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Invariant hyperkähler metrics with a homogeneous complex structure

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Abstract

We give a hyperkähler analogue of the classification of compact homogeneous Kähler manifolds. Namely we show that, for a compact semisimple Lie group G, the complete G-invariant hyperkähler manifolds which are also locally G^c-homogeneous with respect to one of the complex structures are precisely the hyperkähler metrics of Kronheimer, Biquard and Kovalev on coadjoint semisimple orbits of G^c. As a consequence the only complete hyperkähler metrics which are of cohomogeneity one with respect to a triholomorphic action of compact semisimple Lie group are the flat metric on $\mathbb{H}^n$ and the Calabi metric on $T^*\mathbb{C}P^n$.

Introduction

The simplest examples of manifolds carrying a particular geometric structure are the homogeneous ones and any attempt of classification starts (and usually ends) with these. We give two examples: under mild assumptions on the group G, the symplectic manifolds homogeneous with respect to G are coadjoint orbits of G with the standard Kirillov–Kostant–Souriau symplectic form. Similarly the compact simply connected homogeneous Kähler manifolds are the coadjoint orbits of a compact semisimple Lie group (with the standard Kähler form but usually a non-normal metric).

Recently there has been a considerable interest in the quaternionic analogue of Kähler manifolds, the so-called hyperkähler manifolds. By definition a Riemannian manifold M is hyperkähler if its metric is Kähler with respect to three anti-commuting complex structures. This condition forces the dimension of M to be $4n$ and its holonomy to lie inside $Sp(n)$. As a consequence hyperkähler manifolds are Ricci flat and, in particular, the homogeneous ones are flat.

On the other hand, Kähler is to symplectic what hyperkähler is to complex-symplectic. The homogeneous complex-symplectic manifolds have been classified (under the same assumption on the group as in the symplectic case) by Lichnerowicz[**17**] – up to coverings they are coadjoint orbits of complex Lie groups with the complex Kirillov–Kostant–Souriau symplectic form. Therefore it seems to us that the natural analogues of compact homogeneous Kähler manifolds are the hyperkähler manifolds whose hyperkähler structure is invariant under an action of a compact semisimple Lie group G and which are G^c-homogeneous with respect to one of the complex structures.

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By the above result of Lichnerowicz such a manifold is diffeomorphic to a coadjoint orbit of the complexified Lie group G^c and, indeed, these were shown by Kronheimer [**14**, **15**], Biquard [**4**] and Kovalev [**13**] to carry hyperkähler metrics. We shall prove that in the complete case these are all the metrics satisfying the conditions of the previous paragraph (as explained below, in the incomplete case we know that there are others). We can actually relax the assumptions to the local homogeneity with respect to G^c:

THEOREM 1. *Let (M, g, I_1, I_2, I_3) be a connected complete hyperkähler manifold. Suppose that a compact semisimple Lie group G acts on M preserving the metric and the complex structures and suppose further that the tangent space T_mM at any point $m \in M$ is spanned by $\{X_\rho, I_1 X_\rho; \rho \in \mathfrak{g} = \mathrm{Lie}\,(G)\}$, where X_ρ is the Killing vector field corresponding to $\rho \in \mathfrak{g}$.*

Then M is isometric to a coadjoint semisimple orbit of the complexified group G^c with a Kronheimer–Biquard–Kovalev metric.

Remarks. 1. We will show that the isometry is also a biholomorphism with respect to each complex structure I_1, I_2, I_3.

2. The assumption on the way G acts can be weakened [**8**]: G acts isometrically and any $SO(3)$ or $SU(2)$ factor of G acts triholomorphically. Indeed, an isometric action of G induces a representation of G on the space of parallel 2-forms which are precisely the linear combinations of the Kähler forms $\omega_i = g(I_i \cdot, \cdot), i = 1, 2, 3$. It is therefore enough to assume that this 3-dimensional representation is trivial.

3. Without the assumption on completeness, the manifold M will still be locally diffeomorphic to a coadjoint orbit; however, the examples of Belinskiĭ et al. [**1**] show that one cannot expect the statement on isometry to hold, at least in dimension 4. We hope to address the incomplete case in a subsequent work.

The proof of Theorem 1 is based on the study of the moment map flow. The hyperkähler metrics of Kronheimer, Biquard and Kovalev are examples of natural L^2-metrics on certain moduli spaces of $SU(2)$-invariant instantons on $\mathbb{R}^4 \backslash \{0\}$ (it is amusing to observe that the compact homogeneous Kähler metrics are examples of L^2-metrics on moduli spaces of $U(1)$-invariant instantons on $\mathbb{R}^2 \backslash \{0\}$) and are described by Nahm's equations (see Section 2)). The idea behind our proof is to use the three moment maps μ_1, μ_2, μ_3 for the action of G and to show that, along a path of steepest descent for the function $\|\mu\|^2$ (or $\|\mu_3\|^2$), these moment maps satisfy Nahm's equations.

The proof ot Theorem 1 gives us the following result (the 'if' part of (i) was shown by Biquard [**4**]).

PROPOSITION 2. *Let G^c be a complex semisimple Lie group.*

(i) *A Kronheimer–Biquard–Kovalev metric on a coadjoint orbit of G^c is complete if and only if the orbit is semisimple.*

(ii) *If G^c is simple and connected, then Kronheimer–Biquard–Kovalev metrics on semisimple orbits of G^c are irreducible.*

A more precise de Rham decomposition is given in Section 2.

Theorem 1 gives us, in the complete case, a different proof of the recent classification due to Dancer and Swann [**8**] of hyperkähler metrics which are of

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cohomogeneity one with respect to a triholomorphic action of a compact semisimple Lie group:

THEOREM 3. *Let (M, g, I_1, I_2, I_3) be a connected complete hyperkähler manifold. Suppose that g is of cohomogeneity one with respect to a compact semisimple. Lie group G such that any $SO(3)$ or $SU(2)$ factor of G acts triholomorphically. Then (M, g) is isometric either to the flat $\mathbb{H}^n$ or to $T^*\mathbb{C}P^n$ with the Calabi metric.*

Let us remark that the assumption on the action of the 3-dimensional factors is essential, even in dimension greater than 4, as is shown by the example of the higher dimensional analogue of the Taub-NUT metric [**9**].

1. *Hyperkähler metrics on coadjoint orbits*

In this section we briefly describe, after Kronheimer, Biquard and Kovalev, the hyperkähler structures on coadjoint orbits of a complex semisimple Lie group G^c. In what follows we identify the coadjoint orbits with the adjoint ones via an invariant inner product.

The hyperkähler structures on adjoint orbits are given in terms of *Nahm's equations*. These are ordinary differential equations

$$\dot{T}_i + [T_0, T_i] + \tfrac{1}{2} \sum_{j,k=1,2,3} \epsilon_{ijk} [T_j, T_k] = 0, \quad i = 1, 2, 3, \tag{1·1}$$

for smooth Lie algebra valued functions $T_j : [0, +\infty) \to \mathfrak{g}, j = 0, 1, 2, 3$. Here $\mathfrak{g}$ is the real (compact) Lie algebra. One can think of $(T_0(t), T_1(t), T_2(t), T_3(t))$ as components of an anti-self-dual $\mathbb{R}^3$-invariant connection on $\mathbb{R}^4$. It is then clear that the gauge group of smooth functions $g : [0, +\infty) \to G$, acts as follows:

$$T_0 \mapsto \mathrm{Ad}\,(g)\, T_0 - \dot{g} g^{-1}$$
$$T_i \mapsto \mathrm{Ad}\,(g)\, T_i, \quad i = 1, 2, 3.$$

To obtain the hyperkähler metric on a given adjoint orbit one has to choose apropriate boundary conditions. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a subspace $\mathfrak{h}_0$ of $\mathfrak{h}$. Furthermore let $Y \in \mathfrak{g}^c$ be a nilpotent element in the centraliser $C(\mathfrak{h}_0)$ of $\mathfrak{h}_0$ in $\mathfrak{g}^c$. Then for any $\tau \in \mathfrak{h}_0^c$ with $C(\tau) = C(\mathfrak{h}_0)$ the adjoint orbit of $\tau + Y$ is diffeomorphic to $G^c/\mathrm{Stab}\,(\mathfrak{h}_0, Y)$. The family of hyperkähler structures on this manifold is parametrized by triples $(\tau_1, \tau_2, \tau_3) \in \mathfrak{h}_0^3$ such that $C(\tau_1, \tau_2, \tau_3) = C(\mathfrak{h}_0)$. First, because of the equivalence of the G-conjugacy classes of homomorphisms $\rho : \mathfrak{su}(2) \to \mathfrak{g}$ and the G^c orbits of nilpotent elements in $\mathfrak{g}^c$, one can assume that $Y = \varsigma_2 + i\varsigma_3$ with $\varsigma_2, \varsigma_3, \varsigma_1 = -2[\varsigma_2, \varsigma_3]$ elements of $\mathfrak{g}$ generating $\mathfrak{su}(2)$. Now we consider the moduli space of solutions to (2·1) such that

$$\|T_0(t)\| + \sum_1^3 \left\| T_i(t) - \left(\tau_i + \frac{\varsigma_i}{2t}\right) \right\| = O(t^{-1-\eta}), \quad \sum_0^3 \|\dot{T}_i(t)\| = O(t^{-2}), \tag{1·2}$$

for some $\eta > 0$, where the norm is given by the Killing form (in case of a regular orbit one can replace this by the exponential decay). The relevant gauge group consists of gauge transformations $g(t)$ preserving this condition and satisfying $g(0) = 1$. Let us denote this moduli space by $M = M(\tau_1, \tau_2, \tau_3, Y)$. The existence of a hyperkähler structure on the moduli space of solutions to Nahm's equations is a general

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phenomenon and can be described by means of the (infinite-dimensional) hyperkähler quotient construction (cf. [**14**]). The tangent space can be described as the space of solutions to the linearized Nahm's equations and satisfying the condition of being orthogonal (in the L^2-metric) to vectors arising from infinitesimal gauge transformations. In other words the tangent space to M at a solution (T_0, T_1, T_2, T_3) can be identified (cf. [**14**]) with the set of solutions (t_0, t_1, t_2, t_3) to the following system of linear equations:

$$\begin{aligned} &\dot{t}_0+[T_0,t_0]+[T_1,t_1]+[T_2,t_2]+[T_3,t_3]=0,\\ &\dot{t}_1+[T_0,t_1]-[T_1,t_0]+[T_2,t_3]-[T_3,t_2]=0,\\ &\dot{t}_2+[T_0,t_2]-[T_1,t_3]-[T_2,t_0]+[T_3,t_1]=0,\\ &\dot{t}_3+[T_0,t_3]+[T_1,t_2]-[T_2,t_1]-[T_3,t_0]=0. \end{aligned} \tag{1·3}$$

The metric is defined by

$$\|(t_0,t_1,t_2,t_3)\|^2=-\int_0^{+\infty}\sum_0^3\langle t_i,t_i\rangle \tag{1·4}$$

where $\langle\ ,\ \rangle$ is the Killing form on $\mathfrak{g}$. The three anti-commuting complex structures can be seen by writing a tangent vector as $t_0+it_1+jt_2+kt_3$.

In the regular case, i.e. when $C(\tau_2,\tau_3)=C(\mathfrak{h}_0)$, Kronheimer and then Biquard and Kovalev show that $T_2(0)+iT_3(0)$ is conjugate to $\tau_2+i\tau_3+Y$ and the assignment

$$(T_1(t),T_2(t),T_3(t))\mapsto T_2(0)+iT_3(0)$$

defines a biholomorphism between $(M(\tau_1,\tau_2,\tau_3,Y),I_1)$ and the adjoint orbit of $\tau_2+i\tau_3+Y$. We can use the gauge freedom to make $T_0\equiv 0$. The manifold $M(\tau_1,\tau_2,\tau_3,Y)$ can be now described as the space of solutions to the *reduced* Nahm's equations

$$\dot{T}_i+\tfrac{1}{2}\sum_{j,k=1,2,3}\epsilon_{ijk}[T_j,T_k]=0,\quad i=1,2,3, \tag{1·5}$$

such that there exists a $g\in G$ with

$$\sum_{i=1}^3\left\|T_i(t)-\mathrm{Ad}\,(g)\left(\tau_i+\frac{\varsigma_i}{2t}\right)\right\|=O(t^{-1-\eta}),\quad \sum_{i=1}^3\|\dot{T}_i(t)\|=O(t^{-2}), \tag{1·6}$$

for some $\eta>0$.

We will finish this section by relating this construction to the compact homogeneous Kähler manifolds. Let us consider the hyperkähler manifolds $M(\tau_1,0,0,0)$. As Biquard [**4**] shows, this manifold is biholomorphic, with respect to I_1, to $T(G/C(\tau_1))$. Moreover $M(\tau_1,0,0,0)$ admits an isometric S^1-action, given by $T_2+iT_3\mapsto e^{i\theta}(T_2+iT_3)$ which preserves the complex structure I_1. The set of fixed points of this action is therefore a Kähler manifold. It corresponds to the set of points in $M(\tau_1,0,0,0)$ for which $T_2=T_3\equiv 0$, i.e. to the moduli space of solutions to the equation $\dot{T}_1=[T_1,T_0]$. The tangent space at (T_0,T_1) is the set of solutions (t_0,t_1) to

$$\begin{aligned} &\dot{t}_0+[T_0,t_0]+[T_1,t_1]=0,\\ &\dot{t}_1+[T_0,t_1]-[T_1,t_0]=0, \end{aligned} \tag{1·7}$$

and the complex structure sends (t_0,t_1) to $(-t_1,t_0)$. Since by using the gauge freedom we can always achieve $T_0\equiv 0$ and so T_1 is constant, the equations (1·7) are easy to

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solve and it is not difficult to express the complex structure in terms of the Killing vector fields on $G/C(\tau_1)$. We find that this is the canonical complex structure of a compact homogeneous Kähler manifold [**2**].

2. *Proof of Theorem* 1

The key idea of the proof is to use the three moment maps μ_1, μ_2, μ_3 (see below) and show that along a path $m(t)$ of steepest descent for the function $F: M \to \mathfrak{g}$ defined by

$$F(m) = \tfrac{1}{2}\|\mu_2(m)\|^2, \tag{2·1}$$

the functions $T_i(t) = \mu_i(m(t))$ satisfy the Nahm equations.

We recall that whenever a connected compact semisimple Lie group acts isometrically and triholomorphically on a hyperkähler manifold (M, g, I_1, I_2, I_3) there exist [**10**, **11**] unique equivariant moment maps $\mu_i: M \to \mathfrak{g}^* \simeq \mathfrak{g}, i = 1, 2, 3$, such that

$$\langle d\mu_i(v), \rho\rangle = g(I_i X_\rho, v)$$

for any $\rho \in \mathfrak{g}$ (X_ρ denotes the corresponding Killing vector field). Let us make here an observation that, since we assume M to be connected, we can replace G by its component of identity and so we can assume that G is connected. Therefore we have a hyperkähler moment map. The map

$$\mu_2 + i\mu_3: M \to \mathfrak{g}^c \tag{2·2}$$

is I_1-holomorphic and G^c-equivariant (with respect to the local G^c action). Since M is locally G^c-homogeneous with respect to I_1, it follows that $\mu_2 + i\mu_3$ is a submersion. On the other hand, since T_pM is spanned by $I_2 X_\rho, I_3 X_\rho$ at any $p \in M$, the definition of the moment map implies that $\mu_2 + i\mu_3$ is an immersion. Hence $\mu_2 + i\mu_3$ is a local biholomorphism with respect to I_1.

Because of the equivariance of the moment maps we have:

$$d\mu_i(X_\rho) = [\rho, \mu_i] \tag{2·3}$$

for any $\rho \in \mathfrak{g}, i = 1, 2, 3$. This gives further relations:

$$d\mu_1(I_2 X_\rho) = [\rho, \mu_3], \quad d\mu_3(I_2 X_\rho) = -[\rho, \mu_1] \text{ etc.} \tag{2·4}$$

Now we have the key lemma.

LEMMA 2·1. *At any point* $m \in M$

$$I_2 X_{\mu_2(m)} = I_3 X_{\mu_3(m)}$$

and

$$d\mu_i(I_2 X_{\mu_2(m)}) = \tfrac{1}{2} \sum_{j,k=1,2,3} \epsilon_{ijk}[\mu_j(m), \mu_k(m)], \quad i = 1, 2, 3.$$

Proof. Put $v = I_2 X_{\mu_2(m)}, w = I_3 X_{\mu_3(m)}$. Then, from (2·3), $d\mu_2(I_2 v) = d\mu_3(I_3 w) = 0$ and, from (2·4), $d\mu_1(v) = d\mu_1(w) = [\mu_2(m), \mu_3(m)]$. This implies that $d\mu_2(I_2 v) = d\mu_2(I_2 w)$ and $d\mu_3(I_2 v) = d\mu_3(I_2 w)$. Since (μ_2, μ_3) is an immersion we can conclude that $I_2 v = I_2 w$ and so $v = w$. This shows that $I_2 X_{\mu_2(m)} = I_3 X_{\mu_3(m)}$. For the second part of the lemma we observe that (2·3) proves the formula for $i = 1, 3$. Now $d\mu_2(I_2 X_{\mu_1(m)}) = d\mu_2(I_3 X_{\mu_3(m)}) = d\mu_1(X_{\mu_3(m)}) = [\mu_3(m), \mu_1(m)]$. ∎

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This lemma shows, as $I_2 X_{\mu_2} = \frac{1}{2}\nabla\|\mu_2\|^2$, that along a path $m(t)$ of steepest descent for the function (2·1), the functions

$$T_i(t) = \mu_i(m(t)) \tag{2·5}$$

satisfy equations (1·5). We remark here that the gradient flow of $\frac{1}{2}\|\mu_2\|^2$ agrees with that of $\frac{1}{6}(\|\mu_1\|^2+\|\mu_2\|^2+\|\mu_3\|^2)$ and the former function is chosen merely for convenience.

Now we will show that under the completeness assumption such a path will exist for infinite time and end up in M.

LEMMA 2·2. *Under the assumptions of Theorem* 1, *for any* $m\in M$ *the path* $m(t)$ *of steepest descent for the function* (2·1) *with* $m(0) = m$ *exists for all* $t > 0$ *and has a unique limit point in* M.

Proof. First we show the existence of a limit point. Suppose that some path of steepest descent $m(t)$ exists for $t < r$ (r can be $+\infty$). By completeness of M it is sufficient to show that this path has finite length. The length of $m(t)$ is given by $\int_0^r (g(\dot m, \dot m))^{1/2} = \int_0^r |(d/dt)F(m(t))|^{1/2}$. Now $F(m(t)) = \frac{1}{2}\|T_2(t)\|^2$ where $T_1(t), T_2(t), T_3(t)$ satisfy the (reduced) Nahm equations. Hence $(d/dt)F(m(t)) = \langle T_2(t), [T_1(t), T_3(t)]\rangle$. If $T_i(t)$ satisfy (1·5), then $\|T_i(t)\|^2 - \|T_j(t)\|^2$ are constant in t. Therefore $(d/dt)F(m(t)) = O((F(m(t))+F(m(t))^3)^{1/2})$ which shows that $m(t)$ has finite length if $r < +\infty$.

Consider now the case of $r = +\infty$. It follows from results of Biquard that the functions $T_i(t)$ must satisfy (1·6) for some $\tau_i, \varsigma_i, i = 1, 2, 3$, and so $(d^2/dt^2)F(m(t)) = \sum \|[T_j(t), T_k(t)]\|^2 = \sum \|\dot T_i(t)\|^2 = O(t^{-4})$. Since also $(d/dt)F(m(t))(+\infty) = 0$, it follows that $(d/dt)F(m(t)) = O(t^{-3})$.

The set of the limit points of the trajectory of $-\nabla F$ is connected. However, the image of this set under $\mu_2 + i\mu_3$ is a single point. Since $\mu_2 + i\mu_3$ is a local diffeomorphism, the limit point must be unique. ∎

Let us pick now a point $m\in M$ and consider the path $m(t)$ of steepest descent starting at m. The functions (2·5) satisfy the equations (1·5) and it follows from results of Biquard that they must satisfy (1·6) for some $\tau_i, \varsigma_i, i = 1, 2, 3$. From Lemma 2·2 we conclude that there exists an $m_\infty \in M$ with $\mu_i(m_\infty) = \tau_i$. The fact that the map $\mu_2 + i\mu_3$ is a biholomorphism of a neighbourhood of m_∞ in (M, I_1) with a neighbourhood of $\mu_2(m_\infty) + i\mu_3(m_\infty) = \tau_2 + i\tau_3$ in its adjoint G^c orbit implies that we must have $\varsigma_2 = \varsigma_3 = 0$, and so $\varsigma_1 = 0$.

We now claim that the set of G-orbits of critical points of the function F is discrete. Let m'_∞ be a critical point in the neighbourhood of m_∞ and let $\tau'_i = \mu_i(m'_\infty), i = 1, 2, 3$. If m'_∞ is close enough to m_∞, then it is in the same G^c-orbit. In particular $\tau'_2 + i\tau'_3$ is G^c-conjugate to $\tau_2 + i\tau_3$. Since τ'_i's mutually commute they belong to the same Cartan subalgebra of $\mathfrak{g}$ which can be conjugated by a $g\in G$ to a Cartan subalgebra containing the τ_i. Let $m''_\infty = g\cdot m'_\infty$ and $\tau''_i = \mathrm{Ad}(g)\tau'_i$. We still have that $\tau''_2 + i\tau''_3$ is conjugate (by an element of G^c) to $\tau_2 + i\tau_3$. Let $g_1 \in G^c$ be such that $\mathrm{Ad}(g_1)(\tau''_2 + i\tau''_3) = \tau_2 + i\tau_3$. Let us decompose g_1 according to the KAN decomposition with respect to the complexified Cartan subalgebra containing $\tau_1, \tau_2, \tau_3, \tau''_1, \tau''_2, \tau''_3$. It follows that the unipotent part of g_1 acts trivially on $\tau''_2 + i\tau''_3$ and so there is an element k_1 of the compact group such that $\mathrm{Ad}(k_1)\tau''_2 = \tau_2$ and $\mathrm{Ad}(k_1)\tau''_3 = \tau_3$. Since $\mu_2 + i\mu_3$ is a G-equivariant diffeomorphism in a neighbourhood of the G-orbit of m_∞, we have that $m''_\infty = k_1\cdot m_\infty$.

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Our aim is to show that there is a unique G-orbit of critical points of F and this follows from:

LEMMA 2·3. *Consider the subset S of $\mathfrak{g}^3$ consisting of triples A_1, A_2, A_3 such that the solution $T_1(t), T_2(t), T_3(t)$ to equations (1·5) with $T_i(0) = A_i$ exists for all $t \in [0, +\infty]$ and satisfies (1·6) with $\varsigma_i = 0, i = 1, 2, 3$. Then the function $f: S \to \mathfrak{g}^3$ defined by $f(A_1, A_2, A_3) = \lim_{t \to +\infty}(T_1(t), T_2(t), T_3(t))$ is continuous (in the metrics induced from the Killing norm on $\mathfrak{g}^3$).*

Proof. The right-hand side of (1·5) is a locally Lipschitz function on S. Hence, if $A_i^n \to A_i, i = 1, 2, 3$, then the L^1 norm of $T_i^n - T_i$ on $[0, R]$ converges to 0, $i = 1, 2, 3$, for an arbitrary R. Suppose that the limiting values τ_i^n of T_i^n do not converge to the limiting values τ_i of T_i's. The estimates (1·6) can be written in our case as $\|T_i(t) - \tau_i\| \leqslant ct^{-1-\eta}$ and Biquard [**4**, lemma 1·1] shows that, for $\eta = 0$, the constant c is locally independent of the initial values $T_1(0), T_2(0), T_3(0)$. This implies that $\int_0^R \|T_i^n - T_i\| \geqslant \int_{R/2}^R \|T_i^n - T_i\| \geqslant \int_{R/2}^R (\|\tau_i^n - \tau_i\| - c't^{-1})$, where c' does not depend on R or n. If R is large enough, the right-hand side cannot converge to 0 and so we obtain a contradiction. ∎

This lemma together with the connectivity of M show now that there is only one G-orbit of critical points of F on M. Therefore the map Ψ from M to $M(\tau_1, \tau_2, \tau_3, 0)$ sending a point $m \in M$ to the functions (2·5) starting at $(\mu_1(m), \mu_2(m), \mu_3(m))$ is well-defined. This map is a local biholomorphism with respect to I_1, and it is our aim to show that it is also a local biholomorphism with respect to I_2 and I_3.

LEMMA 2·4. *The map $\Psi: M \to M(\tau_1, \tau_2, \tau_3, 0)$ defined above respects the complex structures I_2 and I_3.*

Proof. Consider first how the complex structures I_2 and I_3 are defined on $M(\tau_1, \tau_2, \tau_3, 0)$. The equations (1·3) are linear and so the solutions t_0, t_1, t_2, t_3 are completely determined by their initial values at $t = 0$. Hence the complex structures are simply given by the action of unit imaginary quaternions on $t_0(0) + it_1(0) + jt_2(0) + kt_3(0)$.

If we consider now $M(\tau_1, \tau_2, \tau_3, 0)$ as the space of solutions to the reduced Nahm's equations (1·5), then a tangent vector is a triple (s_1, s_2, s_3) satisfying the linearization of (1·5). The corresponding solution (t_0, t_1, t_2, t_3) to (1·3) is given by adding to (s_1, s_2, s_3) an infinitesimal gauge transformation $(-\dot{\rho}, [\rho, T_1], [\rho, T_2], [\rho, T_3])$, so that (1·3) are satisfied. Since $\rho(0) = 0, s_i(0) = t_i(0)$ for $i = 1, 2, 3$.

Now suppose that $(s_1, s_2, s_3) = d\Psi(v)$ for some $v \in T_m M$. Then $s_i(0) = d\mu_i(v)$. Since $d\Psi$ respects I_1, it follows that $t_0(0) = d\mu_1(\mathrm{I}_1 \psi)$. Since $d\mu_1(I_1 v) = d\mu_2(I_2 v) = d\mu_3(I_3 v)$, $d\Psi$ also respects I_2 and I_3. ∎

It follows now from the uniqueness of the quaternionic Obata connection that the map Ψ preserves the Levi–Civita connection. To obtain the statement on isometry we proceed as follows. Let $\tilde{M}$ be the universal covering of M. Then $\tilde{M}$ carries a natural hyperkähler structure and we still have a map $\tilde{M} \to M(\tau_1, \tau_2, \tau_3, 0)$ which preserves the Levi–Civita connection. Let $\tilde{M} = M_1 \times \ldots \times M_k$ be the de Rham decomposition of $\tilde{M}$. We need now the following proposition on the de Rham decomposition of the Kronheimer-Biquard-Kovalev metrics whose proof we postpone till later.

PROPOSITION 2·5. *Let G be a connected compact semisimple Lie group whose simple*

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factors are $G_i, i = 1, \dots, n$. Let $M(\tau_1, \tau_2, \tau_3, 0)$ be an adjoint semisimple orbit of G^c with a Kronheimer–Biquard–Kovalev metric and let $\pi_i : \mathfrak{g} \to \mathfrak{g}_i, i = 1, \dots, n$, be the projections onto the simple factors. Then the de Rham decomposition of $M(\tau_1, \tau_2, \tau_3, 0)$ is given by

$$M(\tau_1, \tau_2, \tau_3 0) = \prod_{i=1}^{n} M(\pi_i(\tau_1), \pi_i(\tau_2), \pi_i(\tau_3), 0)$$

where the i-th factor is an orbit of G_i. In particular, when G is simple, $M(\tau_1, \tau_2, \tau_3, 0)$ is irreducible.

Since it preserves the Levi–Civita connection, the map $\tilde{M} \to M(\tau_1, \tau_2, \tau_3, 0)$ preserves the de Rham decomposition. Each M_i is mapped into some $M(\pi_i(\tau_1), \pi_i(\tau_2), \pi_i(\tau_3), 0)$. Since they are both simply connected complete irreducible Riemmanian manifolds with the same Levi–Civita connection, this map must be an isometry up to a homothety. However $M(\pi_i(\tau_1), \pi_i(\tau_2), \pi_i(\tau_3), 0)$ with the metric multiplied by a factor r is isometric to $M(r^2\pi_i(\tau_1), r^2\pi_i(\tau_2), r^2\pi_i(\tau_3), 0)$ (in terms of the description of Section 1 the isometry is given by $T_i(t) \mapsto r^2 T_i(r^2 t), i = 1, 2, 3$). Therefore we can conclude that $\tilde{M}$ is isometric to an adjoint semisimple orbit of G^c with a Kronheimer–Biquard–Kovalev metric, i.e. to a manifold $M(\tau_1', \tau_2', \tau_3', 0)$.

Now notice that we also had a map from M to $M(\tau_1, \tau_2, \tau_3, 0)$ and so to $M(\tau_1', \tau_2', \tau_3', 0)$ and that this map is now a local isometry. Since M is complete and semisimple orbits are simply connected, this map must be an isometry. Theorem 1 is now proven except for a proof of Proposition 2·5 which we proceed to give now.

It follows directly from the definition of the manifolds $M(\tau_1, \tau_2, \tau_3, 0)$ that they split as described in Proposition 2·5. Therefore it is enough to show that $M = M(\tau_1, \tau_2, \tau_3, 0)$ is irreducible for a connected simple Lie group G. Suppose it is not. Then, since it is simply connected (we can always realize an adjoint semisimple orbit as an orbit in the universal covering of G^c; the simple connectivity follows then from the long exact homotopy sequence) and complete by result of Biquard [**4**], $M(\tau_1, \tau_2, \tau_3, 0)$ splits into the product of a Euclidean and irreducible hyperkähler factors, say $M_i, i = 1, \dots, k$. Moreover the connected component $I^0(M)$ of the isometry group of M is just the product of $I^0(M_i)$. Now, if $\pi_i : M \to M_i$ denotes the natural projection, then, for any $\phi \in I^0(M)$ and any $m \in M, \pi_i(\phi(m))$ depends only on $\pi_i(m)$ (proof is given in [**12**, proof of theorem VI·3·5]). Therefore, for each $i = 1, \dots, k$, we obtain a natural projection $\pi_i : I^0(M) \to I^0(M_i)$ which is a group homomorphism. Let $G_i = \pi_i(G) (G \subseteq I^0(M))$. Each G_i is simple and $G \subset \prod_{i=1}^{k} G_i$. Moreover each G_i acts triholomorphically on M_i and all the assumptions of Theorem 1 are now satisfied for M_i (with respect to some complex structure). Hence we can still apply the part of the proof of Theorem 1 up to the statement of Proposition 2·5 and conclude that each M_i is mapped into some $M(\tau_1^i, \tau_2^i, \tau_3^i, 0)$ by a map that preserves the Levi–Civita connection, where $M(\tau_1^i, \tau_2^i, \tau_3^i, 0)$ is an adjoint semisimple orbit of G_i (in particular the Euclidean factor cannot occur). This time however each M_i is irreducible and so we can conclude as before that it is isometric to $M(\tau_1^i, \tau_2^i, \tau_3^i, 0)$ where we perhaps multiplied τ_j^i by some positive constant, $j = 1, 2, 3$. Let $\mathfrak{h}^i = \pi_i(\mathfrak{h})$ where $\mathfrak{h}$ is the Cartan algebra of $\mathfrak{g}$ containing τ_1, τ_2, τ_3 and $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ is the restriction of the natural projection. If $\mathfrak{h}_0$ is the subspace of $\mathfrak{h}$ such that $C(\tau_1, \tau_2, \tau_3) = C(\mathfrak{h}_0)$ and $\mathfrak{h}_0^i = \pi_i(\mathfrak{h}_0)$, then we must have $\mathfrak{h}_0 = \bigoplus_{i=1}^{k} \mathfrak{h}_0^i$ (both sides represent the second cohomology of the manifold under consideration; there is also an obvious inclusion). Then however, for every $i = 1, \dots, k, \mathfrak{g} \cap \mathfrak{g}_i \supset \mathfrak{h}_0^i \neq \varnothing$. Since

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$\mathfrak{g} \cap \mathfrak{g}_i$ is an ideal in $\mathfrak{g}$, we obtain a contradiction unless $k = 1$ and $M(\tau_1, \tau_2, \tau_3, 0)$ is irreducible.

Remark. It would be interesting to classify hyperkähler manifolds (M, g, I_1, I_2, I_3) admitting an isometric triholomorphic and locally *quaternionically* transitive compact group action, i.e. such that at each point $m \in M$ the tangent space $T_m M$ is spanned by $\{X_\rho, I_1 X_\rho, I_2 X_\rho, I_3 X_\rho; \rho \in \mathfrak{g}\}$. This class is much larger and includes the hyperkähler metrics on T^*G^c [**16**], $T^*(G^c/K^c)$ [**7**] and on slices $S(\rho) \times G^c$ [**3**].

3. *Proof of Theorem* 3

By Remark 2 of the introduction we can assume that G acts triholomorphically. Since a Killing vector field on a compact Ricci-flat manifold is parallel, a complete hyperkähler manifold of cohomogeneity one has only one singular orbit. The assumption of Theorem 1 about the Killing vector fields is satisfied at any point outside the singular orbit for any complex structure. To handle the singular orbit we need the following lemma due to Andrew Dancer (oral communication).

LEMMA 3·1. *Let M be a hyperkähler manifold of cohomogeneity* 1. *Then, with respect to any complex structure I, the singular orbit B is either complex or $T_p M = T_p B + I(T_p B)$ for any $p \in B$.*

Proof. Let $B = G/H$ be the singular orbit. For a $p \in B$ consider the vector space $(T_p B + I(T_p B)) \cap N_p B$. Since H respects I, this space is an H-module. However the cohomogeneity being one implies that $N_p B$ is an irreducible H-module. Hence either $I(T_p B) = T_p B$ or $T_p B + I(T_p B) = T_p M$. Since for every other point of B there exists a holomorphic isometry of M leaving B invariant and mapping p to that point, the assertion of the lemma is proven. ∎

Therefore if the singular orbit B is not a point, then there is a complex structure, say I_1, such that $T_p M = T_p B + I_1(T_p B)$ for any $p \in B$. Indeed, if not, then B is a compact homogeneous hyperkähler manifold which contradicts the semisimplicity of G. Hence all the assumptions of Theorem 1 are satisfied in this case and we can conclude that M is a semisimple complex adjoint orbit. Moreover, since such an orbit of a non-simple group is just a product of orbits of simple factors, we can assume that G is simple.

Now we can use the argument of [**8**, remark 2·2]: semisimple adjoint orbit is G-equivariantly diffeomorphic to $T(G/K)$ with K compact, and so it is of cohomogeneity one if and only if G/K an isotropic (cf. [**2**]) homogeneous space (with respect to the normal metric). This implies [**2**] that G/K is rank one symmetric. Since G/K also admits a G-invariant Kähler structure, it must be $\mathbb{C}P^n$.

There remains the case when the singular orbit is a point, say p. In this case Lemma 2·1 still holds for $M \backslash \{p\}$ and Lemma 2·2 remains true. The limit point of any path of steepest descent for the function $\frac{1}{2}\|\mu_2\|^2$ is now p. We can still argue as in Section 2 that there is a local diffeomorphism Ψ from $M \backslash \{p\}$ to $M(0, 0, 0, Y)$, for some nilpotent $Y \in \mathfrak{g}^c$, and that this map preserves the Levi–Civita connection. In fact the existence of Ψ is now much simpler to show, since, for a nilpotent orbit, a solution $(T_1(t), T_2(t), T_3(t))$ to Nahm's equations is completely determined by $T_2(0) + iT_3(0)$ (there is only one moduli space of solutions corresponding to each nilpotent orbit).

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Taking the universal cover and using the de Rham decomposition we can conclude that a complete hyperkähler manifold of cohomogeneity one is either locally Euclidean or locally irreducible. Since G is semisimple, locally Euclidean actually implies Euclidean. We will show that M cannot be locally irreducible. If it were, Ψ could be assumed to be a local isometry (and a biholomorphism with repect to any complex structure). Using Proposition 5·5 of [**18**], we can conclude that $M\backslash\{p\}$ itself is isometric to a cone over a 3-Sasakian manifold. Such a metric can be completed by adding a point if and only if it is the flat metric on $\mathbb{R}^{4n}\backslash\{0\}$ [**5**]. Therefore, if the singular orbit is a point, then M is isometric to the flat $\mathbb{H}^n$.

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