3(\mu\alpha^2\pi^2/2)^{\frac{1}{3}}T^{\frac{1}{3}}$ than the case with one end at zero and the other end with zero velocity. Therefore, $\mathbf{r}_\phi$ has to be the second case and so

$$\inf_{\partial\Delta_{T,\phi}} \mathcal{A}_T^0 = \tfrac{3}{2}(\mu\alpha^2\pi^2)^{1/3}T^{1/3}.$$

To conclude the lemma we have to find a path in $\Delta_{T,\phi}$ with $\mathcal{A}_T^0$ value lower than this number.

Set

$$\tilde{\mathbf{r}}_\phi(t) = \left(\frac{\alpha T^2}{\mu\phi^2}\right)^{1/3} e^{(t/T)\phi i}.$$

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This $\tilde{\mathbf{r}}_\phi(t)$ is indeed the circular Keplerian orbit with prime period $2\pi T/\phi$. The calculation for $\mathcal{A}_T^0(\tilde{\mathbf{r}}_\phi)$ is simple:

$$\begin{aligned}\mathcal{A}_T^0(\tilde{\mathbf{r}}_\phi) &= \frac{\phi}{2\pi}\int_0^{2\pi T/\phi} \frac{\mu}{2}|\dot{\tilde{\mathbf{r}}}_\phi|^2 + \frac{\alpha}{|\tilde{\mathbf{r}}_\phi|}\,dt \\ &= \frac{\phi}{2\pi}\cdot 3\left(\frac{\mu\alpha^2\pi^2}{2}\right)^{1/3}\left(\frac{2\pi T}{\phi}\right)^{1/3} \\ &= \frac{3}{2}(\mu\alpha^2\phi^2)^{1/3}T^{1/3} \\ &< \inf_{\partial\Delta_{T,\phi}} \mathcal{A}_T^0.\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.1. Let $\nu = 1 - \cos\phi$, then $\nu \in (0, 2]$ since $\phi \in (0, \pi]$, and

$$\langle \mathbf{r}(0), \mathbf{r}(T)\rangle = r(0)r(T)(1-\nu)$$

for any $\mathbf{r} \in \Gamma_{T,\phi}$. By Proposition 2.1, the functional $\mathcal{A}_T^0$ is coercive on $\Gamma_{T,\phi}$. It is easy to see that $\Gamma_{T,\phi}$ is closed in both the weak and strong topology in $H^1([0, T], \mathbb{C})$. Therefore, $\mathcal{A}_T^0$ attains its infimum on $\Gamma_{T,\phi}$.

Let $\mathbf{r}_\phi \in \bar{\Delta}_{T,\phi}$ be a minimizer of $\mathcal{A}_T^0$ in $\Gamma_{T,\phi}$. When $\phi = \pi$, we can extend $\mathbf{r}_\pi$ to a loop in $\bar{\Lambda}_{2T}$ by concatenating $\mathbf{r}_\pi$ with its complex conjugate. More precisely, the loop

$$\mathbf{R}(t) = \begin{cases} \mathbf{r}_\pi(t) & \text{for } t \in [0, T] \\ \bar{\mathbf{r}}_\pi(2T - t) & \text{for } t \in (T, 2T]\end{cases}$$

belongs to $\bar{\Lambda}_{2T}$. By Gordon's theorem,

$$\mathcal{A}_T^0(\mathbf{r}_\pi) = \frac{1}{2}\int_0^{2T} \frac{\mu}{2}|\dot{\mathbf{R}}|^2 + \frac{\alpha}{|\mathbf{R}|}\,dt \geq \frac{3}{2}(\mu\alpha^2\pi^2)^{1/3}T^{1/3}.$$

The lower bound on the right-hand side is achieved when and only when $\mathbf{r}_\pi$ is half of an elliptical Keplerian orbit (including collision–ejection orbits) with prime period $2T$. This proves (a) and (5) for the case $\phi = \pi$.

Suppose $\phi \in (0, \pi)$. By Lemmas 3.2 and 3.3, there is no loss of generality in assuming that $\mathbf{r}_\phi = re^{i\theta} \in \Delta_{T,\phi}$. This actually implies that $\mathbf{r}_\phi$ is a Keplerian orbit with non-zero angular momentum. In the proof of Lemma 3.3, we selected a portion of the circular Keplerian orbit $\tilde{\mathbf{r}}_\phi$ and showed that the value of $\mathcal{A}_T^0$ is exactly given by the right-hand side of (5). Any other circular Keplerian orbits in $\Delta_{T,\phi}$ that winds around the origin by an angle $2k\pi + \phi$, $k \in \mathbb{Z} \setminus \{0\}$ has greater action than $\tilde{\mathbf{r}}_\phi$. To prove (5) and (b), it remains to show that $\mathbf{r}_\phi$ must be circular.

Choose any admissible variation h for the variable r with $h(0) \neq 0$. From the first variation (Gâteaux variation) of $\mathcal{A}_T^0$ with respect to r, the term $\mu\dot{r}\cdot h$ has to be zero. This implies $\dot{r}(0) = 0$. Similarly, $\dot{r}(T) = 0$. Since $\mathbf{r}_\phi = re^{i\theta} \in \Delta_{T,\phi}$ is a non-degenerate conic section, there are constants $\mathfrak{p} > 0$, $\mathfrak{e} \geq 0$, $\theta_0 \in [0, 2\pi)$ such that

$$\frac{\mathfrak{p}}{r} = 1 + \mathfrak{e}\cos(\theta - \theta_0).$$

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Differentiating the identity with respect to t at $t = 0,\ T$ yields

$$-\mathfrak{e}\sin(-\theta_0)\cdot\dot{\theta}(0) = 0 = -\mathfrak{e}\sin(\phi-\theta_0)\cdot\dot{\theta}(T).$$

The only possibility is $\mathfrak{e} = 0$ because $\phi \in (0,\pi)$ and the angular momentum is non-zero. This shows that the minimizing orbit $\mathbf{r}_\phi$ is a circular Keplerian orbit. □

4. *Binary decompositions and estimates for the action functional*

In this section we use the technique of binary decompositions introduced in [**2**] to estimate the value of the action functional. First we describe the procedure in detail for $N = 4$, and then outline this method for $N = 6$.

4.1. *Four equal masses.* Consider the system (1) with four equal masses $m_1 = m_2 = m_3 = m_4 = 1$. Let $\lambda \in [0,1]$ be some constant to be chosen later. Define

$$K_{ij}^0(\dot{x}) = \frac{1}{12}|\dot{x}_i - \dot{x}_j|^2, \quad K_{ij}^1(\dot{x}) = \frac{1}{3}|\dot{\hat{x}}_{ij}|^2,$$
$$U_{ij}^0(x) = \frac{\lambda}{|x_i - x_j|}, \quad U_{ij}^1(x) = \frac{1-\lambda}{|x_i - x_j|},$$

where $\hat{x}_{ij} = \frac{1}{2}(x_i + x_j)$. We can easily verify that

$$K(\dot{x}) = \sum_{\substack{i,j\\ i<j}}(K_{ij}^0(\dot{x}) + K_{ij}^1(\dot{x})), \quad U(x) = \sum_{\substack{i,j\\ i<j}}(U_{ij}^0(x) + U_{ij}^1(x)).$$

Define

$$\mathcal{A}_{ij}^0(x) = \int_0^T K_{ij}^0(\dot{x}) + U_{ij}^0(x)\,dt,$$
$$\mathcal{A}_{ij}^1(x) = \int_0^T K_{ij}^1(\dot{x}) + U_{ij}^1(x)\,dt,$$
$$\mathcal{A}_T^0(x) = \sum_{\substack{i,j\\ i<j}}\mathcal{A}_{ij}^0, \quad \mathcal{A}_T^1(x) = \sum_{\substack{i,j\\ i<j}}\mathcal{A}_{ij}^1.$$

Then

$$\mathcal{A}_T(x) = \mathcal{A}_T^0(x) + \mathcal{A}_T^1(x).$$

The idea behind this decomposition of the action functional is the following. Each mass m_i can be considered a compound of three particles $\{m_{ij} : j \neq i\}$, each of which has mass $\frac{1}{3}$. For any $i \neq j$, $\{m_{ij}, m_{ji}\}$ constitute a binary pair which acts like a particle–antiparticle pair with a suitable attraction constant so that the Lagrangian for this binary pair is given by $K_{ij}^0(x) + U_{ij}^0(x)$. Fixing any i, the subsystems $\{m_{ij} : j \neq i\}$ are bound so that they all have the same position x_i. The total action due to this system of six binary pairs is $\mathcal{A}_T^0$. We call such a decomposition for this system of masses a *binary decomposition*. On many occasions the action of each binary pair can be estimated by either Gordon's formula (4) or formula (5) in Theorem 3.1, according to the nature of each binary pair. Another way of decomposing $\mathcal{A}_T$ is to fix the center of mass at the origin and

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then express the kinetic energy in terms of mutual velocities using the Leibnitz formula. See Venturelli [**13**] and Zhang and Zhou [**15**] for an application to the three-body problem.

Observe that $2|\hat{x}_{ik} - \hat{x}_{jk}| = |x_i - x_j|$ for any distinct i, j, k, and

$$\sum_{\substack{i,j\\ i<j}} K^1_{ij}(\dot{x}) = \sum_{\substack{i,j\\ i<j}} \frac{1}{3}|\dot{\hat{x}}_{ij}|^2 = \sum_{k=1}^{4} \sum_{\substack{i,j\\ i<j\\ i,j\neq k}} \frac{1}{12}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2)$$

$$= \sum_{\substack{i,j\\ i<j}} \sum_{\substack{k=1\\ k\neq i,j}}^{4} \frac{1}{12}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2),$$

$$\sum_{\substack{i,j\\ i<j}} U^1_{ij}(x) = \sum_{\substack{i,j\\ i<j}} \frac{1-\lambda}{|x_i - x_j|} = \sum_{k=1}^{4} \sum_{\substack{i,j\\ i<j\\ i,j\neq k}} \frac{\frac{1}{4}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}$$

$$= \sum_{\substack{i,j\\ i<j}} \sum_{\substack{k=1\\ k\neq i,j}}^{4} \frac{\frac{1}{4}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}.$$

This is essentially treating centers of mass as real masses and performing a binary decomposition for them. Again, the value of $\mathcal{A}^1_T$ can be estimated by using (4) or (5).

Below we show two examples that demonstrate how these estimates can be carried out. The proof of one of our major results, Theorem 5.1, will call for this lemma several times.

LEMMA 4.1. *Let two distinct indices i, j and any $\tau > 0$, $\phi \in (0, \pi]$ be given.*

(a) *If $x_i - x_j \in \bar{\Lambda}_\tau$, then*

$$\int_0^\tau \frac{1}{12}|\dot{x}_i - \dot{x}_j|^2 + \frac{\lambda}{|x_i - x_j|}\,dt \geq \left(\frac{3\pi\lambda}{2}\right)^{2/3} \tau^{1/3},$$

$$\sum_{\substack{k=1\\ k\neq i,j}}^{4} \int_0^\tau \frac{1}{12}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2) + \frac{\frac{1}{4}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}\,dt \geq \left(\frac{3\pi(1-\lambda)}{4}\right)^{2/3} \tau^{1/3}.$$

(b) *If $x_i - x_j \in \Gamma_{\tau,\phi}$, then*

$$\int_0^\tau \frac{1}{12}|\dot{x}_i - \dot{x}_j|^2 + \frac{\lambda}{|x_i - x_j|}\,dt \geq \left(\frac{3\phi\lambda}{4}\right)^{2/3} \tau^{1/3},$$

$$\sum_{\substack{k=1\\ k\neq i,j}}^{4} \int_0^\tau \frac{1}{12}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2) + \frac{\frac{1}{4}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}\,dt \geq \left(\frac{3\phi(1-\lambda)}{8}\right)^{2/3} \tau^{1/3}.$$

Proof. The first inequality in (a) follows straight from (4). For any $k \neq i, j$, $\hat{x}_{ik} - \hat{x}_{jk}$ also belongs to $\bar{\Lambda}_\tau$ since $\hat{x}_{ik} - \hat{x}_{jk} = \frac{1}{2}(x_i - x_j)$. With this observation, the second inequality in (a) follows easily from (4). The arguments for part (b), where (5) is used, are similar. □

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4.2. *Six equal masses.* Consider the system (1) with six equal masses $m_1 = \cdots = m_6 = 1$. Following the idea in the previous section, this system can be considered as a system of 15 binary pairs, each of which behaves like a particle–antiparticle pair. The binary decomposition is given by the following. Let $\lambda \in [0, 1]$ be some constant. Define

$$K_{ij}^0(\dot{x}) = \frac{1}{20}|\dot{x}_i - \dot{x}_j|^2, \quad K_{ij}^1(\dot{x}) = \frac{1}{5}|\dot{\hat{x}}_{ij}|^2,$$

$$U_{ij}^0(x) = \frac{\lambda}{|x_i - x_j|}, \quad U_{ij}^1(x) = \frac{1-\lambda}{|x_i - x_j|},$$

where $\hat{x}_{ij} = \frac{1}{2}(x_i + x_j)$. Let

$$\mathcal{A}_{ij}^0(x) = \int_0^T K_{ij}^0(\dot{x}) + U_{ij}^0(x)\, dt,$$

$$\mathcal{A}_{ij}^1(x) = \int_0^T K_{ij}^1(\dot{x}) + U_{ij}^1(x)\, dt,$$

$$\mathcal{A}_T^0(x) = \sum_{\substack{i,j \\ i<j}} \mathcal{A}_{ij}^0, \quad \mathcal{A}_T^1(x) = \sum_{\substack{i,j \\ i<j}} \mathcal{A}_{ij}^1.$$

Then

$$\mathcal{A}_T(x) = \mathcal{A}_T^0(x) + \mathcal{A}_T^1(x).$$

Similar to the derivation in the previous section, we have

$$\sum_{\substack{i,j \\ i<j}} K_{ij}^1(\dot{x}) = \sum_{\substack{i,j \\ i<j}} \frac{1}{5}|\dot{\hat{x}}_{ij}|^2 = \sum_{k=1}^{6} \sum_{\substack{i,j \\ i<j \\ i,j\neq k}} \frac{1}{40}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2)$$

$$= \sum_{\substack{i,j \\ i<j}} \sum_{\substack{k=1 \\ k\neq i,j}}^{6} \frac{1}{40}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2),$$

$$\sum_{\substack{i,j \\ i<j}} U_{ij}^1(x) = \sum_{\substack{i,j \\ i<j}} \frac{1-\lambda}{|x_i - x_j|} = \sum_{k=1}^{6} \sum_{\substack{i,j \\ i<j \\ i,j\neq k}} \frac{\frac{1}{8}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}$$

$$= \sum_{\substack{i,j \\ i<j}} \sum_{\substack{k=1 \\ k\neq i,j}}^{6} \frac{\frac{1}{8}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}.$$

The action due to each binary pair can estimated by using (4) or (5).

The proof for Lemma 4.2 below follows easily from (4) and (5) as in the proof of Lemma 4.1. In the proof of Theorem 6.1 we will use this lemma several times.

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LEMMA 4.2. *Let two distinct indices i, j and any $\tau > 0$, $\phi \in (0, \pi]$ be given.*

(a) *If $x_i - x_j \in \bar{\Lambda}_\tau$, then*

$$\int_0^\tau \frac{1}{20}|\dot{x}_i - \dot{x}_j|^2 + \frac{\lambda}{|x_i - x_j|}\,dt \geq 3\left(\frac{\pi^2\lambda^2}{20}\right)^{1/3}\tau^{1/3},$$

$$\sum_{\substack{k=1\\k\neq i,j}}^{6}\int_0^\tau \frac{1}{40}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2) + \frac{\frac{1}{8}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}\,dt \geq 3\left(\frac{\pi^2(1-\lambda)^2}{80}\right)^{1/3}\tau^{1/3}.$$

(b) *If $x_i - x_j \in \Gamma_{\tau,\phi}$, then*

$$\int_0^\tau \frac{1}{20}|\dot{x}_i - \dot{x}_j|^2 + \frac{\lambda}{|x_i - x_j|}\,dt \geq \frac{3}{2}\left(\frac{\phi^2\lambda^2}{10}\right)^{1/3}\tau^{1/3},$$

$$\sum_{\substack{k=1\\k\neq i,j}}^{6}\int_0^\tau \frac{1}{40}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2) + \frac{\frac{1}{8}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}\,dt \geq \frac{3}{2}\left(\frac{\phi^2(1-\lambda)^2}{40}\right)^{1/3}\tau^{1/3}.$$

5. *The four-body problem with equal masses*

In this section we consider a system of four equal masses $m_1 = m_2 = m_3 = m_4 = 1$ moving on $\mathbb{C}$ and define an action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ on $H_{d,T}$ via orthogonal transformations. We show that for any d large enough, there exists a minimizer for the action functional $\mathcal{A}_T$ on $H_{d,T}$ with the prescribed symmetry.

Let G be a group of linear transformations on $H_{d,T}$ generated by a, b and c:

$$(a \cdot x)(t) = (\bar{x}_2, \bar{x}_1, \bar{x}_4, \bar{x}_3)(-t), \tag{6}$$

$$(b \cdot x)(t) = e^{\pi i/d}(x_3, x_4, x_1, x_2)\left(t + \frac{T}{2}\right), \tag{7}$$

$$(c \cdot x)(t) = -(x_2, x_1, x_4, x_3)(t). \tag{8}$$

Observe that G is commutative and a, b, c are all of order 2; therefore, G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. When $d \geq 3$ is an odd integer, this is equivalent to an example in [**7**] for which the theorem therein cannot fully apply. Some simple observations for any path x in $H^G_{d,T}$ are summarized below.

- By (6) and (8), $x_1(0) = -x_2(0)$, $x_3(0) = -x_4(0)$ are aligned on the imaginary axis. $x_1(T/2) = -x_2(T/2)$, $x_3(T/2) = -x_4(T/2)$ are aligned on a straight line through the origin, with the same configuration as at $t = 0$ except that x_1, x_3 are exchanged and so are x_2 and x_4.
- $(\bar{x}_2, \bar{x}_1, \bar{x}_4, \bar{x}_3)(T/4) = e^{\pi i/d}(x_3, x_4, x_1, x_2)(T/4)$. In particular, the configuration at this moment is a rectangle. This excludes all collision-free *self-similar* paths (that is, paths that remain a similar configuration at any instant) from $H^G_{d,T}$.
- $[0, T/4]$ is a fundamental domain of the action.

Figure 1 shows the configuration at $t = 0, T/2, T$ for some paths in $H^G_{d,T}$. The values of $\mathcal{A}_T$ on these paths are lower than the lower bounds for $\mathcal{A}_T$ over collision paths obtained in the proof of Theorem 5.1. These paths were obtained by approximating minimizers of $\mathcal{A}_T$ by Fourier series.

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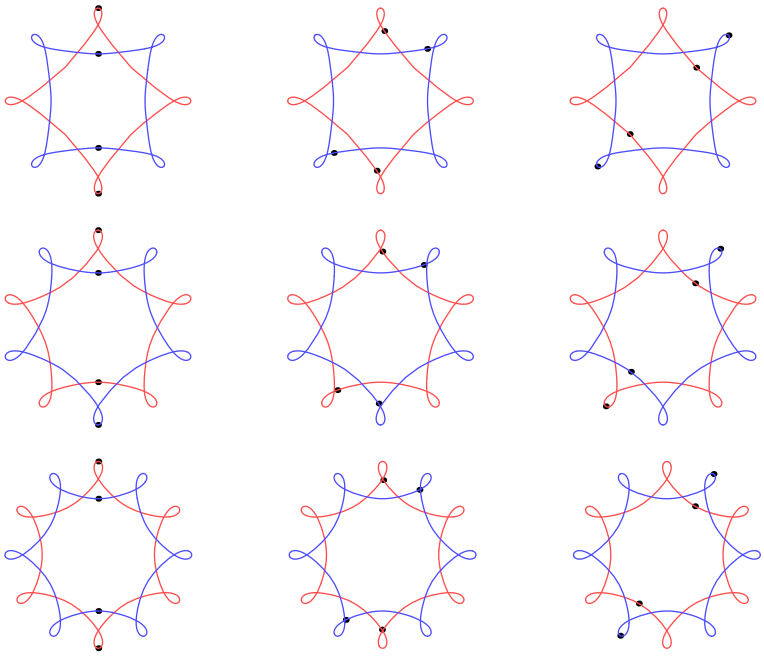


FIGURE 1. Double choreographic solutions with $d = 4, 5, 6$.

THEOREM 5.1. *Consider the system (1) with four equal masses $m_1 = m_2 = m_3 = m_4 = 1$. There exists some $d_0 \leq 3.55$ with the property that, for any $T > 0$ and any $d \geq d_0$, the action functional $\mathcal{A}_T$ attains its infimum on $H^G_{d,T}$ and any minimizer is a classical solution of (1).*

Proof. For any $x \in H^G_{d,T}$,

$$\begin{aligned}\langle x(0), x(T)\rangle &= \langle (x_1, x_2, x_3, x_4)(0), e^{-2\pi i/d}(x_1, x_2, x_3, x_4)(0)\rangle \\ &= \cos\left(\frac{2\pi}{d}\right)|x(0)|^2 \\ &= \cos\left(\frac{2\pi}{d}\right)|x(0)| \cdot |x(T)|.\end{aligned}$$

By (2) and Proposition 2.1, the functional $\mathcal{A}_T$ is coercive on $H^G_{d,T}$ and thus attains its infimum since $H^G_{d,T}$ is weakly closed. The group G consists of orthogonal transformations and $\mathcal{A}_T$ is G-invariant. It follows from Palais' principle of symmetric criticality that minimizers of $\mathcal{A}_T$ on $H^G_{d,T}$ are solutions of (1).

A fundamental domain of the group action is $[0, T/4]$. The configuration at $t = 0$ is collinear and all masses are aligned on the imaginary axis. Suppose $x_j = r_j e^{i\theta_j}$ is the polar form of x_j. By calculating the first variation of $\mathcal{A}_T$ with respect to r_j, we can easily see that $\dot{r}_j(0) = 0$ for any j provided that there is no collision. In this case, by symmetry the minimizer x indeed solves (1) for any $t \in \mathbb{R}$. According to Marchal's theorem, all we

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need to show is that for some $d_0 \le 3.55$, minimizers of $\mathcal{A}_T$ on $H^G_{d,T}$ with $d \ge d_0$ do not have collision(s) on the boundary of the interval $[0, T/4]$.

Let $x \in H^G_{d,T}$ be a path that begins or ends with a collision on $[0, T/4]$. There are three types of collisions as follows.

Case 1. x begins or ends with a total collapse.

By symmetry, $x_i(0) = x_j(0)$ and $x_i(T/2) = x_j(T/2)$ for any $i \ne j$, thus the path $x_i - x_j$ belongs to $\bar{\Lambda}_{T/2}$. Using Lemma 4.1,

$$\begin{aligned}
\mathcal{A}^0_T(x) &= 2\sum_{\substack{i,j\\ i<j}} \int_0^{T/2} \frac{1}{12}|\dot{x}_i - \dot{x}_j|^2 + \frac{\lambda}{|x_i - x_j|}\,dt \\
&\ge 12\left(\frac{3\pi\lambda}{2}\right)^{2/3}\left(\frac{T}{2}\right)^{1/3}, \\
\mathcal{A}^1_T(x) &= 2\sum_{\substack{i,j\\ i<j}} \sum_{\substack{k=1\\ k\ne i,j}}^{4} \int_0^{T/2} \frac{1}{12}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2) + \frac{\frac{1}{4}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}\,dt \\
&\ge 12\left(\frac{3\pi(1-\lambda)}{4}\right)^{2/3}\left(\frac{T}{2}\right)^{1/3}.
\end{aligned}$$

Therefore,

$$\mathcal{A}_T(x) \ge 12\left(\frac{9\pi^2}{32}\right)^{1/3}[2^{2/3}\lambda^{2/3} + (1-\lambda)^{2/3}]T^{1/3}.$$

This is valid regardless of the value of $\lambda \in [0, 1]$. By maximizing the right-hand side over all $\lambda \in [0, 1]$, we obtain

$$\begin{aligned}
\mathcal{A}_T(x) &\ge 12\left(\frac{45\pi^2}{32}\right)^{1/3} T^{1/3} = 3(90\pi^2)^{1/3}T^{1/3} \\
&\approx 28.83823\, T^{1/3}.
\end{aligned}$$

Note that, using the symmetry, Case 1 actually includes several cases:

- $x_1(0) = x_2(0)$ and $x_3(0) = x_4(0)$;
- $x_1(T/4) = x_2(T/4)$;
- $x_3(1/4) = x_4(1/4)$.

The case $x_1(T/4) = x_2(T/4)$ implies $x_3(T/4) = x_4(T/4)$ and *vice versa*.

Case 2. x begins or ends with a double–double collision.

Cases included in this case are:

- $x_1(0) = x_3(0)$ and $x_2(0) = x_4(0)$;
- $x_1(0) = x_4(0)$ and $x_2(0) = x_3(0)$;
- $x_1(T/4) = x_3(T/4)$ and $x_2(T/4) = x_4(T/4)$;
- $x_1(T/4) = x_4(T/4)$ and $x_2(T/4) = x_3(T/4)$.

These cases are all similar. We only treat the case $x_1(0) = x_3(0)$, $x_2(0) = x_4(0)$. By symmetry, this case implies $x_1(T/2) = x_3(T/2)$, $x_2(T/2) = x_4(T/2)$. This says

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that both $x_1 - x_3$ and $x_2 - x_4$ belong to $\bar{\Lambda}_{T/2}$. From (7), we can easily verify $x_1 - x_4$, $x_1 - x_2, x_2 - x_3, x_3 - x_4 \in \Gamma_{T/2,\pi/d}$. By Lemma 4.1,

$$\begin{aligned}
\mathcal{A}_T^0(x) &= 2\sum_{\substack{i,j\\ i<j}} \int_0^{T/2} \frac{1}{12}|\dot{x}_i - \dot{x}_j|^2 + \frac{\lambda}{|x_i - x_j|}\,dt \\
&\geq 2\left(2\left(\frac{3\pi\lambda}{2}\right)^{2/3} + 4\left(\frac{3\pi\lambda}{4d}\right)^{2/3}\right)\left(\frac{T}{2}\right)^{1/3}, \\
&= (12\pi\lambda)^{2/3}\left(\left(\frac{1}{2}\right)^{1/3} + \left(\frac{1}{d}\right)^{2/3}\right)T^{1/3}, \\
\mathcal{A}_T^1(x) &= 2\sum_{\substack{i,j\\ i<j}} \sum_{\substack{k=1\\ k\neq i,j}}^{4} \int_0^{T/2} \frac{1}{12}(|\dot{\hat{x}}_{ik}|^2 + |\dot{\hat{x}}_{jk}|^2) + \frac{\frac{1}{4}(1-\lambda)}{|\hat{x}_{ik} - \hat{x}_{jk}|}\,dt \\
&\geq (6\pi(1-\lambda))^{2/3}\left(\left(\frac{1}{2}\right)^{1/3} + \left(\frac{1}{d}\right)^{2/3}\right)T^{1/3}.
\end{aligned}$$

As in the previous case, by maximizing the sum of the lower bounds for $\mathcal{A}_T^0$ and $\mathcal{A}_T^1$ over $\lambda \in [0, 1]$, we obtain

$$\begin{aligned}
\mathcal{A}_T(x) &\geq 3\left(\frac{20\pi^2}{3}\right)^{1/3}\left(\left(\frac{1}{2}\right)^{1/3} + \left(\frac{1}{d}\right)^{2/3}\right)T^{1/3} =: \alpha(d) \\
&\approx \left(9.61274 + \frac{12.1113}{d^{2/3}}\right)T^{1/3}.
\end{aligned}$$

Case 3. x begins with a single double collision.

Note that it is impossible to have only one double collision at $t = T/4$; cases 1, 2, 3 exhaust all possibilities. There are two possible single–double collisions to begin with; we may assume without loss of generality that $x_3(0) = x_4(0)$. The other case, $x_1(0) = x_2(0)$, is similar.

Our assumption clearly implies that $x_3 - x_4 \in \bar{\Lambda}_T$. From (7),

$$\begin{aligned}
(x_1 - x_3)\left(\frac{T}{2}\right) &= e^{-\pi i/d}(x_3 - x_1)(0) = e^{\pi i((d-1)/d)}(x_1 - x_3)(0); \\
(x_1 - x_4)\left(\frac{T}{2}\right) &= (x_1 + x_3)\left(\frac{T}{2}\right) = e^{-\pi i/d}(x_1 + x_3)(0) = e^{-\pi i/d}(x_1 - x_4)(0).
\end{aligned}$$

Thus $x_1 - x_3 \in \Gamma_{T/2,((d-1)/d)\pi}$ and $x_1 - x_4 \in \Gamma_{T/2,\pi/d}$. By symmetry and Lemma 4.1,

$$\begin{aligned}
\mathcal{A}_T^0(x) = 2\int_0^T \frac{1}{12}|\dot{x}_3 - \dot{x}_4|^2 + \frac{\lambda}{|x_3 - x_4|}\,dt + 4\int_0^{T/2} \frac{1}{12}|\dot{x}_1 - \dot{x}_3|^2 + \frac{\lambda}{|x_1 - x_3|}\,dt \\
+ 4\int_0^{T/2} \frac{1}{12}|\dot{x}_1 - \dot{x}_4|^2 + \frac{\lambda}{|x_1 - x_4|}\,dt
\end{aligned}$$

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$$\geq 2\left(\frac{3\pi\lambda}{2}\right)^{2/3} T^{1/3} + 4\left(\frac{3\pi(d-1)\lambda}{4d}\right)^{2/3}\left(\frac{T}{2}\right)^{1/3} + 4\left(\frac{3\pi\lambda}{4d}\right)^{2/3}\left(\frac{T}{2}\right)^{1/3}$$

$$= 2\left(\frac{3\pi\lambda}{2}\right)^{2/3}\left(1+\left(\frac{d-1}{d}\right)^{2/3}+\left(\frac{1}{d}\right)^{2/3}\right) T^{1/3},$$

$$\mathcal{A}_T^1(x) = 2\sum_{k\neq 3,4}\int_0^T \frac{1}{12}(|\dot{\hat{x}}_{3k}|^2 + |\dot{\hat{x}}_{4k}|^2) + \frac{\frac{1}{4}(1-\lambda)}{|\hat{x}_{3k}-\hat{x}_{4k}|}\,dt$$

$$+ 4\sum_{k\neq 1,3}\int_0^{T/2} \frac{1}{12}(|\dot{\hat{x}}_{1k}|^2 + |\dot{\hat{x}}_{3k}|^2) + \frac{\frac{1}{4}(1-\lambda)}{|\hat{x}_{1k}-\hat{x}_{3k}|}\,dt$$

$$+ 4\sum_{k\neq 1,4}\int_0^{T/2} \frac{1}{12}(|\dot{\hat{x}}_{1k}|^2 + |\dot{\hat{x}}_{4k}|^2) + \frac{\frac{1}{4}(1-\lambda)}{|\hat{x}_{1k}-\hat{x}_{4k}|}\,dt$$

$$\geq 2\left(\frac{3\pi(1-\lambda)}{4}\right)^{2/3}\left(1+\left(\frac{d-1}{d}\right)^{2/3}+\left(\frac{1}{d}\right)^{2/3}\right) T^{1/3}.$$

As before, by adding the lower bound estimates for $\mathcal{A}_T^0$ and $\mathcal{A}_T^1$ and then maximizing over $\lambda \in [0,1]$, we obtain

$$\mathcal{A}_T(x) \geq 3\left(\frac{5\pi^2}{6}\right)^{1/3}\left(1+\left(\frac{d-1}{d}\right)^{2/3}+\left(\frac{1}{d}\right)^{2/3}\right) T^{1/3} =: \beta(d)$$

$$\approx \left(12.1113 + \frac{6.05565}{d^{2/3}}\right) T^{1/3}.$$

To complete the proof it suffices to select a test path $x^{(d)}$ with action lower than the bounds $\alpha(d)$, $\beta(d)$ obtained in Cases 2 and 3. By appropriate scaling in space and time, we may set $T = 1$ without any loss of generality. Let $x^{(d)}$ and $\gamma(d)$ be the test path and function defined in Lemma 5.2. Then

$$\mathcal{A}_T(x^{(d)}) \leq \gamma(d)$$

for any $d \geq 3.55$. The constant term in $\gamma(d)$ is approximately 9.04654. Comparing with the lower bound estimates $\alpha(d)$ and $\beta(d)$ for collision paths, this at least implies that the theorem is valid for all large d. We can easily verify (by Newton's method, for instance) that $\gamma(d)$ is lower than $\min\{\alpha(d), \beta(d)\}$ for all $d \geq 2.78$. □

LEMMA 5.2. *Let* $x^{(d)} = (x_1^{(d)}, x_2^{(d)}, x_3^{(d)}, x_4^{(d)})$ *be defined as follows:*

$$r(t) = \frac{d^{2/3}}{4} + \frac{\cos(2\pi t)}{5},$$

$$\theta(t) = \frac{\sin(2\pi t)}{d^{2/3}} - \frac{2\pi t}{d} + \frac{\pi}{2},$$

$$x_1^{(d)}(t) = r(t)e^{i\theta(t)},$$

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$$x_2^{(d)}(t) = -r(t)e^{i\theta(t)},$$
$$x_3^{(d)}(t) = r\left(t - \frac{1}{2}\right)e^{i(\theta(t-1/2)-\pi/d)},$$
$$x_4^{(d)}(t) = -r\left(t - \frac{1}{2}\right)e^{i(\theta(t-1/2)-\pi/d)}.$$

Then $x^{(d)} \in H^G_{d,1}$ *and*

$$\mathcal{A}_1(x^{(d)}) \le \left(\frac{41}{100} + \frac{1}{2d^{2/3}} - \frac{4}{5d} + \frac{3}{25d^{4/3}} + \frac{4}{25d^2}\right)\pi^2 + 5 + \frac{8.70155}{d^{2/3}} =: \gamma(d)$$

for any $d \ge 3.55$.

Proof. The verification for $x^{(d)} \in H^G_{d,1}$ is straightforward. The calculation for the exact contribution of the kinetic energy to the total action is easy:

$$\int_0^1 K(\dot{x}^{(d)})\,dt = \left(\frac{41}{100} + \frac{1}{2d^{2/3}} - \frac{4}{5d} + \frac{3}{25d^{4/3}} + \frac{4}{25d^2}\right)\pi^2.$$

We need to show that the contribution of the potential energy is bounded from above by $5 + 8.70155/d^{2/3}$, which follows immediately if we can show

$$|x_1^{(d)}(t) - x_2^{(d)}(t)|, \quad |x_3^{(d)}(t) - x_4^{(d)}(t)| \ge \frac{d^{2/3}}{2}, \tag{9}$$
$$|x_1^{(d)}(t) - x_4^{(d)}(t)| = |x_2^{(d)}(t) - x_3^{(d)}(t)| \ge 0.42539d^{2/3}, \tag{10}$$
$$|x_1^{(d)}(t) - x_3^{(d)}(t)| = |x_2^{(d)}(t) - x_4^{(d)}(t)| \ge \frac{2}{5}, \tag{11}$$

for any $t \in [0, \frac{1}{4}]$.

Inequality (9) is obvious. Observe that $\cos s \ge 1 - 2s/\pi$ for $s \in [0, \pi/2]$,

$$\begin{aligned}
|x_1^{(d)}(t) - x_4^{(d)}(t)|^2 &= |x_2^{(d)}(t) - x_3^{(d)}(t)|^2 \\
&= 2\left[\frac{d^{4/3}}{16} + \frac{\cos^2(2\pi t)}{25} + \left(\frac{d^{4/3}}{16} - \frac{\cos^2(2\pi t)}{25}\right)\cos\left(\frac{2\sin(2\pi t)}{d^{2/3}}\right)\right] \\
&\ge 2\left[\frac{d^{4/3}}{16} + \frac{\cos^2(2\pi t)}{25} + \left(\frac{d^{4/3}}{16} - \frac{\cos^2(2\pi t)}{25}\right)\left(1 - \frac{4\sin(2\pi t)}{d^{2/3}}\right)\right] \\
&\ge 2\left[\frac{1}{8} - \frac{1}{4\pi d^{2/3}}\right]d^{4/3} \\
&\ge (0.42539)^2\, d^{4/3}
\end{aligned}$$

for any $d \ge 3.55$. This proves (10).

Now we prove (11) for $|x_1^{(d)}(t) - x_3^{(d)}(t)|$. The estimate for $|x_2^{(d)}(t) - x_4^{(d)}(t)|$ is similar. By using the elementary inequality $\sin s \ge s(1 - s/\pi)$ for $s \in [0, \pi/2]$, one can easily

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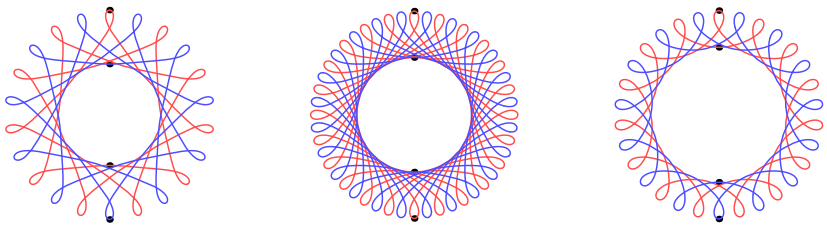


FIGURE 2. Double choreographic solutions with $d = \frac{11}{3}, \frac{24}{5}, \frac{15}{2}$.

verify

$$\begin{aligned}\frac{1}{4\pi}\frac{d}{dt}|x_1^{(d)}(t) - x_3^{(d)}(t)|^2 &= -\frac{\sin(4\pi t)}{25}\left(1+\cos\left(\frac{2\sin(2\pi t)}{d^{2/3}}\right)\right)\\ &\quad + \frac{2\cos(2\pi t)}{d^{2/3}}\left(\frac{d^{4/3}}{16} - \frac{\cos^2(2\pi t)}{25}\right)\sin\left(\frac{2\sin(2\pi t)}{d^{2/3}}\right)\\ &\geq -\frac{2\sin(4\pi t)}{25} + \frac{2\sin(4\pi t)}{d^{4/3}}\left(\frac{d^{4/3}}{16} - \frac{1}{25}\right)\left(1 - \frac{2\sin(2\pi t)}{d^{2/3}\pi}\right)\\ &\geq \frac{2\sin(4\pi t)}{25}\left[-1 + 25\left(\frac{1}{16} - \frac{1}{25d^{4/3}}\right)\left(1 - \frac{2}{\pi d^{2/3}}\right)\right]\\ &\geq 0\end{aligned}$$

for any $t \in [0, \frac{1}{4}]$ and $d \geq 3.5425$. This shows that the infimum of $|x_1^{(d)}(t) - x_3^{(d)}(t)|$ on $[0, \frac{1}{4}]$ is $|x_1^{(d)}(0) - x_3^{(d)}(0)| = \frac{2}{5}$, completing the proof. □

As noted in §2, a minimizer x of $\mathcal{A}_T$ on $H^G_{d,T}$ is periodic if $d \in \mathbb{Q}$; it is quasi-periodic if $d \in \mathbb{R} \setminus \mathbb{Q}$. In the first case, let $d = p/q$ be a reduced fraction. By (7) and (8), the solution is indeed a double choreographic solution with either $\{m_1, m_2\}$ sharing the same orbit when p is even, or $\{m_1, m_4\}$ sharing the same orbit when p is odd (see Figure 2). To see that there are infinitely many distinct solutions of this type, observe that $x_1 - x_3$, $x_2 - x_4 \in \Gamma_{T/2,((d-1)/d)\pi}$, $x_1 - x_2$, $x_1 - x_4$, $x_3 - x_2$, $x_3 - x_4 \in \Gamma_{T/2,\pi/d}$ for any $x \in H^G_{d,T}$. Following the same argument as in the proof of Theorem 5.1, we can easily verify that

$$\inf_{H^G_{d,T}} \mathcal{A}_1 \geq 3\left(\frac{5\pi^2}{6}\right)^{1/3}\left(\left(\frac{d-1}{d}\right)^{2/3} + 2\left(\frac{1}{d}\right)^{2/3}\right) \approx 6.05565 + \frac{12.1113}{d^{3/2}}.$$

For simplicity, consider the special case $d \in \mathbb{N}$, $d \geq 3$. Compare this estimate with $\gamma(d)$ in Lemma 5.2, the total action of a minimizer in $H^G_{d,1}$ over its period d falls inside the interval $(5d, 10d)$, which are all disjoint.

Now suppose $x \in H^G_{d,T} \cap H_{d',T'}$ minimizes $\mathcal{A}_T$ on $H^G_{d,T}$. If T and T' are not rationally dependent, then $\{kT' \bmod T : k \in \mathbb{N}\}$ would be dense in $[0, T]$ and so x would remain the same configuration at any instant. From the G-invariance and Theorem 5.1, this is simply impossible. The fact that T and T' are rationally dependent implies that d and d' are rationally dependent as well. Being 'rationally dependent' is an equivalence relation

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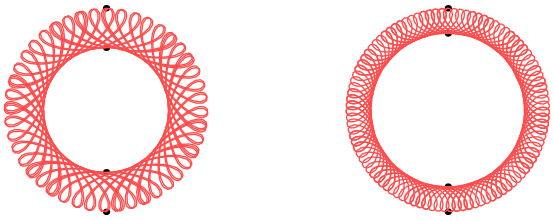


FIGURE 3. Quasi-periodic solutions with $d = 2\pi$ and $d = \sqrt{200}$.

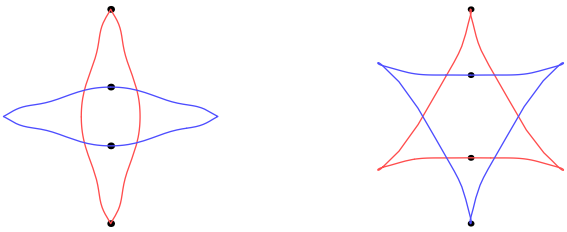


FIGURE 4. Double choreographic solutions with $d = 2$ and $d = 3$.

on $\mathbb{R}$ and thus there are uncountably many distinct quasi-periodic solutions since there are uncountably many such equivalence classes (see Figure 3). To summarize, we have the following corollary.

COROLLARY 5.3. *The four-body problem with equal masses has infinitely many double choreographic solutions and uncountably many quasi-periodic solutions.*

Remark 5.4. If we appeal to numerical methods to calculate the action of test paths, then Lemma 5.2 and the bound for d_0 in Theorem 5.1 can be easily improved. We do not present these numerical results here except for two special cases: $d = 2$ and $d = 3$.

Set $T = 1$. Let $\alpha(3)$, $\beta(3)$ be as in the proof of Theorem 5.1, and $x^{(3)}$ be as in Lemma 5.2 but with $r(t)$, $\theta(t)$ replaced by

$$r(t) = 0.53(1 + 0.413\cos(2\pi t) - 0.045\cos(4\pi t) + 0.01\cos(6\pi t)),$$
$$\theta(t) = 0.485\sin(2\pi t) - 0.105\sin(4\pi t) + 0.027\sin(6\pi t) - \frac{2\pi t}{3} + \frac{\pi}{2}.$$

Then $\mathcal{A}_1(x^{(3)}) \approx 12.5123 < 13.5882 \approx \min\{\alpha(3), \beta(3)\}$. This proves Theorem 5.1 for $d = 3$. Incidentally, this particular case can be proved without using Theorem 3.1; see [**2**]. The graph for the case $d = 3$ in Figure 4 was first obtained by Ferrario and Terracini [**7**].

Similarly, let $x^{(2)}$ be given by Lemma 5.2 with $r(t)$, $\theta(t)$ replaced by

$$r(t) = 0.443(1 + 0.518\cos(2\pi t) - 0.064\cos(4\pi t) + 0.015\cos(6\pi t)),$$
$$\theta(t) = 0.689\sin(2\pi t) - 0.182\sin(4\pi t) + 0.054\sin(6\pi t) - \pi t + \frac{\pi}{2}.$$

Then $\mathcal{A}_1(x^{(2)}) \approx 13.6728 < 13.6853 \approx \min\{\alpha(2), \beta(2)\}$. This proves Theorem 5.1 for $d = 2$ (see Figure 4). These numerical data also suggest that the optimal d_0 in Theorem 5.1 is less than but very close to two.

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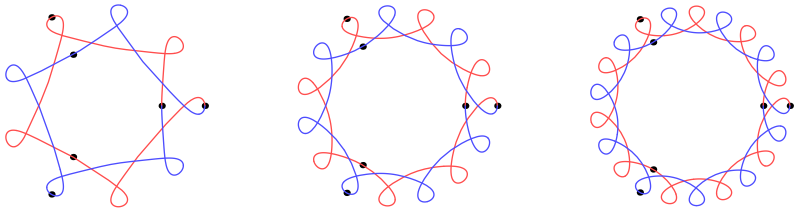


FIGURE 5. Simple choreographic solutions with $d = \frac{9}{2}, \frac{15}{2}, \frac{21}{2}$.

6. *The six-body problem with equal masses*

We now turn to the system (1) with six equal masses $m_1 = \cdots = m_6 = 1$.

Let K be a group of linear transformations on $H_{d,T}$ generated by a and g:

$$(a \cdot x)(t) = (\bar{x}_1, \bar{x}_3, \bar{x}_2, \bar{x}_4, \bar{x}_6, \bar{x}_5)(-t), \tag{12}$$

$$(g \cdot x)(t) = e^{(1/d+2/3)\pi i}(x_6, x_4, x_5, x_3, x_1, x_2)\left(t + \frac{T}{2}\right). \tag{13}$$

The group K is isomorphic to the dihedral group D_6 of order 12 with relations

$$a^2 = g^6 = 1, \quad aga^{-1} = g^5.$$

Note that any path x in $H^K_{d,T}$ is invariant under the transformations

$$(g^3 \cdot x)(t) = e^{\pi i/d}(x_4, x_5, x_6, x_1, x_2, x_3)\left(t + \frac{T}{2}\right), \tag{14}$$

$$(g^4 \cdot x)(t) = e^{2\pi i/3}(x_3, x_1, x_2, x_6, x_4, x_5)(t). \tag{15}$$

Some simple observations for any path x in $H^K_{d,T}$ are summarized below.

- By (12) and (15), $x_1(0)$, $x_4(0)$ are aligned on the real axis and the triangles with vertices $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$ are both equilateral at any instant. The configuration of the system at $t = T/2$ is congruent to the configuration at $t = 0$, except that x_1, x_4 were exchanged, x_2 and x_5 were exchanged, and so were x_3 and x_6.
- $(\bar{x}_1, \bar{x}_3, \bar{x}_2, \bar{x}_4, \bar{x}_6, \bar{x}_5)(T/4) = e^{(1/d+2/3)\pi i}(x_6, x_4, x_5, x_3, x_1, x_2)(T/4)$. In particular, the distance from each x_j to the origin is the same. This together with the previous observation excludes all collision-free self-similar paths from $H^K_{d,T}$.
- $[0, T/4]$ is a fundamental domain of the action.

Figures 5–7 show several simple choreographic, double choreographic and quasi-periodic paths with considerably low action, respectively. The discussion for this example is similar to the previous section and we only prove a result similar to Theorem 5.1 for large d. Two differences are: (i) when $d = 3(2k+1)/2$ with $k \in \mathbb{N}$, for instance, the resulting loop is simple choreographic; (ii) there is one case where the binary decomposition in §4 does not provide satisfactory estimates.

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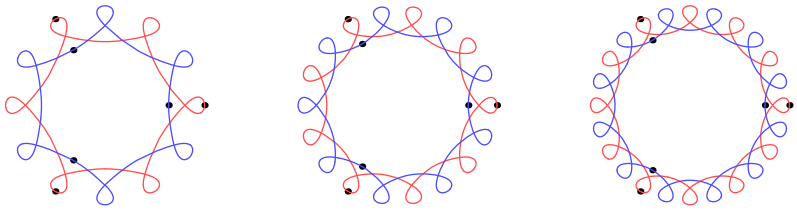


FIGURE 6. Double choreographic solutions with $d = 6, 9, 12$.

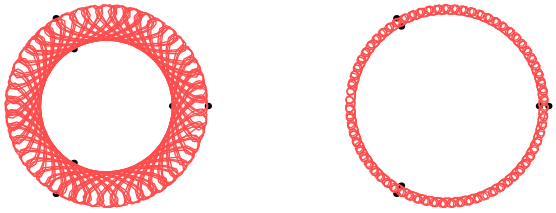


FIGURE 7. Quasi-periodic solutions with $d = 2\pi$ and $d = \sqrt{5000}$.

THEOREM 6.1. *Consider the system (1) with six equal masses $m_1 = \cdots = m_6 = 1$. There exists some $d_0 > 1$ with the property that, for any $T > 0$ and any $d \geq d_0$, the action functional $\mathcal{A}_T$ attains its infimum on $H^K_{d,T}$ and any minimizer is a classical solution of (1).*

Proof. The existence of minimizers for the action functional $\mathcal{A}_T$ on $H^K_{d,T}$ follows by the same arguments as in the proof of Theorem 5.1. A fundamental domain of the group action is $[0, T/4]$. By calculating the first variation of $\mathcal{A}_T$ with respect to the radial component r_j of each x_j, we can easily see that $\dot{r}_j(0) = 0$ for any j provided that there is no collision. In this case, by symmetry the minimizer x solves (1) for any $t \in \mathbb{R}$. According to Marchal's theorem, all we need to show is that the minimizers of $\mathcal{A}_T$ on $H^K_{d,T}$ do not have collision(s) on the boundary of the interval $[0, T/4]$ for every large d.

Let $x \in H^K_{d,T}$ be a path that begins or ends with a collision on $[0, T/4]$. There are three types of collision as follows.

Case 1. x begins or ends with a total collapse.

By symmetry $x_i(0) = x_j(0)$ and $x_i(T/2) = x_j(T/2)$ for any $i \neq j$, thus the path $x_i - x_j$ belongs to $\bar{\Lambda}_{T/2}$. Using Lemma 4.2,

$$\begin{aligned}\mathcal{A}^0_T(x) &= 2\sum_{\substack{i,j\\ i<j}} \int_0^{T/2} \frac{1}{20}|\dot{x}_i - \dot{x}_j|^2 + \frac{\lambda}{|x_i - x_j|}\,dt \\ &\geq 90\left(\frac{\pi^2\lambda^2}{20}\right)^{1/3}\left(\frac{T}{2}\right)^{1/3},\end{aligned}$$

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$$\mathcal{A}_T^1(x) = 2\sum_{\substack{i,j\\ i<j}}\sum_{\substack{k=1\\ k\neq i,j}}^{6}\int_0^{T/2}\frac{1}{40}(|\dot{\hat{x}}_{ik}|^2+|\dot{\hat{x}}_{jk}|^2)+\frac{\frac{1}{8}(1-\lambda)}{|\hat{x}_{ik}-\hat{x}_{jk}|}\,dt$$
$$\geq 90\left(\frac{\pi^2(1-\lambda)^2}{80}\right)^{1/3}\left(\frac{T}{2}\right)^{1/3}.$$

By adding these lower bounds for $\mathcal{A}_T^0$ and $\mathcal{A}_T^1$ and then maximizing over $\lambda\in[0,1]$, we obtain

$$\mathcal{A}_T(x)\geq 45\left(\frac{\pi}{2}\right)^{2/3}T^{1/3}\approx 60.80777T^{1/3}.$$

Case 2. x begins or ends with a triple–double collision.

Assume the case $x_1(0)=x_4(0)$, $x_2(0)=x_5(0)$, $x_3(0)=x_6(0)$. The estimates for all other possibilities are the same. By symmetry this case implies $x_1(T/2)=x_4(T/2)$, $x_2(T/2)=x_5(T/2)$, $x_3(T/2)=x_6(T/2)$. This says that $x_1-x_4, x_2-x_5, x_3-x_6$ belong to $\bar{\Lambda}_{T/2}$. Instead of using the binary decomposition and Lemma 4.2 in §4, we consider a simpler decomposition and apply (4):

$$\begin{aligned}\mathcal{A}_T(x) &> 2\int_0^{T/2}\frac{1}{4}|\dot{x}_1-\dot{x}_4|^2+\frac{1}{|x_1-x_4|}\,dt+2\int_0^{T/2}\frac{1}{4}|\dot{x}_2-\dot{x}_5|^2+\frac{1}{|x_2-x_5|}\,dt\\ &\quad+2\int_0^{T/2}\frac{1}{4}|\dot{x}_3-\dot{x}_6|^2+\frac{1}{|x_3-x_6|}\,dt\\ &\geq 9\pi^{2/3}T^{1/3}=:\alpha(d)\\ &\approx 19.30526T^{\frac{1}{3}}.\end{aligned}$$

Case 3. x begins with a single–triple collision.

There are two possible single–triple collisions to begin with; we may assume without loss of generality that $x_4(0)=x_5(0)=x_6(0)=0$. The other case, $x_1(0)=x_2(0)=x_3(0)$, is similar. From (14), in this case we have

$$\begin{gathered}x_1-x_2,\quad x_2-x_3,\quad x_1-x_3,\quad x_4-x_5,\quad x_5-x_6,\quad x_4-x_6\in\bar{\Lambda}_T,\\ x_1-x_4,\quad x_2-x_5,\quad x_3-x_6\in\Gamma_{T/2,((d-1)/d)\pi},\\ x_1-x_5,\quad x_2-x_4,\quad x_1-x_6,\quad x_3-x_4,\quad x_3-x_5,\quad x_2-x_6\in\Gamma_{T,2\pi/d}.\end{gathered}$$

By Lemma 4.2,

$$\begin{aligned}\mathcal{A}_T^0(x) &\geq 18\left(\frac{\pi^2\lambda^2}{20}\right)^{1/3}T^{1/3}+9\left(\frac{\pi^2\lambda^2(d-1)^2}{10d^2}\right)^{1/3}\left(\frac{T}{2}\right)^{1/3}+9\left(\frac{(2\pi)^2\lambda^2}{10d^2}\right)^{1/3}T^{1/3}\\ &= 18\left(\frac{\pi^2\lambda^2}{20}\right)^{1/3}\left(1+\frac{1}{2}\left(\frac{d-1}{d}\right)^{2/3}+\left(\frac{1}{d}\right)^{2/3}\right)T^{1/3},\\ \mathcal{A}_T^1(x) &\geq 18\left(\frac{\pi^2\lambda^2}{80}\right)^{1/3}\left(1+\frac{1}{2}\left(\frac{d-1}{d}\right)^{2/3}+\left(\frac{1}{d}\right)^{2/3}\right)T^{1/3}.\end{aligned}$$

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As before, by adding the lower bound estimates for $\mathcal{A}_T^0$ and $\mathcal{A}_T^1$ and then maximizing over $\lambda \in [0, 1]$, we obtain

$$\begin{aligned}\mathcal{A}_T(x) &\geq 18\left(\frac{\pi}{4}\right)^{2/3}\left(1+\frac{1}{2}\left(\frac{d-1}{d}\right)^{2/3}+\left(\frac{1}{d}\right)^{2/3}\right)T^{1/3} =: \beta(d)\\ &> 18\left(\frac{\pi}{4}\right)^{2/3}\left(\frac{3}{2}+\frac{1}{2}\left(\frac{1}{d}\right)^{2/3}\right)\\ &\approx \left(22.9839+\frac{9.6613}{d^{2/3}}\right)T^{1/3}.\end{aligned}$$

To complete the proof it suffices to select a test path $x^{(d)}$ with action lower than the bounds $\alpha(d)$, $\beta(d)$ obtained in Cases 2 and 3. By appropriate scaling in space and time, we may set $T = 1$ without any loss of generality. Let $x^{(d)}$ and $\gamma(d)$ be the test path and function defined in Lemma 6.2. Then

$$\begin{aligned}\mathcal{A}_T(x^{(d)}) &< \frac{123\,\pi^2}{200}+\frac{15}{2}+\left(\frac{3\pi^2}{4}+24\sqrt{3}\right)\frac{1}{d^{2/3}}\\ &\approx 13.56981+\frac{48.9714}{d^{2/3}}\end{aligned}$$

for $d \geq 3.55$. The theorem follows by comparing this estimate with the lower bound estimates $\alpha(d)$ and $\beta(d)$ for collision paths. □

LEMMA 6.2. *Let* $x^{(d)} = (x_1^{(d)}, x_2^{(d)}, x_3^{(d)}, x_4^{(d)}, x_5^{(d)}, x_6^{(d)})$ *be defined as follows:*

$$\begin{aligned}r(t) &= \frac{d^{2/3}}{4}+\frac{\cos(2\pi t)}{5},\\ \theta(t) &= \frac{\sin(2\pi t)}{d^{2/3}}-\frac{2\pi t}{d},\\ x_1^{(d)}(t) &= r(t)e^{i\theta(t)},\\ x_2^{(d)}(t) &= r(t)e^{i(\theta(t)+2\pi/3)},\\ x_3^{(d)}(t) &= r(t)e^{i(\theta(t)+4\pi/3)},\\ x_4^{(d)}(t) &= r\left(t-\frac{1}{2}\right)e^{i(\theta(t-1/2)-\pi/d)},\\ x_5^{(d)}(t) &= r\left(t-\frac{1}{2}\right)e^{i(\theta(t-1/2)-\pi/d+2\pi/3)},\\ x_6^{(d)}(t) &= r\left(t-\frac{1}{2}\right)e^{i(\theta(t-1/2)-\pi/d+4\pi/3)}.\end{aligned}$$

Then $x^{(d)} \in H_{d,1}^K$, *and*

$$\mathcal{A}_1(x^{(d)}) < \frac{123\,\pi^2}{200}+\frac{15}{2}+\left(\frac{3\pi^2}{4}+24\sqrt{3}\right)\frac{1}{d^{2/3}} =: \gamma(d)$$

for any $d \geq 3.55$.

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Proof. The verification for $x^{(d)} \in H^K_{d,1}$ is straightforward. The contribution of the kinetic energy to the total action is

$$\begin{aligned}\int_0^1 K(\dot{x}^{(d)})\,dt &= \frac{123\,\pi^2}{200} + \frac{\pi^2}{d^{2/3}}\left(\frac{3}{4} - \frac{6}{5d^{1/3}} + \frac{9}{50d^{2/3}} + \frac{6}{25d^{4/3}}\right)\\ &< \frac{123\,\pi^2}{200} + \frac{3\pi^2}{4d^{2/3}}\\ &\approx 6.06981 + \frac{7.4022}{d^{2/3}}.\end{aligned}$$

It is easy to see that, given any $t \in [0, \frac{1}{4}]$,

$$|x_i^{(d)}(t) - x_j^{(d)}(t)| \geq \sqrt{3}\left(\frac{d^{2/3}}{4} - \frac{1}{5}\right) > \frac{d^{2/3}}{2\sqrt{3}}$$

for $d \geq 3.55$ and for any indices $i < j$ except $i, j = (1, 4), (2, 5), (3, 6)$.

Now we estimate $|x_1^{(d)}(t) - x_4^{(d)}(t)|$. The estimates for $|x_2^{(d)}(t) - x_5^{(d)}(t)|$ and $|x_3^{(d)}(t) - x_6^{(d)}(t)|$ are identical. Use the elementary inequality $\sin s \geq s(1 - s/\pi)$ for $s \in [0, \pi/2]$, and then follow exactly the same argument as in Lemma 5.2,

$$\begin{aligned}\frac{1}{4\pi}\frac{d}{dt}|x_1^{(d)}(t) - x_4^{(d)}(t)|^2 &= -\frac{\sin(4\pi t)}{25}\left(1 + \cos\left(\frac{2\sin(2\pi t)}{d^{2/3}}\right)\right)\\ &\quad + \frac{2\cos(2\pi t)}{d^{2/3}}\left(\frac{d^{4/3}}{16} - \frac{\cos^2(2\pi t)}{25}\right)\sin\left(\frac{2\sin(2\pi t)}{d^{2/3}}\right)\\ &\geq 0\end{aligned}$$

for any $d \geq 3.55$ and any $t \in [0, \frac{1}{4}]$. The infimum of $|x_1^{(d)}(t) - x_4^{(d)}(t)|$ on $[0, \frac{1}{4}]$ is therefore $|x_1^{(d)}(0) - x_4^{(d)}(0)| = \frac{2}{5}$. The lemma follows easily from this estimate. □

Following the arguments at the end of the previous section, Theorem 6.1 has the following consequence.

COROLLARY 6.3. *The six-body problem with equal masses has infinitely many simple and double choreographic solutions and uncountably many quasi-periodic solutions.*

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REFERENCES

[1] K.-C. Chen. Action-minimizing orbits in the parallelogram four-body problem with equal masses. *Arch. Rat. Mech. Anal.* **158** (2001), 293–318.

[2] K.-C. Chen. Binary decompositions for the planar N-body problem and symmetric periodic solutions. *Arch. Rat. Mech. Anal.*, to appear.

https://doi.org/10.1017/S0143385703000245 Published online by Cambridge University Press

[3] A. Chenciner. Action minimizing solutions in the Newtonian n-body problem: from homology to symmetry. *Proceedings of the International Congress of Mathematicians (Beijing, 2002)*, Vol. III, pp. 279–294.

[4] A. Chenciner, J. Gerver, R. Montgomery and C. Simó. Simple choreographic motions of N bodies: a preliminary study. *Geometry, Mechanics and Dynamics*. Springer, New York, 2002, pp. 287–308.

[5] A. Chenciner and R. Montgomery. A remarkable periodic solution of the three-body problem in the case of equal masses. *Ann. Math.* **152** (2000), 881–901.

[6] A. Chenciner and A. Venturelli. Minima de l'intégrale d'action du problème newtonien de 4 corps de masses égales dans $\mathbb{R}^3$: orbites 'hip-hop'. *Celestial Mech. Dynam. Astron.* **77** (2000), 139–152.

[7] D. Ferrario and S. Terracini. On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.* to appear.

[8] W. Gordon. A minimizing property of Keplerian orbits. *Amer. J. Math.* **99** (1977), 961–971.

[9] C. Marchal. The family P_{12} of the three-body problem—the simplest family of periodic orbits, with twelve symmetries per period. *Celestial Mech. Dynam. Astron.* **78** (2000), 279–298.

[10] C. Marchal. How the method of minimization of action avoids singularities. *Celestial Mech. Dynam. Astron.* **83** (2002), 325–353.

[11] R. Palais. The principle of symmetric criticality. *Comm. Math. Phys.* **69** (1979), 19–30.

[12] C. Simó. New families of solutions in N-body problems. *Proceedings of the European Congress of Mathematics (Barcelona, 2000)* (*Progr. Math., 201*). Birkhäuser, Basel, 2001, pp. 101–115.

[13] A. Venturelli. Une caractérisation variationnelle des solutions de Lagrange du probléme plan des trois corps. *C. R. Acad. Sci. Paris Sér. I Math.* **332** (2001), 641–644.

[14] A. Venturelli. Application de la minimisation de l'action au probléme des N corps dans le plan et dans l'espace. *Thesis*, Université de Paris 7, 2002.

[15] S. Zhang and Q. Zhou. A minimizing property of Lagrangian solutions. *Acta Math. Sin.* **17** (2001), 497–500.

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