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Furstenberg systems of pretentious and MRT multiplicative functions

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Abstract. We prove structural results for measure-preserving systems, called Furstenberg systems, naturally associated with bounded multiplicative functions. We show that for all pretentious multiplicative functions, these systems always have rational discrete spectrum and, as a consequence, zero entropy. We obtain several other refined structural and spectral results, one consequence of which is that the Archimedean characters are the only pretentious multiplicative functions that have Furstenberg systems with trivial rational spectrum, another is that a pretentious multiplicative function has ergodic Furstenberg systems if and only if it pretends to be a Dirichlet character, and a last one is that for any fixed pretentious multiplicative function, all its Furstenberg systems are isomorphic. We also study structural properties of Furstenberg systems of a class of multiplicative functions, introduced by Matomäki, Radziwiłł, and Tao, which lie in the intermediate zone between pretentiousness and strong aperiodicity. In a work of the last two authors and Gomilko, several examples of this class with exotic ergodic behavior were identified, and here we complement this study and discover some new unexpected phenomena. Lastly, we prove that Furstenberg systems of general bounded multiplicative functions have divisible spectrum. When these systems are obtained using logarithmic averages, we show that a trivial rational spectrum implies a strong dilation invariance property, called strong stationarity, but, quite surprisingly, this property fails when the systems are obtained using Cesàro averages.

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Contents

1 Introduction 2766
2 Main results 2770
3 Background 2784
4 General results for multiplicative functions 2793
5 Preliminary results for pretentious multiplicative functions 2799
6 Structural results for pretentious multiplicative functions 2806
7 Spectral results for pretentious multiplicative functions and applications 2813
8 Furstenberg systems of MRT functions—Cesàro averages 2823
9 Furstenberg systems of MRT functions—Logarithmic averages 2830
Acknowledgements 2835
A Appendix. Exponential sum estimates 2835
B Appendix. Factor of a product erg × id 2838
C Appendix. Ergodic consequences of a result of Klurman 2841
References 2843

1. *Introduction*

A function $f : \mathbb{N} \to \mathbb{U}$, where $\mathbb{U}$ is the complex unit disc, is called *multiplicative* if

$$f(mn) = f(m) \cdot f(n) \quad \text{whenever } (m, n) = 1.$$

It is called *completely multiplicative* if the previous equation holds for all $m, n \in \mathbb{N}$. (A completely multiplicative function is bounded if and only if it takes values in $\mathbb{U}$; in our discussion, whenever we state that a general multiplicative function is bounded, we mean that it takes values in $\mathbb{U}$.) In recent years, extensive effort has been put into the study of correlations and other statistical properties of bounded multiplicative functions, mostly motivated by problems surrounding the conjectures of Chowla [5], Elliott [11, 12], and Sarnak [40, 41]. An approach that, at times, offers advantages is to associate to the class of bounded multiplicative functions certain measure-preserving systems, called Furstenberg systems, that encode their statistical behavior. One then uses this new framework, in conjunction with the machinery of ergodic theory, to extract interesting and often highly non-trivial conclusions. This has led to some important successes, we refer the reader to the survey [13] for more details.

A structural result for Furstenberg systems for logarithmic averages of general bounded multiplicative functions was given in [16, 17]. It asserts, roughly speaking, that their ergodic components are direct products of systems with algebraic structure (inverse limits of nilsystems) and Bernoulli systems. Obtaining more refined structural results for special multiplicative functions, like the Liouville or the Möbius function, has turned out to be extremely challenging. Although the conjectures of Chowla and Elliott predict

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that Furstenberg systems of these multiplicative functions, and more general bounded multiplicative functions that satisfy strong aperiodicity assumptions (such as those in [37, equation (1.9)]), enjoy strong randomness properties, we are still far from being able to verify this (though see [33, 36–39, 43–46] for progress in this direction). To add to this mystery, recent work in [23] exhibited examples of bounded multiplicative functions that have rather erratic and unexpected statistical behavior, ranging from very structured but non-periodic to completely random, according to the scale of the intervals used to define their statistics.

In this article, we plan to focus on two classes of multiplicative functions that complement the notoriously difficult class of strongly aperiodic ones, and our goal is to obtain a rather complete understanding of their statistical properties by studying their Furstenberg systems. Let us briefly summarize some of our main results; the reader will find their exact statements in §2, and further background and explanations regarding our notation in §3.

Pretentious multiplicative functions. We first focus on the class of pretentious multiplicative functions (see Definition 3.1), which already exhibit interesting structural properties. The simplest examples are the Dirichlet characters, these are the periodic completely multiplicative functions, their non-zero values are always roots of unity, and their Furstenberg systems are rotations on finite cyclic groups. A particular example is given by the sequence

$$\chi_{3,1}(n) := \mathbf{1}_{3\mathbb{Z}+1}(n) - \mathbf{1}_{3\mathbb{Z}+2}(n), \quad n \in \mathbb{N},$$

with (a unique) Furstenberg system isomorphic to a rotation on $\mathbb{Z}/(3\mathbb{Z})$. Other simple examples of pretentious completely multiplicative functions can be obtained by assigning the value -1 to finitely many primes and 1 to the remaining primes. This gives rise to pretentious multiplicative functions that can be approximated in density by periodic sequences, and their Furstenberg systems are ergodic procyclic systems, that is, rotations on inverse limits of cyclic groups. Consider for example the completely multiplicative function defined by

$$f(p^k(pn+j)) := (-1)^k, \quad j = 1, \dots, p-1, \quad k, n \in \mathbb{Z}_+,$$

where p is some fixed prime. Its (unique) Furstenberg system is isomorphic to an ergodic rotation on the inverse limit of the cyclic groups $\mathbb{Z}/(p^s\mathbb{Z})$, $s \in \mathbb{N}$. Things become more complicated when we assign values different than one on infinitely many primes. If we define the completely multiplicative function by

$$f(p) := -1 \quad \text{for } p \in \mathbb{P}' \quad \text{such that} \quad \sum_{p \in \mathbb{P}'} \frac{1}{p} < +\infty,$$

and $f(p) := 1$ on $\mathbb{P} \setminus \mathbb{P}'$, then f turns out to always be pretentious, but the structural properties of its Furstenberg systems are less clear. Additionally, things become even more interesting when f is allowed to take values on the complex unit disc $\mathbb{U}$. For non-zero $t \in \mathbb{R}$,

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consider for example the pretentious completely multiplicative function (often called an *Archimedean character*), defined by

$$f(n) := n^{it}, \quad n \in \mathbb{N}.$$

Then f cannot be approximated in density by periodic sequences, it has uncountably many Furstenberg systems for Cesàro averages, but only one for logarithmic averages, and they are all isomorphic to the identity transformation on the circle with the Lebesgue measure. Another interesting example is given by defining f on the primes by (for $t \in \mathbb{R}$, we let $e(t) := e^{2\pi it}$)

$$f(p) := e(1/\log\log p), \quad p \in \mathbb{P},$$

which leads to a completely multiplicative function that pretends to be 1, but does not have convergent means. We can also get mixed behavior, with all previous aspects present, while still maintaining our pretentiousness assumption. For instance, this is the case when we define the completely multiplicative function on the primes by

$$f(p) := p^{it} \cdot \chi_{3,1}(p) \cdot e(1/\log\log p), \quad p \in \mathbb{P}.$$

For more examples, see §3.2.1. For general pretentious multiplicative functions, it is not at all clear if their Furstenberg systems always have rational discrete spectrum or even zero entropy and, in fact, examples of multiplicative functions in the MRT class (see Definition 2.3) seem to indicate otherwise. It is these and related, more refined questions that we answer in this article.

In Theorems 2.7, 2.8, we show that all Furstenberg systems of pretentious multiplicative functions $f: \mathbb{N} \to \mathbb{U}$, for Cesàro or logarithmic averages, have rational discrete spectrum, and any two Furstenberg systems of a fixed f are isomorphic. (This is not a property shared by general bounded multiplicative functions. For example, every element of the MRT class of multiplicative functions that is studied below has a wide variety of pairwise non-isomorphic Furstenberg systems, the structure of which differs sharply depending on whether we use Cesàro or logarithmic averages. This is well illustrated in Theorems 2.18, 2.19, 2.20.) Furthermore, we show that these systems are ergodic exactly when the multiplicative function pretends to be a Dirichlet character, in which case, the multiplicative function has good subsequential approximations in the Besicovitch norm by periodic sequences. Our results complement those in [**1**, **8**], where rational discrete spectrum was established for a restricted class of pretentious multiplicative functions, see Theorem 2.6 below. Theorem 2.11, together with Theorem 2.1, enable us in several cases to identify the spectrum of all Furstenberg systems of pretentious multiplicative functions, and we show that n^{it} is the only one that has trivial rational spectrum. Lastly, in Theorem 2.12, for completely multiplicative functions that pretend to be Dirichlet characters, we describe explicitly the spectrum of their Furstenberg systems and characterize them up to isomorphism as rotations on procyclic groups. A rather immediate consequence of the previous results is the *a priori* non-obvious fact that if a sequence satisfies the Sarnak or the Chowla–Elliott conjecture, then so does any of its multiples by a pretentious multiplicative function (see Theorems 2.15 and 2.16). Some other consequences, regarding existence and

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vanishing of correlations of pretentious multiplicative functions, are given in Theorems 2.13 and 2.14.

MRT multiplicative functions. We study a class of multiplicative functions that lie in the intermediate zone between pretentiousness and strong aperiodicity. They were introduced in [**37**, Appendix B] to give examples of non-pretentious multiplicative functions with non-vanishing 2-point correlations. They are constructed by imitating the function n^{it} on long intervals of consecutive primes, but using different values of t as the size of the interval grows (the explicit defining properties are given in Definition 2.3). This class was studied in [**23**], where Furstenberg systems with rather exotic and unexpected behavior were identified. In Theorem 2.17, we show that all MRT multiplicative functions are aperiodic (this was known only for a certain range of parameters). In Theorems 2.18–2.20, we give structural results for their Furstenberg systems that complement those in [**23**], covering a wider range of subsequential limits. This enables us to show in Theorem 2.19 that in the case of Cesàro averages, a trivial rational spectrum does not always imply a dilation invariance property known as strong stationarity, contrasting a result for logarithmic averages that holds for all bounded multiplicative functions (see Theorem 2.4). Moreover, although unipotent systems feature in our structural results both in the case of Cesàro and logarithmic averages, the exact structure in each case is sharply different—in the first case, we show in Theorems 2.18 and 2.19 that we get unipotent systems of fixed level, while in the second case, we get mixtures of infinitely many unipotent systems with an unbounded number of levels, see Theorem 2.20.

In the course of proving the previous results, we also establish some structural properties for general bounded multiplicative functions that are of independent interest.

General bounded multiplicative functions. We study spectral properties of Furstenberg systems of general multiplicative functions with values on the complex unit disc for Cesàro and logarithmic averages. In [**16**], it was shown that for logarithmic averages, the spectrum of these systems is a subset of the rationals, and here we give some additional information. In Theorem 2.1, we show that for completely multiplicative functions, the spectrum is a divisible subset of $\mathbb{T}$. These properties are used to identify the spectrum of pretentious completely multiplicative functions. Finally, in Theorem 2.4, we show that if a Furstenberg system for logarithmic averages of a bounded multiplicative function has trivial rational spectrum, then the system is necessarily strongly stationary (see Definition 3.10), a property that has very strong structural consequences, some of which are recorded in Corollary 2.5. As stated before, the MRT class provides examples where this property fails for Furstenberg systems of multiplicative functions defined using Cesàro averages.

1.1. *Notation.* We let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{S}^1$ be the unit circle, and $\mathbb{U}$ be the closed complex unit disc. With $\mathbb{P}$, we denote the set of prime numbers.

With $\mathbb{T}$, we denote the one-dimensional torus $\mathbb{R}/\mathbb{Z}$, and we often identify it with $[0, 1)$. We also often denote elements of $\mathbb{T}$ with real numbers and we are implicitly assuming that these real numbers are taken modulo 1.

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For $t \in \mathbb{R}$, we let $e(t) := e^{2\pi it}$, except in §§8 and 9 (and the related Appendix A), where it is more convenient for us to let $e(t) := e^{it}$.

For $z \in \mathbb{C}$, we denote the real part of z by $\Re(z)$.

For $N \in \mathbb{N}$, we let $[N] := \{1, \dots, N\}$ and if $M \in [1, +\infty)$, we let $[M] = \{1, \dots, \lfloor M \rfloor\}$.

We usually denote sequences on $\mathbb{N}$ or on $\mathbb{Z}$ by $(a(n))$, instead of $(a(n))_{n\in\mathbb{N}}$ or $(a(n))_{n\in\mathbb{Z}}$; the domain of the sequence is going to be clear from the context. Whenever we write (N_k), we assume that N_k is a strictly increasing sequence of positive integers.

If A is a finite non-empty subset of the integers and $a\colon A \to \mathbb{C}$, we let

$$\mathbb{E}_{n\in A}\, a(n) := \frac{1}{|A|}\sum_{n\in A} a(n), \quad \mathbb{E}^{\log}_{n\in A}\, a(n) := \frac{1}{\sum_{n\in A} 1/n}\sum_{n\in A}\frac{a(n)}{n}.$$

Given $a, b\colon \mathbb{N} \to \mathbb{C}$, we write $a(n) \prec b(n)$ if $\lim_{n\to\infty} a(n)/b(n) = 0$.

Throughout the article, the letter f is typically used for multiplicative functions and the letter χ for Dirichlet characters.

2. *Main results*

In this section, we give precise statements of our main results. To ease the exposition, we refer the reader to §3 for the definitions of various notions used in the statements.

2.1. *Spectral results for general completely multiplicative functions.*

We start with some results about Furstenberg systems of general completely multiplicative functions. These results are of independent interest, but they will also be used subsequently to deduce results for pretentious multiplicative functions and to contrast results obtained for certain MRT multiplicative functions.

2.1.1. *Divisibility properties of the spectrum.*

In this subsection, we discuss divisibility properties of the spectrum (see Definition 3.3) of Furstenberg systems of bounded completely multiplicative functions. Note that in all cases, we get stronger results when the Furstenberg systems are defined using logarithmic averages versus Cesàro averages, and it is not clear if equally strong results can be obtained for Cesàro averages (see a related question in §2.5).

THEOREM 2.1. *Let (X, μ, T) be a Furstenberg system of a completely multiplicative function $f\colon \mathbb{N} \to \mathbb{U}$. Let also $\alpha \in \mathrm{Spec}(X, \mu, T)$ and $r \in \mathbb{N}$ such that $f(r) \neq 0$.*

(i) *(Logarithmic averages) If the Furstenberg system is defined using logarithmic averages, then $(\alpha + k)/r \in \mathrm{Spec}(X, \mu, T)$ for some $k \in \{0, \dots, r-1\}$.*

(ii) *(Cesàro averages) If the Furstenberg system is defined using Cesàro averages, then $(\alpha + k)/r \in \mathrm{Spec}(X, \mu_r, T)$ for some $k \in \{0, \dots, r-1\}$, where (X, μ_r, T) is another Furstenberg system of f for Cesàro averages. If (X, μ, T) is ergodic, then we can take $\mu_r = \mu$.*

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Remarks.

- The result fails if we assume multiplicativity and not complete multiplicativity. Take for example $f(n) := (-1)^{n+1}$, $n \in \mathbb{N}$, and $\alpha = 1/2$, $r = 2$, or $f := \mathbf{1}_{\mathbb{Z}\setminus 3\mathbb{Z}} - \mathbf{1}_{3\mathbb{Z}}$, and $\alpha = 1/3$, $r = 3$. Also, in the completely multiplicative case, the result fails if we allow $f(r)$ to take the value 0, consider for example the case of Dirichlet characters.
- We caution the reader that if $\alpha = 1/p$, $p \in \mathbb{P}$, and $f(r) \neq 0$, then the previous result is not going to give us additional values in the spectrum unless p divides r, and it will give us additional values if $r = p$ (see Corollary 2.3).
- If $\mu = \lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, \delta_{T^n f}$, then in part (ii), we can take $\mu_r := \lim_{k\to\infty} \mathbb{E}_{n\in[rN'_k]}\, \delta_{T^n f}$ for any subsequence N'_k of N_k for which the previous weak-star limit exists, where we think of f as a point in $\mathbb{U}^{\mathbb{Z}}$.

Theorem 2.1 is proved in §4.1. The key idea is to study the action of the maps τ_r, defined in equation (10), on eigenfunctions of the system, and show that in the case of Furstenberg systems of completely multiplicative functions, non-zero functions are mapped to non-zero functions by τ_r. This is a consequence of Lemma 4.2, which we combine with Lemma 4.1 to prove Theorem 2.1.

Definition 2.1. A subset A of $\mathbb{T}$ is called *divisible* if for every $\alpha \in A$ and $r \in \mathbb{N}$, there exists $\alpha' \in A$ such that $\alpha = r\alpha'$.

For example, the sets $\{m/2^k : m = 0, \dots, 2^k - 1, k \in \mathbb{N}\}$ and $\mathbb{Q} \cap [0, 1)$ are divisible, and any non-trivial divisible subset of $\mathbb{T}$ has to be infinite.

The following is an immediate consequence of the previous result.

COROLLARY 2.2. *Let $f : \mathbb{N} \to \mathbb{U} \setminus \{0\}$ be a completely multiplicative function.*

(i) *(Logarithmic averages) The spectrum of any Furstenberg system of f for logarithmic averages is a divisible subset of $\mathbb{T}$.*

(ii) *(Cesàro averages) The combined spectrum of all Furstenberg systems of f for Cesàro averages is a divisible subset of $\mathbb{T}$. The same property holds for the spectrum of any fixed Furstenberg system of f as long as the system is ergodic.*

Remark. If f is completely multiplicative but is allowed to take the value 0, then the result fails; consider for example f to be a non-trivial Dirichlet character. The result also fails if f is not allowed to take the value 0 but is only assumed to be multiplicative, see the examples given in the first remark after Theorem 2.1.

Note that if $\alpha = 0$ or $\alpha = m/n$ and $(r, n) = 1$, then the content of Theorem 2.1 is empty. However, if $\alpha = 1/p$ and $r = p^s$ for some $s \in \mathbb{N}$, then we do get non-trivial consequences as the next result shows.

COROLLARY 2.3. *Let (X, μ, T) be a Furstenberg system of a completely multiplicative function $f : \mathbb{N} \to \mathbb{U}$ and suppose that $1/p \in \mathrm{Spec}(X, \mu, T)$ for some $p \in \mathbb{P}$ with $f(p) \neq 0$.*

(i) *(Logarithmic averages) If the Furstenberg system is defined using logarithmic averages, then $q/p^s \in \mathrm{Spec}(X, \mu, T)$ for every $s \in \mathbb{N}$, $q \in \{0, \dots, p^s - 1\}$.*

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(ii) *(Cesàro averages) If the Furstenberg system is defined using Cesàro averages, then* $q/p^s \in \mathrm{Spec}(X, \mu_{p,s}, T)$ *for every* $s \in \mathbb{N}$, $q \in \{0, \dots, p^s - 1\}$, *where* $(X, \mu_{p,s}, T)$ *is another Furstenberg system of f for Cesàro averages. If* (X, μ, T) *is ergodic, then we can take* $\mu_{p,s} = \mu$ *for every* $p \in \mathbb{P}$ *and* $s \in \mathbb{N}$.

Remark. As a consequence, if the rational spectrum of a Furstenberg system of a completely multiplicative function $f : \mathbb{N} \to \{-1, 1\}$ is non-trivial, then it is infinitely generated. This is not the case for general multiplicative functions $f : \mathbb{N} \to \{-1, 1\}$, see the examples in the first remark after Theorem 2.1—they are given by non-constant periodic sequences, and hence their Furstenberg systems have non-trivial finite spectrum (hence are non-divisible).

Proof. We prove the first part. Let $s \in \mathbb{N}$. Since $f(p) \neq 0$, applying part (i) of Theorem 2.1 for $r := p^s$, we deduce that $(1 + kp)/p^{s+1} \in \mathrm{Spec}(X, \mu, T)$ for some $k \in \{0, \dots, p^s - 1\}$. Since $(1 + kp, p^{s+1}) = 1$ and the spectrum is closed under multiplication by an integer, it follows that q/p^s is in the spectrum for all $q \in \{0, \dots, p^s - 1\}$. Since $s \in \mathbb{N}$ is arbitrary, we get the asserted statement.

The second part follows by arguing as in the first part and applying part (ii) of Theorem 2.1. □

Further consequences of these results for the class of pretentious multiplicative functions will be given in Theorems 2.11 and 2.12 below.

2.1.2. *Trivial rational spectrum implies strong stationarity.* It turns out that for Furstenberg systems of bounded multiplicative functions defined using logarithmic averages, a trivial rational spectrum has very strong structural consequences. A notable one is strong stationarity, the dilation invariance property described in Definition 3.10, which also played a crucial role in the description of Furstenberg systems of general bounded multiplicative functions in [**16**].

THEOREM 2.4. *If* (X, μ, T) *is a Furstenberg system for logarithmic averages of a multiplicative function* $f : \mathbb{N} \to \mathbb{U}$, *then the following two conditions are equivalent:*

(i) *the system has trivial rational spectrum;*
(ii) *the system is strongly stationary.*

Remarks.

- The implication (ii) $\Longrightarrow$ (i) follows from [**28**] and holds for general strongly stationary systems. So the more interesting implication is (i) $\Longrightarrow$ (ii), and this makes use of the fact that f is multiplicative.
- Quite surprisingly, as the examples of Theorem 2.19 show, the result is no longer true when the Furstenberg systems are defined using Cesàro averages.

Theorem 2.4 is proved in §4.2 and uses the ergodic limit formulae stated in Theorem 4.3, and a recent result about correlations of multiplicative functions of Tao and Teräväinen [**45**] stated in Theorem 4.5.

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Using the previous result and known results about the structure of strongly stationary systems (see [14, Main Theorem] for part (i) and [16, Proposition 3.12(ii)] for part (ii)), we get the following structural result for Furstenberg systems of bounded multiplicative functions that have trivial rational spectrum.

COROLLARY 2.5. *Let $f: \mathbb{N} \to \mathbb{U}$ be a multiplicative function and (X, μ, T) be a Furstenberg system for logarithmic averages that has trivial rational spectrum. Then:*

(i) *the system (X, μ, T) has trivial spectrum, is strongly stationary, and almost every ergodic component is isomorphic to a direct product of a Bernoulli system and an inverse limit of nilsystems;*

(ii) *the system (X, μ, T) is disjoint from all ergodic systems with zero entropy.*

The bulk of this result can also be deduced from [16, Proposition 3.12(ii)] and [17, Theorem 1.5].

The Liouville function is probably the most noteworthy example of a multiplicative function for which it is not known whether all its Furstenberg systems have trivial rational spectrum (the Chowla conjecture predicts that this is indeed the case), that is, it is not known whether any rational in (0, 1) is on the spectrum of any of its Furstenberg systems for Cesàro or logarithmic averages.

2.2. *Structural results for pretentious multiplicative functions.* Our goal in this section is to give a detailed description of the Furstenberg systems of pretentious multiplicative functions. See Definition 3.1 for the definition of pretentiousness and related background, and §3.2.1 for various motivating examples.

2.2.1. *A known structural result.* We start with a known result that gives substantial information for a rich class of pretentious multiplicative functions. It follows by combining [8, Theorem 6] and [1, Theorem 1.7].

THEOREM 2.6. [1, 8] *Let $f: \mathbb{N} \to \mathbb{U}$ be a multiplicative function and suppose that there exists a Dirichlet character χ such that the series*

$$\sum_{p \in \mathbb{P}} \frac{1}{p}(1 - f(p) \cdot \overline{\chi(p)}) \quad \text{converges.} \tag{1}$$

Then, f is Besicovitch rationally almost periodic for Cesàro averages, it has a unique Furstenberg system with respect to Cesàro averages (hence also for logarithmic), and this unique system is isomorphic to an ergodic procyclic system.

Remarks.

- Note that condition (1) is stronger than saying that f pretends to be χ (see Definition 3.1), for $\chi = 1$, see Example (viii) in §3.2.1.
- The reader will find results that extend various aspects of Theorem 2.6 to all multiplicative functions that pretend to be Dirichlet characters in Theorem 2.8 and Corollary 2.9.

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When f pretends to be 1 and condition (1) is not satisfied (this is the case in Example (viii) of §3.2.1), then it follows from [8, Corollary 2] that the mean value of f on some arithmetic progression does not exist and, as a consequence, f is not Besicovitch rationally almost periodic. Prior to our work, for such multiplicative functions, it was not clear what Furstenberg systems may arise, and it seemed plausible that they do not all have rational discrete spectrum. We show in Theorem 2.7 that this is not the case and in Theorem 2.8, we give more refined information about their structure.

2.2.2. *New structural results.* Our first main result applies to all pretentious multiplicative functions and shows that their Furstenberg systems have rational discrete spectrum. It extends Theorem 2.6, which covers multiplicative functions that satisfy condition (1).

THEOREM 2.7. *All Furstenberg systems of pretentious multiplicative functions for Cesàro or logarithmic averages have rational discrete spectrum. As a consequence, they have zero entropy and they do not have irrational spectrum.*

Remarks.

- The collection of Furstenberg systems of a complex valued pretentious multiplicative function may depend on whether we use Cesàro or logarithmic averages. This is the case, for example, when $f(n) = n^{it}$, $t \neq 0$, see the discussion in Example (vi) of §3.2.1. It is a non-trivial fact though that when f pretends to be a Dirichlet character, its Furstenberg systems for Cesàro and logarithmic averages coincide, see part (iv) of Theorem 2.8 below.
- Establishing that all Furstenberg systems of pretentious multiplicative functions have zero entropy is a non-trivial task on its own. In fact, prior to our work, it seemed plausible that some MRT functions (see Definition 2.3) were pretentious, in which case, we would get using [23, Corollary 2.10] that some pretentious multiplicative function has a Bernoulli Furstenberg system, and hence is of positive entropy.
- It seems likely that this result can also be obtained by studying the formulas for the 2-point correlations given by Klurman in [31]. We opted to take a different approach to also get the more refined properties stated in Theorem 2.8.
- It is a rather straightforward consequence of results of Klurman in [31] that if $f : \mathbb{N} \to \mathbb{U}$ is a non-trivial aperiodic multiplicative function, then some Furstenberg system of f for logarithmic averages has a Lebesgue component (see Corollary C.2 in the appendix), and as a consequence, does not have rational discrete spectrum. Hence, a multiplicative function that takes values in $\mathbb{U}$ is pretentious if and only if all its Furstenberg systems for logarithmic averages have rational discrete spectrum. It follows from [33, Theorem 1.2] that a similar equivalence also holds for Cesàro averages.

Our main result for pretentious multiplicative functions is stated next and gives more refined structural information about their Furstenberg systems. In what follows, when we write $f \sim g$, we mean $\mathbb{D}(f, g) < +\infty$ (see Definition 3.1).

THEOREM 2.8. *Let $f : \mathbb{N} \to \mathbb{U}$ be a pretentious multiplicative function and (X, μ, T) be a Furstenberg system of f for Cesàro or logarithmic averages.*

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(i) *If $f \sim \chi$ for some primitive Dirichlet character χ, then (X, μ, T) is an ergodic procyclic system and it is non-trivial if $f \neq 1$.*

(ii) *If $f \sim n^{it} \cdot \chi$ for some $t \neq 0$, then (X, μ, T) is a non-ergodic system with rational discrete spectrum and its spectrum is non-trivial unless $f(n) = n^{it}$, $n \in \mathbb{N}$, for some $t \in \mathbb{R}$. Furthermore, (X, μ, T) is isomorphic to the direct product of the system $(\mathbb{T}, m_{\mathbb{T}}, \mathrm{id})$ and some Furstenberg system of $\tilde{f} := f \cdot n^{-it} \sim \chi$.*

(iii) *In cases (i) and (ii), any two Furstenberg systems of f for Cesàro or logarithmic averages are isomorphic.*

(iv) *In case (i), a Furstenberg system of f for Cesàro averages along (N_k) is well defined if and only if it is well defined for logarithmic averages, and the two Furstenberg systems are equal (that is, the corresponding T-invariant measures coincide).*

(v) *In case (i), every sequence $N_k \to \infty$ has a subsequence (N_k') along which f is Besicovitch rationally almost periodic for Cesàro and logarithmic averages. In case (ii), there is no sequence $N_k \to \infty$ along which f is Besicovitch rationally almost periodic for Cesàro or logarithmic averages.*

Remarks.

- Note that a pretentious multiplicative function belongs to exactly one of the classes treated in cases (i) and (ii).
- See Proposition 5.6 for additional information regarding a variant of part (v) that does not require to pass to a subsequence.

Section 6 is devoted to the proof of Theorem 2.8 (Theorem 2.7 is an immediate consequence) and the proof is completed in §6.3. The argument uses several ingredients: we start in §5 with some preliminary work that leads to the decomposition result stated in Lemma 5.4, which in turn implies the subsequential Besicovitch rational almost periodicity property stated in Proposition 5.5. This basic tool is then exploited in §6 and together with several arguments of ergodic flavor, leads to the proof of Theorem 2.8.

We deduce from Theorem 2.8 some equivalent characterizations of various classes of pretentious multiplicative functions. The first one concerns multiplicative functions that pretend to be Dirichlet characters.

COROLLARY 2.9. *Let $f : \mathbb{N} \to \mathbb{U}$ be a pretentious multiplicative function. Then, the following properties are equivalent:*

(i) *$f \sim \chi$ for some primitive Dirichlet character χ;*

(ii) *some Furstenberg system of f for Cesàro or logarithmic averages is ergodic;*

(iii) *all Furstenberg systems of f for Cesàro and logarithmic averages are ergodic procyclic systems;*

(iv) *there exists a sequence $N_k \to \infty$ along which f is Besicovitch rationally almost periodic for Cesàro or logarithmic averages;*

(v) *every sequence $N_k \to \infty$ has a subsequence along which f is Besicovitch rationally almost periodic for Cesàro and logarithmic averages.*

Corollary 2.9 is proved in §6.4.

Our second corollary concerns multiplicative functions that are equal to Archimedean characters.

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COROLLARY 2.10. *Let $f:\mathbb{N}\to\mathbb{U}$ be a pretentious multiplicative function. Then, the following properties are equivalent:*

(i) $f(n)=n^{it}$, $n\in\mathbb{N}$, *for some* $t\in\mathbb{R}$;
(ii) *at least one Furstenberg system of f for Cesàro or logarithmic averages has trivial rational spectrum;*
(iii) *all Furstenberg systems of f for Cesàro or logarithmic averages have trivial rational spectrum;*
(iv) *all Furstenberg systems of f for Cesàro or logarithmic averages are identity systems;*
(v) $\lim_{N\to\infty}\mathbb{E}_{n\in[N]}|f(n+1)-f(n)|=0$.

Remark. The equivalence of properties (i) and (v) was established in [31, Theorem 1.8] in a stronger form that does not assume pretentiousness. We give a different argument for this equivalence, with an ergodic flavor, but it only works in the pretentious case. The equivalence of properties (iv) and (v) was established in [23, Proposition 5.1]. So our original contribution to this corollary is the insertion of properties (ii) and (iii) on this set of equivalences.

Corollary 2.10 is proved in §6.5.

2.2.3. *Spectral results.* We give a result that, in conjunction with Theorem 2.1, helps us identify the spectrum of Furstenberg systems of pretentious multiplicative functions.

THEOREM 2.11. *Let $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function that satisfies $f\sim n^{it}\cdot\chi$ for some $t\in\mathbb{R}$ and primitive Dirichlet character χ with conductor q, and (X,μ,T) be a Furstenberg system of f for Cesàro or logarithmic averages. Then, for every $p\in\mathbb{P}$, the following properties are equivalent:*

(i) $1/p\in\operatorname{Spec}(X,\mu,T)$;
(ii) *either* $p\mid q$ *or* $f(p^s)\neq p^{ist}\cdot\chi(p^s)$ *for some* $s\in\mathbb{N}$.

Remarks.

- If $f\sim 1$ and $s\geq 2$, then $f(p^s)\neq 1$ does not always imply $1/p^s\in\operatorname{Spec}(X,\mu,T)$. Take for example $f(n):=\mathbf{1}_{\mathbb{Z}\setminus 3\mathbb{Z}}-\mathbf{1}_{3\mathbb{Z}}$, $n\in\mathbb{N}$. Then, $f(3^2)\neq 1$ and $1/3^2\notin\operatorname{Spec}(X,\mu,T)$.
- If f is completely multiplicative, $f\sim 1$, and $f(p)\notin\{0,1\}$, then combining this result with Theorems 2.8 and 2.1, we get that $q/p^s\in\operatorname{Spec}(X,\mu,T)$ for every $s\in\mathbb{N}$ and $q\in\{0,\dots,p^s-1\}$. However, if $f(p)=1$, then $1/p\notin\operatorname{Spec}(X,\mu,T)$.

The implication (i) $\Longrightarrow$ (ii) of Theorem 2.11 is proved in §7.3 and uses the periodic approximation property of Proposition 5.5 to deduce the result for f from its periodic approximants. The implication (ii) $\Longrightarrow$ (i) is proved in §7.4. It is somewhat more involved and uses part (i) of Theorem 2.8, and a combination of elementary and ergodic considerations.

Our next result provides more refined spectral information for multiplicative functions that pretend to be Dirichlet characters and a complete characterization of the spectrum of their Furstenberg systems. We will use it to justify various claims we make in the examples

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given in §3.2.1. We assume complete multiplicativity to have Theorem 2.1 available for us, and explain in the remarks which consequences carry over to general multiplicative functions.

THEOREM 2.12. *Let $f: \mathbb{N} \to \mathbb{U} \setminus \{0\}$ be a completely multiplicative function with $f \sim \chi$ for some primitive Dirichlet character χ with conductor q and (X, μ, T) be a Furstenberg system for Cesàro or logarithmic averages of f. Then,* $\mathrm{Spec}(X, \mu, T)$ *is equal to the subgroup Λ of $\mathbb{T}$ generated by $\{1/p^s : p \in A, s \in \mathbb{N}\}$, where $A := \{p \in \mathbb{P}:$ either $p \mid q$ or $f(p) \neq \chi(p)\}$.*

Remarks.

- Since ergodic discrete spectrum systems are isomorphic if and only if they have the same spectrum, we deduce from part (i) of Theorem 2.8 and the previous result that if f is as in the statement, then all Furstenberg systems of f are isomorphic to an ergodic rotation on the procyclic group that is the dual group of the subgroup Λ defined in the statement above.
- The conclusion fails for non-completely multiplicative functions, even if they take values on ± 1; see the example in the first remark following Theorem 2.11. Another example is given by the square of the Möbius function (see Example (v) in §3.2.1). The conclusion also fails if f is completely multiplicative, but we allow it to take the value 0, consider for example $f := \mathbf{1}_{2\mathbb{Z}+1}$, which has $1/2$ but not $1/4$ on the spectrum of its Furstenberg system.
- See the claim in §7.5 for a variant of the inclusion $\mathrm{Spec}(X, \mu, T) \subset \Lambda$, which holds for all multiplicative functions that pretend to be Dirichlet characters. Also, our argument gives that even without complete multiplicativity, $\mathrm{Spec}(X, \mu, T)$ contains the subgroup Λ of $\mathbb{T}$ generated by $\{1/p : p \in A\}$, where $A := \{p \in \mathbb{P}:$ either $p \mid q$ or $f(p^s) \neq \chi(p^s)$ for some $s \in \mathbb{N}\}$.
- Using part (ii) of Theorem 2.8 and the previous result, we can explicitly identify the structure of the Furstenberg systems of any pretentious completely multiplicative function that avoids the value 0.

We prove Theorem 2.12 in §7.5. We essentially use the divisibility properties of the spectrum given in §2.1.1, Proposition 6.1, which asserts that in this case, Cesàro and logarithmic averages can be used interchangeably, and part (i) of Theorem 2.8, which asserts the ergodicity of the corresponding Furstenberg systems.

2.3. *Applications of the structural results.* We give some number theoretic consequences of our main results.

2.3.1. *Correlations of pretentious multiplicative functions.* We start with a vanishing property for weighted correlations of pretentious multiplicative functions.

THEOREM 2.13. *Let $f: \mathbb{N} \to \mathbb{U}$ be a multiplicative function with $f \sim \chi$ for some primitive Dirichlet character χ with conductor q. Suppose that either:*

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(i) α *is irrational; or*
(ii) $\alpha = k/p$, $p \in \mathbb{P}$, $(k, p) = 1$, $p \nmid q$, *and* $f(p^s) = \chi(p^s)$ *for every* $s \in \mathbb{N}$.
Then,

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, e(n\alpha) \prod_{j=1}^{\ell} f_j(n + n_j) = 0 \tag{2}$$

for all $n_1, \dots, n_\ell \in \mathbb{Z}$ *and* $f_1, \dots, f_\ell \in \{f, \overline{f}\}$.

Remarks.

- If $f \sim n^{it} \cdot \chi$ for some $t \in \mathbb{R}$ and primitive Dirichlet character χ, then a similar result holds as long as we change the assumption on f in property (ii) to $f(p^s) = p^{ist} \cdot \chi(p^s)$ for every $s \in \mathbb{N}$.
- If α is irrational, and we replace Cesàro with logarithmic averages, then equation (2) holds for all bounded multiplicative functions f (see [17, Corollary 1.4]). It is an open problem whether, in this more general setting, a similar property holds for Cesàro averages—this is only known for $\ell = 1$ [6, 7].

Theorem 2.13 is proved in §7.6.

Our next result establishes existence of certain correlations of pretentious multiplicative functions.

THEOREM 2.14. *Let* $f : \mathbb{N} \to \mathbb{U}$ *be a pretentious multiplicative function. If* $k_1, \dots, k_\ell$ *are integers such that* $\sum_{j=1}^{\ell} k_j = 0$, *then the limit*

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \prod_{j=1}^{\ell} f^{k_j}(n + n_j) \tag{3}$$

exists for all $n_1, \dots, n_\ell \in \mathbb{Z}$, *where we use the notation* $f^k := \overline{f}^{|k|}$ *for* $k < 0$.

Remarks.

- The result fails if we do not assume that $\sum_{j=1}^{\ell} k_j = 0$, take $\ell = 1$, and f any pretentious multiplicative function that does not have a mean value. It also fails if we do not assume that f is pretentious: as was shown in [37, Theorem B.1], there exist aperiodic multiplicative functions f (in fact, all multiplicative functions in the MRT class should work), for which $\limsup_{N\to\infty} |\mathbb{E}_{n\in[N]}\, \overline{f(n)} \cdot f(n+1)| > 0$ and in [23], it was shown that $\liminf_{N\to\infty} |\mathbb{E}_{n\in[N]}\, \overline{f(n)} \cdot f(n+1)| = 0$.
- The result was known for $\ell = 2$ and in this case, an explicit formula for the correlations is given in [31, Theorem 1.5].

Theorem 2.14 is proved in §7.7.

2.3.2. *Chowla and Sarnak conjecture.* To facilitate discussion, we introduce the following notions and refer the reader to Definition 3.9 for the notion of completely deterministic sequences.

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Definition 2.2. We say that a sequence $a: \mathbb{N} \to \mathbb{U}$:

(i) *satisfies the Sarnak conjecture for Cesàro averages along* (N_k) if

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, a(n)\, w(n) = 0$$

for all sequences $w: \mathbb{N} \to \mathbb{U}$ that are completely deterministic along (N_k) (see [9] for the relation of this statement with the original conjecture of Sarnak [41]);

(ii) *satisfies the Chowla–Elliott conjecture for Cesàro averages along* (N_k) if

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, a^{\epsilon_1}(n+n_1)\cdots a^{\epsilon_\ell}(n+n_\ell) = 0 \tag{4}$$

for all $\ell \in \mathbb{N}$, distinct $n_1, \dots, n_\ell \in \mathbb{Z}_+$, and all $\epsilon_1, \dots, \epsilon_\ell \in \{-1, 1\}$, where we set $a^{-1} := \overline{a}$.

Similar definitions apply for logarithmic averages.

We give two results, which, roughly speaking, assert that if a bounded sequence satisfies the Sarnak or the Chowla–Elliot conjecture, then so does any multiple of the sequence by a pretentious multiplicative function. We caution the reader that although these claims are rather easy to prove when the multiplicative function satisfies Theorem 2.6, *a priori*, it is not even clear that such claims are expected to hold for general pretentious multiplicative functions. In fact, since all MRT functions satisfy the Sarnak and the Chowla–Elliott conjectures along some subsequence (N_k) (this is a consequence of the main result in [23]), one has to first nullify the possibility that the MRT class contains a pretentious multiplicative function, which is a non-trivial task on its own.

THEOREM 2.15. *Let $a: \mathbb{N} \to \mathbb{U}$ be a sequence that satisfies the Sarnak conjecture for Cesàro averages along (N_k), and $b := a \cdot f$, where $f: \mathbb{N} \to \mathbb{U}$ is a pretentious multiplicative function. Then, b also satisfies the Sarnak conjecture for Cesàro along (N_k). A similar fact also holds for logarithmic averages.*

We prove Theorem 2.15 in §7.8 by making essential use of Theorem 2.7.

THEOREM 2.16. *Let $a: \mathbb{N} \to \{-1, 1\}$ be a sequence that satisfies the Chowla–Elliott conjecture for Cesàro averages along (N_k), and $b := a \cdot f$, where $f: \mathbb{N} \to \mathbb{U}$ is a pretentious multiplicative function. Then, b also satisfies the Chowla–Elliott conjecture for Cesàro averages along (N_k). A similar fact also holds for logarithmic averages.*

We prove Theorem 2.16 in §7.9 by making essential use of Theorem 2.7.

2.4. *Structural results about MRT multiplicative functions.* We focus here on a class of multiplicative functions defined in [37, Appendix B], and which were used to show that the 2-point Elliott conjecture fails for some aperiodic multiplicative functions. It turns out that this class provides a rich playground on which one can construct examples of multiplicative functions with rather exotic statistical behavior, resulting in Furstenberg systems with unexpected structural properties, not to be found within the class of pretentious or strongly aperiodic multiplicative functions (such as those satisfying [37, equation (1.9)]). Our goal here is to give an explicit description of their Furstenberg systems for Cesàro and

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logarithmic averages for a wider range of parameters than those dealt with in [23]. As a consequence, we get further unanticipated properties for such Furstenberg systems, only to be found on this class of multiplicative functions with 'intermediate randomness' properties.

2.4.1. *The setting.* We reproduce here the definition of the MRT class of completely multiplicative functions from [23, Definition 3.1].

Definition 2.3. We say that the completely multiplicative function $f : \mathbb{N} \to \mathbb{S}^1$ belongs to the *MRT class*, or is an *MRT (multiplicative) function* if the following property is satisfied. There exist strictly increasing sequences of integers (t_m), $t_1 := 1$, and (s_m) such that for each $m \in \mathbb{N}$, the following holds:

(i) $t_m < s_{m+1} < s_{m+1}^2 \le t_{m+1}$;
(ii) $f(p) = p^{is_{m+1}}$ for each prime $p \in (t_m, t_{m+1}]$;
(iii) $|f(p) - p^{is_{m+1}}| \le 1/t_m^2$ for each prime $1 < p \le t_m$.

It is shown in [37, Appendix B] that the MRT class is non-empty, and if $s_{m+1} > e^{t_m}$, then f is an aperiodic multiplicative function. However, this is not a growth assumption that we impose in our defining axioms, nor is it known whether it follows from these axioms. However, it is shown in [23] that the axioms (i)–(iii) imply that for every $K \in \mathbb{N}$, we have $s_{m+1}/t_m^K \to \infty$.

It is a consequence of [23] that every element of the MRT class has uncountably many different Furstenberg systems, and depending on the choice of our averaging intervals $[N_m]$, we may get substantially different structural properties, ranging from the identity system to the Bernoulli system. Our goal here is to give a complete classification of these Furstenberg systems for Cesàro and logarithmic averages, when, roughly speaking, N_m grows as a fractional power of s_{m+1} (that is, $\lim_{m\to\infty} N_m/s_{m+1}^{\beta} = c$ for some $\beta, c > 0$).

We stress that in the structural results that we give below, we fix an arbitrary MRT function f and describe the structure of different Furstenberg systems of this fixed f for Cesàro or logarithmic averages, depending on the choice of the sequence $N_m \to \infty$ that defines the Furstenberg system.

The following is a first non-trivial consequence of our results.

THEOREM 2.17. *All MRT multiplicative functions are aperiodic (non-pretentious).*

This follows at once from Theorem 2.18 (or the structural result obtained in [23]) and Theorem 2.7, since for every MRT function, not all of its Furstenberg systems for Cesàro averages have rational discrete spectrum.

2.4.2. *Furstenberg systems of MRT functions for Cesàro averages.* We first state structural results for Furstenberg systems of MRT functions defined using Cesàro averages. In the next subsection, we cover the case of logarithmic averages, which lead to systems with substantially different structural properties.

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Definition 2.4. For $d \in \mathbb{Z}_+$, we consider the measure-preserving system (Y_d, ν_d, S_d), where $Y_d := \mathbb{T}^{d+1}$, $\nu_d := m_{\mathbb{T}^{d+1}}$, and $S_d \colon \mathbb{T}^{d+1} \to \mathbb{T}^{d+1}$ is the transformation defined by

$$S_d(x_0, \dots, x_d) := (x_0, x_1 + x_0, \dots, x_d + x_{d-1}), \quad x_0, \dots, x_d \in \mathbb{T}.$$

We call it the *level d unipotent system*.

For $d = 0$, we have $S_0(x_0) = x_0$ on $\mathbb{T}$ with $m_{\mathbb{T}}$. For $d = 1$, we have $S_1(x_0, x_1) = (x_0, x_1 + x_0)$ on $\mathbb{T}^2$ with $m_{\mathbb{T}^2}$, and so on. We introduce these systems to reproduce in an ergodic setting the correlations of MRT functions given in Proposition 8.4.

Definition 2.5. Given $a, b \colon \mathbb{N} \to \mathbb{C}$, we write $a(n) \prec b(n)$ if $\lim_{n\to\infty} a(n)/b(n) = 0$.

We start with a result previously obtained in [**23**] when $N_m = \lfloor s_{m+1}^{1/c} \rfloor$ and $c > 0$ is not an integer (the case $c \in \mathbb{N}$ is covered by Theorem 2.19). We use somewhat different techniques to cover the more general case below.

THEOREM 2.18. *For $d \in \mathbb{Z}_+$, let (X, μ_d, T) be the Furstenberg system for Cesàro averages of the MRT function $f \colon \mathbb{N} \to \mathbb{S}^1$, taken along the sequence of intervals $[N_m]$ that satisfies $s_{m+1}^{1/(d+1)} \prec N_m \prec s_{m+1}^{1/d}$ (for $d = 0$, we simply assume that $s_{m+1} \prec N_m \le t_{m+1}$). Then, (X, μ_d, T) has trivial spectrum, is strongly stationary, and is isomorphic to the level d unipotent system $(\mathbb{T}^{d+1}, m_{\mathbb{T}^{d+1}}, S_d)$ given in Definition 2.4.*

Theorem 2.18 is proved in §8.3.

Our method of proof is somewhat different than that used in [**23**], the additional flexibility of our method enables us to give more refined results and also handle subsequent problems concerning logarithmic averages. We continue with a case that exhibits different structural behavior and gives examples of non-strongly stationary systems, a feature not present in the case of logarithmic averages.

Definition 2.6. For $\alpha > 0$ and $d \in \mathbb{Z}_+$, let $S_{\alpha,d} \colon \mathbb{T}^{d+1} \to \mathbb{T}^{d+1}$ be the transformation defined by

$$S_{\alpha,d}(x_0, \dots, x_d) := (x_0, x_1 + g_{\alpha,d}(x_0), x_2 + x_1, \dots, x_d + x_{d-1}), \quad x_0, \dots, x_d \in \mathbb{T},$$

where $g_{\alpha,d} \colon \mathbb{T} \to \mathbb{R}_+$ is defined by $g_{\alpha,d}(x) := 1/(\alpha^d \{x\}^d)$ for $x \neq 0$.

This system helps us reproduce the correlations of MRT functions given in Proposition 8.2 and leads to the following result.

THEOREM 2.19. *For $\alpha > 0$ and $d \in \mathbb{Z}_+$, let $(X, \mu_{\alpha,d}, T)$ be the Furstenberg system for Cesàro averages of the MRT function $f \colon \mathbb{N} \to \mathbb{S}^1$, taken along the sequence of intervals $[N_m]$, where $N_m := \lfloor \alpha s_{m+1}^{1/d} \rfloor$. Then, $(X, \mu_{\alpha,d}, T)$ has trivial spectrum, it is not always*

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strongly stationary (in the sense described in Definition 3.10), and it is isomorphic to the system $(\mathbb{T}^{d+1}, m_{\mathbb{T}^{d+1}}, S_{\alpha,d})$*, where* $S_{\alpha,d}$ *is taken as in Definition 2.6.*

Remarks.

- Note that the previous result contrasts Theorem 2.4, which shows that in the case of logarithmic averages, all Furstenberg systems of bounded multiplicative functions with trivial rational spectrum are strongly stationary.
- If we consider logarithmic averages, we will see in Theorem 2.20 that the corresponding system, when $N_m := \lfloor s_{m+1}^{1/d} \rfloor$, is not isomorphic to the level d unipotent system given in Definition 2.4 but to an infinite 'mixture' of such systems.
- For $d = 1$, taking the limit as $\alpha \to \infty$, we get that the measure $\mu_{\alpha,1}$ on the sequence space converges weak-star to a measure that induces a system isomorphic to the identity transformation on $\mathbb{T}$ with $m_{\mathbb{T}}$. Taking the limit as $\alpha \to 0^+$, we get a system isomorphic to that defined by $T(x, y) = (x, y + x)$ on $\mathbb{T}^2$ with $m_{\mathbb{T}^2}$. A similar thing happens for general $d \in \mathbb{N}$; as $\alpha \to 0+$ or $\alpha \to \infty$, we get respectively the level d and level $d - 1$ unipotent systems given in Definition 2.4. In particular, contrasting [**23**], where a countable family of Furstenberg systems was described, here, we give an uncountable family of explicit Furstenberg systems of f that illustrates the 'continuous' transition from the level $d - 1$ to the level d unipotent system according to the choice of N_m.

Theorem 2.19 is proved in §8.3.

2.4.3. *Furstenberg systems of MRT functions for logarithmic averages.* We describe structural properties of Furstenberg systems of MRT functions defined using logarithmic averages. The reader is advised to compare Theorems 2.18 and 2.19 with Theorem 2.20, and note that for the same choice of sequence of averaging intervals $([N_m])$, the structure of the corresponding Furstenberg systems differs sharply when we use Cesàro versus logarithmic averages.

Definition 2.7. For $d \in \mathbb{Z}_+$, let (Y_d, ν_d, S_d) be the level d unipotent system given in Definition 2.4. For $c > 0$, we define the system (Z_c, ν_c', R_c), where Z_c consists of disjoint copies of Y_d, $d \geq \lfloor c \rfloor$, with the corresponding σ-algebra, R_c is defined to be S_d on each piece Y_d, and

$$\nu_c' := \left(1 - \frac{c}{\lceil c \rceil}\right)\nu_{\lfloor c \rfloor} + c \sum_{d=\lceil c \rceil}^{\infty} \left(\frac{1}{d} - \frac{1}{d+1}\right) \nu_d.$$

The motivation for introducing this rather awkward system is that it helps us reproduce the correlations of MRT functions for logarithmic averages given in Proposition 9.1 and leads to the following result.

THEOREM 2.20. *For* $c > 0$*, let* (X, μ_c, T) *be the Furstenberg system for logarithmic averages of the MRT function* $f : \mathbb{N} \to \mathbb{S}^1$*, taken along the sequence of intervals* $[s_{m+1}^{1/c}]$*. Then,* (X, μ_c, T) *has trivial spectrum, is strongly stationary, and is isomorphic to the system* (Z_c, ν_c', R_c) *given in Definition 2.7.*

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Remark. A similar statement holds if instead of $[s_{m+1}^{1/c}]$, we use any other sequence $[L_{s_{m+1}}]$, where (L_N) is a sequence with fractional degree $1/c$ (see Definition 9.1). We actually prove the result in this more general setting.

Theorem 2.20 is proved in §9.

2.5. *Open problems.* We refer the reader to Definition 3.9 for the various ergodic notions associated with bounded sequences throughout this subsection. We also remark that although the conjectures below are stated for Cesàro averages, similar conjectures can be made for logarithmic averages and we find them equally interesting.

In Theorem 2.7, we established that all pretentious multiplicative functions are completely deterministic for Cesàro and logarithmic averages. We believe that this zero-entropy property characterizes pretentiousness within the class of all bounded multiplicative functions.

CONJECTURE 1. *A non-trivial multiplicative function $f: \mathbb{N} \to \mathbb{U}$ is pretentious if and only if it is completely deterministic for Cesàro averages. (This is not to be confused with the notion of zero topological entropy, which is stronger. See Example (v) in §3.2.1.)*

Remark. It is an immediate consequence of [**23**, Main Theorem] that all multiplicative functions in the MRT class are not completely deterministic for Cesàro averages or logarithmic averages.

The structural results in §2.4 suggest that all Furstenberg systems of MRT functions for Cesàro and logarithmic averages are disjoint from all ergodic zero entropy systems. A consequence of this would be that all MRT functions satisfy the Sarnak conjecture with ergodic deterministic weights. Since the MRT class seems to be the most likely place to look for examples, where the previous property fails, we expect the following to hold.

CONJECTURE 2. *Let $f: \mathbb{N} \to \mathbb{U}$ be an aperiodic multiplicative function. Then,*

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, f(n)\, w(n) = 0$$

for every $w: \mathbb{N} \to \mathbb{U}$ that is completely deterministic and ergodic for Cesàro averages.

Remark. One could replace $w(n)$ with $(F(T^n x))$, for all uniquely ergodic systems (X, T) with zero entropy, continuous functions $F \in C(X)$ and points $x \in X$.

Note that when we use logarithmic averages, the previous property is known to hold under a strong aperiodicity assumption on the multiplicative function f [**17**, Theorem 1.5]. However, even in the case of logarithmic averages, the previous conjecture enlarges the scope of this result to all aperiodic multiplicative functions. We also remark that without the ergodicity assumption on the weight, the conjecture is false as is shown in [**23**, §5.2]. Lastly, for all totally ergodic weights and logarithmic averages, the conjecture is known for every zero mean multiplicative function, this follows from the disjointness of Furstenberg systems of bounded multiplicative functions and totally ergodic systems, which is a consequence of [**16**, Proposition 3.12] and [**17**, Theorem 1.5].

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We also state a conjecture regarding a variant of Sarnak's conjecture on which we do not impose any ergodicity assumptions on the weight w. (A similar conjecture was made independently in [33, Conjecture 2.11].)

CONJECTURE 3. *Let $f:\mathbb{N}\to\mathbb{U}$ be an aperiodic multiplicative function. Then, there exists a subsequence $N_k\to\infty$ such that*

$$\lim_{k\to\infty}\mathbb{E}_{n\in[N_k]}\, f(n)\,w(n)=0$$

for every $w:\mathbb{N}\to\mathbb{U}$ that is completely deterministic for Cesàro averages along (N_k).

As remarked above, if we do not take subsequential limits, then the asserted convergence to 0 fails for some aperiodic multiplicative function f [23, §5.2]. If Conjecture 3 is verified for logarithmic averages along (N_k) for some specific aperiodic multiplicative function f, then a modification of an argument in [44] would give that f satisfies the Chowla–Elliott conjecture for logarithmic averages along (N_k). (One has to verify that the implications Conjecture 1.5 $\Longrightarrow$ Conjecture 1.6 $\Longrightarrow$ Conjecture 1.4 in [44] hold with an arbitrary bounded multiplicative function f in place of λ. The first implication works for arbitrary bounded sequences, so in particular for f. For the second implication, one needs to do some minor adjustments as in the proof of [15, Theorem 1.7]. Also, both implications work without any change when the averages $\mathbb{E}^{\log}_{n\in[N]}$ are replaced by the subsequential averages $\mathbb{E}^{\log}_{n\in[N_k]}$.)

We remark that Theorem C.3 in the appendix offers some support for the last two conjectures, at least for logarithmic averages.

Although in Corollary 2.2 we have established divisibility of the spectrum when a Furstenberg system of a non-zero completely multiplicative function is constructed using logarithmic averages, the corresponding question for Cesàro averages remains open, and it is not clear to us which way the answer should go.

Question. Let (X,μ,T) be a Furstenberg system for Cesàro averages of a completely multiplicative function $f:\mathbb{N}\to\mathbb{U}\setminus\{0\}$. Is it true that the spectrum of the system is always a divisible subset of $\mathbb{T}$?

We caution the reader that Corollary 2.2 covers the case of the combined spectrum of all Furstenberg systems of f for Cesàro averages, and not the spectrum of any fixed Furstenberg system of f (unless it is ergodic). For pretentious multiplicative functions, the answer is yes, since as we show in part (iii) of Theorem 2.8 that any two Furstenberg systems of f for Cesàro or logarithmic averages coincide, so we can use Corollary 2.2.

3. *Background*

In this section, we gather some notions and basic facts from ergodic theory and number theory used throughout this article.

3.1. *Dirichlet characters.*

A *Dirichlet character* χ is a periodic completely multiplicative function and is often thought of as a multiplicative function in $\mathbb{Z}_m$ for some $m\in\mathbb{N}$. In this case, χ takes the value 0 on integers that are not coprime to m, and

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takes values on $\phi(m)$-roots of unity on all other integers, where ϕ is the Euler totient function. The indicator function of the integers that are relatively prime to m is called the *principal character modulo m* and is denoted by $\chi_{0,m}$. A Dirichlet character χ mod m may be determined by a character χ' of strictly smaller modulus $m' \mid m$ by the formula $\chi = \chi' \cdot \chi_{0,m}$. If this is not the case, we say that χ is *primitive*, and its period m is then denoted by q and is called the *conductor of* χ. Note that $\chi = 1$ is the only primitive Dirichlet character with conductor $q = 1$, and also the only Dirichlet character that is primitive and principal. We remark also that for every Dirichlet character χ mod m, there exists a primitive Dirichlet character χ' such that $\chi = \chi' \cdot \chi_{0,m}$.

3.2. *Pretentious multiplicative functions.* Following Granville and Soundararajan [25–27], we define the notion of pretentiousness and a related distance between multiplicative functions.

Definition 3.1. If $f, g\colon \mathbb{N} \to \mathbb{U}$ are multiplicative functions, we define the distance between them as

$$\mathbb{D}^2(f, g) := \sum_{p \in \mathbb{P}} \frac{1}{p}(1 - \Re(f(p) \cdot \overline{g(p)})). \tag{5}$$

We say that:

(i) f *pretends to be* g and write $f \sim g$ if $\mathbb{D}(f, g) < +\infty$;
(ii) f is *pretentious* if $f \sim n^{it} \cdot \chi$ for some $t \in \mathbb{R}$ and primitive Dirichlet character χ;
(iii) f is *aperiodic* (or *non-pretentious*) if it is not pretentious;
(iv) f is *trivial* if $\lim_{N\to\infty} \mathbb{E}_{n\in[N]} |f(n)|^2 = 0$ (in which case f is aperiodic).

In case (ii), the real number t and the primitive Dirichlet character χ (and hence its conductor q) are uniquely determined, meaning, if $n^{it} \cdot \chi \sim n^{it'} \cdot \chi'$ for some $t, t' \in \mathbb{R}$ and primitive Dirichlet characters χ, χ', then $t = t'$ and $\chi = \chi'$ (see for example [35, Proposition 6.2.3]). It is also known that for every non-zero $t \in \mathbb{R}$, we have $n^{it} \not\sim \chi$ for every Dirichlet character χ (see for example [27, Corollary 11.4]). Note that for every Dirichlet character χ, there exists a primitive Dirichlet character χ' such that $\chi \sim \chi'$. So without loss of generality, in our statements, we use primitive Dirichlet characters, which in some cases offers notational advantages.

It follows from [8, Corollary 1] that a multiplicative function $f\colon \mathbb{N} \to \mathbb{U}$ is aperiodic if and only if its Cesàro averages on every infinite arithmetic progression are zero, that is, $\lim_{N\to\infty} \mathbb{E}_{n\in[N]} f(an + b) = 0$ for every $a \in \mathbb{N}, b \in \mathbb{Z}_+$.

It can be shown (see [26] or [27, §4.1]) that $\mathbb{D}$ satisfies the triangle inequality

$$\mathbb{D}(f, g) \le \mathbb{D}(f, h) + \mathbb{D}(h, g)$$

for all $f, g, h\colon \mathbb{N} \to \mathbb{U}$. If f takes values on the unit circle, then we always have $f \sim f$. In general, it may be that $f \not\sim f$, which happens if and only if $\lim_{N\to\infty} \mathbb{E}_{n\in[N]} |f(n)|^2 = 0$, that is, if f is trivial (for a proof, see for example [2, Lemma 2.9]).

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Also, for all $f_1, f_2, g_1, g_2 \colon \mathbb{N} \to \mathbb{U}$, we have (see [25, Lemma 3.1])

$$\mathbb{D}(f_1 f_2, g_1 g_2) \le \mathbb{D}(f_1, g_1) + \mathbb{D}(f_2, g_2).$$

It follows that if $f_1 \sim g_1$ and $f_2 \sim g_2$, then $f_1 f_2 \sim g_1 g_2$.

3.2.1. *Some examples.* To get a sense of the variety of measure-preserving systems that arise as Furstenberg systems of pretentious multiplicative functions and their spectral properties, we provide a rather extensive assortment of examples. The terminology we use is explained in §§3.3 and 3.4. In all cases, the Furstenberg systems can be taken either for Cesàro or logarithmic averages. The properties recorded in Examples (i), (ii), (iv), (vii), (viii) follow from Theorem 2.12, the properties of Example (iii) are straightforward, the properties of Example (v) were established in [4] (see also [41, Theorem 9]), and those of Example (vi) can be found in [23, Corollary 5.5] and [17, §1.3]. Finally, we remark that Theorem 2.6 does not apply to Example (viii), so in this case, even showing that all Furstenberg systems have rational discrete spectrum is a non-trivial task. For the sake of brevity, in the following discussion, when we refer to the spectrum of a multiplicative function, we mean the spectrum of any of its Furstenberg systems.

(i) $f(2) := -1$ and $f(p) := 1$ for $p \neq 2$, and f is completely multiplicative. The spectrum of f is $m/2^s$, $m \in \mathbb{Z}_+$, $s \in \mathbb{N}$, and it has a unique Furstenberg system, which is an ergodic procyclic system.

(ii) More generally, let $f(p_1) = \cdots = f(p_\ell) := -1$ and $f(p) = 1$ for $p \notin \{p_1, \dots, p_\ell\}$, and f is completely multiplicative. Then, the spectrum of f consists of all integer combinations of the numbers $1/p_i^k$, where $k \in \mathbb{N}$, $i \in \{1, \dots, \ell\}$, and it has a unique Furstenberg system, which is an ergodic procyclic system.

(iii) $f := \chi$ is a Dirichlet character, or $f(n) = (-1)^{n+1}$, or $f := \mathbf{1}_{\mathbb{Z}\setminus 3\mathbb{Z}} - \mathbf{1}_{3\mathbb{Z}}$. Then, f has finite rational spectrum and it has a unique Furstenberg system, which is an ergodic cyclic system.

(iv) $f := \tilde{\chi}$ is a modified Dirichlet character, defined as in Lemma 7.4, where χ is a primitive Dirichlet character with conductor q. Then, f has a unique Furstenberg system, which is ergodic and procyclic, and its spectrum consists of all integer combinations of $1/p^k$, $k \in \mathbb{N}$, where $p \in \mathbb{P}$ divides q.

(v) $f := \mu^2$ is the indicator of the square free numbers. Then, f has a unique Furstenberg system, which is ergodic and procyclic, and its spectrum consists of all integer combinations of $1/p^2$, where $p \in \mathbb{P}$. In particular, for any prime p, we have that $1/p^2$ is on the spectrum of this system but $1/p^3$ is not. We also remark that this is an example of a multiplicative function that has positive topological entropy but its unique Furstenberg system has zero entropy.

(vi) $f(n) := n^{it}$ for some $t \neq 0$. Then, f has uncountably many Furstenberg systems, all isomorphic to the identity transformation on $\mathbb{T}$ with the Lebesgue measure. In the case of logarithmic averages, it has a unique Furstenberg system, isomorphic

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to the identity transformation on $\mathbb{T}$ with the Lebesgue measure. In both cases, all Furstenberg systems have trivial rational spectrum.

(vii) $f(p) := e(1/p)$, or $f(p) := 1 - 1/p$, $p \in \mathbb{P}$, and f is completely multiplicative. Then, $f \sim 1$ and f has a unique ergodic procyclic Furstenberg system and its spectrum consists of all rational numbers in $[0, 1)$. If we change the value of f at a single prime to 1, say set $f(3) := 1$, then $1/3$ will no longer be in the spectrum, and the non-zero values in the spectrum will now consist of all rationals $p/q \in [0, 1)$ with $(p, q) = 1$ and $(q, 3) = 1$.

(viii) $f(p) := e(1/\log\log p)$, $p \in \mathbb{P}$, and f is completely multiplicative. Then, $f \sim 1$ but f does not satisfy equation (1), so Theorem 2.6 does not apply. (The reason for this is that if $\theta_p := 1/\log\log p$, $p \in \mathbb{P}$, then $\sum_{p\in\mathbb{P}}(\theta_p^2/p) < +\infty$, while $\sum_{p\in\mathbb{P}}(\theta_p/p) = +\infty$.) We have that f does not have a mean value [8, Corollary 2] and hence has several Furstenberg systems, all of which are ergodic procyclic systems, isomorphic to each other, and their spectrum consists of all the rational numbers in $[0, 1)$.

It follows from [17, Theorem 1.5] that an irrational number cannot be on the spectrum of a Furstenberg system for logarithmic averages of any multiplicative function (pretentious or not) that takes values in $\mathbb{U}$. For pretentious multiplicative functions, an alternative proof follows from Theorem 2.13, and it also applies to Furstenberg systems for Cesàro averages (a case not covered in [17]).

3.2.2. *Mean values of multiplicative functions.* We will need the following notion.

Definition 3.2. We say that a sequence $A\colon \mathbb{N} \to \mathbb{R}_+$ is *slowly varying* if for every $c \in (0, 1)$, we have

$$\lim_{N\to\infty} \sup_{n\in[N^c,N]} |A(n) - A(N)| = 0.$$

For example, the sequence $A(n) = \log\log\log n$, defined for $n \geq 3$, is slowly varying, but the sequence $A(n) = \log\log n$, defined for $n \geq 2$, is not slowly varying.

We will use the following result that can be found in the form stated below in [10, Theorems 6.2 and 6.3]. We will only apply this in the case of pretentious multiplicative functions.

THEOREM 3.1. (Delange–Wirsing–Halász) *Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function.*

(i) *If $f \not\sim n^{it}$ for every $t \in \mathbb{R}$, or $f \sim n^{it}$ for some $t \in \mathbb{R}$ and $f(2^s) = -2^{ist}$ for every $s \in \mathbb{N}$, then*

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, f(n) = 0.$$

(ii) *If $f \sim n^{it}$ for some $t \in \mathbb{R}$ and $f(2^s) \neq -2^{ist}$ for some $s \in \mathbb{N}$, then there exist a non-zero $L \in \mathbb{C}$ and a slowly varying sequence $A\colon \mathbb{N} \to \mathbb{R}_+$ such that*

$$\mathbb{E}_{n\in[N]}\, f(n) = L\, N^{it} e(A(N)) + o_N(1).$$

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Remarks.

- If f is completely multiplicative, then L is always non-zero since we cannot have $f(2^s) = -2^{ist}$ for every $s \in \mathbb{N}$.
- Part (i) also holds for logarithmic averages, this follows using partial summation. Part (ii) fails for logarithmic averages, take for example $f(n) = n^{it}$ for some $t \neq 0$, then the logarithmic averages vanish but not the Cesàro averages (more generally, the logarithmic averages vanish if $f \sim n^{it}$ for some $t \neq 0$). However, if $t = 0$ (equivalently, if $f \sim 1$), then part (ii) continues to hold if we replace $\mathbb{E}_{n\in[N]} f(n)$ with $\mathbb{E}^{\log}_{n\in[N]} f(n)$. To see this, note that since $\lim_{N\to\infty} \sup_{n\in[N^c,N]} |A(n) - A(N)| = 0$ for every $c > 0$, we can use partial summation to deduce the asserted asymptotic for the averages $\mathbb{E}^{\log}_{n\in[N^c,N]} f(n)$ for all $c > 0$. Letting $c \to 0^+$ gives the result for the averages $\mathbb{E}^{\log}_{n\in[N]} f(n)$.

3.3. *Measure-preserving systems.* Throughout the article, we make the typical assumption that all probability spaces $(X, \mathcal{X}, \mu)$ considered are *standard Borel*, meaning, X can be endowed with the structure of a complete and separable metric space, and $\mathcal{X}$ is its Borel σ-algebra. A *measure-preserving system*, or simply *a system*, is a quadruple $(X, \mathcal{X}, \mu, T)$, where $(X, \mathcal{X}, \mu)$ is a probability space and $T\colon X \to X$ is an invertible, measurable, measure-preserving transformation. In general, we omit the σ-algebra $\mathcal{X}$ and write (X, μ, T). The system is *ergodic* if the only T-invariant sets in $\mathcal{X}$ have measure 0 or 1. If $f \in L^\infty(\mu)$ and $n \in \mathbb{Z}$, with $T^n f$, we denote the composition $f \circ T^n$, where for $n \in \mathbb{N}$, we let $T^n := T \circ \cdots \circ T$ (n times), $T^{-n} = (T^{-1})^n$, and $T^0 = \mathrm{id}$.

3.3.1. *Factors and isomorphisms.* A *homomorphism*, also called a *factor map*, from a system $(X, \mathcal{X}, \mu, T)$ onto a system $(Y, \mathcal{Y}, \nu, S)$ is a measurable map $\Phi\colon X \to Y$, such that $\mu \circ \Phi^{-1} = \nu$ and with $S \circ \Phi = \Phi \circ T$ valid for μ-almost every (a.e.) $x \in X$. When we have such a homomorphism, we say that the system $(Y, \mathcal{Y}, \nu, S)$ is a *factor* of the system $(X, \mathcal{X}, \mu, T)$. If the factor map $\Phi\colon X \to Y$ is injective on a T-invariant set of full μ-measure, we say that Φ is an *isomorphism* and that the systems $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ are *isomorphic*.

The *pushforward of* μ *by* Φ is denoted by $\Phi_*\mu$ or $\mu \circ \Phi^{-1}$, and is defined by

$$\int F \, d(\Phi_*\mu) = \int F \circ \Phi \, d\mu$$

for every $F \in L^\infty(\nu)$.

3.3.2. *Spectrum, procyclic, and rational discrete spectrum systems.* The notion of the spectrum and the class of systems with rational discrete spectrum play a crucial role in this article and we define these concepts next.

Definition 3.3. Given a system (X, μ, T), we define its *spectrum*, and denote it by $\mathrm{Spec}(X, \mu, T)$, to be the set of $\alpha \in [0, 1)$ for which there exists a non-zero $g \in L^2(\mu)$ such that $Tg = e(\alpha) \cdot g$. We call any such g an *eigenfunction of the system*.

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Hence, $\alpha \in \mathrm{Spec}(X, \mu, T)$ if it is the phase (taken in $[0, 1)$) of an eigenvalue of the operator $f \mapsto Tf$, acting on $L^2(\mu)$. We will often identify $[0, 1)$ with $\mathbb{T}$, and using this identification, we have that the spectrum of a system is always a subset of $\mathbb{T}$ that is closed under multiplication by integers. When the system is ergodic, then the spectrum is a subgroup of $\mathbb{T}$. We say that $e(\alpha)$ is a *rational eigenvalue* if $\alpha \in \mathbb{Q}$.

Definition 3.4. We say that the system (X, μ, T):

(i) is *trivial* if there exists $x_0 \in X$ such that $Tx = x_0$ for μ-a.e. $x \in X$;
(ii) is *an identity* if $Tx = x$ for μ-a.e. $x \in X$;
(iii) is *cyclic* if it is isomorphic to a rotation on a finite cyclic group;
(iv) has *trivial rational spectrum* if $\mathrm{Spec}(X, \mu, T) \cap \mathbb{Q} = \{0\}$;
(v) has *rational discrete spectrum* if $L^2(\mu)$ is spanned by eigenfunctions with rational eigenvalues;
(vi) is *procyclic* if it is an ergodic system with rational discrete spectrum. (Procyclic systems are often called ergodic odometers in the literature.)

It can be shown that a system has rational discrete spectrum if and only if its ergodic components are procyclic systems. Also, a procyclic system is an inverse limit of cyclic systems, and this happens if and only if it is isomorphic to an ergodic rotation on a procyclic group. Lastly, we remark that procyclic systems are isomorphic if and only if they have the same spectrum, but this is not the case for (non-ergodic) systems with rational discrete spectrum.

3.3.3. *Joinings and disjoint systems.* Following [18], if $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ are two systems, we call a measure ρ on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$ a *joining* of the two systems if ρ is $T \times S$-invariant and its projection onto the X and Y coordinates are the measures μ and ν, respectively. We say that the systems on X and on Y are *disjoint* if the only joining of the systems is the product measure $\mu \times \nu$.

We will use the following well-known facts (the Furstenberg systems of the sequences $a, b\colon \mathbb{N} \to \mathbb{U}$ used below can be taken either for Cesàro or logarithmic averages).

(i) All identity systems are disjoint from all ergodic systems.
(ii) Bernoulli systems are disjoint from all zero entropy systems.
(iii) If $a, b\colon \mathbb{N} \to \mathbb{U}$ are sequences, then any Furstenberg system (see Definition 3.7) of the product sequence $a \cdot b$ is a factor of a joining of some Furstenberg system of a and another Furstenberg system of b.
(iv) The spectrum of an ergodic joining of two discrete spectrum systems is contained in the subgroup of $\mathbb{T}$ generated by the spectrum of the individual systems. (Similarly, we can define joinings of countably many systems and property (iv) extends to this more general setting.)
(v) If all Furstenberg systems of a sequence $a\colon \mathbb{N} \to \mathbb{U}$ are disjoint from all Furstenberg systems of a sequence $b\colon \mathbb{N} \to \mathbb{U}$, then the spectrum of any Furstenberg system of the product sequence $a \cdot b$ is contained in the subgroup generated by the spectrum of a Furstenberg system of the sequence a and another Furstenberg system of the sequence b.

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3.4. *Furstenberg systems of sequences.* We reproduce the notion of a Furstenberg system of a bounded sequence and record some basic related facts that will be used later.

3.4.1. *Sequence space systems and correlations.* We start with two preparatory notions.

Definition 3.5. Let $X := \mathbb{U}^{\mathbb{Z}}$ and denote the elements of X by $x := (x(k))_{k\in\mathbb{Z}}$. A *sequence space system* is a system (X, μ, T), where $T\colon X \to X$ is the *shift transformation,* defined by $(Tx)(k) := x(k+1)$, $k \in \mathbb{Z}$, and μ is a T-invariant measure. We implicitly assume that X is equipped with the product topology. We let $F_0 \in \mathbb{C}(X)$ be defined by $F_0(x) := x(0)$, $x \in X$, and call it the *0th*-coordinate projection.

Remark. Any probability measure on X is uniquely determined by its values on the set of *cylinder functions* $\{\prod_{j=1}^{\ell} T^{n_j} F_j, \ell \in \mathbb{N}, n_1, \dots, n_\ell \in \mathbb{Z}, F_1, \dots, F_\ell \in \{F_0, \overline{F_0}\}\}$, since this set is linearly dense in $C(X)$. Another representation for this set is $\{\prod_{j=1}^{\ell} T^{n_j} F_0^{k_j}, \ell \in \mathbb{N}, n_1, \dots, n_\ell \in \mathbb{Z}, k_1, \dots, k_\ell \in \mathbb{Z}\}$, where we use the notation $F_0^k := \overline{F_0}^{|k|}$ for $k < 0$.

Definition 3.6. Let $([N_k])_{k\in\mathbb{N}}$ be a sequence of intervals with $N_k \to \infty$. We say that a sequence $a\colon \mathbb{Z} \to \mathbb{U}$ *admits correlations along* $([N_k])$ (or simply *along* (N_k)) *for Cesàro averages* if the limits

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]} \prod_{j=1}^{\ell} a_j(n+n_j) \tag{6}$$

exist for all $\ell \in \mathbb{N}$, $n_1, \dots, n_\ell \in \mathbb{Z}$, and all $a_1, \dots, a_\ell \in \{a, \overline{a}\}$.

Remarks.

- Given $a\colon \mathbb{Z} \to \mathbb{U}$, using a diagonal argument, we get that every sequence of intervals $([N_k])$ has a subsequence $([N'_k])$ such that the sequence a admits correlations for Cesàro averages along $([N'_k])$.
- If we are given a one-sided sequence $(a(n))_{n\in\mathbb{N}}$, we extend it to $\mathbb{Z}$ in an arbitrary way; then, the existence and values of the correlations do not depend on the extension.

3.4.2. *Furstenberg systems of sequences.* If a sequence $a\colon \mathbb{Z} \to \mathbb{U}$ admits correlations for Cesàro averages along (N_k), then we use a variant of the correspondence principle of Furstenberg [19, 20] to associate a sequence space system that captures the statistical properties of a along (N_k). We briefly describe this process next.

We consider the sequence $a = (a(n))$ as an element of X. Note that the conjugation closed algebra generated by functions of the form $x \mapsto x(k)$, $x \in X$, for $k \in \mathbb{Z}$, separates points in X. We conclude that if the sequence $a\colon \mathbb{Z} \to \mathbb{U}$ admits correlations along (N_k), then, for all $F \in C(X)$, the following limit exists:

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]} F(T^n a).$$

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Hence, the following weak-star limit exists:

$$\mu := \lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, \delta_{T^n a} \tag{7}$$

and we say that *the point a is generic for μ along $([N_k])$ or (N_k)*.

Definition 3.7. Let $a\colon \mathbb{Z}\to\mathbb{U}$ be a sequence that admits correlations for Cesàro averages along (N_k), and (X, μ, T) be as above.

(i) We call (X, μ, T), with μ given by equation (7), the *Furstenberg system of a along $([N_k])$, or for simplicity (N_k), for Cesàro averages.*

(ii) If $F_0 \in \mathbb{C}(X)$ is the *0th*-coordinate projection, then $F_0(T^n x) = x(n)$ for every $n\in\mathbb{Z}$, and

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]} \prod_{j=1}^{\ell} a_j(n+n_j) = \int \prod_{j=1}^{\ell} T^{n_j} F_j \, d\mu \tag{8}$$

for all $\ell\in\mathbb{N}$, $n_1,\dots,n_\ell\in\mathbb{Z}$, and $a_1,\dots,a_\ell\in\{a,\overline{a}\}$, where for $j=1,\dots,\ell$, the function F_j is F_0, $\overline{F_0}$ if respectively a_j is $a,\overline{a}$. Note that the identities (7) and (8) are equivalent, but it is identity (8) that we will mostly use.

(iii) We say that the sequence a has a *unique* Furstenberg system if a admits correlations on $([N])_{N\in\mathbb{N}}$, or equivalently, if a is generic for a measure along $([N])_{N\in\mathbb{N}}$.

Remarks.

- A sequence $a\colon \mathbb{Z}\to\mathbb{U}$ may have several Furstenberg systems depending on which sequence of intervals $([N_k])$ we use in the evaluation of its correlations. We call any such system *a Furstenberg system of a for Cesàro averages*. For a fixed sequence, these systems may or may not be isomorphic.
- Since the set of measures defining Furstenberg systems of a is known to be connected (in the weak-star topology), either it has a single element or it is uncountable.
- We will sometimes use the following fact. If (X, μ, T) is a Furstenberg system of a sequence $a\colon \mathbb{N}\to\mathbb{U}$ along (N_k), then for every $m\in\mathbb{N}$, the Furstenberg system of the mth power of a along (N_k) is also well defined, and it is a factor of the system (X, μ, T) (the factor map is $\Phi\colon X\to X$ defined by $(\Phi x)(k) := x^m(k)$, $k\in\mathbb{Z}$).

Given a bounded sequence, our goal is to obtain structural properties of their Furstenberg systems. Ultimately, we would like to completely determine them up to isomorphism using as building blocks systems with algebraic structure, such as nilsystems, and systems that enjoy strong randomness properties, such as Bernoulli systems.

Similar notions as in the previous subsection can be defined for logarithmic averages.

Definition 3.8. By replacing Cesàro averages with logarithmic averages in Definitions 3.5–3.7, given a sequence $a\colon \mathbb{N}\to\mathbb{U}$ and a sequence of intervals $([N_k])$ along which a admits correlations for logarithmic averages, we can introduce *Furstenberg systems of a along (N_k) for logarithmic averages*. If (X, μ, T) is such a system, then an identity similar to equation (8) is satisfied with $\mathbb{E}^{\log}_{n\in[N_k]}$ in place of $\mathbb{E}_{n\in[N_k]}$. Equivalently,

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$$\mu = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} \delta_{T^n a},$$

where the limit is a weak-star limit.

Remarks.

- If a has a unique Furstenberg system for Cesàro averages, then the same system is the unique Furstenberg system of a for logarithmic averages.
- In general, the sets of Furstenberg systems for Cesàro averages and logarithmic averages can be disjoint, as the example of $a(n) = n^{it}$, $n \in \mathbb{N}$, for $t \neq 0$ shows, see [23, Corollary 5.5] and [17, §1.3].
- Each ergodic Furstenberg system for logarithmic averages along (N_k) is also a Furstenberg system for Cesàro averages along a possibly different sequence (N'_k) (see [22, Corollary 2.2]).

3.4.3. *Ergodic and completely deterministic sequences.* Using the Furstenberg correspondence principle, we can naturally associate ergodic notions to bounded sequences—we record here two that are used in this article.

Definition 3.9. We say that a sequence $a\colon \mathbb{N} \to \mathbb{U}$ is:

(i) *ergodic for Cesàro averages* if all its Furstenberg systems for Cesàro averages are ergodic;

(ii) *ergodic for Cesàro averages along* (N_k) if all its Furstenberg systems for Cesàro averages along subsequences of (N_k) are ergodic;

(iii) *completely deterministic for Cesàro averages* if all its Furstenberg systems for Cesàro averages have zero entropy (this notion was originally introduced in [29, 47]);

(iv) *completely deterministic for Cesàro averages along* (N_k) if all its Furstenberg systems for Cesàro averages along subsequences of (N_k) have zero entropy.

Similar definitions apply for logarithmic averages.

We will use the fact that if $a, b\colon \mathbb{N} \to \mathbb{U}$ are completely deterministic sequences for Cesàro or logarithmic averages along (N_k), then so is their product $a \cdot b$. To see this, use property (iii) in §3.3.3 and the fact that zero entropy systems are closed under joinings and factors.

3.4.4. *Strong stationarity.* Lastly, we define the notion of strong stationarity that was introduced by Furstenberg and Katznelson in [21], which turns out to be relevant for the structural analysis of measure-preserving systems associated with non-pretentious multiplicative functions.

Definition 3.10. A sequence space system (X, μ, T) is *strongly stationary* if

$$\int \prod_{j=1}^{\ell} T^{n_j} F_j \, d\mu = \int \prod_{j=1}^{\ell} T^{rn_j} F_j \, d\mu \tag{9}$$

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for all $\ell, r \in \mathbb{N}$, $n_1, \dots, n_\ell \in \mathbb{Z}$, and $F_1, \dots, F_\ell \in \{F_0, \overline{F_0}\}$, where $F_0 \in L^\infty(\mu)$ is the 0th-coordinate projection. (Equivalently, for every $r \in \mathbb{N}$, the dilation map $\tau_r : X \to X$ in equation (10) is μ-preserving.) A sequence $a : \mathbb{N} \to \mathbb{U}$ is *strongly stationary for Cesàro averages* if all its Furstenberg systems for Cesàro averages are strongly stationary.

Similar definitions apply for logarithmic averages.

3.5. *Besicovitch rationally almost periodic sequences.* We will use the following variant of the classical notion of Besicovitch rational almost periodicity.

Definition 3.11. Let $N_k \to \infty$ be a sequence of integers. Following [1], we say that a sequence $a : \mathbb{N} \to \mathbb{U}$ is:

(i) *Besicovitch rationally almost periodic for Cesàro averages along* (N_k) if for every $\varepsilon > 0$, there exists a periodic sequence $a_\varepsilon : \mathbb{N} \to \mathbb{U}$ such that

$$\limsup_{k\to\infty} \mathbb{E}_{n\in[N_k]} |a(n) - a_\varepsilon(n)|^2 \le \varepsilon;$$

(ii) *Besicovitch rationally almost periodic for logarithmic averages along* (N_k) if for every $\varepsilon > 0$, there exists a periodic sequence $a_\varepsilon : \mathbb{N} \to \mathbb{U}$ such that

$$\limsup_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} |a(n) - a_\varepsilon(n)|^2 \le \varepsilon.$$

If we make no reference to N_k, we mean that the statement holds for every sequence $N_k \to \infty$, in other words, we can replace $\limsup_{k\to\infty} \mathbb{E}_{n\in[N_k]}$ with $\limsup_{N\to\infty} \mathbb{E}_{n\in[N]}$, and similarly for logarithmic averages.

It is easy to verify that if a sequence is Besicovitch rationally almost periodic for Cesàro (or logarithmic) averages, then it has a mean value on every infinite arithmetic progression and its correlations for Cesàro (or logarithmic) averages exist. As a consequence, it has a unique Furstenberg system for Cesàro (or logarithmic) averages. We will use the next result that gives structural information for Furstenberg systems of such sequences.

THEOREM 3.2. [1, Theorem 3.12] *Let $a : \mathbb{N} \to \mathbb{U}$ be a sequence that is Besicovitch rationally almost periodic for Cesàro averages along some subsequence (N_k). Then:*

(i) *the sequence a has a unique Furstenberg system (X, μ, T) for Cesàro averages along (N_k);*

(ii) *the system (X, μ, T) is an ergodic procyclic system.*

Furthermore, a similar statement holds for logarithmic averages.

The argument in [1] is given only for Cesàro averages, but exactly the same argument also works for logarithmic averages.

4. *General results for multiplicative functions*

4.1. *Divisibility property of the spectrum—Proof of Theorem 2.1.* In this subsection, we prove Theorem 2.1. We will need some preparatory results. In particular, it will be crucial to get some understanding of how the dilation maps τ_r, which we are about to define, act

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on eigenfunctions of Furstenberg systems of multiplicative functions. This is the context of Lemmas 4.1 and 4.2. These two lemmas combined enable us to prove Theorem 2.1.

Definition 4.1. Let T be the shift transformation on the sequence space $X = \mathbb{U}^{\mathbb{Z}}$.

(i) For $r \in \mathbb{N}$, we let $\tau_r \colon X \to X$ be the *dilation by r map* defined by

$$(\tau_r x)(j) := x(rj), \quad j \in \mathbb{Z}. \tag{10}$$

It satisfies the commutation relation

$$T\tau_r = \tau_r T^r, \tag{11}$$

meaning $T(\tau_r x) = \tau_r(T^r x)$ for every $x \in X$.

(ii) For $z \in \mathbb{U}$, we define the map $M_z \colon X \to X$ by

$$(M_z x)(j) := z \cdot x(j), \quad j \in \mathbb{Z}. \tag{12}$$

It satisfies the commutation relation

$$TM_z = M_z T. \tag{13}$$

LEMMA 4.1. (Proof of [28, Lemma 2.3]) *Let (X, T) be the sequence space with the shift transformation and $\tau_r \colon X \to X$ be the dilation map defined in equation (10). Let $\chi \colon X \to \mathbb{U}$ be a function, such that for some $\alpha \in [0, 1)$, we have $\chi(Tx) = e(\alpha)\,\chi(x)$ for every $x \in X$. Then,*

$$\chi(\tau_r x) = \chi_0(x) + \cdots + \chi_{r-1}(x), \quad x \in X,$$

for some $\chi_0, \ldots, \chi_{r-1}$ that are linear combinations of the functions $\chi \circ \tau_r \circ T^j$, $j = 0, \ldots, r-1$, and satisfy

$$\chi_k(Tx) = e((\alpha + k)/r)\,\chi_k(x), \quad k = 0, \ldots, r-1,\ x \in X.$$

Remark. Note that if X is equipped with the product topology and χ is Borel measurable, then also χ_k is Borel measurable for $k = 0, \ldots, r-1$.

Proof. For $k = 0, \ldots, r-1$, let $\chi_k := (1/r)\sum_{j=0}^{r-1} e(-j(k+\alpha)/r)\,\chi \circ \tau_r \circ T^j$. Then, the asserted properties follow by a direct computation using our assumption and the commutation relation (11). □

We caution the reader that when we consider a sequence space system (X, μ, T), we may have $\chi \neq 0$ (with respect to μ) but $\tau_r \chi = \tau_r \chi_0 = \cdots = \tau_r \chi_{r-1} = 0$ (with respect to μ), in which case, the content of Lemma 4.1 is practically empty. (For example, let $\mu := (\delta_{x_0} + \delta_{x_1})/2$, where $x_i := \mathbf{1}_{2\mathbb{Z}+i}$, $i = 0, 1$, and $\chi(x) := \mathbf{1}_{\{x_0\}}(x) - \mathbf{1}_{\{x_1\}}(x)$, $x \in X$. Then, $\chi(Tx) = -\chi(x)$ for μ-a.e. $x \in X$ and $\chi \neq 0$ with respect to μ. However, one easily verifies that $\chi(\tau_2 x_0) = \chi(\tau_2 x_1) = 0$, and hence $\chi \circ \tau_2 = \chi \circ \tau_2 \circ T = 0$ with respect to μ. Note that the system (X, μ, T) is the Furstenberg system of the multiplicative function $f(n) := (-1)^{n+1}$, $n \in \mathbb{N}$.) For Furstenberg systems of completely multiplicative functions though, Lemma 4.1 can be combined with the next result that alleviates this problem, to get interesting consequences.

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LEMMA 4.2. *Let (X, μ, T) be a Furstenberg system for logarithmic averages of a completely multiplicative function $f : \mathbb{N} \to \mathbb{U}$. Then, for every $r \in \mathbb{N}$, we have*

$$(M_{f(r)})_*\mu \le r \cdot (\tau_r)_*\mu.$$

Furthermore, a similar statement holds if a Furstenberg system is defined using Cesàro averages and is ergodic.

Remark. For Cesàro averages, we get the following more explicit result. If we let $\mu := \lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, \delta_{T^n f}$ and $\mu_r := \lim_{k\to\infty} \mathbb{E}_{n\in[rN_k]}\, \delta_{T^n f}$, and we assume that both weak-star limits exist (which can always be arranged by passing to a subsequence of (N_k)), then

$$(M_{f(r)})_*\mu \le r \cdot (\tau_r)_*\mu_r.$$

Proof. It suffices to show that for every $F \in C(X)$ with $F \ge 0$, we have

$$\int M_{f(r)} F \, d\mu \le r \int \tau_r F \, d\mu. \tag{14}$$

Suppose that the Furstenberg system of f is taken along (N_k). Thinking of f as the element $(f(k))$ of the sequence space X, we get that μ is the weak-star limit

$$\mu = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]}\, \delta_{T^n f} = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k/r]}\, \delta_{T^n f}, \tag{15}$$

where the second identity holds because of the logarithmic averaging. Note also that for every $z \in \mathbb{C}$ and $r \in \mathbb{N}$, we have $M_z F, \tau_r F \in C(X)$, whenever $F \in C(X)$.

It follows from these facts and equation (13) that

$$\int M_{f(r)} F \, d\mu = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k/r]}\, (F \circ M_{f(r)})(T^n f) = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k/r]}\, F(T^n M_{f(r)} f).$$

Since f is completely multiplicative, we have

$$(M_{f(r)} f)(n) = f(r) f(n) = f(rn) = (\tau_r f)(n). \tag{16}$$

Hence, using the commutation relation (11), we get that the last limit equals

$$\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k/r]}\, F(\tau_r T^{rn} f) \le r \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]}\, F(\tau_r T^n f) = r \int \tau_r F \, d\mu,$$

where to get the upper bound, we used that $F \ge 0$ and the elementary estimate (for $a(n) := F(\tau_r T^n f)$, $n \in \mathbb{N}$)

$$\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k/r]}\, a(rn) \le r \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]}\, a(n),$$

which is valid for every sequence $a : \mathbb{N} \to \mathbb{R}_+$ such that the previous limits exist. This establishes equation (14) and completes the proof.

Suppose now that the Furstenberg system (X, μ, T) is ergodic and μ is defined by the weak-star limit

$$\mu = \lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, \delta_{T^n f}$$

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for some sequence $N_k \to \infty$. We claim that for every $r \in \mathbb{N}$, the weak-star limit below exists and we have

$$\mu = \lim_{k\to\infty} \mathbb{E}_{n\in[N_k/r]}\, \delta_{T^n f}. \tag{17}$$

Assuming the claim, we use equation (17) as our starting point in place of equation (15) and repeat the previous argument verbatim to get the desired conclusion. To prove the claim, after passing to a subsequence, we can assume that the second limit also exists (if the identity fails, then it would also fail on a subsequence along which the second weak-star limits exist). Let μ' be the measure by this limit (the limit on the right-hand side in equation (17)). Then, μ' is T-invariant and satisfies $\mu' \le r\mu$, and hence it is absolutely continuous with respect to μ. Since the system (X, μ, T) is ergodic, we deduce that $\mu = \mu'$, completing the proof of the claim. □

Proof of Theorem 2.1. We prove part (i). By assumption, there exists $\chi \in L^\infty(\mu)$ such that $\chi \neq 0$ with respect to μ and $T\chi = e(\alpha)\chi$. After redefining the function χ on a set of μ-measure 0, we can assume that χ is defined for every $x \in X$ and satisfies the identity

$$\chi(Tx) = e(\alpha)\ \chi(x) \quad \text{for every } x \in X. \tag{18}$$

(If $Z := \{x \in X\colon \chi(Tx) \neq e(\alpha)\ \chi(x)\}$ and $Z' := \bigcup_{n\in\mathbb{Z}} T^n Z$, then $\mu(Z') = 0$. Define $\tilde\chi := \chi \cdot \mathbf{1}_{X\setminus Z'}$, then $\tilde\chi(Tx) = e(\alpha)\ \tilde\chi(x)$ for every $x \in X$ and $\tilde\chi \neq 0$ with respect to μ.)

Let $r \in \mathbb{N}$ be such that $f(r) \neq 0$, let $X_r := M_{f(r)}X$. Note that X_r is a T-invariant subset of X and the map $M_{f(r)}\colon X \to X_r$ is a homeomorphism. We also define $\tilde\chi\colon X \to X$ by

$$\tilde\chi(x) := \chi(M_{f(r)}^{-1}x)\cdot \mathbf{1}_{X_r}(x), \quad x \in X,$$

where $M_{f(r)}^{-1}x$ is defined in an arbitrary way on $X \setminus X_r$ (this does not affect the definition of $\tilde\chi(x)$).

We claim that $\tau_r\tilde\chi \neq 0$ with respect to μ. Indeed, we have

$$\mu(\tau_r\tilde\chi \neq 0) = ((\tau_r)_*\mu)(\tilde\chi \neq 0).$$

By Lemma 4.2, the right-hand side is at least

$$\frac{1}{r}((M_{f(r)})_*\mu)(\tilde\chi \neq 0) = \frac{1}{r}\mu(\chi \neq 0) > 0,$$

where the last equality holds since

$$M_{f(r)}\tilde\chi(x) = \chi(M_{f(r)}^{-1}\circ M_{f(r)}x)\cdot \mathbf{1}_{X_r}(M_{f(r)}x) = \chi(x) \quad \text{for every } x \in X.$$

This proves the claim.

Using equations (13), (18), and the fact that $\mathbf{1}_{X_r}(Tx) = \mathbf{1}_{X_r}(x)$ for every $x \in X$, we get

$$\tilde\chi(Tx) = e(\alpha)\ \tilde\chi(x) \quad \text{for every } x \in X.$$

By Lemma 4.1, we have

$$\tilde\chi(\tau_r x) = \chi_0(x) + \cdots + \chi_{r-1}(x) \quad \text{for every } x \in X,$$

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for some $\chi_0, \dots, \chi_{r-1}$ that are linear combinations of the functions $\tilde{\chi} \circ \tau_r \circ T^j$, $j = 0, \dots, r-1$, and satisfy

$$\chi_k(Tx) = e((\alpha + k)/r)\ \chi_k(x) \quad \text{for every } x \in X,$$

for $k = 0, \dots, r-1$.

Since $\tau_r \tilde{\chi} \neq 0$ with respect to μ, we have $\chi_k \neq 0$ with respect to μ for some $k \in \{0, \dots, r-1\}$. It follows that $(\alpha + k)/r \in \mathrm{Spec}(X, \mu, T)$.

We prove part (ii). If (X, μ, T) is a Furstenberg system for Cesàro averages of f and $\alpha \in \mathrm{Spec}(X, \mu, T)$, then, using the remark following Lemma 4.2 and the previous argument, we get that $(\alpha + k)/r \in \mathrm{Spec}(X, \mu_r, T)$ for some $k \in \{0, \dots, r-1\}$. Since (X, μ_r, T) is also a Furstenberg system of f, we get the first asserted statement.

Lastly, suppose that that the system (X, μ, T) is ergodic. Then, arguing as we did in Lemma 4.2 to arrive at equation (17), we get that if

$$\mu' := \lim_{k\to\infty} \mathbb{E}_{n\in[N_k/r]}\ \delta_{T^n f},$$

then the weak-star limit exists and $\mu = \mu'$. Then, $\alpha \in \mathrm{Spec}(X, \mu', T)$ and, arguing as before, we have that $(\alpha + k)/r \in \mathrm{Spec}(X, \mu'_r, T)$ for some $k \in \{0, \dots, r-1\}$. Since $\mu'_r = \mu$, the second asserted statement follows. □

4.2. *Strong stationarity—Proof of Theorem 2.4.* We will also use the following result from [**14**, Theorem 6.4] (it follows immediately since the ergodic components of systems with trivial rational spectrum are totally ergodic).

THEOREM 4.3. *Let (X, μ, T) be a system with trivial rational spectrum. Then,*

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \int \prod_{j=1}^{\ell} T^{jn} F_j\, d\mu = \lim_{N\to\infty} \mathbb{E}_{n\in[N]} \int \prod_{j=1}^{\ell} T^{j(rn+k)} F_j\, d\mu$$

for all $\ell, r \in \mathbb{N}$, $k \in \{0, \dots, r-1\}$, and $F_1, \dots, F_\ell \in L^\infty(\mu)$. Equivalently, for every rational $\alpha \in (0, 1) \cap \mathbb{Q}$, we have

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \left(e(n\alpha) \cdot \int \prod_{j=1}^{\ell} T^{jn} F_j\, d\mu \right) = 0$$

for all $\ell \in \mathbb{N}$ and $F_1, \dots, F_\ell \in L^\infty(\mu)$.

We deduce from this the following.

COROLLARY 4.4. *Let (X, μ, T) be a system with trivial rational spectrum and for some fixed $\ell \in \mathbb{N}$ and $F_1, \dots, F_\ell \in L^\infty(\mu)$, let the sequence $C : \mathbb{N} \to \mathbb{U}$ be defined by*

$$C(r) := \int \prod_{j=1}^{\ell} T^{jr} F_j\, d\mu, \quad r \in \mathbb{N}.$$

If C is the limit of periodic sequences in the uniform norm, then C is constant.

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Proof. By Theorem 4.3, we have that

$$\lim_{R\to\infty} \mathbb{E}_{r\in[R]}\, e(r\alpha)\cdot C(r) = 0 \tag{19}$$

for every $\alpha \in (0,1)\cap\mathbb{Q}$. The same also holds for irrational $\alpha \in (0,1)$ since C is a uniform limit of periodic sequences. Hence, equation (19) holds for all $\alpha \in (0,1)$ and the theory of Bohr almost periodic sequences implies that C is constant. (This follows from classical results of Besicovitch [3], which are proved for functions but apply equally well to sequences. Alternatively, one could use [2, Theorem 2.7] with the uniform norm in place of the Besicovitch norm.) □

We will also use the following result of Tao and Teräväinen [45].

THEOREM 4.5. [45] *Let $f_1,\dots,f_\ell\colon \mathbb{N}\to\mathbb{U}$ be arbitrary multiplicative functions. Consider a subsequence $N_k\to\infty$ along which the limits below exist:*

$$C(r) := \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} \prod_{j=1}^{\ell} f_j(n+jr)$$

for every $r\in\mathbb{N}$. Then, the sequence C is a uniform limit of periodic sequences.

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. The implication (ii) $\Longrightarrow$ (i) is a general fact that holds for all strongly stationary systems and follows from [28] (the proof is given for ergodic systems, but the same argument applies for general systems). We prove now the more interesting implication (i) $\Longrightarrow$ (ii), which is a very particular property of Furstenberg systems for logarithmic averages of bounded multiplicative functions.

Let $N_k\to\infty$ be the subsequence along which the Furstenberg system (X,μ,T) of f for logarithmic averages is defined. Recall that by Definition 3.10, we have to verify that identity (9) holds. Let $\ell\in\mathbb{N}$ and $f_1,\dots,f_\ell\in\{f,\overline{f}\}$, $n_1,\dots,n_\ell\in\mathbb{Z}$. For $F_1,\dots,F_\ell\in\{F_0,\overline{F_0}\}$, we define the sequence

$$C(r) := \int \prod_{j=1}^{\ell} T^{rn_j} F_j\, d\mu, \quad r\in\mathbb{N}. \tag{20}$$

By equation (8), we have

$$C(r) = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} \prod_{j=1}^{\ell} f_j(n+rn_j), \quad r\in\mathbb{N},$$

where if F_j is F_0, $\overline{F_0}$, then respectively f_j is f, $\overline{f}$. By Theorem 4.5, the sequence C is a uniform limit of periodic sequences. Since, additionally, the system (X,μ,T) has trivial rational spectrum, by Corollary 4.4, the sequence C is constant. We have thus established that identity (9) holds for all $\ell, r\in\mathbb{N}$ and $F_1,\dots,F_\ell\in\{F_0,\overline{F_0}\}$. Hence, the system (X,μ,T) is strongly stationary. □

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5. *Preliminary results for pretentious multiplicative functions*

The main goal of this section is to do some preparatory work that will be used in the next two sections to prove our main structural results for Furstenberg systems of pretentious multiplicative functions. The main ingredient needed for later use is Proposition 5.5, which establishes a key subsequential Besicovitch rational almost periodicity property for all multiplicative functions that pretend to be Dirichlet characters. This fact is proved via the decomposition result of Lemma 5.4, which in turn follows from the estimate in Lemma 5.3 that we prove in the next subsection.

5.1. *Preliminary estimates.*

Our main goal in this subsection is to establish the estimate in Lemma 5.3, which we were not able to find in the form needed in the literature. See though [**31**, Proposition 2.3] and [**32**, Lemma 2.5] for closely related concentration inequalities.

LEMMA 5.1. *Let $f: \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim 1$. For every $p \in \mathbb{P}$, we can write $f(p) = r_p\, e(\theta_p)$ for some $r_p \in [0, 1]$ and $\theta_p \in [-1/2, 1/2)$. Then,*

$$\sum_{p\in\mathbb{P}} \frac{1-r_p}{p} < +\infty \quad \text{and} \quad \sum_{p\in\mathbb{P}} \frac{\theta_p^2}{p} < +\infty.$$

Proof. Since $f \sim 1$, we have

$$\sum_{p\in\mathbb{P}} \frac{1-r_p\cos(\theta_p)}{p} < +\infty.$$

From this, we get immediately that the first series converges. If we use this fact and add and subtract $\cos(\theta_p)$ into the numerator of the last series, we deduce that the series $\sum_{p\in\mathbb{P}}((1-\cos(\theta_p))/p)$ converges, from which the convergence of the second series readily follows (we crucially use here that θ_p is in $[-1/2, 1/2)$ and not in $[0, 1)$). □

LEMMA 5.2. *Let $f: \mathbb{N} \to \mathbb{S}^1$ be a multiplicative function such that for every $p \in \mathbb{P}$, we have $f(p^s) := e(\theta_p)$ for some $\theta_p \in [-1/2, 1/2)$ and all $s \in \mathbb{N}$. For $N \in \mathbb{N}$, let*

$$A(N) := \sum_{p\in\mathbb{P}\cap[N]} \frac{\theta_p}{p}. \tag{21}$$

Then, for some universal constant $C > 0$, we have

$$\mathbb{E}_{n\in[N]}|f(n) - e(A(N))|^2 \le C\left(\sum_{p\in\mathbb{P}\cap[N]} \frac{\theta_p^2}{p} + \frac{\log\log N}{\log N}\right). \tag{22}$$

Proof. For convenience, we use probabilistic language. The reader can translate the argument to conventional number theoretic language by replacing X_p with $\theta_p \cdot \mathbf{1}_{p\mathbb{Z}}$ and $\mathbb{E}_N$ with $\mathbb{E}_{n\in[N]}$ throughout.

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For a fixed $N \in \mathbb{N}$, we consider the finite probability space that consists of the interval $[N]$ together with the probability measure that assigns mass $1/N$ to each point in $[N]$. For a prime $p \in [N]$, we define the random variable $X_p \colon [N] \to \{0, \theta_p\}$ by

$$X_p(n) := \begin{cases} \theta_p & \text{if } p \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

These random variables are relevant for our problem because if we let

$$S_N := \sum_{p \in \mathbb{P} \cap [N]} X_p,$$

then from the definition of f on powers of primes and its multiplicativity, we have

$$f(n) = e(S_N(n)), \quad n \in [N].$$

Note that if

$$a(N) := \mathbb{E}_N(S_N),$$

then

$$\mathbb{E}_{n \in [N]} |f(n) - e(a(N))|^2 \le 4\pi^2 \, \mathbb{E}_{n \in [N]} |S_N(n) - a(N)|^2 = 4\pi^2 \cdot \mathrm{Var}_N(S_N), \tag{23}$$

where to get the first estimate, we used that $|e(x) - e(y)| = 2\pi \left| \int_x^y e(t)\, dt \right| \le 2\pi |x - y|$.

So, our problem reduces to getting an upper bound for $\mathrm{Var}_N(S_N)$. We start with some easy computations that give

$$\mathbb{E}_N(X_p) = \frac{\theta_p}{N} \left\lfloor \frac{N}{p} \right\rfloor, \tag{24}$$

$$\mathrm{Var}_N(X_p) = \mathbb{E}_N(X_p^2) - (\mathbb{E}_N(X_p))^2 \le \frac{\theta_p^2}{p},$$

$$\mathrm{Cov}_N(X_p, X_q) = \mathbb{E}_N(X_p \cdot X_q) - \mathbb{E}_N(X_p) \cdot \mathbb{E}_N(X_q) = \frac{\theta_p \, \theta_q}{N} \left(\left\lfloor \frac{N}{pq} \right\rfloor - \frac{1}{N} \left\lfloor \frac{N}{p} \right\rfloor \cdot \left\lfloor \frac{N}{q} \right\rfloor \right).$$

To get the last equality, we assume that $p \neq q$ and use that $\mathbb{E}_N(X_p \cdot X_q) = \theta_p \, \theta_q / N \, \lfloor N/pq \rfloor$, which follows from the primality of p, q.

Now, let us assume first that all θ_p values have the same sign. Then, using the inequality

$$\left\lfloor \frac{N}{pq} \right\rfloor - \frac{1}{N} \left\lfloor \frac{N}{p} \right\rfloor \cdot \left\lfloor \frac{N}{q} \right\rfloor \le \frac{1}{p} + \frac{1}{q},$$

we get for all $p \neq q$,

$$\mathrm{Cov}_N(X_p, X_q) \le \frac{\theta_p \, \theta_q}{N} \left(\frac{1}{p} + \frac{1}{q} \right).$$

Using these estimates and the identity

$$\mathrm{Var}_N(S_N) = \sum_{p \in \mathbb{P} \cap [N]} \mathrm{Var}_N(X_p) + \sum_{p,q \in \mathbb{P} \cap [N], p \neq q} \mathrm{Cov}_N(X_p \cdot X_q),$$

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we deduce that

$$\mathrm{Var}_N(S_N) \le \sum_{p\in\mathbb{P}\cap[N]} \frac{\theta_p^2}{p} + \frac{1}{N} \sum_{p,q\in\mathbb{P}\cap[N],p\neq q} \theta_p\,\theta_q\Big(\frac{1}{p}+\frac{1}{q}\Big).$$

Since $\theta_p \in [-1/2, 1/2)$, $p \in \mathbb{P}$, we can bound the second sum by $\sum_{p,q\in\mathbb{P}\cap[N],p\neq q} 1/p$, which in turn is bounded by $C_1\ (N/\log N)\ \log\log N$ for some universal constant C_1. Hence,

$$\mathrm{Var}_N(S_N) \le \sum_{p\in\mathbb{P}\cap[N]} \frac{\theta_p^2}{p} + C_1\,\frac{\log\log N}{\log N}. \tag{25}$$

The bound (22), with $a(N)$ in place of $A(N)$, then follows by combining equations (23) and (25). To get equation (22), as stated, note that by equation (24),

$$a(N) = \sum_{p\in\mathbb{P}\cap[N]} \frac{\theta_p}{N}\left\lfloor \frac{N}{p}\right\rfloor$$

and

$$|a(N) - A(N)| \le \frac{C_2}{\log N}$$

for some universal constant $C_2 > 0$.

This completes the proof in the case where all the θ_p values have the same sign.

In the general case, we can decompose f as a product $f_+ \cdot f_-$, where f_+ and f_- are multiplicative functions defined by

$$f_+(p^s) := \begin{cases} e(\theta_p) & \text{if } \theta_p \ge 0,\\ 1 & \text{otherwise,}\end{cases} \quad\text{and}\quad f_-(p^s) := \begin{cases} e(\theta_p) & \text{if } \theta_p < 0,\\ 1, & \text{otherwise,}\end{cases}$$

for every $p \in \mathbb{P}$ and every $s \in \mathbb{N}$. Now, let

$$A_+(N) := \sum_{\substack{p\in\mathbb{P}\cap[N],\\ \theta_p\ge 0}} \frac{\theta_p}{p} \quad\text{and}\quad A_-(N) := \sum_{\substack{p\in\mathbb{P}\cap[N],\\ \theta_p< 0}} \frac{\theta_p}{p}.$$

Then, we have for all $n \in [N]$,

$$\begin{aligned} |f(n) - e(A(N))| &= |f_+(n)\cdot f_-(n) - e(A_+(N))\cdot e(A_-(N))|\\ &\le |f_+(n) - e(A_+(N))| + |f_-(n) - e(A_-(N))|. \end{aligned}$$

It follows that

$$\mathbb{E}_N|f(n) - e(A(N))|^2 \le 2\cdot(\mathbb{E}_N|f_+(n) - e(A_+(N))|^2 + \mathbb{E}_N|f_-(n) - e(A_-(N))|^2),$$

and we get the conclusion by applying the preceding analysis to f_+ and f_-. □

The next estimate will be key for us.

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LEMMA 5.3. *Let $f\colon \mathbb{N}\to\mathbb{S}^1$ be a multiplicative function such that $f\sim 1$. Suppose that for every $p\in\mathbb{P}$, we have $f(p^s):=e(\theta_p)$ for some $\theta_p\in[-1/2,1/2)$ and all $s\in\mathbb{N}$. Then, there exist a slowly varying sequence $A\colon\mathbb{N}\to\mathbb{R}_+$ (see Definition 3.2), defined as in equation (21), and a universal constant $C>0$, such that*

$$\limsup_{N\to\infty}\mathbb{E}_{n\in[N]}|f(n)-e(A(N))|^2\le C\sum_{p\in\mathbb{P}}\frac{\theta_p^2}{p}.$$

Furthermore, the last estimate also holds with $\mathbb{E}^{\log}_{n\in[N]}$ in place of $\mathbb{E}_{n\in[N]}$ (without changing $A(N)$).

Remark. Recall that, by Lemma 5.1, we have $\sum_{p\in\mathbb{P}}(\theta_p^2/p)<+\infty$. The lemma will be used when $\sum_{p\in\mathbb{P}}\theta_p^2/p$ is small, to deduce that $\limsup_{N\to\infty}\mathbb{E}_{n\in[N]}|f(n)-e(A(N))|^2$ is small.

Proof. We first work with Cesàro averages. In this case, the estimate follows immediately from Lemma 5.2 and it remains to show that the sequence in equation (21) is slowly varying. So let $c\in(0,1)$. Using equation (21) and the Cauchy–Schwarz inequality, we get

$$\sup_{n\in[N^c,N]}|A(n)-A(N)|\le\sum_{p\in\mathbb{P}\cap[N^c,N]}\frac{|\theta_p|}{p}\le(B_N\cdot C_N)^{\frac12},$$

where

$$B_N:=\sum_{p\in\mathbb{P}\cap[N^c,N]}\frac{\theta_p^2}{p},\quad C_N:=\sum_{p\in\mathbb{P}\cap[N^c,N]}\frac{1}{p}.$$

We have $\sum_{p\in\mathbb{P}}(\theta_p^2/p)<+\infty$ and this implies $\lim_{N\to\infty}B_N=0$. We also have $\lim_{N\to\infty}C_N=\log(1/c)$, and hence $\lim_{N\to\infty}(B_N\cdot C_N)=0$.

Lastly, we deal with logarithmic averages. Let $c\in(0,1)$. Since the sequence $A(N)$ is slowly varying, it can be treated as a constant on intervals $[N^c,N]$ for N large enough, and using partial summation, we deduce

$$\limsup_{N\to\infty}\mathbb{E}^{\log}_{n\in[N^c,N]}|f(n)-e(A(N))|^2\le C\sum_{p\in\mathbb{P}}\frac{\theta_p^2}{p}.$$

Letting $c\to0^+$, we get the desired estimate. □

5.2. *A decomposition.* If a multiplicative function pretends to be a Dirichlet character, then it is possible to decompose it into a product of two terms, one is a multiplicative function for which Theorem 2.6 is satisfied and the other is a multiplicative function that is approximately constant in density (Lemma 5.3 will allow us to conclude this).

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LEMMA 5.4. *Let $f: \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim \chi$ for some primitive Dirichlet character χ. Then, for every $\varepsilon > 0$, there exists a slowly varying sequence $A_\varepsilon: \mathbb{N} \to \mathbb{R}_+$ such that the following holds. We can decompose f as*

$$f = f_{1,\varepsilon} \cdot f_{2,\varepsilon}, \tag{26}$$

where:

(i) *$f_{1,\varepsilon}: \mathbb{N} \to \mathbb{U}$ is a multiplicative function such that*

$$\sum_{p \in \mathbb{P}} \frac{1}{p}(1 - f_{1,\varepsilon}(p) \cdot \overline{\chi(p)}) \quad \text{converges}; \tag{27}$$

(ii) *$f_{2,\varepsilon}: \mathbb{N} \to \mathbb{S}^1$ is a multiplicative function such that*

$$\limsup_{N \to \infty} \mathbb{E}_{n \in [N]} |f_{2,\varepsilon}(n) - e(A_\varepsilon(N))|^2 \leq \varepsilon. \tag{28}$$

Furthermore, if $f \sim 1$ and for some $p_0 \in \mathbb{P}$ we have $f(p_0^s) = 1$ for every $s \in \mathbb{N}$, then we can also ensure that $f_{1,\varepsilon}$ in part (i) satisfies $f_{1,\varepsilon}(p_0^s) = 1$ for every $s \in \mathbb{N}$. (This will be used in the proof of the implication (i) $\Longrightarrow$ (ii) *of Theorem 2.11 in §7.3.)*

Lastly, in part (ii), we can replace $\mathbb{E}_{n \in [N]}$ with $\mathbb{E}^{\log}_{n \in [N]}$ (without changing $A_\varepsilon(N)$).

Remark. If $f \sim 1$, then we can take $A_\varepsilon(N) := \sum_{p \in \mathbb{P} \cap [P_\varepsilon, N]} \theta_p/p$, where P_ε satisfies $\sum_{p \in \mathbb{P}, p > P_\varepsilon} (\theta_p^2/p) \leq \varepsilon/C$ and C is the universal constant defined in Lemma 5.3.

Proof. We give the argument for Cesàro averages, the proof is similar for logarithmic averages since we can use the variant of Lemma 5.3 that covers logarithmic averages.

Suppose first that $f \sim 1$. Let $\varepsilon > 0$. We can write $f(p) = r_p\, e(\theta_p)$ for some $r_p \in [0, 1]$ and $\theta_p \in [-1/2, 1/2)$, $p \in \mathbb{P}$. Since $f \sim 1$, we have by Lemma 5.1 that

$$\sum_{p \in \mathbb{P}} \frac{1 - r_p}{p} < +\infty \quad \text{and} \quad \sum_{p \in \mathbb{P}} \frac{\theta_p^2}{p} < +\infty. \tag{29}$$

Let $P_\varepsilon \in \mathbb{N}$ be such that

$$\sum_{p \in \mathbb{P}, p > P_\varepsilon} \frac{\theta_p^2}{p} < \frac{\varepsilon}{C}, \tag{30}$$

where $C > 0$ is the universal constant defined in Lemma 5.3. We define the multiplicative function $f_{1,\varepsilon}: \mathbb{N} \to \mathbb{U}$ by

$$f_{1,\varepsilon}(p^s) := \begin{cases} f(p^s) & \text{for } p \leq P_\varepsilon,\ s \in \mathbb{N}, \\ f(p^s)\, e(-\theta_p) & \text{for } p > P_\varepsilon,\ s \in \mathbb{N}, \end{cases} \tag{31}$$

and the multiplicative function $f_{2,\varepsilon}: \mathbb{N} \to \mathbb{S}^1$ by

$$f_{2,\varepsilon}(p^s) := \begin{cases} 1 & \text{for } p \leq P_\varepsilon,\ s \in \mathbb{N}, \\ e(\theta_p) & \text{for } p > P_\varepsilon,\ s \in \mathbb{N}. \end{cases}$$

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Then, equation (26) is clearly satisfied and equation (29) implies that equation (27) holds (with $\chi = 1$).

It remains to verify equation (28). If we set $\theta_p := 0$ for all primes $p \le P_\varepsilon$, then we have $f_{2,\varepsilon}(p^s) = e(\theta_p)$ for all primes $p \in \mathbb{P}$ and $s \in \mathbb{N}$, and by equation (30), we have

$$\sum_{p\in\mathbb{P}} \frac{\theta_p^2}{p} \le \frac{\varepsilon}{C}. \tag{32}$$

Combining this with Lemma 5.3, we get that there exists a slowly varying sequence $A_\varepsilon : \mathbb{N} \to [0, 1)$ such that equation (28) holds.

We consider now the general case, where $f \sim \chi$ for some primitive Dirichlet character χ. Let $\tilde{\chi} : \mathbb{N} \to \mathbb{U}$ be the multiplicative function defined by

$$\tilde{\chi}(p^s) := \begin{cases} \chi(p^s) & \text{if } \chi(p^s) \neq 0, \\ 1 & \text{if } \chi(p^s) = 0. \end{cases}$$

We introduce this variant of χ because it can be inverted. Since $\tilde{\chi}(p) = \chi(p)$ for all but finitely many primes p, we have $\tilde{f} := f \cdot \overline{\tilde{\chi}} \sim 1$. The previous case gives a decomposition

$$\tilde{f} = \tilde{f}_{1,\varepsilon} \cdot \tilde{f}_{2,\varepsilon},$$

where $\tilde{f}_{1,\varepsilon}$ satisfies property (i) with $\chi = 1$, and $\tilde{f}_{2,\varepsilon}$ satisfies property (ii) for some slowly varying sequence $A_\varepsilon : \mathbb{N} \to \mathbb{R}_+$. It follows that

$$f = f_{1,\varepsilon} \cdot f_{2,\varepsilon},$$

where $f_{1,\varepsilon} := \tilde{f}_{1,\varepsilon} \cdot \tilde{\chi}$ and $f_{2,\varepsilon} := \tilde{f}_{2,\varepsilon}$. Then, $f_{2,\varepsilon}$ clearly satisfies property (ii). Also, $f_{1,\varepsilon}(p) \cdot \overline{\chi(p)} = \tilde{f}_{1,\varepsilon}(p)$ for all but finitely many primes p, and hence $f_{1,\varepsilon}$ satisfies property (i) for this χ.

Lastly, suppose that $f \sim 1$ and for some $p_0 \in \mathbb{P}$, we have $f(p_0^s) = 1$ for every $s \in \mathbb{N}$. Then, $e(\theta_{p_0}) = 1$ and by equation (31), we have $f_{1,\varepsilon}(p_0^s) = 1$ for every $s \in \mathbb{N}$. This completes the proof. □

5.3. *Besicovitch rational almost periodicity along a subsequence.* The goal of this subsection is to show that if a multiplicative function pretends to be a Dirichlet character, then it has strong rational almost periodicity properties, in the sense described in Propositions 5.5 and 5.6.

For the purpose of studying Furstenberg systems of pretentious multiplicative functions, we will only use Proposition 5.5, which is better adapted to our needs.

PROPOSITION 5.5. *Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim \chi$ for some primitive Dirichlet character χ. Then, there exist periodic sequences $f_m : \mathbb{N} \to \mathbb{U}$, $m \in \mathbb{N}$, such that every sequence $N_k \to \infty$ has a subsequence $N_k' \to \infty$ for which the following holds. For every $m \in \mathbb{N}$, there exists $\alpha_m \in [0, 1)$ such that*

$$\limsup_{k\to\infty} (\mathbb{E}_{n\in[N_k']} |f(n) - e(\alpha_m) \cdot f_m(n)|^2 + \mathbb{E}^{\log}_{n\in[N_k']} |f(n) - e(\alpha_m) \cdot f_m(n)|^2) \le 1/m. \tag{33}$$

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In particular, f is Besicovitch rationally almost periodic for Cesàro and logarithmic averages along (N_k').

Remark. We stress that although the choices of α_m depend on the subsequence (N_k'), the choice of the periodic sequences f_m depend only on f and on m. Also, one could use in place of the periodic sequences f_m the multiplicative functions $f_{1,1/m}$ defined in Lemma 5.4 (for $\varepsilon := 1/m$). These functions satisfy equation (27), and moreover, if for some $p_0 \in \mathbb{P}$ we have $f(p_0^s) = 1$ for every $s \in \mathbb{N}$, then we can also ensure that f_m satisfies $f_m(p_0^s) = 1$ for every $s \in \mathbb{N}$.

Proof. For $\varepsilon > 0$, let $f = f_{1,\varepsilon} \cdot f_{2,\varepsilon}$ be the decomposition given by Lemma 5.4 for the slowly varying sequence $(A_\varepsilon(N))$. Using a diagonal argument, we can find a subsequence (N_k') of (N_k) such that the limit $\lim_{k\to\infty} e(A_{1/m}(N_k'))$ exists for every $m \in \mathbb{N}$. Let $\alpha_m \in [0, 1)$ be such that

$$e(\alpha_m) = \lim_{k\to\infty} e(A_{1/m}(N_k')), \quad m \in \mathbb{N}.$$

Then, equation (28) implies that

$$\limsup_{k\to\infty}(\mathbb{E}_{n\in[N_k']}|f_{2,1/m}(n) - e(\alpha_m)|^2 + \mathbb{E}^{\log}_{n\in[N_k']}|f_{2,1/m}(n) - e(\alpha_m)|^2) \le \frac{2}{m}$$

for every $m \in \mathbb{N}$. Since $f = f_{1,1/m} \cdot f_{2,1/m}$ and $f_{1,1/m}$ is 1-bounded, we deduce that

$$\limsup_{k\to\infty}(\mathbb{E}_{n\in[N_k']}|f(n) - e(\alpha_m)\cdot f_{1,1/m}(n)|^2 + \mathbb{E}^{\log}_{n\in[N_k']}|f(n) - e(\alpha_m)\cdot f_{1,1/m}(n)|^2) \le \frac{2}{m}$$

for every $m \in \mathbb{N}$. The sequences $f_{1,1/m}$ satisfy property (27), and hence they are Besicovitch rationally almost periodic for Cesàro averages by Theorem 2.6. It follows that there exist periodic sequences $f_m : \mathbb{N} \to \mathbb{U}$, such that

$$\limsup_{N\to\infty}(\mathbb{E}_{n\in[N]}|f_m(n) - f_{1,1/m}(n)|^2 + \mathbb{E}^{\log}_{n\in[N]}|f_m(n) - f_{1,1/m}(n)|^2) \le \frac{1}{m}.$$

Combining the above, we get the asserted estimate with a multiple of $1/m$ in place of $1/m$. Lastly, note that the choices of $f_{1,1/m}$ and f_m do not depend on the subsequence N_k', but only on f and on m. This completes the proof. □

Next, we give a similar result that avoids passing to subsequences. It is of independent interest and not needed for any other result in this article.

PROPOSITION 5.6. *Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim \chi$ for some primitive Dirichlet character χ, and $N_k \to \infty$ be a sequence of integers such that the limit*

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, f(n)\cdot\overline{\chi(n)} \quad \textit{exists.}$$

Suppose that $f(2^s)\cdot\overline{\chi(2^s)} \neq -1$ for some $s \in \mathbb{N}$. (The condition is always satisfied if f is completely multiplicative.) Then, for every $\varepsilon > 0$, there exists $\alpha_\varepsilon \in [0, 1)$ such that

$$\limsup_{k\to\infty} \mathbb{E}_{n\in[N_k]}|f(n) - e(\alpha_\varepsilon)\cdot f_\varepsilon(n)|^2 \le \varepsilon$$

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for some periodic sequence $f_\varepsilon\colon \mathbb{N}\to\mathbb{U}$ that depends only on f and ε, and not on the subsequence N_k. Furthermore, a similar statement holds if in the assumption and the conclusion, we replace $\mathbb{E}_{n\in[N_k]}$ with $\mathbb{E}^{\log}_{n\in[N_k]}$.

Proof. We first give the argument for Cesàro averages. For $\varepsilon>0$, let $f=f_{1,\varepsilon}\cdot f_{2,\varepsilon}$ be the decomposition given by Lemma 5.4 for the slowly varying sequence $A_\varepsilon\colon\mathbb{N}\to\mathbb{R}_+$ that satisfies

$$\limsup_{N\to\infty}\mathbb{E}_{n\in[N]}|f_{2,\varepsilon}(n)-e(A_\varepsilon(N))|^2\le\varepsilon. \tag{34}$$

Using that $f_{1,\varepsilon}$ and χ are 1-bounded, we deduce

$$\limsup_{k\to\infty}|\mathbb{E}_{n\in[N_k]}\ f(n)\cdot\overline{\chi(n)}-e(A_\varepsilon(N_k))\cdot\mathbb{E}_{n\in[N_k]}\ f_{1,\varepsilon}(n)\cdot\overline{\chi(n)}|^2\le\varepsilon.$$

Let $L:=\lim_{k\to\infty}\mathbb{E}_{n\in[N_k]}\ f(n)\cdot\overline{\chi(n)}$ and $L_\varepsilon:=\lim_{k\to\infty}\mathbb{E}_{n\in[N_k]}\ f_{1,\varepsilon}(n)\cdot\overline{\chi(n)}$; the first limit exists by our assumption, and the second because $f_{1,\varepsilon}$ satisfies equation (27); hence Theorem 2.6 is applicable. Furthermore, since $f\cdot\overline{\chi}\sim 1$ and we do not have $f(2^s)\cdot\overline{\chi(2^s)}=-1$ for every $s\in\mathbb{N}$, Theorem 3.1 gives that $L\neq 0$. It follows that

$$\limsup_{k\to\infty}|e(-A_\varepsilon(N_k))-L_\varepsilon/L|^2\le\varepsilon/|L|^2.$$

Hence (since $|e(-A_\varepsilon(N_k))|=1$), there exists $\alpha_\varepsilon\in[0,1)$ such that

$$\limsup_{k\to\infty}|e(A_\varepsilon(N_k))-e(\alpha_\varepsilon)|^2\le 4\varepsilon/|L|^2.$$

Using this, the identity $f=f_{1,\varepsilon}\cdot f_{2,\varepsilon}$, the fact that $|L|\le 1$, and again the estimate (34), we get

$$\limsup_{k\to\infty}\mathbb{E}_{n\in[N_k]}|f(n)-e(\alpha_\varepsilon)\cdot f_{1,\varepsilon}(n)|^2\le 10\varepsilon/|L|^2.$$

The sequences $f_{1,\varepsilon}$ satisfy property (27), and hence they are Besicovitch rationally almost periodic for Cesàro averages by Theorem 2.6. The result follows easily from this.

The proof is similar for logarithmic averages since we can use the variant of Lemma 5.4 that covers logarithmic averages and, also in the case of logarithmic averages, we get that $L\neq 0$ by the variant of Theorem 3.1 that covers logarithmic averages (see the second remark following the theorem). □

6. *Structural results for pretentious multiplicative functions*

The goal of this section is to prove Theorem 2.8, which is the main structural result for Furstenberg systems of pretentious multiplicative functions given in §2.2.2. In §6.1, we do some preparatory work that enables us, for multiplicative functions that pretend to be Dirichlet characters, to work interchangeably with Furstenberg systems defined using Cesàro or logarithmic averages. In §6.2, we show that certain correlation limits of multiplicative functions that pretend to be Dirichlet characters exist, and conclude that all Furstenberg systems of such multiplicative functions for Cesáro or logarithmic averages are isomorphic (a property that we will later extend to all pretentious multiplicative functions).

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In §6.3, we use these preparatory results in conjunction with other ergodic considerations to conclude the proof of Theorem 2.8.

6.1. *Cesàro and logarithmic correlations agree when $f \sim \chi$.* The goal of this subsection is to show in Proposition 6.2 that if a multiplicative function pretends to be a Dirichlet character, then we can use interchangeably Cesàro and logarithmic averages in the definition of its Furstenberg systems. This is a consequence of the following correlation identity.

PROPOSITION 6.1. *Let $f: \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim \chi$ for some primitive Dirichlet character χ. Then,*

$$\lim_{N\to\infty} \left| \mathbb{E}_{n\in[N]} \prod_{j=1}^{\ell} f^{k_j}(n+n_j) - \mathbb{E}^{\log}_{n\in[N]} \prod_{j=1}^{\ell} f^{k_j}(n+n_j) \right| = 0 \tag{35}$$

for all $\ell \in \mathbb{N}$, $n_1, \dots, n_\ell, k_1, \dots, k_\ell \in \mathbb{Z}$, where we let $f^k := \overline{f}^{|k|}$ for $k < 0$.

Remark. The result fails if $f = n^{it}$, $t \neq 0$, and $\ell = 1, k_1 = 1$. In this case, the logarithmic averages converge to 0, but the Cesàro averages behave like $N^{it}/(1+it)$. It can actually be seen that it fails for all completely multiplicative functions that satisfy $f \sim n^{it}$ for some $t \neq 0$.

Proof. Arguing by contradiction, suppose that the conclusion fails. Then, for some choice of $\ell \in \mathbb{N}, n_1, \dots, n_\ell, k_1, \dots, k_\ell \in \mathbb{Z}$, there exists $N_k \to \infty$ along which the difference of the averages in equation (35) is bounded away from zero. By Proposition 5.5, there exists a subsequence (N_k') of (N_k), such that for every $m \in \mathbb{N}$, we have

$$\limsup_{k\to\infty} (\mathbb{E}_{n\in[N_k']} |f(n) - g_m(n)|^2 + \mathbb{E}^{\log}_{n\in[N_k']} |f(n) - g_m(n)|^2) \leq 1/m \tag{36}$$

for some periodic sequence $g_m: \mathbb{N} \to \mathbb{U}$. Since equation (35) clearly holds when we replace f with the periodic sequence g_m, using equation (36), we deduce that equation (35) also holds for f as long as we average over the sequence of intervals $[N_k']$. This is a contradiction, since we have assumed that along the sequence (N_k), the difference of the averages in equation (35) is bounded away from zero, and (N_k') is a subsequence of (N_k). □

We immediately deduce the following.

PROPOSITION 6.2. *Let $f: \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim \chi$ for some primitive Dirichlet character χ. Let also $N_k \to \infty$. Then, the Furstenberg system of f for Cesàro averages along (N_k) is well defined if and only if it is well defined for logarithmic averages along (N_k), and the two Furstenberg systems are equal (meaning the corresponding T-invariant measures agree).*

Proof. Suppose that the Furstenberg system of f for Cesàro averages is defined along the sequence (N_k) (the argument is similar if we assume it is defined for logarithmic averages). Then, Proposition 6.1 implies that the Furstenberg system for logarithmic averages is also

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well defined along (N_k) and the measures that define the two Furstenberg systems coincide, since they agree on all cylinder functions defined in the remark after Definition 3.5. □

6.2. *Correlation limits and isomorphism when $f \sim \chi$.* The goal of this subsection is to show in Proposition 6.4 that different Furstenberg systems of a fixed multiplicative function that pretends to be a Dirichlet character are isomorphic. This is a consequence of the correlation identities stated in the following lemma.

LEMMA 6.3. *Let $f\colon \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim \chi$ for some primitive Dirichlet character χ. If $N_{k,1} \to \infty$ and $N_{k,2} \to \infty$ are sequences, then there exist subsequences $(N'_{k,1})$ and $(N'_{k,2})$ and $\alpha \in [0, 1)$, such that the following holds. For all $\ell \in \mathbb{N}$ and $n_1, \dots, n_\ell, k_1, \dots, k_\ell \in \mathbb{Z}$, both correlation limits below exist and we have*

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N'_{k,1}]} \prod_{j=1}^{\ell} (e(\alpha)\cdot f)^{k_j}(n+n_j) = \lim_{k\to\infty} \mathbb{E}_{n\in[N'_{k,2}]} \prod_{j=1}^{\ell} f^{k_j}(n+n_j), \tag{37}$$

where we let $f^k := \overline{f}^{|k|}$ for $k < 0$. A similar identity also holds for logarithmic averages.

Remark. More generally, if $f \sim n^{it}\cdot\chi$ for some $t \in \mathbb{R}$ and primitive Dirichlet character χ, then we get a similar result with the identity (37) replaced by the identity

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N'_{k,1}]} n^{it'} \cdot \prod_{j=1}^{\ell} (e(\alpha)\cdot f)^{k_j}(n+n_j) = \lim_{k\to\infty} \mathbb{E}_{n\in[N'_{k,2}]} n^{it'} \cdot \prod_{j=1}^{\ell} f^{k_j}(n+n_j), \tag{38}$$

where $t' := -t\cdot\sum_{j=1}^{\ell} k_j$. To see this, use equation (37) for the multiplicative function $f\cdot n^{-it} \sim \chi$ together with the fact that $\lim_{n\to\infty}((n+h)^{it} - n^{it}) = 0$ for every $h \in \mathbb{N}$.

Proof. We give the argument for Cesàro averages; a similar argument applies to logarithmic averages.

By Proposition 5.5 (and the remark following it), for every $m \in \mathbb{N}$, there exist periodic sequences f_m, subsequences $(N'_{k,1})$ of $(N_{k,1})$ and $(N'_{k,2})$ of $(N_{k,2})$, and $\alpha_{1,m}, \alpha_{2,m} \in [0, 1)$, such that

$$\limsup_{k\to\infty} \mathbb{E}_{n\in[N'_{k,1}]} |f(n) - e(\alpha_{1,m})\cdot f_m(n)|^2 \le 1/m \tag{39}$$

and

$$\limsup_{k\to\infty} \mathbb{E}_{n\in[N'_{k,2}]} |f(n) - e(\alpha_{2,m})\cdot f_m(n)|^2 \le 1/m. \tag{40}$$

(Note that the function f_m is the same in both cases.) By working with further subsequences of $(N'_{k,1})$ and $(N'_{k,2})$ along which the sequences $e(\alpha_{1,m})$ and $e(\alpha_{2,m})$ converge, we can assume that equations (39) and (40) hold with $e(\alpha_1)$ in place of $e(\alpha_{1,m})$ and $e(\alpha_2)$ in place of $e(\alpha_{2,m})$ for some $\alpha_1, \alpha_2 \in [0, 1)$. We let

$$\alpha := \alpha_2 - \alpha_1. \tag{41}$$

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Using a diagonal argument, we can find further subsequences that we denote again by $(N'_{k,1})$ and $(N'_{k,2})$ such that all the correlation limits in equation (37) exist. Using equations (39) and (40), we see that it suffices to verify equation (37) with $f(n)$ replaced by $e(\alpha_1)\cdot f_m(n)$ on the first average and by $e(\alpha_2)\cdot f_m(n)$ on the second average. Taking into account equation (41), it suffices to verify that

$$\lim_{k\to\infty}\left|\mathbb{E}_{n\in[N'_{k,1}]}\prod_{j=1}^{\ell} f_m^{k_j}(n+n_j)-\mathbb{E}_{n\in[N'_{k,2}]}\prod_{j=1}^{\ell} f_m^{k_j}(n+n_j)\right|=0$$

for all $\ell\in\mathbb{N}$ and $n_1,\dots,n_\ell,k_1,\dots,k_\ell\in\mathbb{Z}$. Since the sequence f_m is periodic, both averages have limits and these limits coincide, therefore, the asserted identity follows. □

PROPOSITION 6.4. *Let $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function such that $f\sim\chi$ for some primitive Dirichlet character χ. Then, any two Furstenberg systems of f for Cesàro or logarithmic averages are isomorphic.*

Remark. With substantial additional effort, the result will be extended to arbitrary pretentious multiplicative functions in the next subsection, where we prove part (iii) of Theorem 2.8.

Proof. By Proposition 6.2, we have that Furstenberg systems for Cesàro and logarithmic averages coincide, and hence it suffices to treat the case of Cesàro averages. Let (X,μ,T) and (X,μ',T) be two Furstenberg systems for Cesàro averages taken along the sequences $N_{k,1}\to\infty$ and $N_{k,2}\to\infty$, respectively. Using Lemma 6.3 and equation (8) to translate equation (37) into identities for the two Furstenberg systems, we get that there exists $\alpha\in[0,1)$ such that

$$\int\prod_{j=1}^{\ell} T^{n_j}(e(\alpha)\cdot F_0)^{k_j}\,d\mu=\int\prod_{j=1}^{\ell} T^{n_j}F_0^{k_j}\,d\mu' \tag{42}$$

holds for all $\ell\in\mathbb{N}$ and $n_1,\dots,n_\ell,k_1,\dots,k_\ell\in\mathbb{Z}$, where as usual F_0 is the 0th-coordinate projection. This easily implies that the two systems are isomorphic with the isomorphism given by the map $M_{e(\alpha)}:X\to X$ defined by equation (12). Indeed, $M_{e(\alpha)}$ is bijective, satisfies the commutation relation (13), and since $F_0\circ M_{e(\alpha)}=e(\alpha)\cdot F_0$, equation (42) implies that $\mu'=(M_{e(\alpha)})_*\mu$, (the two measures agree on the linearly dense subset of $C(X)$ given by the cylinder functions, see remark after Definition 3.5). □

6.3. *Proof of Theorems 2.7–2.8.* Theorem 2.7 is an immediate consequence of Theorem 2.8, so we focus on the proof of the latter.

6.3.1. *Proof of part (i).* Let $f\sim\chi$ for some primitive Dirichlet character χ. Suppose that the Furstenberg system is defined using Cesàro averages along the sequence (N_k). By Proposition 5.5, there exists a subsequence (N'_k) of (N_k) such that f is Besicovitch rationally almost periodic for Cesàro averages along (N'_k). By Theorem 3.2, the system (X,μ,T) is an ergodic procyclic system.

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It remains to prove that if $f \neq 1$, then the system is non-trivial. Suppose first that $f \sim 1$. Since $f \neq 1$, there exist $p \in \mathbb{P}$ and $s \in \mathbb{N}$ such that $f(p^s) \neq 1$. Then, Theorem 2.11 applies and gives that $1/p \in \mathrm{Spec}(X, \mu, T)$. However, if $f \not\sim 1$, then the conductor q of the primitive Dirichlet character χ satisfies $q > 1$. Then, for every $p \in \mathbb{P}$ with $p \mid q$, we have by Theorem 2.11 that $1/p \in \mathrm{Spec}(X, \mu, T)$. In either case, we get that the system is non-trivial.

A similar argument applies to logarithmic averages since Proposition 5.5, Theorems 3.2 and 2.11 also apply to logarithmic averages. Alternatively, we can use Proposition 6.4, which implies that in this case, Furstenberg systems taken with respect to Cesàro and logarithmic averages coincide.

6.3.2. *Proof of part (ii).* We give all the arguments for Cesàro averages. The proof carries over to logarithmic averages, since we have proved variants for all the results needed for logarithmic averages.

Before embarking on the proof of the various claims, we do some preparatory work. Let $f \sim n^{it} \cdot \chi$ for some $t \neq 0$ and some primitive Dirichlet character χ, and suppose that (X, μ, T) is a Furstenberg system of f for Cesàro averages, defined along the sequence (N_k). Then, $f = \tilde{f} \cdot n^{it}$, where $\tilde{f} := f \cdot n^{-it} \sim \chi$. Upon passing to a subsequence (N_k') of (N_k), we get that the Furstenberg systems of the sequences $\tilde{f}$ and n^{it}, taken along (N_k'), are both well defined. We denote the Furstenberg system of $\tilde{f}$ along (N_k') by (Y, ν, S) ($Y := \mathbb{U}^{\mathbb{Z}}$ and S is again the shift map) and recall that, by part (i), it is isomorphic to an ergodic procyclic system. Furthermore, by [23, Corollary 5.5], for $t \neq 0$, the Furstenberg system of n^{it} along (N_k') defines a continuous measure on X that is supported on the diagonal set $\{(z)_{k\in\mathbb{Z}} \colon z \in \mathbb{S}^1\}$, on which T acts as an identity. Hence, the Furstenberg system of n^{it} can be identified with the system $(Z := \mathbb{S}^1, \lambda, \mathrm{id})$, where λ is a continuous measure on $\mathbb{S}^1$ (in fact, λ is equivalent and not equal to the Lebesgue measure, but we shall not use this). Since $f = \tilde{f} \cdot n^{it}$, it follows that the system (X, μ, T) is a factor of a joining of the systems (Y, ν, S) and $(Z, \lambda, \mathrm{id})$, with factor map

$$\pi(y, z) := M_z(y), \quad y \in Y,\ z \in \mathbb{S}^1, \tag{43}$$

where $M_z(y)(k) := z \cdot y(k),\ k \in \mathbb{Z}$. (We will make use of the form of π only at the end of our argument to get the product structure of the measure μ). Since the two systems are disjoint, we obtain that

$$(X, \mu, T) \text{ is a factor of the direct product of } (Y, \nu, S) \text{ and } (Z, \lambda, \mathrm{id}). \tag{44}$$

After this preparation, we are now ready to prove the claimed properties.

Rational discrete spectrum. Since both systems (Y, ν, S) and $(Z, \lambda, \mathrm{id})$ have rational discrete spectrum, so does their direct product. By equation (44), the same holds for (X, μ, T).

Non-trivial rational spectrum. We show that if $\tilde{f} \neq 1$, then the spectrum of the Furstenberg system (X, μ, T) of f is non-trivial (hence, if the spectrum is trivial, then $f(n) = n^{it}$, $n \in \mathbb{N}$, for some $t \in \mathbb{R}$). Arguing by contradiction, suppose that it is trivial. Then, since it has rational discrete spectrum, it is an identity system. Moreover, as remarked above, the Furstenberg system of n^{-it} is an identity system. Since $\tilde{f} = f \cdot n^{-it}$, we deduce

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that the Furstenberg system (Y, ν, S) of $\tilde{f}$ is a factor of a joining of two identity systems, and thus, it is an identity system. This contradicts part (i), which implies that since $\tilde{f} \sim \chi$ and $\tilde{f} \neq 1$, the system (Y, ν, S) has non-trivial spectrum (since it is a non-trivial ergodic procyclic system).

Non-ergodicity. (Although a different proof for this fact can be given using the 'product structure' result proved later in this subsection, we choose to also give this shorter and more direct argument.) We show that the system (X, μ, T) is non-ergodic. We first reduce to the case where $\chi = 1$. Since the non-zero values of χ are roots of unity of fixed order, there exists $q \in \mathbb{N}$ such that $\chi^q \sim 1$. Then, the Furstenberg system of f^q along (N'_k) is also well defined and ergodic if (X, μ, T) was ergodic (since it defines a factor of (X, μ, T), see the second remark after Definition 3.7). Moreover, $f^q \sim n^{iqt} \cdot \chi^q \sim n^{iqt}$. So, it suffices to show that the Furstenberg system of f^q along (N'_k) is non-ergodic. Furthermore, by considering f^{2q} in place of f^q if needed, we can assume that $f(2) \neq -2^{it}$. Thus, we have reduced matters to the case $\chi = 1$ and $f(2) \neq -2^{it}$.

Recall that we have $f = \tilde{f} \cdot n^{it}$, where $\tilde{f} = f \cdot n^{-it}$, and hence $\tilde{f} \sim 1$ and $\tilde{f}(2) \neq -1$. Moreover, equation (44) holds. We want to show that the system (X, μ, T) is non-ergodic, so arguing by contradiction, suppose that it is ergodic. Then,

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, f(m+n) \cdot \overline{f(m)} = |\lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, f(m)|^2. \tag{45}$$

(By equation (8), this is equivalent to the identity $\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \int T^n F_0 \cdot \overline{F_0}\, d\mu = |\int F_0\, d\mu|^2$, which holds by the von Neumann ergodic theorem.) Using the disjointness of the Furstenberg system of (n^{it}) and $(\tilde{f}(n))$, we get that the right-hand side equals

$$|\lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, m^{it}|^2 \cdot |\lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, \tilde{f}(m)|^2.$$

However, for every $n \in \mathbb{N}$, we have $\lim_{m\to\infty} (m+n)^{it} \cdot m^{-it} = 1$, and hence the left-hand side in equation (45) equals

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, \tilde{f}(m+n) \cdot \overline{\tilde{f}(m)} = |\lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, \tilde{f}(m)|^2,$$

where the last identity follows as before from the ergodicity of the Furstenberg system of $\tilde{f}$ along (N'_k) that was proved in part (i) of Theorem 2.8. Combining the above identities, we deduce that

$$|\lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, m^{it}|^2 \cdot |\lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, \tilde{f}(m)|^2 = |\lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, \tilde{f}(m)|^2.$$

Since $\tilde{f} \sim 1$ and $\tilde{f}(2) \neq -1$, by Theorem 3.1, we have $\lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, \tilde{f}(m) \neq 0$, and hence the previous identity implies that

$$|\lim_{k\to\infty} \mathbb{E}_{m\in[N'_k]}\, m^{it}| = 1.$$

However, one easily verifies (see the remark after Proposition 6.1) that this can only happen if $t = 0$, which is a contradiction.

Product structure. Recall that equation (44) holds and our plan is to use it to apply Proposition B.1 in the appendix. To this end, we need to verify the following.

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Claim. There exists at least one $z \in \mathbb{S}^1$ such that the measure ν is not invariant under M_z.

Arguing by contradiction, suppose that the claim fails. Then, for every $F \in C(Y)$, we have

$$\int F(z \cdot y)\, d\nu = \int F(y)\, d\nu \quad \text{for every } z \in \mathbb{S}^1.$$

Let F_0 be the 0th-coordinate projection on $Y = \mathbb{U}^{\mathbb{Z}}$. Applying the previous identity for $F := F_0^m$, $m \in \mathbb{N}$, and using that $F_0(zy) = z\, y(0) = zF_0(y)$, we deduce

$$(z^m - 1) \int F_0^m\, d\nu = 0 \quad \text{for every } z \in \mathbb{S}^1.$$

Hence, $\int F_0^m\, d\nu = 0$ for every $m \in \mathbb{N}$. Recall that (Y, ν, S) is the Furstenberg system for Cesàro averages of $\tilde{f}$ along the sequence (N_k'). We deduce using equation (8) that

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k']}\, (\tilde{f}(n))^m = 0 \quad \text{for every } m \in \mathbb{N}. \tag{46}$$

Since $\tilde{f} \sim \chi$ for some Dirichlet character χ, there exists $q \in \mathbb{N}$ such that $\tilde{f}^q \sim 1$. Furthermore, by considering $2q$ in place of q, if needed, we can assume that $\tilde{f}(2)^q \neq -1$. Keeping these two facts in mind, we get that equation (46), for $m := q$, contradicts Theorem 3.1, which claims that we cannot have a vanishing subsequential limit in this case. This completes the proof of the claim.

Hence, Proposition B.1 in the appendix gives that there exists $r \in \mathbb{N}$ such that the system (X, μ, T) is isomorphic to $(Y, \nu, S) \times (\mathbb{S}^1, \lambda_r, \mathrm{id})$, where λ_r is the pushforward of the continuous measure λ by $z \mapsto z^r$. (If $f(n) = n^i \cdot \chi$, where $\chi := \mathbf{1}_{4\mathbb{Z}+1} - \mathbf{1}_{4\mathbb{Z}+3}$, then $r = 2$ and the group G that appears in the proof of Proposition B.1 is equal to $\{-1, 1\}$.) Lastly, note that since λ_r is a continuous measure, and any two identity transformations on Lebesgue spaces with continuous probability measures are isomorphic (see for example [**30**, Theorem 17.41]), the system $(\mathbb{S}^1, \lambda_r, \mathrm{id})$ is isomorphic to the system $(\mathbb{T}, m_{\mathbb{T}}, \mathrm{id})$. This completes the proof.

6.3.3. *Proof of part (iii).* Suppose that $f \sim n^{it} \cdot \chi$ for some non-zero $t \in \mathbb{R}$ and primitive Dirichlet character χ. Let (X, μ_1, T) and (X, μ_2, T) be two Furstenberg systems of f for Cesàro or logarithmic averages (not necessarily both taken using Cesàro or logarithmic averages). Using part (ii), we get that for $i = 1, 2$, the system (X, μ_i, T) is isomorphic to the direct product of the system $(\mathbb{T}, m_{\mathbb{T}}, \mathrm{id})$ and the system (X, μ_i', T), which is some Furstenberg system of $\tilde{f} := f \cdot n^{-it}$ (with respect to Cesàro or logarithmic averages). Since $\tilde{f} \sim \chi$, by Proposition 6.4, we have that the systems (X, μ_1', T) and (X, μ_2', T) are isomorphic. Combining the above, we get that the systems (X, μ_1, T) and (X, μ_2, T) are isomorphic. This completes the proof.

6.3.4. *Proof of part (iv).* Follows immediately from Proposition 6.2.

6.3.5. *Proof of part (v).* In case (i), the asserted property follows from Proposition 5.5. In case (ii), suppose that there exists a sequence $N_k \to \infty$ along which f is Besicovitch

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rationally almost periodic for Cesàro or logarithmic averages. Upon passing to a subsequence, we can assume that the Furstenberg system of f for Cesàro or logarithmic averages along (N_k) is well defined. Then, by Theorem 3.2, this system is ergodic. This contradicts the non-ergodicity established in part (ii).

6.4. *Proof of Corollary 2.9.* To establish the equivalence of properties (i), (ii), and (iii), we argue as follows. The implication (i) $\Longrightarrow$ (iii) follows from part (i) of Theorem 2.8. The implication (iii) $\Longrightarrow$ (ii) is obvious. The implication (ii) $\Longrightarrow$ (i) follows from part (ii) of Theorem 2.8.

To establish the equivalence of properties (i), (iv), and (v), we argue as follows. The implication (i) $\Longrightarrow$ (v) follows from the first assertion in part (v) of Theorem 2.8. The implication (v) $\Longrightarrow$ (iv) is obvious. The implication (iv) $\Longrightarrow$ (i) follows from the second assertion in part (v) of Theorem 2.8.

6.5. *Proof of Corollary 2.10.* The implication (i) $\Longrightarrow$ (v) is obvious. The implication (v) $\Longrightarrow$ (iv) is simple and follows from [**23**, Corollary 5.5] and [**17**, §1.3]. The implications (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) are trivial. It remains to establish the implication (ii) $\Longrightarrow$ (i). Namely, we want to show that if some Furstenberg system of f for Cesàro or logarithmic averages has trivial rational spectrum, then $f(n) = n^{it}$, $n \in \mathbb{N}$, for some $t \in \mathbb{R}$. This follows from part (ii) of Theorem 2.8.

7. *Spectral results for pretentious multiplicative functions and applications*

In this section, we prove Theorems 2.11–2.16.

7.1. *Necessary conditions for spectrum.* The next result gives a necessary condition that often helps us decide when a certain number belongs to the spectrum of some Furstenberg system of a bounded sequence. Furthermore, it gives an identity that we will use to prove Theorem 2.13 when we happen to know that this number is not in the spectrum.

PROPOSITION 7.1. *Let (X, μ, T) be the Furstenberg system for Cesàro averages of a sequence $a\colon \mathbb{N} \to \mathbb{U}$ taken along a sequence (N_k). If $\alpha \in (0, 1)$ is such that $\alpha \notin \mathrm{Spec}(X, \mu, T)$, then*

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, e(-n\alpha) \prod_{j=1}^{\ell} a_j(n + n_j) = 0 \tag{47}$$

for all $\ell \in \mathbb{N}$, $n_1, \dots, n_\ell \in \mathbb{N}$, and $a_1, \dots, a_\ell \in \{a, \overline{a}\}$. A similar statement also holds when we consider logarithmic averages.

Proof. Arguing by contradiction, suppose that the conclusion fails for some $\ell \in \mathbb{N}$, $n_1, \dots, n_\ell \in \mathbb{N}$, and $a_1, \dots, a_\ell \in \{a, \overline{a}\}$. Using van der Corput's lemma (the variant needed follows from [**34**, Lemma 3.1]), we get

$$\lim_{H\to\infty} \mathbb{E}_{h\in[H]}\, e(-h\alpha) \Bigg(\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]} \prod_{j=1}^{\ell} a_j(n + n_j) \prod_{j=1}^{\ell} \overline{a_j}(n + h + n_j) \Bigg) \neq 0.$$

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(Note that all the limits as $k \to \infty$ exist because a is assumed to admit correlations along (N_k)). Using equation (8) to translate this equation to the Furstenberg system, we get that

$$\lim_{H\to\infty} \mathbb{E}_{h\in[H]}\, e(-h\alpha) \int F \cdot T^h \overline{F}\, d\mu \neq 0,$$

where $F := \prod_{j=1}^{\ell} T^{n_j} F_j$ for some $F_j \in \{F_0, \overline{F_0}\}$ and F_0 is the 0th-coordinate projection. (The existence of the limit follows by applying the mean ergodic theorem to an appropriate product system.) This implies that α is on the spectrum of (X, μ, T), in fact,

$$G := \lim_{H\to\infty} \mathbb{E}_{h\in[H]}\, e(-h\alpha)\, T^h \overline{F},$$

where the limit is taken in $L^2(\mu)$, is a non-zero $e(\alpha)$-eigenfunction. This contradicts our assumption $\alpha \notin \mathrm{Spec}(X, \mu, T)$ and completes the proof. (Alternatively, one can argue that if the conclusion fails, then we get a non-trivial joining of the system with the rotation by α on the circle. Now, if the limit in equation (47) is non-zero, then the conditional expectation (with respect to the Furstenberg system in the joining) of the eigenfunction (on the circle) corresponding to α is non-zero. Hence, α belongs to the spectrum of the system (X, μ, T).) □

The following simpler condition often suffices to deduce that a certain rational number belongs to the spectrum of a Furstenberg system of a bounded sequence (we caution the reader though that the condition is far from necessary).

COROLLARY 7.2. *Let (X, μ, T) be the Furstenberg system for Cesàro averages of a sequence $a\colon \mathbb{N} \to \mathbb{U}$ taken along $N_k \to \infty$. Suppose that for some $q \geq 2$ and $r \in \{0, \dots, q-1\}$, we have*

$$\lim_{k\to\infty} (\mathbb{E}_{n\in[N_k/q]}\, a(qn+r) - \mathbb{E}_{n\in[N_k]}\, a(n)) \neq 0.$$

Then, p/q belongs to the spectrum of (X, μ, T) for some $p \in \{1, \dots, q-1\}$. A similar statement holds for logarithmic averages, and in this context, we can replace $\mathbb{E}^{\log}_{n\in[N_k/q]}$ with $\mathbb{E}^{\log}_{n\in[N_k]}$.

Proof. Our assumption is equivalent to

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, (1 - q\, \mathbf{1}_{q\mathbb{N}+r}(n))\, a(n) \neq 0. \tag{48}$$

Upon substituting the identity

$$q\, \mathbf{1}_{q\mathbb{N}+r}(n) = \sum_{p=0}^{q-1} e\left(p \cdot \frac{n-r}{q}\right)$$

in equation (48), we deduce that

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, e\left(-n \cdot \frac{p}{q}\right) a(n) \neq 0$$

for some $p \in \{1, \dots, q-1\}$. It follows from Proposition 7.1 (we only need to appeal to the $\ell = 1$ case) that p/q is in the spectrum of the system. □

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7.2. *Preparation for the proof of Theorem 2.11.* If χ is a primitive Dirichlet character with conductor q, then it has a unique Furstenberg system, which is periodic with minimal period q, and its spectrum is spanned by multiples of $1/q$. The next result shows that if $f \sim \chi$ for some primitive Dirichlet character χ, then the spectrum of every Furstenberg system of f is at least as large as the spectrum of the Furstenberg system of χ.

PROPOSITION 7.3. *Let (X, μ, T) be the Furstenberg system for Cesàro averages of a multiplicative function $f: \mathbb{N} \to \mathbb{U}$ that satisfies $f \sim \chi$ for some primitive Dirichlet character χ with conductor $q \geq 2$. Then, for every $p \in \mathbb{P}$ with $p \mid q$, we have $1/p \in \mathrm{Spec}(X, \mu, T)$.*

Proof. We assume that the Furstenberg system is defined along a sequence (N_k) and by passing to a further subsequence, we can assume that the Furstenberg system of all other multiplicative functions defined subsequently are also taken along (N_k).

Suppose first that $q = p^k$ for some $p \in \mathbb{P}$. Since $f \sim \chi$, we have $f \cdot \overline{\chi} \sim 1$. If $f(2^s)\overline{\chi(2^s)} \neq -1$ for some $s \in \mathbb{N}$, then by Theorem 3.1, we have

$$\limsup_{k\to\infty} |\mathbb{E}_{n\in[N_k]}\, f(n) \cdot \overline{\chi(n)}| > 0.$$

Since χ is a primitive Dirichlet character with conductor $q > 1$, it is non-principal and, as a consequence, has mean value zero. Since χ is periodic with period q, using its Fourier expansion in $\mathbb{Z}_q$, we deduce that

$$\limsup_{k\to\infty} |\mathbb{E}_{n\in[N_k]}\, f(n) \cdot e(n \cdot r/q)| > 0$$

for some $r \in \{1, \dots, q-1\}$. It follows from Proposition 7.1 that $r/q \in \mathrm{Spec}(X, \mu, T)$, and since $q = p^k$ and the spectrum is closed under integer multiplication, we deduce that $1/p \in \mathrm{Spec}(X, \mu, T)$. Suppose now that $f(2^s) \cdot \overline{\chi(2^s)} = -1$ for every $s \in \mathbb{N}$. Then, $\chi(2) \neq 0$, and hence $p \neq 2$. We let $\tilde{f} := \mathbf{1}_{2\mathbb{Z}+1} \cdot f \sim \chi$, then $\tilde{f} \cdot \overline{\chi} \sim 1$ and $(\tilde{f} \cdot \overline{\chi})(2) = 0 \neq -1$, and the previous case gives that $1/p$ belongs to the spectrum of the Furstenberg system of $\tilde{f}$ along (N_k). Since $\tilde{f} := \mathbf{1}_{2\mathbb{Z}+1} \cdot f$, the Furstenberg system of $\tilde{f}$ along (N_k) is a factor of a joining of the Furstenberg system of f along (N_k), which by Theorem 2.8 is an ergodic procyclic system, and the Furstenberg system of $\mathbf{1}_{2\mathbb{Z}+1}$, which is periodic with period two. It follows that $1/p$ is an integer combination of $1/2$ and elements in $\mathrm{Spec}(X, \mu, T)$. Since $p \neq 2$, we deduce that $1/p \in \mathrm{Spec}(X, \mu, T)$.

We consider now the general case where q is not a prime power (our assumption that χ is primitive is crucially used here). If $q = p_1^{k_1} \cdots p_\ell^{k_\ell}$, $\ell \geq 2$, is the prime factorization of q, then it is a standard fact that χ can be decomposed as $\chi = \chi_1 \cdots \chi_\ell$, where $\chi_1, \dots, \chi_\ell$ are primitive Dirichlet characters with conductors $p_1^{k_1}, \dots, p_\ell^{k_\ell}$, respectively. Note that then $f \cdot \overline{\chi_1} \cdots \overline{\chi_{\ell-1}} \sim \chi_\ell$, and by the first case, we get that $1/p_\ell$ belongs to the spectrum of the Furstenberg system of $f \cdot \overline{\chi_1} \cdots \overline{\chi_{\ell-1}}$ along (N_k), which is a factor of a joining of the system (X, μ, T), which by Theorem 2.8 is an ergodic procyclic system, and the Furstenberg system of $\overline{\chi_1} \cdots \overline{\chi_{\ell-1}}$. We deduce that $1/p_\ell$ is an integer combination of elements of the form $1/p_i^{k_i}$, $i = 1, \dots, \ell-1$ (these elements span the spectrum

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of $\overline{\chi_1}\cdots\overline{\chi_{\ell-1}}$), and elements in $\mathrm{Spec}(X,\mu,T)$. Since $p_j \neq p_\ell$ for $j=1,\dots,\ell-1$, it follows that $1/p_\ell \in \mathrm{Spec}(X,\mu,T)$. Similarly, we get that $1/p_i \in \mathrm{Spec}(X,\mu,T)$ for $i=1,\dots,\ell-1$. This completes the proof. □

We will also need the following simple fact.

LEMMA 7.4. *If χ is a primitive Dirichlet character with conductor $q\geq 2$, we define the completely multiplicative function $\tilde{\chi}$ on prime numbers p by*

$$\tilde{\chi}(p) := \begin{cases} \chi(p), & p\nmid q,\\ 1, & p\mid q. \end{cases}$$

Then, $\tilde{\chi}$ has a unique Furstenberg system (X,μ,T) and for $p_0\in\mathbb{P}$, we have $1/p_0\in \mathrm{Spec}(X,\mu,T)$ if and only if $p_0\mid q$.

Proof. The uniqueness of the Furstenberg system of $\tilde{\chi}$ will be established in the second part of the proof. For the moment, we assume that (X,μ,T) is some Furstenberg system of $\tilde{\chi}$.

Since $\tilde{\chi}\sim\chi$ and χ is a primitive Dirichlet character with conductor q and $p_0\mid q$, we have by Proposition 7.3 that $1/p_0\in(X,\mu,T)$.

Suppose that $p_0\in\mathbb{P}$ is such that $p_0\nmid q$, we shall show that $1/p_0\notin\mathrm{Spec}(X,\mu,T)$. We define the multiplicative function

$$\tilde{\chi}_m(n) := \begin{cases} \tilde{\chi}(n) & \text{if } q^m\nmid n,\\ 0 & \text{if } q^m\mid n. \end{cases}$$

One easily verifies that $\tilde{\chi}_m$ is periodic with period q^m; hence it has a unique Furstenberg system with spectrum a subset of the subgroup generated by $1/q^m$, and

$$\lim_{m\to\infty}\limsup_{N\to\infty}\mathbb{E}_{n\in[N]}|\tilde{\chi}_m(n)-\tilde{\chi}(n)|=0.$$

By [1, Lemma 3.17], $\tilde{\chi}$ has a unique Furstenberg system, and it is a factor of a joining of the periodic systems generated by $\tilde{\chi}_m$, $m\in\mathbb{N}$. Hence, $\mathrm{Spec}(X,\mu,T)$ is contained in the subgroup $\{j/q^m\colon j=0,1\dots q^m-1, m\in\mathbb{N}\}$ of $\mathbb{T}$. Since $p_0\nmid q$, we deduce that $1/p_0\notin\mathrm{Spec}(X,\mu,T)$, completing the proof. □

Lastly, we will use the following elementary fact.

LEMMA 7.5. *Let G_1,G_2 be subgroups of $\mathbb{T}$ consisting of rational numbers. If $p\in\mathbb{P}$ satisfies $1/p\notin G_j$ for $j=1,2$, then $1/p$ does not belong to the subgroup generated by G_1 and G_2.*

Proof. We argue by contradiction. Suppose that $1/p$ belongs to the subgroup generated by G_1 and G_2. Then, for $j=1,2$, there exist $a_j,b_j\in\mathbb{N}$ such that $a_j/b_j\in G_j$, $(a_j,b_j)=1$, and such that $1/p=a_1/b_1+a_2/b_2$. Since $1/p$ does not belong to the subgroup G_1 and we also have $a_1/b_1\in G_1$ and $(a_1,b_1)=1$, we easily deduce that p does not divide b_1.

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Similarly, we get that p does not divide b_2. However, we have $b_1 b_2 = p(a_1 b_2 + a_2 b_1)$, and hence p divides either b_1 or b_2, which is a contradiction. This completes the proof. □

7.3. *Proof of the implication* (i) $\Longrightarrow$ (ii) *of Theorem 2.11.* By Proposition 6.2, Furstenberg systems for Cesàro and logarithmic averages coincide, so we only explain the argument for Cesàro averages. Suppose that the Furstenberg system (X, μ, T) is taken along the sequence $N_k \to \infty$.

Before proceeding further, let us make the following simple observation. Suppose that $p_1, \dots, p_k \in \mathbb{P}$, $s_1, \dots, s_k \in \mathbb{N}$, and define the multiplicative function f by setting $f(p_i^s) := \alpha_{i,s} \in \mathbb{U}$ for $i = 1, \dots, k$ and $s = 1, \dots, s_i$, and $f(p^s) := 1$ for all other prime powers. Then, $f(n)$ is completely determined from the knowledge of all remainders of n modulo $p_i^{s_i+1}$, $i = 1, \dots, k$, and hence

$$f : \mathbb{N} \to \mathbb{U} \text{ is periodic with period } \prod_{i=1}^{k} p_i^{s_i+1}. \tag{49}$$

In our argument below, we will approximate f by periodic multiplicative functions of the previous form and deduce properties for f from those of its periodic approximants.

Suppose first that $f \sim 1$ and $p_0 \in \mathbb{P}$ is such that $f(p_0^s) = 1$ for every $s \in \mathbb{N}$. Our goal is to show that $1/p_0 \notin \mathrm{Spec}(X, \mu, T)$.

By Proposition 5.5 (see the remark following the result), there exist a subsequence (N_k') of (N_k) and constants $\alpha_m \in [0, 1)$, $m \in \mathbb{N}$, such that

$$\limsup_{k\to\infty} \mathbb{E}_{n\in[N_k']} |f(n) - e(\alpha_m) \cdot f_m(n)|^2 \le 1/m, \tag{50}$$

where $f_m : \mathbb{N} \to \mathbb{U}$, $m \in \mathbb{N}$, are defined as in Lemma 5.4 (for $\varepsilon := 1/m$) and satisfy property (i) of Lemma 5.4 (with $\chi := 1$). Furthermore, since $f(p_0^s) = 1$ for every $s \in \mathbb{N}$, we can also assume that

$$f_m(p_0^s) = 1 \quad \text{for every } m, s \in \mathbb{N}. \tag{51}$$

Moreover, since for $m \in \mathbb{N}$ the sequence f_m satisfies equation (27) with $\chi = 1$, as shown in the proof of [8, Lemma 4] (we crucially use here that $f \sim 1$), we have (in fact, we could replace $\limsup_{k\to\infty} \mathbb{E}_{n\in[N_k']}$ with $\lim_{N\to\infty} \mathbb{E}_{n\in[N]}$)

$$\limsup_{k\to\infty} \mathbb{E}_{n\in[N_k']} |f_m(n) - f_m'(n)|^2 \le 1/m, \tag{52}$$

where for every $m \in \mathbb{N}$, the multiplicative functions $f_m' : \mathbb{N} \to \mathbb{U}$ are defined by

$$f_m'(p^s) := \begin{cases} f_m(p^s) & \text{for } p \le r_m, s \in \mathbb{N}, \\ 1 & \text{for all other prime powers,} \end{cases}$$

for $r_m \in \mathbb{N}$ sufficiently large that we can arrange to form an increasing sequence. Note that equation (51) implies

$$f_m'(p_0^s) = 1 \quad \text{for every } m, s \in \mathbb{N}. \tag{53}$$

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Let $f'_{m,l}\colon \mathbb{N}\to\mathbb{U}$ be multiplicative functions defined by

$$f'_{m,l}(p^s) := \begin{cases} f'_m(p^s) & \text{for } p\le r_m, s\le l,\\ 1 & \text{for all other prime powers.}\end{cases}$$

Then, equation (53) implies that

$$f'_{m,l}(p_0^s)=1 \quad \text{for every } m,l,s\in\mathbb{N}. \tag{54}$$

Combining equations (49) and (54), we get that for every $m,l\in\mathbb{N}$, the sequence $n\mapsto f'_{m,l}(n)$ is periodic with period $P_{m,l}:=\prod_{p\le r_m,p\neq p_0}p^{l+1}$. Arguing as in the proof of [8, Lemma 4], we get

$$\limsup_{k\to\infty}\mathbb{E}_{n\in[N'_k]}|f'_m(n)-f'_{m,l_m}(n)|^2\le 1/m \tag{55}$$

for $l_m\in\mathbb{N}$ sufficiently large that we can arrange to form an increasing sequence. Combining equations (50), (52), (55), we get

$$\lim_{m\to\infty}\limsup_{k\to\infty}\mathbb{E}_{n\in[N'_k]}|f(n)-e(\alpha_m)\cdot f'_{m,l_m}(n)|^2=0. \tag{56}$$

Now, for every $m\in\mathbb{N}$, the sequence $(e(\alpha_m)\cdot f'_{m,l_m}(n))_{n\in\mathbb{N}}$ is periodic with period

$$P_m:=\prod_{p\le r_m,p\neq p_0}p^{l_m+1}.$$

Hence, the spectrum of its Furstenberg system (X,μ_m,T) is contained in the subgroup G_m of $\mathbb{T}$ generated by $1/P_m$. Since p_0 does not divide P_m, we have that $1/p_0\notin G_m$. In summary,

$$\mathrm{Spec}(X,\mu_m,T)\subset G_m \quad\text{and}\quad 1/p_0\notin G_m \quad \text{for every } m\in\mathbb{N}. \tag{57}$$

Note also that because (r_m) and (l_m) are increasing sequences, we have that P_m divides P_{m+1}, and hence (G_m) is an increasing sequence of rational subgroups.

By equation (56) and [1, Lemma 3.17], the system (X,μ,T) is a factor of a joining of the periodic systems (X,μ_m,T), $m\in\mathbb{N}$. Hence, $\mathrm{Spec}(X,\mu,T)$ is contained in the subgroup generated by the sequence of subgroups $\mathrm{Spec}(X,\mu_m,T)$, $m\in\mathbb{N}$ (see remark (iv) at the end of §3.3), which in turn, by equation (57), is contained in the subgroup generated by the increasing sequence of rational subgroups G_m, $m\in\mathbb{N}$. By equation (57), we have that $1/p_0\notin G_m$ for every $m\in\mathbb{N}$, and hence $1/p_0\notin\mathrm{Spec}(X,\mu,T)$.

Next, we deal with the case where $f\sim\chi$ for some primitive Dirichlet character χ with conductor $q\ge 2$. Let $p_0\in\mathbb{P}$ be such that $p_0\nmid q$ and $f(p_0^s)=\chi(p_0^s)$ for every $s\in\mathbb{N}$. Our goal is to show that $1/p_0\notin\mathrm{Spec}(X,\mu,T)$. We take $\tilde{\chi}$ as in Lemma 7.4 and let $\tilde{f}:=f\cdot\overline{\tilde{\chi}}$. Then, $\tilde{f}\sim 1$ and $\tilde{f}(p_0^s)=1$ for every $s\in\mathbb{N}$ (we used that $p_0\nmid q$ here). Hence, by the previous part, we have that $1/p_0$ does not belong to the spectrum of the Furstenberg system of $\tilde{f}$ taken along a subsequence (N'_k) of (N_k). Furthermore, by Lemma 7.4, it follows that $1/p_0$ does not belong to the spectrum of the (unique) Furstenberg system of $\tilde{\chi}$. Since $f=\tilde{f}\cdot\tilde{\chi}$ (we used that $|\tilde{\chi}|=1$ here), arguing as before, and using Lemma 7.5, we get that $1/p_0\notin(X,\mu,T)$. This completes the proof.

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Finally, to deal with the case $f \sim n^{it} \cdot \chi$ for some $t \in \mathbb{R}$ and primitive Dirichlet character χ, we can combine the previous part with part (ii) of Theorem 2.8. Alternatively, given the previous part, one can get a more direct proof by arguing as in §7.4.4 below.

7.4. *Proof of the implication* (ii) $\Longrightarrow$ (i) *of Theorem 2.11.* We assume that $f \sim \chi$ for some primitive Dirichlet character χ with conductor q and we have either $p \nmid q$ or $f(p^s) \neq \chi(p^s)$ for some $p \in \mathbb{P}$ and $s \in \mathbb{N}$. Our goal is to show that $1/p \in \mathrm{Spec}(X, \mu, T)$. We do this in three steps. The bulk of the proof is contained in the first step, where we work under the additional hypothesis $f \sim 1$ and $f(2^{s_0}) \neq -1$ for some $s_0 \in \mathbb{N}$. In the second step, we use the first one to cover the complementary case $f \sim 1$ and $f(2^s) = -1$ for every $s \in \mathbb{N}$. Finally, in the third step, we use the first two steps to cover the general case $f \sim n^{it} \cdot \chi$ for some $t \in \mathbb{R}$ and primitive Dirichlet character χ.

7.4.1. *The case $f \sim 1$ and $f(2^{s_0}) \neq -1$ for some $s_0 \in \mathbb{N}$.* We assume that $f \sim 1$ and $f(2^{s_0}) \neq -1$ for some $s_0 \in \mathbb{N}$, which is in addition to our hypothesis that $f(p^s) \neq 1$ for some $s \in \mathbb{N}$. Note first that since $f \sim 1$, Proposition 6.2 implies that the Furstenberg systems of f for Cesàro and logarithmic averages coincide. So, we only have to treat the case of logarithmic averages, and this turns out to offer a substantial advantage since it enables us to replace $\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k/p]}$ with $\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]}$ throughout.

So, let (X, μ, T) be a Furstenberg system of f for logarithmic averages taken along the sequence $N_k \to \infty$. Note that then, the limit $L := \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} f(n)$ exists and since $f \sim 1$ and $f(2^{s_0}) \neq -1$ for some $s_0 \in \mathbb{N}$, we get by Theorem 3.1 that $L \neq 0$ (see the remarks following the theorem).

Arguing by contradiction, suppose that $1/p \notin \mathrm{Spec}(X, \mu, T)$. By Corollary 7.2, this implies

$$L = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} f(pn + j) = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} f(n) \quad \text{for every } j \in \{0, \dots, p-1\}. \tag{58}$$

Using these identities, our goal is to show that $f(p^s) = 1$ for every $s \in \mathbb{N}$, which would contradict our hypothesis.

For convenience, we use the convention

$$\mathbb{E}^{\log}_{n\in\mathbf{N}} a(n) := \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} a(n)$$

whenever the limit exists. In what follows, we will use the following basic property of logarithmic averages. If $a : \mathbb{N} \to \mathbb{U}$ is such that the limit $\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} a(n)$ exists, then we also have that the limit $\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[pN_k]} a(n)$ exists and the two limits are equal. Using this fact to justify the first identity below and equation (58) to justify the second, we get

$$\begin{aligned} \mathbb{E}^{\log}_{n\in\mathbf{N}} f(n) &= \frac{1}{p}\mathbb{E}^{\log}_{n\in\mathbf{N}} f(pn) + \frac{1}{p}\sum_{j=1}^{p-1} \mathbb{E}^{\log}_{n\in\mathbf{N}} f(pn + j) \\ &= \frac{1}{p}\mathbb{E}^{\log}_{n\in\mathbf{N}} f(pn) + L \cdot \frac{p-1}{p}. \end{aligned}$$

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Repeating this process one more time to evaluate $(1/p)\mathbb{E}^{\log}_{n\in\mathbf{N}}\, f(pn)$, we get

$$\frac{1}{p}\mathbb{E}^{\log}_{n\in\mathbf{N}}\, f(pn) = \frac{1}{p^2}\mathbb{E}^{\log}_{n\in\mathbf{N}}\, f(p^2n) + \frac{1}{p^2}\sum_{j=1}^{p-1}\mathbb{E}^{\log}_{n\in\mathbf{N}}\, f(p(pn+j))$$
$$= \frac{1}{p^2}\mathbb{E}^{\log}_{n\in\mathbf{N}}\, f(p^2n) + L\cdot\frac{p-1}{p}\cdot\frac{f(p)}{p},$$

where to get the second identity, we used that $f(p(pn+j)) = f(p)\, f(pn+j)$ for $j = 1, \dots, p-1$ and equation (58). Combining these two identities gives

$$\mathbb{E}^{\log}_{n\in\mathbf{N}}\, f(n) = \frac{1}{p^2}\mathbb{E}^{\log}_{n\in\mathbf{N}}\, f(p^2n) + L\cdot\frac{p-1}{p}\left(1+\frac{f(p)}{p}\right).$$

Repeating this process $M-2$ more times, we deduce

$$\mathbb{E}_{n\in\mathbf{N}}\, f(n) = \frac{1}{p^M}\mathbb{E}^{\log}_{n\in\mathbf{N}}\, f(p^Mn) + L\cdot\frac{p-1}{p}\sum_{s=0}^{M-1}\frac{f(p^s)}{p^s}.$$

Letting $M\to\infty$ gives

$$L = \mathbb{E}_{n\in\mathbf{N}}\, f(n) = L\cdot\frac{p-1}{p}\cdot\sum_{s=0}^{\infty}\frac{f(p^s)}{p^s}.$$

Since $L\neq 0$, we deduce that

$$\sum_{s=1}^{\infty}\frac{f(p^s)}{p^s} = \frac{1}{p-1}. \tag{59}$$

Since $\sum_{s=1}^{\infty}(1/p^s) = 1/(p-1)$ and $\Re(f(p^s))\in[-1,1]$, comparing real parts in equation (59), we get $\Re(f(p^s)) = 1$ for every $s\in\mathbb{N}$. Since $f(p^s)\in\mathbb{U}$, we deduce that $f(p^s) = 1$ for every $s\in\mathbb{N}$, completing the proof in this first case.

7.4.2. *The case $f\sim 1$ and $f(2^s) = -1$ for every $s\in\mathbb{N}$.*. We assume that $f\sim 1$ and $f(2^s) = -1$ for every $s\in\mathbb{N}$. Let $p\in\mathbb{P}$ be such that $f(p^{s_0})\neq 1$ for some $s_0\in\mathbb{N}$ and our goal is to show that $1/p\in\mathrm{Spec}(X,\mu,T)$. Suppose that the Furstenberg system of f for logarithmic averages is taken along $N_k\to\infty$.

We first show that $1/2\in\mathrm{Spec}(X,\mu,T)$. Since $f(2^s) = -1$ for every $s\in\mathbb{N}$, by Theorem 3.1, we have $\lim_{k\to\infty}\mathbb{E}^{\log}_{n\in[N_k]}\, f(n) = 0$. However, the multiplicative function

$$\tilde{f} := f\cdot\mathbf{1}_{2\mathbb{Z}+1} \tag{60}$$

also satisfies $\tilde{f}\sim 1$, and in addition satisfies $\tilde{f}(2)\neq -1$. Hence, Theorem 3.1 gives that for a subsequence (N_k') of (N_k) along which the next limit exists, we have

$$0\neq\lim_{k\to\infty}\mathbb{E}^{\log}_{n\in[N_k']}\,\tilde{f}(n) = \lim_{k\to\infty}\mathbb{E}^{\log}_{n\in[N_k'/2]}\, f(2n+1) = \lim_{k\to\infty}\mathbb{E}^{\log}_{n\in[N_k']}\, f(2n+1),$$

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where the last identity holds because we use logarithmic averages. Hence,

$$\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N'_k]} f(2n+1) \neq \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N'_k]} f(n).$$

Since (N'_k) is a subsequence of (N_k), Corollary 7.2 implies that $1/2 \in \mathrm{Spec}(X, \mu, T)$.

Suppose now that $p \neq 2$, we will show that $1/p \in \mathrm{Spec}(X, \mu, T)$. We define again the multiplicative function $\tilde{f}$ as in equation (60). It satisfies $\tilde{f} \sim 1$ and $\tilde{f}(p^{s_0}) \neq 1$, together with $\tilde{f}(2) \neq -1$. So, by the case treated in §7.4.1, we have that if $(X, \tilde{\mu}, T)$ is the Furstenberg system of $\tilde{f}$ along a subsequence (N'_k) of (N_k), then $1/p \in \mathrm{Spec}(X, \tilde{\mu}, T)$. Now, notice that since $\tilde{f} = f \cdot \mathbf{1}_{2\mathbb{Z}+1}$, we have that $(X, \tilde{\mu}, T)$ is a factor of a joining of the system (X, μ, T), which is an ergodic procyclic system by part (i) of Theorem 2.8, and the ergodic rotation on two elements. Hence, $1/p$ belongs to the subgroup spanned by $\mathrm{Spec}(X, \mu, T)$ and $1/2$. Since (X, μ, T) is ergodic, $\mathrm{Spec}(X, \mu, T)$ is a group, and as we showed just above, $1/2 \in \mathrm{Spec}(X, \mu, T)$. Combining these facts, we deduce that $1/p \in \mathrm{Spec}(X, \mu, T)$.

7.4.3. *The case $f \sim \chi$.* We now treat the case where $f \sim \chi$ for some primitive Dirichlet character χ with conductor q. If $p \in \mathbb{P}$ satisfies $p \mid q$, then by Proposition 7.3, we have that $1/p \in \mathrm{Spec}(X, \mu, T)$. So it remains to show that if $p \in \mathbb{P}$ is such that $p \nmid q$ and $f(p^{s_0}) \neq \chi(p^{s_0})$ for some $s_0 \in \mathbb{N}$, then $1/p \in \mathrm{Spec}(X, \mu, T)$. Suppose that the Furstenberg system (X, μ, T) of f for logarithmic averages is taken along $N_k \to \infty$.

We define the multiplicative function

$$\tilde{f} := f \cdot \overline{\chi}.$$

Since $f \sim \chi$, we have $\tilde{f} \sim 1$. Furthermore, since $f(p^{s_0}) \neq \chi(p^{s_0})$ and $|\chi(p^{s_0})| = 1$, we have $\tilde{f}(p^{s_0}) \neq 1$. Hence, if (N'_k) is a subsequence of (N_k) along which the Furstenberg system $(X, \tilde{\mu}, T)$ of $\tilde{f}$ for logarithmic averages is defined, the case treated in §7.4.2 implies that $1/p \in \mathrm{Spec}(X, \tilde{\mu}, T)$.

Since $\tilde{f} = f \cdot \overline{\chi}$, the system $(X, \tilde{\mu}, T)$ is a factor of a joining of the system (X, μ, T) and the Furstenberg system of χ, the first is an ergodic procyclic system by part (i) of Theorem 2.8 and the second is a periodic system with period q. Hence, $1/p$ belongs to the subgroup generated by $G_1 := \mathrm{Spec}(X, \mu, T)$ and the set $G_2 := \{j/q : j = 0, \dots, q-1\}$. Note that G_1, G_2 are rational subgroups of $\mathbb{T}$ and since $p \nmid q$, we have $1/p \notin G_2$. To conclude the proof, we note that if $1/p \notin G_1$, then since we also have $1/p \notin G_2$, we get by Lemma 7.5 that $1/p$ does not belong to the subgroup generated by G_1 and G_2, which is a contradiction. We deduce that $1/p \in G_1 = \mathrm{Spec}(X, \mu, T)$, which completes the proof.

7.4.4. *The case $f \sim n^{it} \cdot \chi$.* Assuming the previous case, the result follows easily by part (ii) of Theorem 2.8, but we give a more direct argument below.

Suppose that $f \sim n^{it} \cdot \chi$ for some $t \neq 0$ and primitive Dirichlet character χ with conductor q. Let $p \in \mathbb{P}$ be such that either $p \mid q$ or $f(p^s) \neq p^{ist} \cdot \chi(p^s)$ for some $s \in \mathbb{N}$ and our goal is to show that $1/p \in \mathrm{Spec}(X, \mu, T)$. Suppose that the Furstenberg system (X, μ, T) of f for Cesàro averages is taken along $N_k \to \infty$, the argument is similar for logarithmic averages.

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Let $\tilde{f} := f \cdot n^{-it}$. Then, our assumptions imply that $\tilde{f} \sim \chi$ and either $p \mid q$ or $\tilde{f}(p^s) \neq \chi(p^s)$ for some $s \in \mathbb{N}$. Hence, if (N_k') is a subsequence of (N_k) along which the Furstenberg system $(X, \tilde{\mu}, T)$ of $\tilde{f}$ for Cesàro averages is defined, the case treated in §7.4.3 implies that $1/p \in \mathrm{Spec}(X, \tilde{\mu}, T)$.

Since $\tilde{f} := f \cdot n^{-it}$, the system $(X, \tilde{\mu}, T)$ is a factor of a joining of the system (X, μ, T) and the Furstenberg system of n^{-it}. The first system is an ergodic procyclic system by part (i) of Theorem 2.8 and the second is isomorphic to an identity transformation in $\mathbb{T}$ by [**23**, Corollary 5.5]. Hence, the two systems are disjoint, and since the second system has trivial spectrum, it follows that $\mathrm{Spec}(X, \tilde{\mu}, T)$ is contained in $\mathrm{Spec}(X, \mu, T)$. Since $1/p \in \mathrm{Spec}(X, \tilde{\mu}, T)$, we deduce that $1/p \in \mathrm{Spec}(X, \mu, T)$, which completes the proof.

7.5. *Proof of Theorem 2.12.* Since $f \sim \chi$ for some primitive Dirichlet character χ with conductor q, Proposition 6.2 implies that Furstenberg systems of f for Cesàro and logarithmic averages coincide. We are going to work with logarithmic averages to have part (i) of Corollary 2.3 available to us.

The inclusion $\mathrm{Spec}(X, \mu, T) \subset \Lambda$ follows from the following stronger fact that applies to general multiplicative functions that pretend to be Dirichlet characters.

Claim. Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim \chi$ for some primitive Dirichlet character χ with conductor q. If

$$A := \{p \in \mathbb{P} : \text{either } p \mid q \text{ or } f(p^s) \neq \chi(p^s) \text{ for some } s \in \mathbb{N}\}$$

and Λ is the subgroup generated by $\{1/p^s : p \in A, s \in \mathbb{N}\}$, then $\mathrm{Spec}(X, \mu, T) \subset \Lambda$.

Let us see how we prove the claim. Since, by the first part, (X, μ, T) is an ergodic procyclic system, $\mathrm{Spec}(X, \mu, T)$ is a subgroup of the rationals in $\mathbb{T}$. Any such subgroup is generated by elements of the form $1/p^s$ for $p \in \mathbb{P}$ and $s \in \mathbb{N}$ (see for example [**42**]). By Theorem 2.11, if $p \notin A$, then $1/p \notin \mathrm{Spec}(X, \mu, T)$, and hence $1/p^s \notin \mathrm{Spec}(X, \mu, T)$ for every $s \in \mathbb{N}$. Combining the above facts, we deduce that $\mathrm{Spec}(X, \mu, T)$ is contained in the subgroup generated by elements of the form $1/p^s$ for $p \in A$, $s \in \mathbb{N}$, that is, $\mathrm{Spec}(X, \mu, T) \subset \Lambda$. This proves the claim.

So, it remains to establish the inclusion $\Lambda \subset \mathrm{Spec}(X, \mu, T)$. Since $f \sim \chi$ and for $p \in A$ we either have $p \mid q$ or $f(p) \neq \chi(p)$, Theorem 2.11 implies that $1/p \in \mathrm{Spec}(X, \mu, T)$ for every $p \in A$. Using this, our assumption that $f(p) \neq 0$, and the fact that f is completely multiplicative, we deduce from part (i) of Corollary 2.3 that $1/p^s \in \mathrm{Spec}(X, \mu, T)$ for every $p \in A$ and $s \in \mathbb{N}$. By part (i) of Theorem 2.8, the system (X, μ, T) is ergodic, and hence its spectrum is a subgroup of $\mathbb{T}$. Combining the above, we get $\Lambda \subset \mathrm{Spec}(X, \mu, T)$, which completes the proof.

7.6. *Proof of Theorem 2.13.* By Proposition 7.1, it suffices to show that under the stated assumptions, we have $\alpha \notin \mathrm{Spec}(X, \mu, T)$ for every Furstenberg system (X, μ, T) of f for Cesàro averages. In case (i), this follows from the fact that α is irrational and $\mathrm{Spec}(X, \mu, T)$

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contains only rational values, which is an immediate consequence of Theorem 2.7. In case (ii), this follows from Theorem 2.11.

7.7. *Proof of Theorem 2.14.* We use Lemma 6.3, in its more general version mentioned in the remark after the lemma, and follow the notation there. Note that since $\sum_{j=1}^{\ell} k_j = 0$, in equation (38), we have $t' = 0$ and $\prod_{j=1}^{\ell} (e(\alpha))^{k_j} = 1$. The result then follows from the fact that a sequence $a\colon \mathbb{N} \to \mathbb{U}$ is convergent if any two subsequences of a have further subsequences whose difference converges to 0.

7.8. *Proof of Theorem 2.15.* Suppose that $a\colon \mathbb{N} \to \mathbb{U}$ satisfies the Sarnak conjecture for Cesàro averages along (N_k), and $b = a \cdot f$, where $f\colon \mathbb{N} \to \mathbb{U}$ is a pretentious multiplicative function. Let $w\colon \mathbb{N} \to \mathbb{U}$ be a completely deterministic sequence along (N_k). By Theorem 2.7, the sequence f is completely deterministic. It follows from the remark made after Definition 3.9 that the sequence $f \cdot w$ is also completely deterministic along (N_k). Hence, our assumption on a gives that

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, a(n)\, f(n)\, w(n) = 0.$$

This shows that the sequence $a \cdot f$ satisfies the Sarnak conjecture for Cesàro averages along (N_k) and completes the proof. A similar argument works for logarithmic averages since Theorem 2.7 also applies to this setting.

7.9. *Proof of Theorem 2.16.* Suppose that $a\colon \mathbb{N} \to \{-1, 1\}$ satisfies the Chowla–Elliott conjecture for Cesàro averages along (N_k), and $b = a \cdot f$, where $f\colon \mathbb{N} \to \mathbb{U}$ is a pretentious multiplicative function. It suffices to show that

$$\lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, (a \cdot f)^{\epsilon_1}(n + n_1) \cdots (a \cdot f)^{\epsilon_\ell}(n + n_\ell) = 0$$

for all $\ell \in \mathbb{N}$, distinct $n_1, \ldots, n_\ell \in \mathbb{N}$, and $\epsilon_1, \ldots, \epsilon_\ell \in \{-1, 1\}$ (recall our notation $f^{-1} := \overline{f}$). If this fails, then for some $\ell \in \mathbb{N}$, distinct $n_1, \ldots, n_\ell \in \mathbb{Z}_+$, and $\epsilon_1, \ldots, \epsilon_\ell \in \{-1, 1\}$, we have convergence to a non-zero constant, along a subsequence (N_k') of (N_k) on which the Furstenberg systems of a and f are well defined. Our assumption implies that the Furstenberg system of a along (N_k') is Bernoulli and by Theorem 2.7, the Furstenberg system of f along (N_k') has zero entropy. (Crucially here, we use that a takes values in $\{-1, 1\}$, if a took values on the unit circle we would have to assume that equation (4) holds for all $\epsilon_1, \ldots, \epsilon_\ell \in \mathbb{Z}$ not all of them 0.) Hence, the two systems are disjoint. This implies that the correlation along (N_k') of $a \cdot f$ that we assumed to be non-zero, is equal to the product of the individual correlations of a and f, and hence it is zero since a satisfies the Chowla–Elliott conjecture along (N_k'). This is a contradiction. A similar arguments works for logarithmic averages since Theorem 2.7 also applies to this setting.

8. *Furstenberg systems of MRT functions—Cesàro averages*

In this section, we will prove structural results for Furstenberg systems of MRT multiplicative functions (see Definition 2.3) when these systems are defined using Cesàro averages. In particular, we will prove Theorems 2.18 and 2.19.

Throughout this section and the next one, we use the convention $e(t) := e^{it}$ for $t \in \mathbb{R}$.

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8.1. *Correlations of MRT functions for Cesàro averages.* We will use the following result from [23, Lemmas 3.2 and 3.3] to rewrite correlations of an MRT multiplicative function in a convenient form.

LEMMA 8.1. *Let $f:\mathbb{N}\to\mathbb{S}^1$ be an MRT multiplicative function. Following the terminology in Definition 2.3, let $N_m\to\infty$ with $N_m\le t_{m+1}$, $m\in\mathbb{N}$. Then,*

$$\lim_{m\to\infty}\frac{|n\in[N_m]\colon |f(n)-n^{is_{m+1}}|>t_m^{-1}|}{N_m}=0.$$

We also have the same conclusion when we consider the logarithmic density of the set $\{n\in[N_m]\colon |f(n)-n^{is_{m+1}}|>t_m^{-1}\}$.

Remark. The argument was given in [23] for the standard density, but it also applies for the logarithmic density.

Let (N_m) be a strictly increasing sequence of integers with $t_m\le N_m\le t_{m+1}$. In the computation of the correlations of the function f on the interval $[N_m]$, the previous lemma allows us to replace $f(n)$ with $n^{is_{m+1}}=e(s_{m+1}\log n)$. We deduce that

$$\lim_{m\to\infty}\mathbb{E}_{n\in[N_m]}\prod_{j=1}^{\ell}f^{k_j}(n+n_j)=\lim_{m\to\infty}\mathbb{E}_{n\in[N_m]}\,e\Big(s_{m+1}\Big(\sum_{j=1}^{\ell}k_j\log(n+n_j)\Big)\Big)\tag{61}$$

holds for all $\ell\in\mathbb{N}$, $k_1,\dots,k_\ell,n_1,\dots,n_\ell\in\mathbb{Z}$. Depending on how we choose N_m in relation with s_{m+1}, we get different Furstenberg systems. In the following subsections, we will use this formula to compute the correlations of all MRT multiplicative functions for various choices of N_m and determine the structure of their corresponding Furstenberg systems for Cesàro averages. This will lead to a proof of Theorems 2.18 and 2.19.

8.1.1. *The case $N_m:=\lfloor\alpha s_{m+1}^{1/d}\rfloor$.* The goal of this subsection is to compute the correlations of an MRT function when we average over intervals $[N_m]$ that satisfy $N_m=\lfloor\alpha s_{m+1}^{1/d}\rfloor$ for some $d\in\mathbb{Z}_+$ and $\alpha>0$. This is the context of the next result.

PROPOSITION 8.2. *Let $\alpha>0$, $d\in\mathbb{N}$. For every $\ell\in\mathbb{N}$, $k_1,\dots,k_\ell,n_1,\dots,n_\ell\in\mathbb{Z}$, let i_0 be the minimum $i\in\mathbb{Z}_+$ such that $\sum_{j=1}^{\ell}k_jn_j^i\neq 0$ and $i_0:=+\infty$ if no such i exists. Then,*

$$\lim_{m\to\infty}\mathbb{E}_{n\in[\alpha s_{m+1}^{1/d}]}\prod_{j=1}^{\ell}f^{k_j}(n+n_j)=\begin{cases}0 & \text{if } 0\le i_0<d,\\ \int_0^1(G_{\alpha,d}(x))^{\sum_{j=1}^{\ell}k_jn_j^d}\,dx & \text{if } i_0\ge d,\end{cases}$$

where

$$G_{\alpha,d}(x):=e(1/(\alpha^dx^d))\quad \text{for } x\in(0,1).$$

Remark. Note that the function defined by $x\mapsto e(1/(\alpha x^d))=e^{i/(\alpha^dx^d)}$ for $x\neq 0$, and, say, $0\mapsto 0$, is Riemann integrable in $[0,1]$, since it is bounded and Riemann integrable on $[a,1]$ for every $a\in[0,1]$.

Proposition 8.2 is an immediate consequence of equation (61) and the next result.

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LEMMA 8.3. *Let $\alpha > 0$, $d \in \mathbb{N}$. For every $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell, n_1, \dots, n_\ell \in \mathbb{Z}$, let i_0 be the minimum $i \in \mathbb{Z}_+$ such that $\sum_{j=1}^{\ell} k_j n_j^i \neq 0$ and $i_0 := +\infty$ if no such i exists. Then,*

$$\lim_{N\to\infty} \mathbb{E}_{n\in[\alpha N^{1/d}]}\, e\Bigg(N\Bigg(\sum_{j=1}^{\ell} k_j \log(n+n_j)\Bigg)\Bigg) = \begin{cases} 0 & \text{if } 0 \le i_0 < d, \\ \int_0^1 (G_{\alpha,d}(x))^{\sum_{j=1}^{\ell} k_j n_j^d}\, dx & \text{if } i_0 \ge d, \end{cases}$$

where

$$G_{\alpha,d}(x) := e(1/(\alpha^d x^d)) \quad \text{for } x \in (0,1).$$

Proof. Let

$$A := \lim_{N\to\infty} \mathbb{E}_{n\in[\alpha N^{1/d}]}\, e\Bigg(N\Bigg(\sum_{j=1}^{\ell} k_j \log(n+n_j)\Bigg)\Bigg).$$

For convenience, we approximate $\log(n+a)$ with $\log n$ plus a polynomial. Since

$$\left|\log(n+a) - \log n - \sum_{i=1}^{d} (-1)^{i-1} \frac{a^i}{n^i}\right| \le \frac{C_a}{M_N^{d+1}} \quad \text{for all } n \ge M_N,$$

we have

$$\lim_{N\to\infty} \max_{n\in[N^c, N^{1/d}]} N\left|\log(n+a) - \log n - \sum_{i=1}^{d} (-1)^{i-1} \frac{a^i}{n^i}\right| = 0$$

for every $c \in \mathbb{R}_+$ such that $1/(d+1) < c < 1/d$. So, to compute A, we can replace $\log(n+n_j)$ with $\log n - \sum_{i=1}^{d} (-1)^{i-1} n_j^i / n^i$ throughout. Hence,

$$A = \lim_{N\to\infty} \mathbb{E}_{n\in[\alpha N^{1/d}]}\, e\Bigg(N\Bigg(\sum_{j=1}^{\ell} k_j \log n + \sum_{i=1}^{d} (-1)^{i-1} \sum_{j=1}^{\ell} k_j n_j^i\, n^{-i}\Bigg)\Bigg).$$

Suppose first that $i_0 < d$. Then, $\sum_{j=1}^{\ell} k_j n_j^{i_0} \neq 0$ and by combining Corollary A.4 (it applies for this i_0 since $N^{1/d} \prec N^{1/i_0}$) and Lemma 6.3, we get that $A = 0$.

Suppose now that $i_0 \ge d$. Then, $\sum_{j=1}^{\ell} k_j n_j^i = 0$ for $i = 0, \dots, d-1$, and

$$A = \lim_{N\to\infty} \mathbb{E}_{n\in[\alpha N^{1/d}]}\, e(KN/n^d),$$

where

$$K := \sum_{j=1}^{\ell} k_j n_j^d. \tag{62}$$

Then,

$$A = \lim_{N\to\infty} \mathbb{E}_{n\in[\alpha N^{1/d}]}\, e(K\alpha^{-d}(\lfloor \alpha N^{1/d} \rfloor + \epsilon(N))^d / n^d),$$

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where $\epsilon(N) \in \{0, 1\}$. Note that the error we make by replacing $(\lfloor \alpha N^{1/d} \rfloor + \epsilon(N))^d$ with $(\lfloor \alpha N^{1/d} \rfloor)^d$ is bounded by $CN^{(d-1)/d}$ for some $C > 0$, and this is much smaller than n^d for $n \in [N^c, N^{1/d}]$ whenever $1/d - 1/d^2 < c < 1/d$. We deduce that $\epsilon(N)$ can be ignored in the computation of the above limit without affecting the value of A. Hence,

$$A = \lim_{N\to\infty} \mathbb{E}_{n\in[\alpha N^{1/d}]}\, e(K\alpha^{-d}\lfloor \alpha N^{1/d} \rfloor^d / n^d),$$

where K is as in equation (62). More conveniently,

$$A = \lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, e(K\alpha^{-d}N^d / n^d),$$

assuming that the last limit exists, which is something we shall prove shortly. The last limit can be rewritten as a limit of Riemann sums

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, (G_{\alpha,d}(n/N))^K,$$

where $G_{\alpha,d}\colon (0, 1] \to \mathbb{S}^1$ is the Riemann integrable function in the statement. Hence,

$$A = \int_0^1 (G_{\alpha,d}(x))^K\, dx.$$

This completes the proof. □

8.1.2. *The case* $s_{m+1}^{1/(d+1)} \prec N_m \prec s_{m+1}^{1/d}$. The goal of this subsection is to compute the correlations of an MRT function when we average over intervals $[N_m]$ that satisfy $s_{m+1}^{1/(d+1)} \prec N_m \prec s_{m+1}^{1/d}$ for some $d \in \mathbb{Z}_+$. This is the context of the next result.

PROPOSITION 8.4. *Let $d \in \mathbb{Z}_+$ be fixed. If $d = 0$, suppose that $s_{m+1} \prec N_m \le t_{m+1}$, and if $d \in \mathbb{N}$, suppose that $s_{m+1}^{1/(d+1)} \prec N_m \prec s_{m+1}^{1/d}$. For every $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell, n_1, \dots, n_\ell \in \mathbb{Z}$, let i_0 be the minimum $i \in \mathbb{Z}_+$ such that $\sum_{j=1}^{\ell} k_j n_j^i \neq 0$ and $i_0 := +\infty$ if no such i exists. Then,*

$$\lim_{m\to\infty} \mathbb{E}_{n\in[N_m]} \prod_{j=1}^{\ell} f^{k_j}(n + n_j) = \begin{cases} 0 & \text{if } 0 \le i_0 \le d, \\ 1 & \text{if } i_0 > d. \end{cases}$$

Proposition 8.4 is an immediate consequence of equation (61) and the next result.

LEMMA 8.5. *Let $d \in \mathbb{Z}_+$ be fixed. Suppose that the sequence (L_N) of positive integers satisfies $N \prec L_N$ if $d = 0$, and $N^{1/(d+1)} \prec L_N \prec N^{1/d}$ if $d \in \mathbb{N}$. For every $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell, n_1, \dots, n_\ell \in \mathbb{Z}$, let i_0 be the minimum $i \in \mathbb{Z}_+$ such that $\sum_{j=1}^{\ell} k_j n_j^i \neq 0$ and $i_0 := +\infty$ if no such i exists. Then,*

$$\lim_{N\to\infty} \mathbb{E}_{n\in[L_N]}\, e\Big(N\Big(\sum_{j=1}^{\ell} k_j \log(n + n_j)\Big)\Big) = \begin{cases} 0 & \text{if } 0 \le i_0 \le d, \\ 1 & \text{if } i_0 > d. \end{cases}$$

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Proof. Let $d \in \mathbb{Z}_+$ and A be the limit we aim to compute. As in the proof of Lemma 8.3, we start by approximating $\log(n+a)$ with $\log n$ plus a polynomial. Since

$$\left|\log(n+a) - \log n - \sum_{i=1}^{d}(-1)^{i-1}\frac{a^i}{n^i}\right| \le \frac{C_a}{M_N^{d+1}} \quad \text{for all } n \ge M_N,$$

and $N^{1/(d+1)} \prec L_N$, we have

$$\lim_{N\to\infty} \max_{n\in[cL_N, L_N]} N\left|\log(n+a) - \log n - \sum_{i=1}^{d}(-1)^{i-1}\frac{a^i}{n^i}\right| = 0$$

for every positive $c \in (0,1)$. So using Lemma A.3, we get that to compute A, we can replace $\log(n+n_j)$ with $\log n - \sum_{i=1}^{d}(-1)^{i-1}n_j^i/n^i$ throughout. (This is a trivial but crucial reduction that does not work for logarithmic averages and leads to different correlations and Furstenberg systems.) Hence,

$$A = \lim_{N\to\infty} \mathbb{E}_{n\in[L_N]}\, e\left(N\left(\sum_{j=1}^{\ell} k_j \log n + \sum_{i=1}^{d}(-1)^{i-1}\sum_{j=1}^{\ell} k_j n_j^i\, n^{-i}\right)\right). \tag{63}$$

We also get a similar identity with $d+1$ in place of d.

Suppose first that $i_0 \le d$. We have $\sum_{j=1}^{\ell} k_j n_j^{i_0} \neq 0$ and we get by combining identity (63) with Corollary A.4 (it applies for this i_0 since $L_N \prec N^{1/d}$, and hence $L_N \prec N^{1/i_0}$) and Lemma 6.3 that $A = 0$.

Suppose now that $i_0 > d$. Then, $\sum_{j=1}^{\ell} k_j n_j^i = 0$ for $i = 0, \dots, d$. In this case, using identity (63) with $d+1$ in place of d, we have

$$A = \lim_{N\to\infty} \mathbb{E}_{n\in[L_N]}\, e(K\, N/n^{d+1}),$$

where $K := (-1)^d \sum_{j=1}^{\ell} k_j n_j^{d+1}$. Our growth assumption $N^{1/(d+1)} \prec L_N$ implies that for all $c \in (0,1)$, we have

$$\lim_{N\to\infty} \max_{n\in[cL_N, L_N]} (N/n^{d+1}) = 0.$$

It follows that

$$\lim_{N\to\infty} \mathbb{E}_{n\in[cL_N, L_N]}\, e(K\, N/n^{d+1}) = 1$$

for all $c \in (0,1)$. Hence, $A = 1$ by Lemma A.3. (This last conclusion fails badly for logarithmic averages, which is the reason why similar computations for logarithmic averages lead to very different expressions.) This completes the proof. □

8.2. *Ergodic models of MRT functions for Cesàro correlations.* Having computed the correlations of MRT functions for certain ranges of the parameter N_m, we would like to identify simple measure-preserving systems and functions that reproduce these correlations. When we manage to do this, it will be an easy matter to show isomorphism between this system and the Furstenberg system of the MRT function.

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8.2.1. *The case $N_m := \lfloor \alpha s_{m+1}^{1/d} \rfloor$.*. The next lemma identifies systems and functions that reproduce the correlations of MRT functions when $N_m := \lfloor \alpha s_{m+1}^{1/d} \rfloor$.

LEMMA 8.6. *For $\alpha > 0$ and $d \in \mathbb{Z}_+$, let $(\mathbb{T}^{d+1}, S_{\alpha,d}, m_{\mathbb{T}^{d+1}})$ be the system given in Definition 2.6. Let also $F\colon \mathbb{T}^{d+1} \to \mathbb{S}^1$ be defined by*

$$F(x_0, \dots, x_d) := e(x_d).$$

For every $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell, n_1, \dots, n_\ell \in \mathbb{Z}$, let i_0 be the minimum $i \in \mathbb{Z}_+$ such that $\sum_{j=1}^{\ell} k_j n_j^i \neq 0$ and $i_0 := +\infty$ if no such i exists. Then,

$$\int \prod_{j=1}^{\ell} S_{\alpha,d}^{n_j} F^{k_j} \, dm_{\mathbb{T}^{d+1}} = \begin{cases} 0 & \text{if } 0 \le i_0 < d, \\ \int_0^1 (G_{\alpha,d}(x))^{\sum_{j=1}^{\ell} k_j n_j^d} \, dx & \text{if } i_0 \ge d, \end{cases} \tag{64}$$

where

$$G_{\alpha,d}(x) := e(1/(\alpha^d x^d)) \quad \text{for } x \in (0,1).$$

Proof. For $x = (x_0, \dots, x_d) \in \mathbb{T}^{d+1}$, direct computation gives that

$$F(S_{\alpha,d}^n x) = e\Big(\sum_{i=0}^{d-1} \binom{n}{i} x_{d-i} + \binom{n}{d} g_{\alpha,d}(x_0) \Big), \quad n \in \mathbb{N}.$$

Hence, for $x \in \mathbb{T}^{d+1}$, we have

$$\prod_{j=1}^{\ell} F^{k_j}(S_{\alpha,d}^{n_j} x) = e\Big(\sum_{i=0}^{d-1} c_i \cdot x_{d-i} \Big) \cdot (G_{\alpha,d}(x_0))^{c_d},$$

where

$$c_i := \sum_{j=1}^{\ell} k_j \binom{n_j}{i}, \quad i = 0, \dots, d.$$

Note that $c_i = 0$ for $i = 0, \dots, d-1$ if and only if $\sum_{j=1}^{\ell} k_j n_j^i = 0$ for $i = 0, \dots, d-1$, equivalently when $i_0 \ge d$, and in this case, we have $c_d = \sum_{j=1}^{\ell} k_j n_j^d$. It follows that the integral in equation (64) is equal to $\int_0^1 (G_{\alpha,d}(x))^{\sum_{j=1}^{\ell} k_j \binom{n_j}{d}} \, dx$ if $i_0 \ge d$, and is equal to 0 otherwise, that is, when $i_0 < d$. This completes the proof. □

8.2.2. *The case $s_{m+1}^{1/(d+1)} \prec N_m \prec s_{m+1}^{1/d}$.* Repeating the computation in the proof of Lemma 8.6 with the function x_0 in place of $g_{\alpha,d}(x_0)$ gives the following result.

LEMMA 8.7. *For $d \in \mathbb{Z}_+$, let $(\mathbb{T}^{d+1}, S_d, m_{\mathbb{T}^{d+1}})$ be the level d unipotent system given in Definition 2.4. Let also $F\colon \mathbb{T}^{d+1} \to \mathbb{S}^1$ be defined by*

$$F(x_0, x_1, \dots, x_d) := e(x_d).$$

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For every $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell, n_1, \dots, n_\ell \in \mathbb{Z}$, let i_0 be the minimum $i \in \mathbb{Z}_+$ such that $\sum_{j=1}^{\ell} k_j n_j^i \neq 0$ and $i_0 := +\infty$ if no such i exists. Then,

$$\int \prod_{j=1}^{\ell} S_d^{n_j} F^{k_j} \, dm_{\mathbb{T}^{d+1}} = \begin{cases} 0 & \text{if } 0 \le i_0 \le d, \\ 1 & \text{if } i_0 > d. \end{cases}$$

8.3. *Furstenberg systems of MRT functions for Cesàro averages.* We are now ready to prove our main results regarding Furstenberg systems of MRT functions for Cesàro averages.

Proof of Theorem 2.19. We first show that the system $(X, \mu_{\alpha,d}, T)$ is isomorphic to the system $(\mathbb{T}^{d+1}, m_{\mathbb{T}^{d+1}}, S_{\alpha,d})$.

Recall that $X = \mathbb{U}^{\mathbb{Z}}$. Let $F_0 \colon X \to \mathbb{S}^1$ be the 0th-coordinate projection and $F \colon \mathbb{T}^{d+1} \to \mathbb{S}^1$ be as in Lemma 8.6. Combining Proposition 8.2 with Lemma 8.6, we deduce that

$$\lim_{m\to\infty} \mathbb{E}_{n\in[\alpha s_{m+1}^{1/d}]} \prod_{j=1}^{\ell} f^{k_j}(n + n_j) = \int \prod_{j=1}^{\ell} S_{\alpha,d}^{n_j} F^{k_j} \, dm_{\mathbb{T}^{d+1}}.$$

Hence,

$$\int \prod_{j=1}^{\ell} T^{n_j} F_0^{k_j} \, d\mu_{\alpha,d} = \int \prod_{j=1}^{\ell} S_{\alpha,d}^{n_j} F^{k_j} \, dm_{\mathbb{T}^{d+1}} \tag{65}$$

holds for every $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell, n_1, \dots, n_\ell \in \mathbb{Z}$.

Next, we define the map $\Phi \colon \mathbb{T}^{d+1} \to X$ by

$$\Phi(y) := (F(S_{\alpha,d}^n y))_{n\in\mathbb{Z}}, \quad y \in \mathbb{T}^{d+1}.$$

We clearly have $\Phi \circ S_{\alpha,d} = T \circ \Phi$. Moreover, a direct computation shows that the map Φ is injective. Lastly, since $F = F_0 \circ \Phi$, identity (65) implies that

$$\int \prod_{j=1}^{\ell} T^{n_j} F_0^{k_j} \, d\mu_{\alpha,d} = \int \prod_{j=1}^{\ell} T^{n_j} F_0^{k_j} \, d(m_{\mathbb{T}^{d+1}} \circ \Phi^{-1}).$$

Hence, $\mu_{\alpha,d} = m_{\mathbb{T}^{d+1}} \circ \Phi^{-1}$, since a linearly dense subset of $C(X)$ has the same integral with respect to the two measures (see second remark after Definition 3.5). This establishes the asserted isomorphism.

Since the system $(\mathbb{T}^{d+1}, m_{\mathbb{T}^{d+1}}, S_{\alpha,d})$ has trivial rational spectrum, so does the system $(X, \mu_{\alpha,d}, T)$.

To prove that the system $(X, \mu_{\alpha,d}, T)$ is not strongly stationary, it suffices to show that

$$\int \overline{F_0} \cdot T F_0 \, d\mu \neq \int \overline{F_0} \cdot T^r F_0 \, d\mu$$

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for some $r \in \mathbb{N}$. Equivalently, by applying equations (65) (for $\ell = 2$, $n_1 = 0$, $n_2 = r$, $k_1 = -1$, $k_2 = 1$) and (64), it suffices to show that for fixed α, d we have

$$\int_0^1 e(1/(\alpha^d x^d))\, dx \neq \int_0^1 e(r/(\alpha^d x^d))\, dx$$

for some non-zero $r \in \mathbb{Z}$. Direct computation shows that this is the case, for example, when $\alpha = d = 1$ and $r = 2$. □

Proof of Theorem 2.18. We combine Proposition 8.4 with Lemma 8.7, and argue as in the proof of Theorem 2.19. □

9. *Furstenberg systems of MRT functions—Logarithmic averages*

In this section, we will prove structural results for Furstenberg systems of MRT multiplicative functions (see Definition 2.3) when these systems are defined using logarithmic averages. In particular, we will prove Theorem 2.20. In the case of logarithmic averages, the correlations of MRT functions turn out to be different than those for Cesàro averages, and this leads to substantially different Furstenberg systems.

9.1. *Correlations of MRT functions for logarithmic averages.*

The main goal of this subsection is to give in Proposition 9.1 an explicit description of the correlations for logarithmic averages of MRT functions when we average over sequences of intervals that grow as fractional powers of s_{m+1}, a notion that we define next. Again, using Lemma 8.1, our starting point is the following identity:

$$\lim_{m\to\infty} \mathbb{E}^{\log}_{n\in[N_m]} \prod_{j=1}^{\ell} f^{k_j}(n+n_j) = \lim_{m\to\infty} \mathbb{E}^{\log}_{n\in[N_m]} e\Big(s_{m+1}\Big(\sum_{j=1}^{\ell} k_j \log(n+n_j)\Big)\Big), \quad (66)$$

which holds for all $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell, n_1, \dots, n_\ell \in \mathbb{Z}$. As in the case of Cesàro averages, depending on how we choose N_m in relation with s_{m+1}, we get different Furstenberg systems, and the next notion will help us identify which sequences N_m give rise to which systems.

Definition 9.1. We define the *fractional degree* of a sequence of positive real numbers $L_N \to +\infty$ to be

$$\lim_{N\to\infty} \frac{\log L_N}{\log N}$$

if the limit exists (could be $+\infty$).

Remarks.

- Note that

$$\lim_{N\to\infty} \frac{\log L_N}{\log N} = \begin{cases} 0 & \text{if } L_N \prec N^{\varepsilon} \quad \text{for every } \varepsilon > 0, \\ c \in (0,+\infty) & \text{if } N^{c-\varepsilon} \prec L_N \prec N^{c+\varepsilon} \quad \text{for every } \varepsilon > 0, \\ +\infty & \text{if } N^d \prec L_N \quad \text{for every } d \in \mathbb{N}. \end{cases}$$

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- If $c \in (0, +\infty)$, then the sequence (N^c) has fractional degree c. More generally, the same holds for the sequence $(N^c(\log N)^b + L'_N)$ for every $b \in \mathbb{R}$ and every (L'_N) with $L'_N \prec N^c$.

PROPOSITION 9.1. *Let $N_m := s_{m+1}^{1/c}$ for some $c > 0$. For every $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell$, $n_1, \dots, n_\ell \in \mathbb{Z}$, let i_0 be the minimum $i \in \mathbb{Z}_+$ such that $\sum_{j=1}^{\ell} k_j n_j^i \neq 0$ and $i_0 := +\infty$ if no such i exists. Then,*

$$\lim_{m\to\infty} \mathbb{E}^{\log}_{n\in[N_m]} \prod_{j=1}^{\ell} f^{k_j}(n + n_j) = \begin{cases} 0 & \text{if } 0 \le i_0 \le c, \\ 1 - c/i_0 & \text{if } c < i_0 < +\infty, \\ 1 & \text{if } i_0 = +\infty. \end{cases}$$

Furthermore, the same identity holds if (L_N) is a sequence of positive real numbers with fractional degree $1/c$ and $N_m := L_{s_{m+1}}$.

Remark. By taking $c := 1/2$ (or any other $c < 1$), we get the following non-vanishing property of the 2-point correlations of f:

$$\lim_{m\to\infty} \mathbb{E}^{\log}_{n\in[s_{m+1}^2]} \overline{f(n)} \cdot f(n + h) = \frac{1}{2}$$

for every non-zero $h \in \mathbb{Z}$.

Proposition 9.1 is an immediate consequence of identity (66) and the next result.

LEMMA 9.2. *Let (L_N) be a sequence of positive real numbers with fractional degree $1/c$ for some $c \in (0, +\infty)$. For every $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell, n_1, \dots, n_\ell \in \mathbb{Z}$, let i_0 be the minimum $i \in \mathbb{Z}_+$ such that $\sum_{j=1}^{\ell} k_j n_j^i \neq 0$ and $i_0 := +\infty$ if no such i exists. Then,*

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N]} e\Big(N\Big(\sum_{j=1}^{\ell} k_j \log(n + n_j)\Big)\Big) = \begin{cases} 0 & \text{if } 0 \le i_0 \le c, \\ 1 - c/i_0 & \text{if } c < i_0 < +\infty, \\ 1 & \text{if } i_0 = +\infty. \end{cases}$$

Remarks.

- For example, if A is the value of the limit on the left-hand side, then for $c \in [1, 2)$ (in which case, L_N has fractional degree in $(1/2, 1]$), we have $A = 0$ if either $\sum_{j=1}^{\ell} k_j \neq 0$ or $\sum_{j=1}^{\ell} k_j n_j \neq 0$ (since $i_0 \le 1 \le c$); we have $A = 1 - c/2$ if $\sum_{j=1}^{\ell} k_j = \sum_{j=1}^{\ell} k_j n_j = 0$ and $\sum_{j=1}^{\ell} k_j n_j^2 \neq 0$ (since $c < i_0 = 2$), and so on.
- Note that the limit A remains the same if we replace $n_1, \dots, n_\ell$ with $rn_1, \dots, rn_\ell$, for every $r \in \mathbb{N}$ and $\ell \in \mathbb{N}$, $n_1 \dots, n_\ell \in \mathbb{Z}$. Hence, the Furstenberg system of the multiplicative function that has such correlations is strongly stationary. This is unlike the case of Cesàro averages, where for $L_N := N^{1/k}$, $k \in \mathbb{N}$, we got that the corresponding correlations given in Lemma 8.6 were not dilation invariant.

Proof. To have a specific example in mind, in the course of the proof, the reader may find it convenient to assume that $L_N = N^{1/c}$, where $c > 0$. We denote by A the limit of the averages we want to compute.

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We shall use the following basic estimate that follows by applying the Taylor–Lagrange theorem for the function $\log x$. For every $a \in \mathbb{R}_+$, $d \in \mathbb{N}$, and $M_N > 0$, we have

$$\Big|\log(n+a) - \log n - \sum_{i=1}^{d}(-1)^{i-1}\frac{a^i}{n^i}\Big| \le \frac{C_a}{M_N^{d+1}} \quad \text{for all } n \ge M_N. \tag{67}$$

The utility of this approximation is that it enables us to connect various exponential sums that appear below to those treated in Corollary A.6.

Case 1 ($i_0 \le c$). If $i_0 \le c$, we claim that $A = 0$. Let $\varepsilon > 0$. We also take $\varepsilon < 1$. We apply equation (67) for $d_\varepsilon := [1/\varepsilon]$, $a := n_j$, $j = 1, \dots, \ell$, and use that $\lim_{N\to\infty}(N/M_N^{d_\varepsilon+1}) = 0$ for $M_N := N^\varepsilon$. We get that

$$\lim_{N\to\infty}\max_{n\in[N^\varepsilon, L_N]} N\Big|\sum_{j=1}^{\ell} k_j \log(n+n_j) - \sum_{j=1}^{\ell} k_j \log n + \sum_{i=1}^{d_\varepsilon}(-1)^{i-1}\sum_{j=1}^{\ell} k_j n_j^i\, n^{-i}\Big| = 0.$$

Using this and the estimate $\mathbb{E}^{\log}_{n\in[N]} \mathbf{1}_{[1,N^\varepsilon]}(n) \le \varepsilon$, we get that for

$$A_\varepsilon := \lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N]}\, e\Big(N\Big(\sum_{j=1}^{\ell} k_j \log n + \sum_{i=1}^{d_\varepsilon}(-1)^{i-1}\sum_{j=1}^{\ell} k_j n_j^i\, n^{-i}\Big)\Big),$$

we have

$$|A - A_\varepsilon| \le \varepsilon.$$

So, to show that $L = 0$, it suffices to show that for every $\varepsilon > 0$, we have $A_\varepsilon = 0$. Equivalently, it suffices to show that (for every $\varepsilon > 0$)

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N]}\, e\Big(N\Big(\sum_{j=1}^{\ell} k_j \log n + \sum_{i=1}^{d_\varepsilon}(-1)^{i-1}\sum_{j=1}^{\ell} k_j n_j^i\, n^{-i}\Big)\Big) = 0.$$

Recall that the defining property of i_0 implies that $\sum_{j=1}^{\ell} k_j n_j^i = 0$ for $i = 0, \dots, i_0 - 1$. Since $c \ge i_0$ and the fractional degree of L_N is $1/c$ (so positive in particular), we have $N^\gamma \prec L_N \prec N^{(1/i_0)+\varepsilon}$ for some $\gamma > 0$ and every $\varepsilon > 0$. Hence, Corollary A.6 applies and gives $A = 0$.

Case 2 ($i_0 > c$). We claim that $A = 1 - c/i_0$ if i_0 is finite and $L = 1$ if $i_0 = +\infty$. Suppose first that i_0 is finite.

We decompose

$$A = A_1 + A_2,$$

where

$$A_1 := \lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N]} \mathbf{1}_{[1,N^{1/i_0}]}(n)\, e\Big(N\Big(\sum_{j=1}^{\ell} k_j \log(n+n_j)\Big)\Big)$$

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and

$$A_2 := \lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[N^{1/c}]} \mathbf{1}_{[N^{1/i_0},L_N]}(n)\, e\Big(N\Big(\sum_{j=1}^{\ell} k_j \log(n+n_j)\Big)\Big).$$

(Note that since L_N has fractional degree $1/c > 1/i_0$, we have $N^{1/i_0} \prec L_N$ and the logarithmic averages over $[L_N]$ and $[N^{1/c}]$ coincide.)

Case 2a (Computation of A_1). We claim that $A_1 = 0$. Indeed, we have

$$A_1 = c\cdot i_0^{-1}\cdot \lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[N^{1/i_0}]} e\Big(N\Big(\sum_{j=1}^{\ell} k_j \log(n+n_j)\Big)\Big)$$

and by Case 1, the last limit is 0.

Case 2b (Computation of A_2). It remains to show that $A_2 = 1 - c/i_0$. For $\varepsilon > 0$ such that $1/i_0 + \varepsilon < 1/c$, let

$$A_{2,\varepsilon} := \lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[N^{1/c}]} \mathbf{1}_{[N^{1/i_0+\varepsilon},N^{1/c}]}(n)\, e\Big(N\Big(\sum_{j=1}^{\ell} k_j \log(n+n_j)\Big)\Big). \tag{68}$$

Since $A_2 = \lim_{\varepsilon\to 0^+} A_{2,\varepsilon}$, it suffices to compute $A_{2,\varepsilon}$ for these values of $\varepsilon > 0$.

We apply equation (67) for $d := i_0 - 1$ (which is ≥ 0 since $i_0 \geq 1$), $a = n_j$, $j = 1,\dots,\ell$, and $M_N := N^{1/i_0+\varepsilon}$ for ε small enough so that $M_N \prec L_N$. Note that since $\lim_{N\to\infty}(N/M_N^{d+1}) = 0$, we have

$$\lim_{N\to\infty} \max_{n\in[N^{1/i_0+\varepsilon},L_N]} N\Big|\log(n+a) - \log n - \sum_{i=1}^{i_0-1}(-1)^{i-1}\frac{a^i}{n^i}\Big| = 0.$$

(If $i_0 = 1$, the sum over i is empty.) For $j = 1,\dots,\ell$, we apply these identities for $a := n_j$, multiply them by k_j, and add them up. We deduce using the defining property of i_0 ($\sum_{j=1}^{\ell} k_j n_j^i = 0$ for $i = 0,\dots,i_0-1$) that

$$\lim_{N\to\infty} \max_{n\in[N^{1/i_0+\varepsilon},L_N]} N\Big|\sum_{j=1}^{\ell} k_j \log(n+n_j)\Big| = 0.$$

Hence,

$$\lim_{N\to\infty} \max_{n\in[N^{1/i_0+\varepsilon},L_N]} \Big|e\Big(N\sum_{j=1}^{\ell} k_j \log(n+n_j)\Big) - 1\Big| = 0.$$

If we combine this with equation (68), we get (we use here that the fractional degree of L_N is $1/c$ and the fact that we use logarithmic averages)

$$A_{2,\varepsilon} = \lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N]} \mathbf{1}_{[N^{1/i_0+\varepsilon},L_N]}(n) = 1 - c/i_0 - c\varepsilon.$$

Hence, $A_2 = \lim_{\varepsilon\to 0^+} A_{2,\varepsilon} = 1 - c/i_0$. This completes the proof of the Case 2 when i_0 is finite.

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If $i_0 = +\infty$ (in this case, $A = A_2$), then the previous argument gives for every $i_0 \in \mathbb{N}$ the following identity:

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N]} \mathbf{1}_{[N^{1/j_0},L_N]}(n)\, e\Big(N\Big(\sum_{j=1}^{\ell} k_j \log(n+n_j)\Big)\Big) = 1 - c/j_0.$$

Letting $j_0 \to +\infty$ gives that $A = A_2 = 1$, as required. □

9.2. *Ergodic models of MRT functions for logarithmic correlations.* As was the case in §8.2, our goal is to identify simple measure-preserving systems and functions that reproduce the correlations in the number theory setting obtained in Proposition 9.1. With a bit of guesswork, we get that the systems in Definition 2.7 help us do the job; in fact, the next lemma motivated their definition.

LEMMA 9.3. *Let $c > 0$. For $d \geq \lfloor c \rfloor$, let (Y_d, ν_d, S_d) be the level d unipotent system given in Definition 2.4 and $F_d\colon \mathbb{T}^{d+1} \to \mathbb{S}^1$ be defined by*

$$F_d(x_0, \dots, x_d) := e(x_d).$$

Let also the system (Z_c, ν_c', R_c) be as in Definition 2.7 and $G_c \in L^\infty(\nu_c')$ be defined by

$$G_c := \sum_{d=\lfloor c \rfloor}^{\infty} \mathbf{1}_{Y_d} \cdot F_d.$$

Lastly, for every $\ell \in \mathbb{N}$, $k_1, \dots, k_\ell, n_1, \dots, n_\ell \in \mathbb{Z}$, let i_0 be the minimum $i \in \mathbb{Z}_+$ such that $\sum_{j=1}^{\ell} k_j n_j^i \neq 0$ and $i_0 := +\infty$ if no such i exists. Then,

$$\int \prod_{j=1}^{\ell} R_c^{n_j} G_c^{k_j}\, d\nu_c' = \begin{cases} 0 & \text{if } 0 \leq i_0 \leq c, \\ 1 - c/i_0 & \text{if } c < i_0 < +\infty, \\ 1 & \text{if } i_0 = +\infty. \end{cases}$$

Proof. We have that

$$\int \prod_{j=1}^{\ell} R_c^{n_j} G_c^{k_j}\, d\nu_c' = \Big(1 - \frac{c}{\lceil c \rceil}\Big) C_{\lfloor c \rfloor} + c \sum_{d=\lceil c \rceil}^{\infty} \Big(\frac{1}{d} - \frac{1}{d+1}\Big) C_d,$$

where for $d \in \mathbb{Z}_+$, we let

$$C_d := \int \prod_{j=1}^{\ell} S_d^{n_j} F_d^{k_j}\, d\nu_d.$$

By Lemma 8.7, we have

$$C_d = \begin{cases} 0 & \text{if } 0 \leq i_0 \leq d, \\ 1 & \text{if } i_0 > d. \end{cases}$$

The asserted identity follows by combining the previous identities and simple direct computation. □

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9.3. *Furstenberg systems of MRT functions for logarithmic averages.* Combining the previous results, it is now easy to prove our main result regarding the structure of Furstenberg systems of MRT functions for logarithmic averages.

Proof of Theorem 2.20. We combine Proposition 9.1 with Lemma 9.3, and argue as in the proof of Theorem 2.19. □

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A. *Appendix. Exponential sum estimates*

In this appendix, we gather some simple facts and exponential sum estimates used in the proofs of the results regarding the structure of Furstenberg systems of MRT multiplicative functions.

A.1. *Cesàro averages.* The next result is an immediate consequence of [24, Theorem 2.9].

THEOREM A.1. (Kuzmin–van der Corput) *Let $q \geq 2$ be an integer and $c \in (0, 1)$. Suppose that C_1, C_2 are constants (depending on q and c only) and $h \in C^q([1, +\infty))$ be a function, such that for some $L, M > 0$, we have*

$$C_1 \, M \, L^{-r} \leq \max_{x \in [cL, L]} |h^{(r)}(x)| \leq C_2 \, M \, L^{-r}$$

for $r = 1, \dots, q$. Then, there exists a positive constant C_3, depending only on c, C_1, C_2 (and not on L, M), such that

$$|\mathbb{E}_{n \in [cL, L]} \, e(h(n))| \leq C_3((M/L^q)^{1/Q} + 1/M),$$

where $Q := 4 \cdot 2^{q-2} - 2$.

We are only going to use the following consequence that applies to a special class of functions g appearing in our arguments.

COROLLARY A.2. *Let $g(x) := c_0 \, \log x + \sum_{i=1}^{k} c_i x^{-i}$ for $x \in \mathbb{R}_+$. Let also i_0 be the minimum $i \in \{0, \dots, k\}$ such that $c_i \neq 0$. Then, for every $q \geq 2$ and $c \in (0, 1)$, there exists a constant $C_{c,q,g}$ such that the following holds. If $L > 0$ is large enough (depending only on c, q, g), then for every $N > 0$, we have*

$$|\mathbb{E}_{n \in [cL, L]} \, e(N \, g(n))| \leq C_{c,q,g}((N/L^{q+i_0})^{1/Q} + L^{i_0}/N),$$

where $Q := 4 \cdot 2^{q-2} - 2$.

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Proof. Let $N > 0$, $c > 0$, and $h_N(x) := Ng(x)$. We have

$$h_N^{(r)}(x) = N\Big(\sum_{i=0}^{k} c_i\, c_{r,i}\, x^{-i-r}\Big)$$

for every $r \in \mathbb{N}$ for some non-zero constants $c_{0,r}, \dots, c_{k,r}$. Bounding the derivatives of the function inside the parenthesis (which does not depend on N), we get that there exist $L_0 \in \mathbb{N}$ and $C_1, C_2 > 0$, depending only on c, g, r but not on N, such that for all $L \geq L_0$, we have (we also use the defining property of i_0 here)

$$C_1\, NL^{-i_0-r} \leq \max_{x\in[cL,L]} |h_N^{(r)}(x)| \leq C_2\, NL^{-i_0-r}$$

for $r = 1, \dots, q$. Using Theorem A.1 for $h := h_N$ and $M := NL^{-i_0}$, we get that the asserted estimate holds for all $L \geq L_0$. □

LEMMA A.3. *Let $L_N \to \infty$ be a sequence of positive integers. For $N \in \mathbb{N}$, let $a_N\colon [L_N] \to \mathbb{U}$ be finite sequences such that for every small enough $c > 0$, we have*

$$\lim_{N\to\infty} \mathbb{E}_{n\in[cL_N,L_N]}\, a_N(n) = 0.$$

Then,

$$\lim_{N\to\infty} \mathbb{E}_{n\in[L_N]}\, a_N(n) = 0.$$

Remark. We crucially use here that $(1/L_N)\sum_{n\in[1,cL_N]} |a_N(n)| \leq c$ and this converges to 0 as $c \to 0^+$. An analogous property fails for logarithmic averages, so this convergence criterion cannot be used for logarithmic averages.

Combining the above, we get the following qualitative result that we will use in subsequent sections.

COROLLARY A.4. *Let $g(x) = c_0 \log x + \sum_{i=1}^{k} c_i x^{-i}$ be non-zero and i_0 be the minimum $i \in \{0, \dots, k\}$ such that $c_i \neq 0$. Let also (L_N) be a sequence of positive real numbers that satisfies $N^\gamma \prec L_N \prec N^{1/i_0}$ for some $\gamma > 0$. Then,*

$$\lim_{N\to\infty} \mathbb{E}_{n\in[L_N]}\, e(N\, g(n)) = 0.$$

Remark. If $i_0 = 0$, then our only growth assumption is $N^\gamma \prec L_N$ for some $\gamma > 0$.

Proof. Let $c \in (0, 1)$. Choose $q \geq 2$ such that $(q + i_0)\gamma > 1$. Using Corollary A.2 and our growth assumption on L_N, we get that

$$\lim_{N\to\infty} \mathbb{E}_{n\in[cL_N,L_N]}\, e(N\, g(n)) = 0.$$

We deduce from this using Lemma A.3 that the asserted convergence to 0 holds. □

A.2. *Logarithmic averages.* The next lemma will enable us to deduce convergence results for logarithmic averages of variable sequences from corresponding results for Cesàro averages once we have some additional uniformity.

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LEMMA A.5. *For $N \in \mathbb{N}$, let L_N be a sequence of positive integers that satisfies $N^\gamma \prec L_N$ for some $\gamma > 0$, and let $a_N : [L_N] \to \mathbb{U}$ be finite sequences. Suppose that for all small enough $c, \delta > 0$, we have*

$$\lim_{N\to\infty} \sup_{n\in[L_N^\delta, L_N^{1-\delta}]} |\mathbb{E}_{k\in[cn,n]}\, a_N(k)| = 0. \tag{A.1}$$

Then,

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N]}\, a_N(n) = 0. \tag{A.2}$$

Proof. We first note that our assumption (A.1) implies

$$\lim_{N\to\infty} \sup_{n\in[L_N^\delta, L_N^{1-\delta}]} |\mathbb{E}_{k\in[n]}\, a_N(k)| = 0. \tag{A.3}$$

Indeed, one immediately verifies that the difference between the two expressions is bounded by a function of c that converges to 0 as $c \to 0^+$.

Note also that to show equation (A.2), it suffices to show that for all small enough $\delta > 0$, we have

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N^\delta, L_N^{1-\delta}]}\, a_N(n) = 0. \tag{A.4}$$

Indeed, because of the logarithmic averaging, the difference between this limit and that in equation (A.2) is bounded by a function of δ that converges to 0 as $\delta \to 0^+$.

Now let $\varepsilon > 0$ and $\delta > 0$ be small enough. Equation (A.3) implies that there exists N_0 such that for $N > N_0$, we have

$$\sup_{n\in[L_N^\delta, L_N^{1-\delta}]} |\mathbb{E}_{k\in[n]}\, a_N(k)| \le \varepsilon. \tag{A.5}$$

Using partial summation, we get for $N \ge N_0$ that

$$\left| \sum_{n\in[L_N^\delta, L_N^{1-\delta}]} \frac{a_N(n)}{n} \right| \le 2 + \sum_{n\in[L_N^\delta, L_N^{1-\delta}]} \left| \frac{1}{n^2} \sum_{k=1}^{n} a_N(k) \right| \le 2 + \sum_{n\in[L_N^\delta, L_N^{1-\delta}]} \frac{\varepsilon}{n},$$

where the last estimate follows from equation (A.5). Note also that since $N^\gamma \prec L_N$, we have $\lim_{N\to\infty} \sum_{n\in[L_N^\delta, L_N^{1-\delta}]} (1/n) = +\infty$. We deduce that

$$\limsup_{N\to\infty} |\mathbb{E}^{\log}_{n\in[L_N^\delta, L_N^{1-\delta}]}\, a_N(n)| \le \varepsilon.$$

Since ε is arbitrary, this implies equation (A.4) and completes the proof. □

COROLLARY A.6. *Let $g(x) := c_0 \log x + \sum_{i=1}^{k} c_i x^{-i}$ be non-zero and i_0 be the minimum $i \in \{0, \dots, k\}$ such that $c_i \neq 0$. Let also (L_N) be a sequence of positive real numbers that satisfies $N^\gamma \prec L_N \prec N^{(1/i_0)+\varepsilon}$ for some $\gamma > 0$ and for every $\varepsilon > 0$. Then,*

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N]}\, e(N\, g(n)) = 0.$$

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Remarks.

- If $i_0 = 0$, then our only growth assumption is $N^\gamma \prec L_N$ for some $\gamma > 0$.
- If $c \in (0, 1/i_0]$, then the assumptions are satisfied when $L_N := N^c$ or $L_N := N^c \log N$. If $i_0 = 0$, then the assumptions are satisfied when $L_N = N^c$ for all $c \in (0, +\infty)$. Note that by Corollary A.4, if we use Cesàro averages, then a similar result holds for $c \in (0, 1/i_0)$, but not for $c = 1/i_0$. For example, although $\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[N]}\, e(N/n) = 0$, we have $\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, e(N/n) \neq 0$.

Proof. Let $c \in (0, 1)$ and $\delta > 0$ be arbitrary. Let $\gamma > 0$ be such that $N^\gamma \prec L_N$. Choose $q \geq 2$ such that $(q + i_0)\gamma\delta > 1$ and $\varepsilon > 0$ such that $(1 + \varepsilon)(1 - \delta) < 1$. Using Corollary A.2 and our growth assumption on L_N, we get that there exists a constant $C_{c,q}$ such that for all large enough N, we have

$$\sup_{L_N^\delta \leq n \leq L_N^{1-\delta}} |\mathbb{E}_{k\in[cn,n]}\, e(Ng(n))| \leq C_{c,q}(N/N^{(q+i_0)\gamma\delta})^{1/Q} + N^{(1+\varepsilon)(1-\delta)}/N),$$

where $Q := 4 \cdot 2^{q-2} - 2$ if $i_0 \geq 1$, and a similar estimate with 1 in place of $N^{(1+\varepsilon)(1-\delta)}$ if $i_0 = 0$. Since $(q + i_0)\gamma\delta > 1$ and $(1 + \varepsilon)(1 - \delta) < 1$, we get

$$\lim_{N\to\infty} \sup_{L_N^\delta \leq n \leq L_N^{1-\delta}} |\mathbb{E}_{k\in[cn,n]}\, e(Ng(n))| = 0.$$

Applying Lemma A.5, we deduce

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[L_N]}\, e(N\, g(n)) = 0$$

completing the proof. □

B. *Appendix. Factor of a product* erg × id

In this appendix, we prove an ergodic result that was used in the proof of part (ii) of Theorem 2.8.

Our setting is the following. We consider two systems (Y, ν, S) and $(Z, \lambda, \mathrm{id})$, where:

- $Y := \mathbb{U}^{\mathbb{Z}}$, $S\colon Y \to Y$ is the shift map, and ν is an ergodic S-invariant probability measure;
- $Z := \mathbb{S}^1$, and λ is an arbitrary Borel probability measure on $\mathbb{S}^1$.

For $z \in \mathbb{S}^1$, we recall that $M_z\colon Y \to Y$ denotes the coordinatewise multiplication by z

$$(M_z y)(k) := (z \cdot y(k)), \quad k \in \mathbb{Z}. \tag{B.2}$$

We define the factor (X, μ, T) of the direct product $(Y, \nu, S) \times (Z, \lambda, \mathrm{id})$ as follows:

$$X := \mathbb{U}^{\mathbb{Z}} \quad \text{and} \quad T\colon X \to X \text{ is also the shift map,}$$

and the factor map $\pi\colon Y \times \mathbb{S}^1 \to X$ is given by

$$\pi(y, z) := M_z(y), \quad y \in Y,\ z \in \mathbb{S}^1, \tag{B.3}$$

where M_z is as in equation (B.2), and

$$\mu := \pi_*(\nu \otimes \lambda) \tag{B.4}$$

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is the pushforward of the product measure $\nu \otimes \lambda$ by the factor map π. Using the previous notation and assumptions, we have the following result.

PROPOSITION B.1. *Suppose that there exists $z \in \mathbb{S}^1$ such that the measure ν is not invariant by M_z. Then, there exists $r \in \mathbb{N}$, such that (X, μ, T) is isomorphic to the direct product $(Y, \nu, S) \times (Z, \lambda_r, \mathrm{id})$, where λ_r is the pushforward of λ by $z \mapsto z^r$.*

Proof. Our plan is as follows. We first define the positive integer $r \in \mathbb{N}$ that appears in the conclusion of our statement and then to each $x \in X$, we associate unique elements $\tilde{y} \in Y$ and $\tilde{z} \in Z$ such that $M_{\tilde{z}}(\tilde{y}) = x$. Once this is done, we prove that the map $\Phi\colon X \to Y \times Z$, defined by $\Phi(x) := (\tilde{y}, \tilde{z}^r)$, is the required isomorphism.

Definition of r. Let

$$G := \{z \in Z\colon \nu \text{ is invariant by } M_z\}. \tag{B.5}$$

Since G is a closed subgroup of $\mathbb{S}^1$ and by assumption $G \neq \mathbb{S}^1$, there exists $r \in \mathbb{N}$ such that

$$G = \{\zeta_r^k,\ k = 0, \dots, r-1\},$$

where ζ_r is an order r root of unity.

Definition of $\tilde{y}(x)$ and $\tilde{z}(x)$. Using the ergodicity of the measure ν and the fact that the map M_z preserves $C(Y)$, we deduce that for ν-almost all $y \in Y$ and for all $z \in Z$, we have

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, \delta_{M_z S^n y} = (M_z)_* \nu,$$

where $(M_z)_*\nu$ denotes the pushforward of ν by M_z. It follows from this and equations (B.3), (B.4), that for μ-a.e. $x \in X$, the following limit exists:

$$\mu_x := \lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, \delta_{T^n x},$$

and for $\nu \otimes \lambda$-a.e. $(y, z) \in Y \times Z$, we have $\mu_{M_z(y)} = (M_z)_*\nu$. Next, for μ-a.e. $x \in X$, we let

$$H_x := \{w \in \mathbb{S}^1\colon (M_{w^{-1}})_*\mu_x = \nu\}.$$

Then, for $\nu \otimes \lambda$-a.e. $(y, z) \in Y \times Z$, using equations (B.3) and (B.4), we get

$$H_{M_z(y)} = \{w \in \mathbb{S}^1\colon (M_{w^{-1}z})_*\nu = \nu\} = zG.$$

This establishes that for $\nu \otimes \lambda$-a.e. $(y, z) \in Y \times Z$, the set zG is uniquely determined by the element $x := M_z(y) \in X$. Hence, for μ-a.e. $x \in X$, we can define

$$\tilde{z}(x) := \text{ the unique element of } zG \text{ that has argument in } [0, 2\pi/r).$$

Let us justify that the map $x \mapsto \tilde{z}(x)$ is measurable. There is a Borel subset $X_0 \subset X$ of full μ measure on which $x \mapsto \mu_x$ is Borel measurable, and such that for all $x \in X_0$, μ_x is of the form $(M_z)_*\nu$ for some $z \in \mathbb{S}^1$. Set

$$\mathbb{S}^1_r := \{e(\theta) : 0 \le \theta < 2\pi/r\} \subset \mathbb{S}^1.$$

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Then, the graph of $x \mapsto \tilde{z}(x)$, $x \in X_0$, is the preimage of $\{\nu\}$ by the Borel map $(x, z) \mapsto (M_{z^{-1}})_*\mu_x$, $(x, z) \in X_0 \times \mathbb{S}^1_r$. So this graph is Borel measurable in $X_0 \times \mathbb{S}^1_r$, and by [30, Theorem 14.12], the map $x \mapsto \tilde{z}(x)$ is Borel measurable. We also set, for $x \in X_0$,

$$\tilde{y}(x) := M_{\tilde{z}(x)^{-1}}(x) \in Y.$$

Definition of Φ. We define the map $\Phi\colon X_0 \to Y \times Z$ as follows:

$$\Phi(x) := (\tilde{y}(x), \tilde{z}(x)^r), \tag{B.6}$$

where $r \in \mathbb{N}$, $\tilde{y}(x) \in Y$, $\tilde{z}(x) \in Z$ are defined as above. The map Φ is measurable and we are going to show next that it establishes an isomorphism between the systems (X, μ, T) and $(Y, \nu, S) \times (Z, \lambda_r, \mathrm{id})$.

Φ *is an isomorphism.* We have to establish the following three claims.

Claim 1. Φ is injective.

Note that for μ-a.e. $x \in X$, we have

$$x = M_{\tilde{z}(x)}(\tilde{y}(x)). \tag{B.7}$$

Replacing if necessary X_0 by a smaller Borel set (but still of full measure), we can assume that the above identity and equation (B.6) hold on X_0. Suppose that $\Phi(x) = \Phi(x')$ for some $x, x' \in X_0$. Then, $\tilde{y}(x) = \tilde{y}(x')$ and $\tilde{z}^r(x) = \tilde{z}^r(x')$. Since both $\tilde{z}(x)$, $\tilde{z}(x')$ have argument in $[0, 2\pi/r)$, we deduce that $\tilde{z}(x) = \tilde{z}(x')$. Then, equation (B.7) gives that $x = x'$, which establishes that the map Φ is injective on X_0.

Claim 2. We have $\Phi \circ T = (S \times \mathrm{id}) \circ \Phi$.

Equivalently, using equation (B.6), it suffices to verify that for μ-a.e. $x \in X$, we have

$$(\tilde{y}(Tx), \tilde{z}^r(Tx)) = (S\tilde{y}(x), \tilde{z}^r(x)). \tag{B.8}$$

Since $\mu_{Tx} = \mu_x$, we have

$$\tilde{z}(Tx) = \tilde{z}(x)$$

and

$$\tilde{y}(Tx) = M_{\tilde{z}(x)^{-1}}(Tx) = SM_{\tilde{z}(x)^{-1}}(x) = S\tilde{y}(x).$$

Combining these two identities, we get equation (B.8).

Claim 3. We have $\Phi_*\mu = \nu \otimes \lambda_r$.

Note first that for $\nu \otimes \lambda$-a.e. $(y, z) \in Y \times Z$, we have $\tilde{z}(M_z(y))^{-1} = z^{-1}w$ for some $w \in G$, an r-root of unity depending only on z, and

$$\tilde{y}(M_z(y)) = M_{\tilde{z}(M_z(y))^{-1}z}(y) = M_w(y).$$

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Since $w \in G$, we get by equation (B.5) that for a fixed $z \in Z$, the pushforward of ν by $y \mapsto \tilde{y}(M_z(y))$ is still ν. This means that, for all $f \in C(Y)$, we have for all $z \in Z$,

$$\int_Y f(\tilde{y}(M_z(y)))\, d\nu(y) = \int_Y f(y)\, d\nu(y).$$

However then, integrating the above identity with respect to z, we get

$$\int_X f(\tilde{y}(x))\, d\mu(x) = \int_Z \left(\int_Y f(\tilde{y}(M_z(y)))\, d\nu(y) \right) d\lambda(z) = \int_Y f(y)\, d\nu(y),$$

which shows that the pushforward of μ by $x \mapsto \tilde{y}(x)$ is also ν.

Let us consider now the pushforward of μ by $x \mapsto \tilde{z}(x)^r$. Observe that for $\nu \otimes \lambda$-a.e. $(y, z) \in Y \times Z$, we have $\tilde{z}(M_z(y))G = zG$, and hence

$$\tilde{z}(M_z(y))^r = z^r.$$

Therefore, for an arbitrary continuous $g \in C(\mathbb{S}^1)$, we have

$$\int_X g(\tilde{z}(x)^r)\, d\mu(x) = \int_{Y\times Z} g(\tilde{z}(M_z(y))^r)\, d(\nu \otimes \lambda) = \int_Z g(z^r)\, d\lambda(z) = \int_Z g(z)\, d\lambda_r(z).$$

We conclude that the pushforward of μ by the map $x \mapsto \tilde{z}^r(x)$ is λ_r.

Combining the above, we have that the measure $\Phi_*\mu$ is a joining of (Y, ν, S) and $(Z, \lambda_r, \mathrm{id})$. Since, by assumption, the system (Y, ν, S) is ergodic and the system $(Z, \lambda_r, \mathrm{id})$ is an identity system, the two systems are disjoint. Hence, $\Phi_*\mu = \nu \otimes \lambda_r$, as required. □

Our argument gives that r depends only on ν and the factor generated by the map $x \mapsto \tilde{z}(x)$ from (X, μ, T) to $(Z, \lambda_r, \mathrm{id})$ is non-trivial (in which case, (X, μ, T) is non-ergodic) if and only if λ is not concentrated on the group of r-roots of unity. This is always the case if λ is a continuous measure.

C. *Appendix. Ergodic consequences of a result of Klurman*

In this last appendix, we justify some remarks made immediately after Theorem 2.7 and Conjecture 3.

The next result follows immediately from an argument given by Klurman [31] and crucially uses results about 2-point correlations of Tao [43]. We sketch the argument for completeness.

PROPOSITION C.1. [31, Proof of Lemma 4.3] *Let $f : \mathbb{N} \to \mathbb{U}$ be an aperiodic multiplicative function. Then, there exist $N_k \to \infty$ such that*

$$\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} \overline{f(n)} \cdot f(n+j) = 0 \quad \textit{for every } j \in \mathbb{N}. \tag{C.3}$$

Remarks.

- As shown in [37, Theorem B.1], some aperiodic multiplicative functions may have non-vanishing 2-point correlations (see also the remark following Proposition 9.1).
- More elaborate arguments of [33] give stronger results, see for example Theorem 1.2 therein.

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Proof. We first claim that for every $m \in \mathbb{N}$, there exists a subsequence $(N_{m,k})_{k\in\mathbb{N}}$ of $(N_k)_{k\in\mathbb{N}}$ (which also depends on f) such that

$$\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_{m,k}]} \overline{f(n)} \cdot f(n+j) = 0 \quad \text{for } j = 1, \dots, m.$$

Indeed, suppose that the conclusion fails. Then, there exist $m \in \mathbb{N}$, $\varepsilon > 0$, and $N_0 = N_0(m, \varepsilon)$, such that for all $N \geq N_0$, we have

$$|\mathbb{E}^{\log}_{n\in[N]} \overline{f(n)} \cdot f(n + j_N)| \geq \varepsilon$$

for some $j_N \in [m]$. This fact, combined with the argument used to prove the second part of [31, Lemma 4.3] gives, without any change, that f is pretentious, which is a contradiction.

Using the previous claim, we get that for every $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that

$$|\mathbb{E}^{\log}_{n\in[N_k]} \overline{f(n)} \cdot f(n+j)| \leq 1/k \quad \text{for } j = 1, \dots, k.$$

Furthermore, we can assume that the sequence $(N_k)_{k\in\mathbb{N}}$ is strictly increasing. The result follows. □

COROLLARY C.2. *If $f : \mathbb{N} \to \mathbb{U}$ is a non-trivial aperiodic multiplicative function, then at least one of its Furstenberg systems for logarithmic averages has a Lebesgue component. In fact, if F_0 is the 0th-coordinate projection, on some Furstenberg system (X, μ, T) of f, we have*

$$\int \overline{F_0} \cdot T^j F_0 \, d\mu = 0 \quad \textit{for every } j \in \mathbb{N}. \tag{C.4}$$

Proof. Using equation (8), we see that $F_0 \neq 0$ (μ-almost everywhere) is equivalent to $\mathbb{E}_{n\in\mathbb{N}} |f(n)|^2 \neq 0$, which in turn is equivalent to the non-triviality assumption for f.

Let (N_k) be the subsequence given by Proposition C.1. Pick a subsequence (N'_k) along which f admits correlations for logarithmic averages and let (X, μ, T) be the corresponding Furstenberg system of f. Then, equations (8) and (C.3) imply that equation (C.4) holds. □

By combining Corollary C.2 with the main argument in [17], we deduce the following subsequential variant of the logarithmically averaged Sarnak conjecture for ergodic weights that applies to all aperiodic multiplicative functions (for multiplicative functions that satisfy stronger aperiodicity properties, [17, Theorem 1.1] covers a more general result).

THEOREM C.3. *Let $f : \mathbb{N} \to \mathbb{U}$ be an aperiodic multiplicative function. Then, there exists a subsequence $N_k \to \infty$ such that for every deterministic, ergodic sequence $w : \mathbb{N} \to \mathbb{U}$, we have*

$$\lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]} f(n)\, w(n) = 0.$$

Remark. As remarked in [23, §5.2], there exists an aperiodic multiplicative function $f : \mathbb{N} \to \mathbb{S}^1$ and a non-ergodic deterministic sequence $w : \mathbb{N} \to \mathbb{U}$ such that the averages $\mathbb{E}^{\log}_{n\in[N]} f(n)\, w(n)$ do not converge to 0 as $N \to \infty$. So passing to a subsequence is needed

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if one wants to allow w to be an arbitrary deterministic weight sequence (not necessarily ergodic). However, see Conjectures 2 and 3 for variants that use relaxed assumptions.

To prove Theorem C.3, one works with the subsequence $N_k \to \infty$ provided by Corollary C.2 and repeats the proof of [17, Theorem 1.1] given in [17, §5.1]. The point is that Corollary C.2 serves as a substitute for the 2-point correlation result of Tao [43] (the latter is not applicable to all aperiodic multiplicative functions).

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