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MULTIPLICATIVE FUNCTIONS ADDITIVE ON GENERALISED DECAGONAL NUMBERS

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Abstract

We characterise all multiplicative functions that are additive on generalised decagonal numbers. By analysing properties of decagonal numbers, we show that such functions are not uniquely determined. Our result extends previous work on multiplicative functions additive on polygonal numbers of lower orders.

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1. Introduction

Multiplicative functions play a central role in number theory, and a fundamental problem is to understand how their behaviour is constrained by conditions imposed on special sets of integers. A classical direction, initiated by Spiro [11], is to impose additive relations on structured subsets of the natural numbers and study the extent to which such conditions determine the function. For instance, Spiro [11] showed that a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfying $f(p + q) = f(p) + f(q)$ for primes p and q is, under mild hypotheses, the identity function. Related problems have since been studied for other structured sets.

Let $k \geq 2$, $S \subseteq \mathbb{N}$, and let $\mathcal{F}$ be a class of arithmetic functions. For $f \in \mathcal{F}$, consider the functional equation

$$f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k) \tag{1.1}$$

for arbitrary $x_1, \ldots, x_k \in S$. One then asks whether such a relation determines f. We say that S is a k-additive uniqueness set for $\mathcal{F}$ if a function $f \in \mathcal{F}$ satisfying condition (1.1) is uniquely determined. In recent years, this problem has been studied in connection with polygonal numbers. In particular, results have been obtained for triangular numbers [2, 8], squares [1, 7], generalised pentagonal and hexagonal numbers [3, 5, 6], and generalised octagonal numbers [4]. More recently, the 2-additive uniqueness of generalised nonagonal numbers was considered in [9].

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The generalised s-gonal numbers are given by

$$P_n^s = \frac{(s-2)n^2 - (s-4)n}{2}, \quad n \in \mathbb{Z},\ n \neq 0.$$

We consider the case $s = 10$, the generalised decagonal numbers

$$\begin{aligned}\mathcal{D} &= \{D_n = n(4n-3) : n \in \mathbb{Z}, n \neq 0\} \\ &= \{1, 7, 10, 22, 27, 45, 52, 76, 85, 115, 126, 162, 175, \ldots\}.\end{aligned}$$

We study multiplicative functions satisfying

$$f(D_n + D_m) = f(D_n) + f(D_m)$$

for all nonzero integers m and n, and determine all such functions.

Since $D_n \equiv 1 \pmod 3$ if $3 \nmid n$, while $9 \mid D_n$ if $3 \mid n$, integers of the form $3d$ with $3 \nmid d$ cannot be expressed as sums of two elements of $\mathcal{D}$. This suggests that $f(3)$ may be arbitrary.

Define a function $g : \mathbb{N} \to \mathbb{C}$ by

$$g(n) = \begin{cases} g(3) \cdot \dfrac{n}{3} & \text{if } 3 \parallel n, \\ n & \text{otherwise.} \end{cases}$$

The main result of this work is as follows.

THEOREM 1.1. *If a multiplicative function f satisfies*

$$f(a + b) = f(a) + f(b)$$

for all nonzero generalised decagonal numbers a and b, then $f = g$.

2. Proof of Theorem 1.1

We prove Theorem 1.1 by induction, beginning with small values of n. If $N = ab$ with $a, b \geq 2$ and $\gcd(a, b) = 1$, then $f(N) = f(a)f(b)$. Thus, it suffices to show that $f(3^k) = 3^k$ for $k \geq 2$ and $f(p^k) = p^k$ for primes $p \neq 3$.

We repeatedly use the relation $f(D_n + D_m) = f(D_n) + f(D_m)$ for suitable m and n. In general, we cannot write $f(m(4m-3)) = f(m)f(4m-3)$ and we must account for the divisibility of the factors.

We first evaluate $f(n)$ for small n.

LEMMA 2.1. *For $n \leq 32$, we have $f(n) = g(n)$.*

PROOF. We have $f(1) = 1$ and $f(2) = 2f(1) = 2$. Note that

$$\begin{aligned} f(4)f(5) = f(20) &= f(10) + f(10) = 2f(2)f(5) = 4f(5), \\ f(11) &= f(1) + f(10) = 1 + 2f(5). \end{aligned}$$

Thus, $f(5) = 0$ if and only if $f(11) = 1$.

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If $f(5) = 0$, then $f(11) = 1$, so $f(23) = f(1) + f(2)f(11) = 3$. However, this contradicts

$$f(2)f(23) = f(46) = f(1) + f(9)f(5) = 1.$$

Hence, $f(5) \neq 0$ and $f(4) = 4$. Moreover,

$$\begin{aligned} 1 + f(9)f(5) = f(2)f(23) &= 2(f(1) + f(2)f(11)) \\ &= 2(1 + 2f(11)) = 2 + 4(1 + 2f(5)) = 6 + 8f(5). \end{aligned} \tag{2.1}$$

From

$$f(2)f(5) + f(9)f(5) = f(10) + f(45) = f(55) = f(5)f(11) = f(5)(1 + 2f(5)),$$

we obtain

$$f(9) = 2f(5) - 1.$$

Thus,

$$1 + f(9)f(5) = 1 + 2f(5)^2 - f(5). \tag{2.2}$$

Combining (2.1) and (2.2) gives

$$(2f(5) + 1)(f(5) - 5) = 0,$$

so $f(5) = -1/2$ or $f(5) = 5$.

From

$$\begin{aligned} f(5)f(11) &= f(45) + f(10) = f(5)f(9) + f(5)f(2), \\ f(23) &= f(22) + f(1) = 2f(11) + 1, \\ f(4)f(29) &= f(116) = f(115) + f(1) = f(5)f(23) + 1, \\ f(29) &= f(7) + f(22) = f(7) + 2f(11), \end{aligned}$$

we obtain

$$\begin{aligned} &f(5) = 5, \quad f(11) = 11, \quad f(9) = 9, \quad f(23) = 23, \quad f(29) = 29, \quad f(7) = 7, \\ &f(5) = -\frac{1}{2}, \quad f(11) = 0, \quad f(9) = -2, \quad f(23) = 1, \quad f(29) = \frac{1}{8}, \quad f(7) = \frac{1}{8}. \end{aligned}$$

The latter case leads to a contradiction. Indeed,

$$\begin{aligned} f(7)f(11) &= f(76) + f(1) = 4f(19) + 1, \\ f(7)f(19) &= f(126) + f(7) = f(2)f(7)f(9) + f(7). \end{aligned}$$

Using

$$\begin{aligned} f(17) &= f(7) + f(2)f(5), \\ f(32) &= f(2)f(11) + f(2)f(5), \\ f(4)f(7) &= f(28) = f(27) + 1, \\ f(32)f(11) &= f(351) + f(1) = f(27)f(13) + 1, \end{aligned}$$

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$$f(8)f(25) = f(85) + f(115) = f(5)f(17) + f(5)f(23),$$
$$f(16)f(11) = f(7)f(25) + 1,$$
$$f(2)f(31) = f(10) + f(4)f(13),$$

we conclude that $f(n) = g(n)$ for $n \leq 32$. □

We now extend this to higher powers of 3.

LEMMA 2.2. *Let $k \geq 2$. If $f(n) = n$ for $n \leq 3^{k-1}$ such that $n \not\equiv \pm 3 \pmod 9$, then $f(3^k) = 3^k$.*

PROOF. We already know that $f(9) = 9$.

Suppose $k = 2r$, with $r \geq 2$. Then,

$$D_{3^r} + D_{-3^r} = 3^{r+1}(4 \cdot 3^{r-1} - 1) + 3^{r+1}(4 \cdot 3^{r-1} + 1) = 8 \cdot 3^{2r}.$$

Since $3^{r+1} \leq 3^{2r-1}$ and $4 \cdot 3^{r-1} \pm 1 < 3^{2r-1}$, it follows that $f(3^{2r}) = 3^{2r}$.

Suppose $k = 2r + 1$. Since we have $f(27) = 27$, we may assume $r \geq 2$. Note that $3^{2r} \equiv \pm 1 \pmod{10}$. Assume that $3^k = 3(10m + 1)$ for some integer $m \geq 8$. Consider

$$D_{m+1} + D_{-3m} = (m + 1)(4m + 1) + 9m(4m + 1) = (10m + 1)(4m + 1). \tag{2.3}$$

Since $9 \mid 10m + 1$, we have $m \equiv -1 \pmod 9$ and $4m + 1 \equiv -3 \pmod 9$. Thus,

$$\gcd(m + 1, 4m + 1) = \gcd(9m, 4m + 1) = \gcd(10m + 1, 4m + 1) = 3.$$

Rewriting (2.3),

$$(3m + 3)\left(\frac{4m + 1}{3}\right) + 27m\left(\frac{4m + 1}{3}\right) = 3^k\left(\frac{4m + 1}{3}\right).$$

Since $27, m, 3m + 3, (4m + 1)/3 < 10m + 1 = 3^{2r}$, we obtain $f(3^k) = 3^k$.

A similar argument using

$$D_{-m+1} + D_{3m} = (m - 1)(4m - 1) + 9m(4m - 1) = (10m - 1)(4m - 1)$$

shows that if $3^k = 3(10m - 1)$ for some integer $m > 1$, then $f(3^k) = 3^k$. □

We now turn to powers of 5.

LEMMA 2.3. *Let $k \geq 2$. If $f(n) = n$ for $n < 5^k$ such that $n \not\equiv \pm 3 \pmod 9$, then $f(5^k) = 5^k$.*

PROOF. We already know that $f(5) = 5$ and $f(25) = 25$.

Suppose $k = 2r$, with $r \geq 2$. Consider

$$D_{5^r} + D_{-5^r} = 5^r(4 \cdot 5^r - 3) + 5^r(4 \cdot 5^r + 3) = 8 \cdot 5^{2r}.$$

Since $5^r, 4 \cdot 5^r \pm 3 < 5^{r+1} \leq 5^{2r-1}$, it follows that $f(5^{2r}) = 5^{2r}$.

Suppose $k = 2r + 1$, where $r \geq 1$. Write $5^{2r} = 4m + 1$ for some $m \geq 6$ with $5 \nmid m$. Consider

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$$D_{-3m} + D_{5m+2} = 9m(4m+1) + (5m+2)(20m+5) = 2(4m+1)(17m+5),$$

where $9m, 4m+1, 5m+2, 17m+5 < 20m+5 = 5^k$.

If $3 \mid m$, we write $m = 3^u d$ for some $d \geq 1$, $3 \nmid d$. Then, $f(9m) = f(3^{u+2})f(d) = 3^{u+2}d = 9m$, and since $3 \nmid (4m+1)(5m+2)(17m+5)$, we have $f(a) = a$ for $a \in \{4m+1, 5m+2, 17m+5\}$.

If $3 \nmid m$, then $f(9m) = f(9)f(m) = 9m$. If $m \equiv 2 \pmod 3$, then

$$5^{2r} = 4m + 1 \equiv 0 \pmod 3,$$

which is a contradiction. Thus, $m \equiv 1 \pmod 3$ and $3 \nmid (4m+1)(5m+2)(17m+5)$.

In all cases, it follows that $f(5^k) = 5^k$. □

Before proceeding to the next case, we record the following useful observation.

LEMMA 2.4.

(1) *The only nonnegative solution to*

$$2^x + 3 = 5^y \tag{2.4}$$

is $(x, y) = (1, 1)$.

(2) *The only nonnegative solutions to*

$$2^x - 3 = 5^y \tag{2.5}$$

are $(x, y) = (2, 0), (3, 1), (7, 3)$.

PROOF. It is easy to show that (2.4) has the unique solution $(x, y) = (1, 1)$. Indeed, assuming $x \geq 2$ and reducing the equation modulo 4 leads to a contradiction. For (2.5), it is readily verified that $(x, y) = (2, 0), (3, 1)$ and $(7, 3)$ are solutions. By [10, Theorem 4], the equation has at most one solution in positive integers (x, y) with $y > 1$. Hence, these constitute all solutions. □

We now consider powers of 2.

LEMMA 2.5. *Let* $k \geq 1$. *If* $f(n) = n$ *for* $n \leq 2^{k-1}$ *such that* $3 \nmid n$, *then* $f(2^k) = 2^k$.

PROOF. Since $f(2^k) = 2^k$ for $1 \leq k \leq 5$, we may assume $k \geq 6$.

Suppose $k = 2r + 1$, with $r \geq 3$. Consider

$$D_{2^{r-1}} + D_{-2^{r-1}} = 2^{r-1}(2^{r+1} - 3) + 2^{r-1}(2^{r+1} + 3) = 2^{2r+1},$$

with $2^{r-1} < 2^{r+1} - 3 < 2^{r+1} + 3 < 2^{k-1}$. Thus, $f(2^k) = 2^k$.

Suppose k is even. Then, $2^{k+1} \equiv \pm 3 \pmod 5$.

If $2^{k+1} \equiv 3 \pmod 5$, consider

$$D_{2^{k-1}} + D_{2^{k-1}} = 2^{k-1}(2^{k+1} - 3) + 2^{k-1}(2^{k+1} - 3) = 2^k(2^{k+1} - 3).$$

By Lemma 2.4, the only nonnegative solutions of $2^x - 3 = 5^y$ are $(x, y) \in \{(2, 0), (3, 1), (7, 3)\}$. In addition, $q \nmid 2^{k+1} - 3$ for $q = 2, 3$. We write $2^{k+1} - 3 = 5^u d$ with $u \geq 1$, $d \geq 7$ and $5 \nmid d$. Then,

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$$2^{k-1}5^u d + 2^{k-1}5^u d = 2^k 5^u d,$$

where

$$d = \frac{2^{k+1}-3}{5^u} < \frac{2^{k+1}}{5^u} < 2^{k-1} \quad \text{and} \quad 5^u = \frac{2^{k+1}-3}{d} < \frac{2^{k+1}}{7} < 2^{k-1}.$$

Thus, $f(2^k) = 2^k$.

If $2^{k+1} \equiv -3 \pmod 5$, a similar argument using

$$D_{-2^{k-1}} + D_{-2^{k-1}} = 2^{k-1}(2^{k+1}+3) + 2^{k-1}(2^{k+1}+3) = 2^k(2^{k+1}+3),$$

and the fact that $2^x + 3 = 5^y$ has the unique solution $(x, y) = (1, 1)$ yields $f(2^k) = 2^k$. □

LEMMA 2.6. *Let $p \geq 7$ be prime and $k \geq 2$. If $f(n) = n$ for $n < p^k$ with $n \not\equiv \pm 3 \pmod 9$, then $f(p^k) = p^k$.*

PROOF. Suppose $k = 2r$, with $r \geq 1$. Consider

$$D_{p^r} + D_{-p^r} = p^r(4p^r - 3) + p^r(4p^r + 3) = 8p^r,$$

with $p^r, 4p^r \pm 3 < p^{r+1} < p^k$. Hence, $f(p^k) = p^k$.

Suppose k is odd, so $p^k \equiv \pm 1, \pm 3 \pmod{10}$.

Assume that $p^k = 10m + 1$ for some positive integer m. Since we already know that $f(q) = q$ for $q = 11, 31$, we may assume that $m \geq 4$. Consider

$$D_{m+1} + D_{-3m} = (m+1)(4m+1) + 9m(4m+1) = (10m+1)(4m+1),$$

with $m+1, 4m+1, 9m < 10m+1 = p^k$. If $m \equiv 2 \pmod 3$, then $p^k = 10m + 1 \equiv 0 \pmod 3$, which is contrary to our assumption on p. Hence, $m \not\equiv 2 \pmod 3$ and

$$\gcd(m+1, 4m+1) = \gcd(9m, 4m+1) = \gcd(4m+1, 10m+1) = 1.$$

Thus, $f(p^k) = p^k$.

By a similar argument using the equation

$$D_{-m+1} + D_{3m} = (m-1)(4m-1) + 9m(4m-1) = (10m-1)(4m-1),$$

it can be shown that if $p^k = 10m - 1$, then $f(p^k) = p^k$.

Next, assume that $p^k = 10m + 3$ for some positive integer m. Since we have $f(q) = q$ for $q = 13, 23$, we may assume that $m \geq 3$. Consider

$$D_{-m} + D_{-3m} = m(4m+3) + 9m(4m+1) = 4m(10m+3),$$

where $m, 4m, 4m+1, 4m+3, 9m < 10m+3 = p^k$. Since $3 \nmid m$,

$$\gcd(m, 4m+3) = \gcd(9m, 4m+1) = \gcd(4m, 10m+3) = 1,$$

so $f(p^k) = p^k$.

By a similar argument using the equation

$$D_m + D_{3m} = m(4m-3) + 9m(4m-1) = 4m(10m-3),$$

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it can be shown that if $p^k = 10m - 3$, then $f(p^k) = p^k$. This completes the proof of the lemma and of the main theorem. □

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