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# INJECTIVE LINEAR TRANSFORMATIONS WITH EQUAL GAP AND DEFECT

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**Abstract**

Let $V$ be an infinite-dimensional vector space over a field $F$ and let $I(V)$ be the inverse semigroup of all injective partial linear transformations on $V$. Given $\alpha \in I(V)$, we denote the domain and the range of $\alpha$ by $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$, and we call the cardinals $g(\alpha) = \operatorname{codim}\operatorname{dom}\alpha$ and $d(\alpha) = \operatorname{codim}\operatorname{im}\alpha$ the 'gap' and the 'defect' of $\alpha$. We study the semigroup $A(V)$ of all injective partial linear transformations with equal gap and defect and characterise Green's relations and ideals in $A(V)$. This is analogous to work by Sanwong and Sullivan ['Injective transformations with equal gap and defect', *Bull. Aust. Math. Soc.* **79** (2009), 327–336] on a similarly defined semigroup for the set case, but we show that these semigroups are never isomorphic.

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## 1. Introduction

Let$V$ bea vector space over a field $F$ and suppose that $\dim V = p$. We let $P(V)$ denote the set of all *partial linear transformations* of $V$, that is, all linear transformations $\alpha : A \to B$ where $A, B$ are subspaces of $V$. As for partial transformations of a set, we denote the *domain* and the *range* of $\alpha \in P(V)$ by $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$, respectively, and we define the *composition* $\alpha \circ \beta$ of $\alpha, \beta \in P(V)$ to be the linear transformation with domain $U = (\operatorname{im}\alpha \cap \operatorname{dom}\beta)\alpha^{-1}$ such that, for all $u \in U$,

$$u(\alpha \circ \beta) = (u\alpha)\beta.$$

To simplify notation, we often write $\alpha \circ \beta$ as $\alpha\beta$. Clearly, $U$ is a subspace of $V$ and $\alpha \circ \beta \in P(V)$ if $\alpha, \beta \in P(V)$. Also, $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ for all $\alpha, \beta, \gamma \in P(V)$, so $(P(V), \circ)$ is a semigroup. Let $I(V)$ denote the set of all injective partial linear transformations of $V$. It is easy to see that $I(V)$ is an inverse subsemigroup of $P(V)$ and its idempotents are precisely the identity maps $\mathrm{id}_U$ on the subspaces $U$ of $V$. The partial linear map with domain $\{0\}$ in $V$, which we denote by 0, is a zero for the semigroup $I(V)$.

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If $\alpha \in P(V)$, we write

$$r(\alpha) = \dim \operatorname{im} \alpha, \quad g(\alpha) = \operatorname{codim} \operatorname{dom} \alpha, \quad d(\alpha) = \operatorname{codim} \operatorname{im} \alpha.$$

As usual, these are called the *rank*, *gap* and *defect* of $\alpha$, respectively.

In this paper, we consider the set of all injective partial linear transformations on $V$ with equal gap and defect,

$$A(V) = \{\alpha \in I(V) : g(\alpha) = d(\alpha)\}.$$

If $V$ is finite-dimensional, then $A(V) = I(V)$. Therefore, throughout the rest of this paper, except when specified otherwise, $V$ is a vector space with dimension $p \geq \aleph_0$.

In Section 2, we show that $A(V)$ is an inverse subsemigroup of $I(V)$ which is almost never isomorphic to its set analogue (studied by Sanwong and Sullivan in [7]). In Section 3, we describe Green's relations on $A(V)$, and using this we characterise all ideals of $A(V)$. Finally, in Section 4, we show that the semigroups of injective partial linear transformations with equal gap and defect are always 0-quasiresiduated and we describe the right and left quasiresiduals for each pair of nonzero mappings in $A(V)$.

## 2. Basic results

As an abbreviation, we write $\{e_i\}$ to denote a subset $\{e_i : i \in I\}$ of $V$, taking as understood that the subscript $i$ belongs to some (unmentioned) index set $I$. (This is comparable with [1, Volume 2, page 241].) The subspace $A$ of $V$ generated by a linearly independent subset $\{e_i\}$ of $V$ is denoted by $\langle e_i \rangle$ and we write $\dim A = |I|$. Often it is necessary to define some $\alpha \in P(V)$ by first choosing a linearly independent subset $\{e_i\}$ of $V$ and some $\{a_i\} \subseteq V$ and then letting $e_i\alpha = a_i$ for each $i$ and extending this action by linearity to the whole of $\operatorname{dom} \alpha = \langle e_i \rangle$. To abbreviate matters, we simply say, given $\{e_i\}$ and $\{a_i\}$ within context, that $\alpha \in P(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

For all other notation and terminology in semigroup theory, we refer the reader to [1, 2].

PROPOSITION 2.1. *The set $A(V)$ is an inverse subsemigroup of $I(V)$ with identity $\mathrm{id}_V$. Moreover, the general linear group $G(V)$ on $V$ is the group of units in $A(V)$.*

PROOF. Let $\alpha, \beta \in A(V)$, so $g(\alpha) = d(\alpha)$ and $g(\beta) = d(\beta)$. Suppose $\operatorname{im} \alpha \cap \operatorname{dom} \beta = \langle v_i \rangle$ and write $\operatorname{im} \alpha = \langle v_i \rangle \oplus \langle u_j \rangle$, $\operatorname{dom} \beta = \langle v_i \rangle \oplus \langle b_k \rangle$ and $V = \langle v_i \rangle \oplus \langle u_j \rangle \oplus \langle b_k \rangle \oplus \langle c_\ell \rangle$. Clearly, $d(\alpha) = |K| + |L|$ and $g(\beta) = |J| + |L|$. Let $v_i\alpha^{-1} = a_i$ and $u_j\alpha^{-1} = a_j$ for every $i$ and every $j$. Also, write $v_i\beta = w_i$ and $b_k\beta = z_k$ for every $i$ and every $k$. Then

$$\alpha\beta = \begin{pmatrix} a_i \\ w_i \end{pmatrix}$$

and $d(\alpha\beta) = |K| + d(\beta) = |K| + g(\beta) = |K| + (|J| + |L|) = |J| + (|K| + |L|) = |J| + d(\alpha) = |J| + g(\alpha) = g(\alpha\beta)$. Therefore, $A(V)$ is a semigroup.

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Since $\operatorname{dom}(\alpha^{-1}) = \operatorname{im}\alpha$ and $\operatorname{im}(\alpha^{-1}) = \operatorname{dom}\alpha$, it follows that $g(\alpha^{-1}) = d(\alpha)$ and $d(\alpha^{-1}) = g(\alpha)$. Thus, if $\alpha \in A(V)$, then $\alpha^{-1} \in A(V)$, and so $A(V)$ is an inverse semigroup. Clearly, $G(V) \subseteq A(V)$. Conversely, if $\alpha$ is a unit in $A(V)$, then $\alpha\beta = \mathrm{id}_V$ for some $\beta \in A(V)$, and so $\operatorname{dom}\alpha = V$. Thus, $d(\alpha) = g(\alpha) = 0$ and $\alpha \in G(V)$. In other words, $G(V)$ is the group of units in $A(V)$. □

In passing, we observe that $A(V)$ is not a maximal inverse subsemigroup of $I(V)$. In [7], the authors proved the correspondent result for sets and their proof holds verbatim in the linear context (see [7, page 329]).

In fact, in [7], the authors considered a set analogue of $A(V)$, defined as follows. Let $X$ be a set and denote by $I(X)$ the symmetric inverse semigroup on $X$. If $\alpha \in I(X)$, we let $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$ denote the domain and range of $\alpha$, respectively, and we say that $g(\alpha) = |X \setminus \operatorname{dom}\alpha|$ and $d(\alpha) = |X \setminus \operatorname{im}\alpha|$ are the *gap* and the *defect* of $\alpha$, respectively. Let $A(X)$ denote the set of all one-to-one partial transformations of $X$ with equal gap and defect. As for the linear case, $A(X) = I(X)$ if $X$ is finite, and for infinite $X$, $A(X)$ is an inverse subsemigroup of $I(X)$ with identity $\mathrm{id}_X$.

In [3], it is shown that $I(X)$ and $I(V)$ are isomorphic only when $|X| = 1 = \dim V$ and $|F| = 2$. Here we use a similar idea to show that $A(X)$ and $A(V)$ are almost never isomorphic for any set $X$ and any vector space $V$.

It is well known that the set $E(S)$ of all idempotents in a semigroup $S$ is partially ordered by

$$e \leq f \quad \text{if and only if} \quad e = ef = fe \quad (\text{for all } e, f \in E(S)). \tag{2.1}$$

Recall that every idempotent in $I(V)$ is of the form $\mathrm{id}_U$, where $U$ is a subspace of $V$, and clearly all idempotents in $I(V)$ belong to $A(V)$. It is not difficult to see that

$$id_U \leq \mathrm{id}_W \iff \mathrm{id}_U = \mathrm{id}_U\,\mathrm{id}_W \iff U \subseteq W.$$

As usual, we write $\mathrm{id}_U < \mathrm{id}_W$ if $\mathrm{id}_U \leq \mathrm{id}_W$ and $\mathrm{id}_U \neq \mathrm{id}_W$, that is, if $U$ is a proper subspace of $W$. Clearly, $\mathrm{id}_U < \mathrm{id}_W$ if and only if $\mathrm{id}_U = \mathrm{id}_U\,\mathrm{id}_W$ but $\mathrm{id}_W \neq \mathrm{id}_W\,\mathrm{id}_U$.

We say that $\mathrm{id}_W$ *covers* $\mathrm{id}_U$ in $A(V)$, and write $\mathrm{id}_U \ll \mathrm{id}_W$, if $\mathrm{id}_U < \mathrm{id}_W$ and $\mathrm{id}_U < \mathrm{id}_T < \mathrm{id}_W$ does not occur for any idempotent $\mathrm{id}_T \in A(V)$. Clearly, this is equivalent to saying that there exists some $w \notin U$ such that $W = U \oplus \langle w \rangle$.

Suppose that $W = U \oplus \langle w \rangle$ and $\mathrm{id}_U < \mathrm{id}_T < \mathrm{id}_W$ for some idempotent $\mathrm{id}_T \in A(V)$. Let $a \in T \setminus U$. Then $a \in W \setminus U$, and so $a = b + kw$ for some $b \in U$ and some nonzero scalar $k$. Thus $w = k^{-1}(a - b)$ and hence $w \in T$. But this implies that $W = U \oplus \langle w \rangle \subseteq T$ and $\mathrm{id}_W = \mathrm{id}_T$, which is a contradiction.

Conversely, suppose that $\mathrm{id}_W$ covers $\mathrm{id}_U$. Since $\mathrm{id}_U < \mathrm{id}_W$, $U$ is a proper subspace of $W$. Therefore, there exists $w \in W \setminus U$, so $U \cap \langle w \rangle = \{0\}$ and $U \oplus \langle w \rangle \subseteq W$. Clearly, $\mathrm{id}_{U \oplus \langle w \rangle}$ is an idempotent in $A(V)$ such that $\mathrm{id}_U < \mathrm{id}_{U \oplus \langle w \rangle} \leq \mathrm{id}_W$. From our assumption on $\mathrm{id}_U$ and $\mathrm{id}_W$, it follows that $\mathrm{id}_W = \mathrm{id}_{U \oplus \langle w \rangle}$, and hence $W = U \oplus \langle w \rangle$, as required.

We say that $\mathrm{id}_U \ll \mathrm{id}_W \ll \mathrm{id}_T$ is an *idempotent chain of length* 2 in $A(V)$ if $\mathrm{id}_W$ covers $\mathrm{id}_U$ and $\mathrm{id}_T$ covers $\mathrm{id}_W$. Note that $A(V)$ contains idempotent chains of length 2 only when $\dim V \geq 2$.

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Our aim now is to use this notion of 'idempotent chain' in $A(V)$ to show that this semigroup cannot be isomorphic to any $A(X)$. To do this, we need to obtain similar algebraic conditions on the latter semigroup.

Every nonzero idempotent in $I(X)$ is of the form $\mathrm{id}_Y$, where $Y$ is a nonempty subset of $X$, and clearly it belongs to $A(X)$. Let $\mathrm{id}_\emptyset = \emptyset$, the empty mapping in $I(X)$. It is not difficult to see that idempotents in $A(X)$ are partially ordered by

$$\mathrm{id}_A \preceq \mathrm{id}_B \iff \mathrm{id}_A = \mathrm{id}_A\,\mathrm{id}_B \iff A \subseteq B.$$

Also, $\mathrm{id}_A \prec \mathrm{id}_B$ if and only if $A \subsetneq B$, and this is equivalent to saying that $\mathrm{id}_A = \mathrm{id}_A\,\mathrm{id}_B$ but $\mathrm{id}_B \neq \mathrm{id}_B\,\mathrm{id}_A$. By analogy with the linear case, we say that $\mathrm{id}_B$ *covers* $\mathrm{id}_A$, and write $\mathrm{id}_A \ll \mathrm{id}_B$, if $\mathrm{id}_A \prec \mathrm{id}_B$ and there is no idempotent $\mathrm{id}_Y$ such that $\mathrm{id}_A \prec \mathrm{id}_Y \prec \mathrm{id}_B$, that is, if $B = A \cup \{x\}$ for some $x \notin A$. In addition, if $\mathrm{id}_A \ll \mathrm{id}_B \ll \mathrm{id}_C$, we call it an *idempotent chain of length* 2 in $A(X)$. Clearly, in this case, $A$ is a subset of $X$ and there are distinct $x, y \in X \setminus A$ such that $B = A \cup \{x\}$ and $C = A \cup \{x, y\}$.

Note that, by the remarks above, the notion of 'cover' and, thereby, the notion of 'chain' are preserved under any isomorphism $\Phi$ from $A(X)$ onto $A(V)$. In effect, we may conclude that if such an isomorphism exists, then any idempotent chain of length 2 in $A(X)$ corresponds to an idempotent chain of length 2 in $A(V)$, and conversely. Next, we see that the latter is not true if $\dim V \geq 2$: that is, we show that some idempotent chains of length 2 in $A(V)$ do not correspond to any idempotent chain of length 2 in $A(X)$.

To do this, suppose that $\dim V \geq 2$ and let $u, v$ be linearly independent vectors in $V$. Clearly,

$$\begin{aligned}
\mathrm{id}_{\{0\}} &\ll \mathrm{id}_{\langle u\rangle} \ll \mathrm{id}_{\langle u,v\rangle},\\
\mathrm{id}_{\{0\}} &\ll \mathrm{id}_{\langle v\rangle} \ll \mathrm{id}_{\langle u,v\rangle},\\
\mathrm{id}_{\{0\}} &\ll \mathrm{id}_{\langle u+v\rangle} \ll \mathrm{id}_{\langle u,v\rangle},
\end{aligned}$$

where $\langle u\rangle$, $\langle v\rangle$ and $\langle u + v\rangle$ are distinct one-dimensional subspaces of $V$.

If $\Phi : A(X) \to A(V)$ is an isomorphism and $\mathrm{id}_A, \beta_1, \beta_2, \beta_3, \mathrm{id}_B \in A(X)$ are such that $\mathrm{id}_A\,\Phi = \mathrm{id}_{\{0\}}$, $\mathrm{id}_B\,\Phi = \mathrm{id}_{\langle u,v\rangle}$ and $\beta_1\Phi = \mathrm{id}_{\langle u\rangle}$, $\beta_2\Phi = \mathrm{id}_{\langle v\rangle}$ and $\beta_3\Phi = \mathrm{id}_{\langle u+v\rangle}$, then $\mathrm{id}_A \prec\!\!\prec \beta_i \prec\!\!\prec \mathrm{id}_B$ in $A(X)$ for $i = 1, 2, 3$ and so, by a previous remark, either $\beta_3 = \beta_1$ or $\beta_3 = \beta_2$. But this implies that $\langle u + v\rangle = \langle u\rangle$ or $\langle u + v\rangle = \langle v\rangle$, which is a contradiction. Therefore, there is no isomorphism between $A(X)$ and $A(V)$ when $\dim V \geq 2$.

On the other hand, if $\dim V = 1$ and $|X| \geq 2$, it is easy to see that $A(V)$ admits no idempotent chains of length 2 but $A(X)$ has at least two chains of length 2: $\mathrm{id}_\emptyset \prec\!\!\prec \mathrm{id}_{\{x\}} \prec\!\!\prec \mathrm{id}_{\{x,y\}}$ and $\mathrm{id}_\emptyset \prec\!\!\prec \mathrm{id}_{\{x\}} \prec\!\!\prec \mathrm{id}_{\{x,y\}}$, where $x, y$ are two distinct elements of $X$. Thus, there is no isomorphism between $A(X)$ and $A(V)$ in this case either.

By [3, Theorem 1], we have the following result.

THEOREM 2.2. *The semigroups $A(X)$ and $A(V)$ are isomorphic only when $|X| = 1 = \dim V$ and $|F| = 2$.*

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## 3. Green's relations and ideals

The characterisation of the Green's relations on $I(V)$ is well known. Namely, for every $\alpha, \beta \in I(V)$: $\alpha \mathcal{L} \beta$ if and only if $\operatorname{im}\alpha = \operatorname{im}\beta$; $\alpha \mathcal{R} \beta$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$; $\alpha \mathcal{D} \beta$ if and only if $r(\alpha) = r(\beta)$, and $\mathcal{D} = \mathcal{J}$ in $I(V)$.

By Hall's theorem [2, Proposition 2.4.2], the $\mathcal{L}$ and $\mathcal{R}$ relations on $A(V)$ are the restrictions of the corresponding relations on $I(V)$ to $A(V)$, since this semigroup is inverse. The corresponding assertions for $\mathcal{D}$ and $\mathcal{J}$ are not true as we proceed to see.

PROPOSITION 3.1. *Let $\alpha, \beta \in A(V)$. Then:*

(a) $\beta = \lambda\alpha$ *for some* $\lambda \in A(V)$ *if and only if* $\operatorname{im}\beta \subseteq \operatorname{im}\alpha$;
(b) $\beta = \alpha\mu$ *for some* $\mu \in A(V)$ *if and only if* $\operatorname{dom}\beta \subseteq \operatorname{dom}\alpha$;
(c) $\beta = \lambda\alpha\mu$ *for some* $\lambda, \mu \in A(V)$ *if and only if* $r(\beta) \leq r(\alpha)$ *and* $d(\beta) \geq d(\alpha)$; *and*
(d) $\mathcal{D} = \mathcal{J}$.

PROOF. By the remark before this proposition, it suffices to show (c) and (d). To see that (c) holds, suppose that $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in A(V)$. Clearly, $r(\beta) \leq r(\alpha)$. Write $\operatorname{im}\lambda \cap \operatorname{dom}\alpha = \langle a_i \rangle$. Also let $\operatorname{im}\lambda = \langle a_i \rangle \oplus \langle z_m \rangle$ and $\operatorname{dom}\alpha = \langle a_i \rangle \oplus \langle a_n \rangle$. For every $i$ and every $n$, let $a_i\alpha = v_i$ and $a_n\alpha = v_n$, respectively. Since $(\operatorname{im}\lambda \cap \operatorname{dom}\alpha)\alpha = \langle v_i \rangle$, if $\langle v_i \rangle = \langle v_j \rangle \oplus \langle v_k \rangle$ where $\langle v_j \rangle = \langle v_i \rangle \cap \operatorname{dom}\mu$, then $\langle a_i \rangle = \langle a_j \rangle \oplus \langle a_k \rangle$ where $a_j$ and $a_k$ are such that $a_j\alpha = v_j$ and $a_k\alpha = v_k$. Let $\langle v_n \rangle \cap \operatorname{dom}\mu = \langle v_r \rangle$ and $\langle v_n \rangle = \langle v_r \rangle \oplus \langle v_s \rangle$. Also, let $\operatorname{dom}\mu = \langle v_j \rangle \oplus \langle v_r \rangle \oplus \langle v_\ell \rangle$ and $V = \langle v_j \rangle \oplus \langle v_k \rangle \oplus \langle v_r \rangle \oplus \langle v_s \rangle \oplus \langle v_\ell \rangle \oplus \langle v_t \rangle$.

It is not difficult to see that

$$\begin{aligned} d(\beta) = d(\lambda\alpha\mu) &= \operatorname{codim}\langle v_j\mu \rangle = \dim\langle v_r\mu \rangle + \dim\langle v_\ell\mu \rangle + d(\mu) \\ &= \dim\langle v_r \rangle + \dim\langle v_\ell \rangle + g(\mu) = |R| + |L| + |K| + |S| + |T| \geq |L| + |T| = d(\alpha). \end{aligned}$$

Conversely, suppose that $r(\beta) \leq r(\alpha)$ and $d(\beta) \geq d(\alpha)$, and write

$$\alpha = \begin{pmatrix} a_i & a_j \\ v_i & v_j \end{pmatrix}, \quad \beta = \begin{pmatrix} b_i \\ u_i \end{pmatrix}.$$

Define $\lambda, \mu$ in $I(V)$ by

$$\lambda = \begin{pmatrix} b_i \\ a_i \end{pmatrix}, \quad \mu = \begin{pmatrix} v_i \\ u_i \end{pmatrix}.$$

Clearly, $\lambda\alpha\mu = \beta$. To see that $\lambda, \mu \in A(V)$, suppose that $r(\alpha) < p$. Then $d(\alpha) = p$, and hence $g(\alpha) = p$. Also, $r(\beta) \leq r(\alpha) < p$ implies that $d(\beta) = p = g(\beta)$. Thus,

$$d(\lambda) = |J| + g(\alpha) = |J| + p = p = g(\beta) = g(\lambda)$$

and $\lambda \in A(V)$. Similarly,

$$g(\mu) = |J| + d(\alpha) = |J| + p = p = d(\beta) = d(\mu),$$

so $\mu \in A(V)$.

If $r(\alpha) = p$ and $r(\beta) < p$, then $|J| = p$, and so $\lambda, \mu \in A(V)$, as before. Now suppose that $r(\alpha) = r(\beta) = p$. Since $g(\beta) = d(\beta) \geq d(\alpha) = g(\alpha)$, we may choose $J$ such that

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$g(\beta) = |J| + g(\alpha)$. Then,

$$d(\lambda) = |J| + g(\alpha) = g(\beta) = g(\lambda)$$

and

$$g(\mu) = |J| + d(\alpha) = |J| + g(\alpha) = g(\beta) = d(\beta) = d(\mu),$$

and $\lambda, \mu \in A(V)$, as required.

To see that (d) holds, suppose that $\alpha \mathcal{J} \beta$ in $A(V)$. By (c), $r(\alpha) = r(\beta)$ and $d(\alpha) = d(\beta)$. Write

$$\alpha = \begin{pmatrix} a_i \\ v_i \end{pmatrix}, \quad \beta = \begin{pmatrix} b_i \\ u_i \end{pmatrix},$$

and define $\gamma$ in $I(V)$ by

$$\gamma = \begin{pmatrix} b_i \\ v_i \end{pmatrix}.$$

Since $g(\gamma) = g(\beta) = d(\beta) = d(\alpha) = d(\gamma)$, we have $\gamma \in A(V)$. By (a) and (b), $\alpha \mathcal{L} \gamma \mathcal{R} \beta$, and hence $\alpha \mathcal{D} \beta$ in $A(V)$. In other words, we have just proved that $\mathcal{J} \subseteq \mathcal{D}$ and the desired result follows. □

For each $r$ and $d$ with $1 \leq r \leq p'$ and $0 \leq d \leq p$ (where $k'$ denotes the successor of a cardinal $k$), let

$$A_{r,d} = \{\alpha \in A(V) : \ r(\alpha) < r \ \text{ and } \ d(\alpha) \geq d\}.$$

By the previous proposition, given $\alpha \in A_{r,d}$ and $\lambda, \mu \in A(V)$,

$$r(\lambda\alpha\mu) \leq r(\alpha) < r \quad \text{and} \quad d(\lambda\alpha\mu) \geq d(\alpha) \geq d,$$

so $\lambda\alpha\mu \in A_{r,d}$ and $A_{r,d}$ is an ideal of $A(V)$.

Clearly, $A_{1,p} = \{0\}$ and $A_{p',0} = A(V)$. For each $r$ with $1 \leq r \leq p$, if $\alpha \in A(V)$ is such that $r(\alpha) < r$, then $d(\alpha) = p$. Thus, for every $d$ such that $0 \leq d \leq p$,

$$A_{r,d} = A_{r,p} = \{\alpha \in I(V) : \ r(\alpha) < r\} = I_r$$

(note that $r(\alpha) = \dim(\operatorname{dom} \alpha)$ and this implies that $g(\alpha) = p = d(\alpha)$). Moreover, if $r_1, r_2$ are such that $1 \leq r_1 < r_2 \leq p'$, then

$$A_{r_1,p} \subseteq A_{r_2,p},$$

and it is not difficult to see that this containment is proper. In fact, if we write $V = \langle e_i \rangle \oplus \langle e_j \rangle$ with $|I| = r_1$ and $|J| = p$ (which is possible since $p \geq \aleph_0$), then it follows that $\mathrm{id}_{\langle e_i \rangle} \in A_{r_2,p} \setminus A_{r_1,p}$.

Now, let $d_1, d_2$ be such that $0 \leq d_1 < d_2 \leq p$. Clearly,

$$A_{p',d_2} \subseteq A_{p',d_1}.$$

Also, this containment is proper: if $V = \langle e_\ell \rangle \oplus \langle e_k \rangle$ with $|L| = d_1$ and $|K| = p$, then $\mathrm{id}_{\langle e_k \rangle} \in A_{p',d_1} \setminus A_{p',d_2}$. In other words, we have just seen that the ideals $A_{r,d}$ form a chain,

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that is,

$$\{0\}=A_{1,p}\subsetneq A_{2,p}\subsetneq\cdots\subsetneq A_{p,p}\subsetneq A_{p',p}\subseteq\cdots\subseteq A_{p',\aleph_0}\subsetneq\cdots\subsetneq A_{p',1}\subsetneq A_{p',0}=A(V) \quad (3.1)$$

(where all containments are proper if $p$ is strictly greater than $\aleph_0$).

THEOREM 3.2. *The ideals of $A(V)$ are exactly the sets $A_{r,d}$ with $1\le r\le p'$ and $0\le d\le p$. Moreover, $A_{r,d}$ is principal if and only if $r=k'$ for some cardinal $k$.*

PROOF. Let $I$ be an ideal of $A(V)$ and let $r$ be the minimum cardinal $s$ such that $r(\alpha)<s$ for every $\alpha\in I$ and $d$ be the minimum cardinal $t$ such that $t=d(\beta)$ for some $\beta\in I$. If $r(\alpha)=p$ for some $\alpha\in I$, then $I\subseteq A_{p',d}$. Let $\delta\in I$ be such that $d(\delta)=d$. Clearly, $r(\delta)\le p$. If $r(\delta)<p$, then $d=p=r(\alpha)$. Also, $d\le d(\alpha)$ (by the choice of $d$), and so $d(\alpha)=p$. Given $\beta\in A_{p',d}$, $r(\beta)\le p=r(\alpha)$ and $d(\beta)\ge d=p=d(\alpha)$. By Proposition 3.1, $\beta=\lambda\alpha\mu$ for some $\lambda,\mu\in A(V)$. On the other hand, if $r(\delta)=p$, then, for every $\beta\in A_{p',d}$, $r(\beta)\le p=r(\delta)$ and $d(\beta)\ge d=d(\delta)$, and again by Proposition 3.1, $\beta=\lambda\delta\mu$ for some $\lambda,\mu\in A(V)$. In both cases, $\beta\in I$ (since $I$ is an ideal of $A(V)$ and $\alpha,\delta\in I$), so $I=A_{p'.d}$.

Now suppose that $r(\alpha)<p$ for every $\alpha\in I$. Then $d(\alpha)=p$ for every $\alpha\in I$, and so $I\subseteq A_{r,p}$. Conversely, if $\beta\in A_{r,p}$ then there exists $\epsilon\in I$ such that $r(\beta)\le r(\epsilon)$: otherwise, $r(\alpha)<r(\beta)$ for every $\alpha\in I$, which contradicts the choice of $r$. Since $r(\beta)\le r(\epsilon)<p$, it follows that $d(\beta)=p=d(\alpha)$ and, by Proposition 3.1, $\beta=\lambda\epsilon\mu$ for some $\lambda,\mu\in A(V)$. Thus, $\beta\in I$ and $I=A_{r,p}$.

As previously remarked, if $r\le p$, then

$$A_{r,d}=\{\alpha\in I(V):\ r(\alpha)<r\}=I_r$$

for every $d$ such that $0\le d\le p$, and it is well known that this is an ideal of $I(V)$. Moreover, $I_r$ is principal if and only if $r=k'$ for some cardinal $k$. Thus, it suffices to show that $A_{p',d}$ is principal for every $d$ with $0\le d\le p$. Write $V=\langle e_i\rangle\oplus\langle e_j\rangle$ with $|I|=p$ and $|J|=d$. Clearly, $\mathrm{id}_{\langle e_i\rangle}\in A_{p',d}$. Given $\alpha\in A_{p',d}$, it follows that $r(\alpha)\le p=r(\mathrm{id}_{\langle e_i\rangle})$ and $d(\alpha)\ge d=d(\mathrm{id}_{\langle e_i\rangle})$. By Proposition 3.1, $\alpha=\lambda\,\mathrm{id}_{\langle e_i\rangle}\,\mu$ for some $\lambda,\mu\in A(V)$. Therefore,

$$A_{p',d}\subseteq A(V)\,\mathrm{id}_{\langle e_i\rangle}\,A(V).$$

Since $A_{p',d}$ is an ideal of $A(V)$ and $\mathrm{id}_{\langle e_i\rangle}\in A_{p',d}$, the equality follows. Thus, $A_{p',d}$ is principal. □

Observe that we have indirectly proved that

$$A_{r_1,d_1}\subseteq A_{r_2,d_2}\quad\text{if and only if}\quad r_1\le r_2\text{ and }d_1\ge d_2.$$

We now consider the Rees quotient semigroups corresponding to the ideals in (3.1). First, we consider the semigroups $A_{r',p}/A_{r,p}$ with $1\le r<p$. By a remark before Theorem 3.2, $A_{r',p}/A_{r,p}=I_{r'}/I_r$, so it is not difficult to see that each nonzero element in this semigroup has rank $r$ and defect $p$. Therefore, by Proposition 3.1, given $\alpha,\beta\in I_{r'}\setminus I_r$, it follows that $\alpha\,\mathcal{J}\,\beta$ and so $A_{r',p}/A_{r,p}$ is 0-simple.

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If $r = p$, the nonzero elements of the quotient semigroup $A_{r',p}/A_{r,p}$ are exactly the mappings in $A(V)$ with rank and defect $p$, and $A_{r',p}/A_{r,p}$ is also 0-simple.

If $r$ is infinite, let $\epsilon$ be a nonzero idempotent in $A_{r',p}/A_{r,p}$. Then $\epsilon = \mathrm{id}_W$, where $W$ is a subspace of $V$ with dimension $r$. Write $W = \langle e_i \rangle \oplus \langle f_i \rangle$, with $|I| = r$. If $U = \langle e_i \rangle$, then $r(\mathrm{id}_U) = r$ and so $\mathrm{id}_U$ is a nonzero idempotent in $A_{r',p}/A_{r,p}$ such that $\mathrm{id}_U \leq \mathrm{id}_W = \epsilon$. Clearly, $\mathrm{id}_U \neq \epsilon$, and hence $A_{r',p}/A_{r,p}$ has no primitive idempotents. In other words, we have just proved that $A_{r',p}/A_{r,p}$ is not completely 0-simple for every $\aleph_0 \leq r \leq p$.

On the other hand, if $r$ is finite, then every nonzero idempotent in $A_{r',p}/A_{r,p}$ is primitive, and so $A_{r',p}/A_{r,p}$ is completely 0-simple.

Finally, we consider the semigroups $A_{p',d}/A_{p',d'}$ for $0 \leq d < p$. Clearly, since $d < p$, each nonzero element of this semigroup has defect $d$ and rank $p$, and so $\alpha \mathcal{J} \beta$ for every $\alpha, \beta \in A_{p',d} \setminus A_{p',d'}$. Thus, $A_{p',d}/A_{p',d'}$ is 0-simple.

Now, if $d$ is infinite, let $\epsilon$ be a nonzero idempotent in $A_{p',d}/A_{p',d'}$. Then there exists a subspace $W$ of $V$ such that $\epsilon = \mathrm{id}_W$. Since $r(\epsilon) = p$ and $d(\epsilon) = d$, we can write $W = \langle e_i \rangle \oplus \langle f_k \rangle$ and $V = \langle e_i \rangle \oplus \langle f_k \rangle \oplus \langle g_k \rangle$, with $|I| = p > d = |K|$. Thus, $U = \langle e_i \rangle$ is a proper subspace of $W$ and $\mathrm{id}_U$ is such that $r(\mathrm{id}_U) = |I| = p$ and $d(\mathrm{id}_U) = |K| + |K| = d + d = d$ (since $d$ is infinite). Hence, $\mathrm{id}_U$ is a nonzero idempotent in $A_{p',d}/A_{p',d'}$ such that $\mathrm{id}_U \leq \mathrm{id}_W$ and $\mathrm{id}_U \neq \mathrm{id}_W$. Therefore, $A_{p',d}/A_{p',d'}$ has no primitive idempotents, so it is not completely 0-simple.

For finite $d$, let $\epsilon, \delta$ be two nonzero idempotents in $A_{p',d}/A_{p',d'}$ such that $\delta \leq \epsilon$. Then there exist subspaces $U, W$ of $V$ such that $\delta = \mathrm{id}_U$, $\epsilon = \mathrm{id}_W$ and $U \leq W$. Since $d(\delta) = d(\epsilon) = d < \aleph_0$ and $d(\delta) = \dim(W/U) + d(\epsilon)$, it follows that $\dim(W/U) = 0$, and hence $U = W$. Thus, $\delta = \epsilon$ and every nonzero idempotent in $A_{p',d}/A_{p',d'}$ is primitive. Therefore, $A_{p',d}/A_{p',d'}$ is completely 0-simple.

## 4. Quasiresiduals in $A(V)$

In 1952, Vagner extended the order in (2.1) from $E(S)$ to all of $S$ when $S$ is inverse, that is,

$$a \leq b \quad \text{if and only if} \quad a = eb \text{ for some } e \in E(S).$$

Three decades later, Hartwig and Nambooripad, independently, generalised this order relation to the class of regular semigroups. In [4], the author proved that, for any semigroup $S$, the relation $\leq_S$ defined by

$$a \leq_S b \quad \text{if and only if} \quad a = xa = xb = by \text{ for some } x, y \in S^1$$

is a partial order, and it coincides with the Hartwig–Nambooripad order when $S$ is regular.

Given $a, b \in S$, the *right* (*left*) *quasiresidual of $a$ by $b$ with respect to* $\leq_S$ is defined as the set

$$\langle a : b \rangle = \{x \in S \mid bx \leq_S a\} \qquad (\text{respectively, } \langle a :: b \rangle = \{x \in S \mid xb \leq_S a\}).$$

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We say that $S$ is *right* (*left*) *quasiresiduated* if $\langle a : b\rangle \neq \emptyset$ (respectively, $\langle a :: b\rangle \neq \emptyset$) for every $a, b$ in $S$. The semigroup $S$ is called *quasiresiduated* if it is both right and left quasiresiduated.

Clearly, if $S$ is a semigroup with zero, then zero belongs to any right and any left quasiresidual (since zero is the least element of $(S, \leq_S)$), and so $S$ is trivially quasiresiduated. Therefore, we say that a semigroup $S$ with zero is *right* (*left*) *0-quasiresiduated* if, for every $a, b \in S\setminus\{0\}$, $\langle a : b\rangle \neq \{0\}$ (respectively, $\langle a :: b\rangle \neq \{0\}$). A right and left 0-quasiresiduated semigroup is called *0-quasiresiduated*.

According to Mitsch [6, page 376], 'right (left) 0-quasiresiduation can be seen as an order-theoretical generalisation of the right (left) 0-simplicity of a semigroup'. In [5], the author studies the 0-quasiresiduation of several classes of semigroups of (linear) transformations and, in particular, proves that $I(V)$ is left and right 0-quasiresiduated when $V$ is finite-dimensional but, for the infinite case, it is neither right nor left 0-quasiresiduated. In fact, similar results hold for most of the classes studied in [5]: for finite sets and finite-dimensional vector spaces, the classes of semigroups with zero considered are left and/or right 0-quasiresiduated, but these properties are not satisfied in the infinite case.

As observed by Mitsch in [6, page 377], inverse semigroups with zero are neither right nor left 0-quasiresiduated in general. In this section, we show that $A(V)$ is always 0-quasiresiduated. To do this, we prove two lemmas, where we describe right and left quasiresiduals for nonzero $\alpha, \beta \in A(V)$.

LEMMA 4.1. *Given* $\alpha, \beta \in A(V)\setminus\{0\}$,

$$\langle \alpha : \beta\rangle = \{\gamma \in A(V) : U = \operatorname{im}\beta \cap \operatorname{dom}\gamma \subseteq W\beta \text{ and } \gamma|_U = (\beta^{-1}\alpha)|_U\},$$

*where* $W = \operatorname{dom}\alpha \cap \operatorname{dom}\beta$.

PROOF. Assume that $\alpha, \beta \in A(V)\setminus\{0\}$ and write $\operatorname{dom}\alpha \cap \operatorname{dom}\beta = W$. If $\gamma \in \langle \alpha : \beta\rangle$, then there exists a subspace $Z$ of $V$ such that

$$\beta\gamma = \alpha\,\mathrm{id}_Z,$$

by [2, Proposition 5.2.1]. By definition, $\operatorname{dom}(\beta\gamma) \subseteq \operatorname{dom}\beta$ and $\operatorname{dom}(\alpha\,\mathrm{id}_Z) \subseteq \operatorname{dom}\alpha$. Therefore, $\operatorname{dom}(\beta\gamma) = \operatorname{dom}(\alpha\,\mathrm{id}_Z) \subseteq \operatorname{dom}\alpha \cap \operatorname{dom}\beta = W$. If $W = \{0\}$, then

$$(\operatorname{im}\beta \cap \operatorname{dom}\gamma)\beta^{-1} = \operatorname{dom}(\beta\gamma) = \{0\},$$

and $\operatorname{im}\beta \cap \operatorname{dom}\gamma = \{0\} = W\beta$. Clearly, $0\gamma = 0 = 0(\beta^{-1}\alpha)$, and so $\gamma|_{\{0\}} = (\beta^{-1}\alpha)|_{\{0\}}$.

Now suppose $W \neq \{0\}$, and write $W = \langle v_\ell\rangle$, $\operatorname{dom}\alpha = \langle v_\ell\rangle \oplus \langle v_i\rangle$, $\operatorname{dom}\beta = \langle v_\ell\rangle \oplus \langle v_j\rangle$ and $V = \langle v_\ell\rangle \oplus \langle v_i\rangle \oplus \langle v_j\rangle \oplus \langle v_k\rangle$, with $|L| > 0$. For each $\ell$, let $v_\ell\alpha = a_\ell$ and $v_\ell\beta = b_\ell$. It is easy to see that

$$\beta^{-1}\alpha = \begin{pmatrix} b_\ell \\ a_\ell \end{pmatrix}.$$

Also, we have $(\operatorname{im}\beta \cap \operatorname{dom}\gamma)\beta^{-1} \subseteq \langle v_\ell\rangle$ and so $U = \operatorname{im}\beta \cap \operatorname{dom}\gamma \subseteq \langle v_\ell\rangle\beta = W\beta$. Given $u \in U$, there exists $v \in \operatorname{dom}\beta$ such that $v\beta = u$. Thus, $u\gamma = (v\beta)\gamma = v(\beta\gamma) =$

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$v(\alpha\,\mathrm{id}_Z)$. Also, $v \in \mathrm{dom}(\beta\gamma) = \mathrm{dom}(\alpha\,\mathrm{id}_Z) = (\mathrm{im}\,\alpha \cap Z)\alpha^{-1}$ and so $v\alpha \in \mathrm{im}\,\alpha \cap Z$. Hence,

$$u\gamma = (v\alpha)\,\mathrm{id}_Z = v\alpha = (u\beta^{-1})\alpha = u(\beta^{-1}\alpha).$$

In other words, we have just shown that $\gamma|_U = (\beta^{-1}\alpha)|_U$.

Conversely, if $\gamma \in A(V)$ is such that $U = \mathrm{im}\,\beta \cap \mathrm{dom}\,\gamma \subseteq W\beta$ and $\gamma|_U = (\beta^{-1}\alpha)|_U$, then it is not difficult to see that $U\beta^{-1} \subseteq \mathrm{dom}\,\alpha$ and $\beta\gamma = \mathrm{id}_{U\beta^{-1}}\,\alpha$, so $\gamma \in \langle \alpha : \beta \rangle$. □

LEMMA 4.2. *Given $\alpha, \beta \in A(V)\backslash\{0\}$,*

$$\langle \alpha :: \beta \rangle = \{\delta \in A(V) : U = \mathrm{im}\,\delta \cap \mathrm{dom}\,\beta \subseteq W\beta^{-1} \text{ and } \delta|_{U\delta^{-1}} = (\alpha\beta^{-1})|_{U\delta^{-1}}\},$$

*where $W = \mathrm{im}\,\alpha \cap \mathrm{im}\,\beta$.*

PROOF. Let $\alpha, \beta \in A(V)\backslash\{0\}$ and write $W = \mathrm{im}\,\alpha \cap \mathrm{im}\,\beta$. If $\delta \in \langle \alpha :: \beta \rangle$, then there exists a subspace $Z$ of $V$ such that $\delta\beta = \mathrm{id}_Z\,\alpha$. Clearly,

$$(\mathrm{im}\,\delta \cap \mathrm{dom}\,\beta)\beta = \mathrm{im}(\delta\beta) = \mathrm{im}(\mathrm{id}_Z\,\alpha) \subseteq \mathrm{im}\,\beta \cap \mathrm{im}\,\alpha = W,$$

so $U = \mathrm{im}\,\delta \cap \mathrm{dom}\,\beta \subseteq W\beta^{-1}$. Also, $U\delta^{-1} = \mathrm{dom}(\mathrm{id}_Z\,\alpha) = Z \cap \mathrm{dom}\,\alpha$. Given $v \in U\delta^{-1}$,

$$(v\delta)\beta = v(\delta\beta) = v(\mathrm{id}_Z\,\alpha) = (v\,\mathrm{id}_Z)\alpha = v\alpha.$$

Since $\beta$ is one-to-one, $v\delta = (v\alpha)\beta^{-1}$, so

$$v\delta|_{U\delta^{-1}} = v\delta = (v\alpha)\beta^{-1} = v(\alpha\beta^{-1}) = v(\alpha\beta^{-1})|_{U\delta^{-1}}.$$

Thus, $\delta|_{U\delta^{-1}} = (\alpha\beta^{-1})|_{U\delta^{-1}}$.

Conversely, let $\delta \in A(V)$ be such that $U = \mathrm{im}\,\delta \cap \mathrm{dom}\,\beta \subseteq W\beta^{-1}$ and $\delta|_{U\delta^{-1}} = (\alpha\beta^{-1})|_{U\delta^{-1}}$. Then $\mathrm{dom}(\delta\beta) = U\delta^{-1} \subseteq \mathrm{dom}\,\alpha$ and, given $v \in U\delta^{-1}$, it is easy to see that $v\delta \in \mathrm{dom}\,\beta$ and so

$$v(\delta\beta) = (v\delta)\beta = (v\delta|_{U\delta^{-1}})\beta = (v(\alpha\beta^{-1})|_{U\delta^{-1}})\beta = (v(\alpha\beta^{-1}))\beta = v\alpha = v(\mathrm{id}_{U\delta^{-1}}\,\alpha).$$

Therefore, $\delta\beta = \mathrm{id}_{U\delta^{-1}}\,\alpha$ and $\delta \in \langle \alpha :: \beta \rangle$. □

By Lemmas 4.1 and 4.2, we have a new example of 0-quasiresiduated semigroups.

PROPOSITION 4.3. *Given an arbitrary vector space $V$, $A(V)$ is always 0-quasiresiduated.*

PROOF. Assume that $\alpha, \beta \in A(V)\backslash\{0\}$. If $\mathrm{dom}\,\alpha \cap \mathrm{dom}\,\beta = \{0\}$, then

$$d(\beta) = g(\beta) = \mathrm{codim}\,\mathrm{dom}\,\beta \geq \dim \mathrm{dom}\,\alpha > 0$$

(since $\alpha \neq 0$). Thus, if $V = \mathrm{im}\,\beta \oplus U$, it is easy to see that $\beta\,\mathrm{id}_U = 0 = 0\alpha$, so $\mathrm{id}_U \in \langle \alpha : \beta \rangle \backslash\{0\}$. On the other hand, if $\mathrm{dom}\,\alpha \cap \mathrm{dom}\,\beta \neq \{0\}$ and we use the same notation as in the second paragraph of the proof of Lemma 4.1,

$$\gamma = \begin{pmatrix} b_\ell \\ a_\ell \end{pmatrix}$$

is nonzero and it belongs to the right quasiresidual of $\alpha$ by $\beta$.

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Now, if $\operatorname{im}\alpha \cap \operatorname{im}\beta = \{0\}$, then $g(\beta) = d(\beta) \geq r(\alpha) > 0$. Thus, $V = \operatorname{dom}\beta \oplus U'$ for some $U' \neq \{0\}$, and $\mathrm{id}_{U'}\beta = 0 = 0\alpha$. Hence, $\mathrm{id}_{U'} \in \langle \alpha :: \beta \rangle \setminus \{0\}$. If $\operatorname{im}\alpha \cap \operatorname{im}\beta \neq \{0\}$, write $\operatorname{im}\alpha \cap \operatorname{im}\beta = \langle c_r \rangle$. For each $r$, let $cr\alpha^{-1} = v_r$ and $cr\beta^{-1} = w_r$. Then

$$\delta = \begin{pmatrix} v_r \\ w_r \end{pmatrix} \neq 0$$

belongs to the left quasiresidual of $\alpha$ by $\beta$. □

It is easily verified that, for every $\alpha \in A(V)\setminus\{0\}$ such that $g(\alpha) = 0 = d(\alpha)$, we have $\alpha A(V) = A(V) = A(V)\alpha$. On the other hand, if $g(\alpha) \neq 0$ (and hence $d(\alpha) \neq 0$), then $\alpha$ is both a left and a right zerodivisor. So Proposition 4.3 is, in fact, a consequence of [6, Proposition 5.1], but here we have described right and left quasiresiduals for each pair $\alpha, \beta$ in $A(V) \setminus \{0\}$.

As for the linear case, the semigroup $A(X)$ of all injective partial transformations of a set $X$ with equal gap and defect satisfies the conditions in [6, Proposition 5.1], so we may conclude that $A(X)$ is always 0-quasiresiduated.

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