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Radiation fields and hyperbolic scattering theory

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Section 0. *Introduction*

Let $u(x,t)$, where $x\in\mathbb{R}^3$, $t\in\mathbb{R}$, be a C^∞ solution of the wave equation on $\{|x|>a\}\times\mathbb{R}$ for some $a>0$, and suppose that $u=0$ for $|x|>a$, $t<0$. One then also has $u=0$ for $|x|>a+t$, $t>0$, so that $u(.,t)$ can be thought of as a wave expanding into a previously undisturbed medium. One can now ask for a description of the asymptotic behaviour of u as $|x|\to\infty$. It turns out that there is a $v_0(\theta,\tau)\in C^\infty(S^2\times\mathbb{R})$ such that

$$\lim_{r\to\infty}(ru(r\theta,r+\tau))=v_0(\theta,\tau)$$

in the topology of $C^\infty(S^2\times\mathbb{R})$. This limit may be called the radiation field of the expanding wave u. (See (2) and the earlier papers quoted there.)

In section 1 of this paper, the existence of such radiation fields is established for a class of hyperbolic equations that may be called perturbed wave equations. If $\Sigma_{i,j=1}^n h_{ij}(x)\,dx_i\,dx_j$ is a C^∞ Riemannian metric on $\mathbb{R}^n$, then $\mathbb{R}^n\times\mathbb{R}$, equipped with the metric

$$ds^2=dt^2-\sum_{i,j=1}^n h_{ij}\,dx_i\,dx_j, \tag{0·1}$$

becomes a Lorentzian manifold whose d'Alembertian is

$$\Box u=\frac{\partial^2u}{\partial t^2}-\Delta u=\frac{\partial^2u}{\partial t^2}-h^{-\frac12}\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(h^{\frac12}h^{ij}\frac{\partial u}{\partial x_j}\right), \tag{0·2}$$

where (h^{ij}) is the inverse of the matrix (h_{ij}), and $h=\det h_{ij}$. When $h_{ij}=\delta_{ij}$ (Kronecker deltas) for $i,j=1,2,\ldots,n$, then $\Box u=0$ is the wave equation. The equations considered here are characterized by $h_{ij}=\delta_{ij}+O(|x|^{-2})$, $i,j=1,2,\ldots,n$, as $|x|\to\infty$, and a regularity condition at infinity (see (1·4) below). For such equations one can show that, if $u\in C^\infty(\{|x|>a\}\times\mathbb{R})$ for some $a>0$ and

$$\Box u=0 \quad\text{on}\quad \{|x|>a\}\times\mathbb{R}, \quad u=0 \quad\text{on}\quad \{|x|>a\}\times(-\infty,t_0) \tag{0·3}$$

for some $t_0\in\mathbb{R}$, then the radiation field

$$\lim_{r\to\infty}(r^{(n-1)/2}u(r\theta,r+\tau)), \quad (\theta,\tau)\in S^{n-1}\times\mathbb{R}, \tag{0·4}$$

exists. (There is a distribution version of this, which is not discussed here.) The proof is effected by combining a transformation of variables which is a variant of the 'conformal method' in General Relativity (9) with standard hyperbolic estimates (compare (1)).

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There is a connection between radiation fields and the hyperbolic scattering theory of Lax and Phillips. Consider the initial-value problem

$$\Box u = 0 \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R}, \quad u = f_0(x), \quad u_t \equiv \partial u/\partial t = f_1(x) \quad \text{when} \quad t = 0, \tag{0·5}$$

where f_0 and f_1 are in $C_0^\infty(\mathbb{R}^n)$. The solution at time t gives a map

$$C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n),$$

$$f = (f_0, f_1) \mapsto U(t)f = (u(.,t), u_t(.,t)) \tag{0·6}$$

which satisfies an energy identity,

$$\|f\|_{H_E} = \|U(t)f\|_{H_E} \tag{0·7}$$

where

$$\|f\|_{H_E}^2 = \left(\frac{1}{2}\int\left(\sum_{i,j=1}^{n} h^{ij}\frac{\partial f_0}{\partial x_i}\frac{\partial f_0}{\partial x_j} + f_1^2\right)h^{\frac{1}{2}}\,dx\right)^{\frac{1}{2}}. \tag{0·8}$$

Because of this, the maps $\{U(t): t \in \mathbb{R}\}$ extend in a natural way to a group of unitary operators on the energy space H_E which is obtained as the completion of

$$C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$$

with respect to the Hilbert norm (0·8). (It is assumed that $n \geqslant 3$; some minor modifications are needed for $n = 1$ and $n = 2$.) The main ingredient of Lax–Phillips theory is a representation of $\{U(t)\}$ as translation on $L_2(\mathbb{R}; N)$ where N is an auxiliary Hilbert space. For the standard wave equation, say on $\mathbb{R}^3 \times \mathbb{R}$, it is well known that one can take $N = L_2(S^2)$ and that the translation representer of initial data f is essentially just the radiation field of u_t up to reflection in the τ variable, say $\tilde{f}(\theta, -s)$ where

$$\tilde{f}(\theta, \tau) = \lim_{r\to\infty} (ru_t(r\theta, r+\tau))$$

((6), p. 108, theorem 2·4).

The extension of this relation to the perturbed wave equations $\Box u = 0$, $\Box u$ given by (0·2) subject to (1·4) below, is explored in sections 2 and 3. Define the radiation field map for C_0^∞ data f by

$$f \mapsto \tilde{f}(\theta, \tau) = \lim_{r\to\infty} (r^{(n-1)/2} u_t(r\theta, r+\tau)); \tag{0·9}$$

the existence of this limit follows from the results in section 1. One can then show that, with the energy norm defined by (0·8),

$$\|\tilde{f}\|_{L_2(S^{n-1}\times\mathbb{R})} \leqslant \|f\|_{H_E}. \tag{0·10}$$

The map $f \mapsto \tilde{f}$ therefore extends by continuity to a map $\mathscr{R}: H_E \to L_2(S^{n-1} \times \mathbb{R})$ which also satisfies (0·10), and it is immediate that

$$\mathscr{R} \circ U(s)f = \mathscr{R}f(\theta, \tau + s)$$

so that $\mathscr{R}$ is an intertwining operator between $\{U(t)\}$ and the group of backward translations in τ on $L_2(S^{n-1} \times \mathbb{R})$. It is an immediate corollary of the proof of (0·10) that one has equality for the standard wave equation.

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It is shown in section 3 that one has equality in (0·10) if and only if the kernel of $\mathscr{R}$ is trivial, and that the resulting isometry $\mathscr{R}$ is then surjective. When this is the case, there is also a backward radiation field map, and a scattering operator can be defined. These results all extend to Dirichlet problems (or Neumann problems) on exterior domains. When the perturbation of the basic metric $dt^2 - |dx|^2$ has compact support, then it follows from the work of Lax and Phillips that $\mathscr{R}$ is injective. One thus obtains a 'concrete' approach to hyperbolic scattering theory by means of radiation fields.

The material in section 1 was presented as part of a set of lectures given at the University of Lund in 1977; I want to thank Lars Gårding for giving me the opportunity to visit Lund, and Lars Hörmander for several discussions.

Section 1

The class of perturbed wave equations considered in this paper is defined as follows. Let $(h_{ij}(x))$, where $x \in \mathbb{R}^n$ and $1 \leqslant i, j \leqslant n$, be a positive definite symmetric matrix with C^∞ entries, so that $\Sigma h_{ij}\, dx_i\, dx_j$ is a C^∞ Riemannian metric on $\mathbb{R}^n$. The Laplace–Beltrami operator associated with this is

$$\Delta u = h^{-\frac{1}{2}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(h^{\frac{1}{2}} h^{ij} \frac{\partial u}{\partial x_j}\right),$$

where $h = \det h_{ij}$ and $(h^{ij}) = (h_{ij})^{-1}$. Then, with $t \in \mathbb{R}$,

$$ds^2 = dt^2 - \sum_{i,j=1}^{n} h_{ij}\, dx_i\, dx_j \tag{1·1}$$

is a C^∞ Lorentzian metric on $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$, whose d'Alembertian is

$$\Box u = \frac{\partial^2 u}{\partial t^2} - \Delta u, \tag{1·2}$$

where $u = u(x, t)$ can be a function or a distribution.

For $0 \neq x \in \mathbb{R}^n$ we shall generally write

$$|x| = r \in \mathbb{R}^+, \quad x/|x| = \theta \in S^{n-1}, \quad \rho = r^{-1} \in \mathbb{R}^+, \tag{1·3}$$

so that $x = r\theta$, and (r, θ) are spherical polar coordinates. The basic hypothesis on the h_{ij} is

$$\begin{cases} \text{there exist } \epsilon_{ij}(\rho, \theta) \in C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1}) \text{ such that} \\ h_{ij}(x) = \delta_{ij} + \rho^2 \epsilon_{ij}(\rho, \theta), \quad \rho = r^{-1}, \quad \text{for} \quad x \neq 0, \quad i, j = 1, \ldots, n, \end{cases} \tag{1·4}$$

where the δ_{ij} are the Kronecker deltas. Note that the ϵ_{ij} are assumed to be smooth up to $\rho = 0$, so that (1·4) specifies the behaviour of the metric 'at infinity'.

Let $a \in \mathbb{R}^+$. By definition, an *expanding wave* will be a function

$$u(x, t) \in C^\infty(\{|x| > a\} \times \mathbb{R})$$

such that

$$\begin{cases} \Box u = 0 \quad \text{for} \quad |x| > a, \quad \text{and there is a } t_0 \in \mathbb{R} \\ \text{such that } u = 0 \quad \text{if} \quad |x| > a,\ t < t_0. \end{cases} \tag{1·5}$$

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As we shall be concerned with asymptotic properties of expanding waves, the magnitude of a is immaterial; the essential point is that there must be some $a > 0$ for which (1·5) holds.

The principal result of this section is the following:

(1·6) THEOREM. *Assume that* (1·4) *holds, and that u is an expanding wave satisfying* (1·5). *Put, for $r > a$,*

$$r^{(n-1)/2}u(r\theta, t) = v(\rho, \theta, \tau), \tag{1·7}$$

where

$$\rho = r^{-1}, \quad \theta \in S^{n-1}, \quad \tau = t - r, \tag{1·8}$$

so that $v \in C^\infty((0, a^{-1}) \times S^{n-1} \times \mathbb{R})$. Then v extends by continuity to a function of class $C^\infty([0, a^{-1}) \times S^{n-1} \times \mathbb{R})$.

The proof requires several steps. The first of these is

(1·9) LEMMA. *Let $u(x, t) \in C^2(\mathbb{R}^{n+1})$ and let v be defined by* (1·7) *and* (1·8), *with $r > 0$.* Then

$$\Box u = \rho^{(n+3)/2} P v, \tag{1·10}$$

where P is a hyperbolic operator on $\{(\rho, \theta, \tau): \rho \geqslant 0, \theta \in S^{n-1}, \tau \in \mathbb{R}\}$ with C^∞ coefficients.

Proof. This is essentially a computation. But it can be simplified and clarified by means of the 'conformal method' used in the theory of relativity (9). Let M denote the manifold $(\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}$ equipped with the Lorentzian metric (1·1), and let $\tilde{M}$ denote the same underlying manifold with the Lorentzian metric

$$d\tilde{s}^2 = r^{-2}\, ds^2, \tag{1·11}$$

so that M and $\tilde{M}$ are conformal Lorentzian manifolds. There is a standard relation between the d'Alembertians of such manifolds, which in the present case can be derived as follows. Let $X_1, \ldots, X_{n-1}$ be local coordinates on an open set $\Omega \subset S^{n-1}$, and put $\rho = X_n$, $\tau = X_{n-1}$, so that $(X_1, \ldots, X_{n+1}) \in \mathbb{R}^{n+1}$ are local coordinates on $\mathbb{R}^+ \times \Omega \times \mathbb{R}$ for both M and $\tilde{M}$. Write

$$ds^2 = \sum_{\mu,\nu=1}^{n+1} g_{\mu\nu} dX_\mu dX_\nu, \quad d\tilde{s}^2 = \sum_{\mu,\nu=1}^{n+1} \tilde{g}_{\mu\nu} dX_\mu dX_\nu; \tag{1·12}$$

clearly

$$g_{\mu\nu} = \rho^{-2}\tilde{g}_{\mu\nu}, \quad |g|^{\frac{1}{2}} = \rho^{-n-1}|\tilde{g}|^{\frac{1}{2}}, \quad g^{\mu\nu} = \rho^2 \tilde{g}^{\mu\nu}.$$

Denote the pull-back of $u(x, t)$ under the map $X \mapsto (x, t)$ by $u^*(X)$. Then

$$\Box u = |g|^{-\frac{1}{2}} \sum_{\mu,\nu=1}^{n+1} \frac{\partial}{\partial X_\mu}\left(|g|^{\frac{1}{2}} g^{\mu\nu} \frac{\partial u^*}{\partial X_\nu}\right) = \rho^{n+1}|\tilde{g}|^{-\frac{1}{2}} \sum_{\mu,\nu=1}^{n+1} \frac{\partial}{\partial X_\mu}\left(\rho^{1-n}|\tilde{g}|^{\frac{1}{2}} \tilde{g}^{\mu\nu} \frac{\partial u^*}{\partial X_\nu}\right).$$

As $u^* = \rho^{(1-n)/2} v$, this gives

$$\Box u = \rho^{n+1}|\tilde{g}|^{-\frac{1}{2}} \sum_{\mu,\nu=1}^{n+1} \frac{\partial}{\partial X_\mu}\left(|\tilde{g}|^{\frac{1}{2}} \tilde{g}^{\mu\nu}\left((\rho^{(1-n)/2} \frac{\partial V}{\partial X_\nu} - v \frac{\partial}{\partial X_\nu} \rho^{(1-n)/2}\right)\right) = \rho^{(n+3)/2} P v,$$

and this is coordinate independent, and of the form (1·10) with

$$P v = \tilde{\Box} v + \gamma v, \quad \gamma = -\rho^{(n-1)/2} \tilde{\Box} \rho^{(1-n)/2}. \tag{1·13}$$

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In terms of (r, θ, t), the metric (1·1) is, in view of (1·4),

$$ds^2 = dt^2 - dr^2 - r^2 |d\theta|^2 - r^{-2} \sum_{i,j=1}^{n} \epsilon_{ij}(r\, d\theta_i + \theta_i\, dr)\,(r\, d\theta_j + \theta_j\, dr).$$

As $t = r + \tau$, $r = \rho^{-1}$, it follows from (1·11) that

$$d\tilde{s}^2 = \rho^2 d\tau^2 - 2\, d\rho\, d\tau - |d\theta|^2 - \sum_{i,j=1}^{n} \epsilon_{ij}(\rho, \theta)\,(\rho\, d\theta_i - \theta_i\, d\rho)\,(\rho\, d\theta_j - \theta_j\, d\rho). \tag{1·14}$$

By the hypothesis (1·4), the conformal metric is seen to be smooth up to $\rho = 0$, and it is easy to check that it is non-degenerate, hence Lorentzian, at $\rho = 0$. So the coefficients of the d'Alembertian $\tilde{\Box}$ are also smooth up to $\rho = 0$, and the principal symbol of P is strictly hyperbolic for $\rho \geqslant 0$.

It remains to show that $\gamma(\rho, \theta) \in C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1})$. It follows from (1·13) and the relation between the two metrics that

$$\gamma = -\rho^{(n+3)/2} \Box \rho^{(1-n)/2} = -\rho^{(n+3)/2} \Delta \rho^{(1-n)/2}.$$

A straightforward computation, which is omitted, shows that

$$\gamma(\rho, \theta) = \tfrac{1}{4}(n-1)(n-3) + \rho^2 \gamma_1(\rho, \theta), \quad \gamma_1 \in C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1}), \tag{1·15}$$

and so the lemma is proved. It is worth observing that for the wave equation ($h_{ij} = \delta_{ij}$) one has $\gamma_1 = 0$, and

$$Pv = -2\frac{\partial^2 v}{\partial\rho\, \partial\tau} - \frac{\partial}{\partial\rho}\left(\rho^2 \frac{\partial v}{\partial\rho}\right) - \Delta_s v + \tfrac{1}{4}(n-1)(n-3)\, v,$$

where Δ_s is the Laplace–Beltrami operator on S^{n-1}.

Lemma 1·9 shows that if u is an expanding wave, and v is defined by (1·7), then $Pv = 0$ on $(0, a^{-1}) \times S^{n-1} \times \mathbb{R}$. To prove Theorem 1·6, one can now appeal to energy estimates for P. It is convenient to have these in covariant form. Such estimates are known (4). These estimates will now be stated; their proof is outlined in the Appendix.

Let $K \subset \tilde{M}$ be a compact set. A C^∞ function $S\colon K \to \mathbb{R}$ will be called *space-like on* K if, in local coordinates (cf. (1·12)),

$$\langle \nabla S, \nabla S \rangle = \nabla^\mu S\, \nabla_\mu S = \tilde{g}^{\mu\nu} \frac{\partial S}{\partial X_\mu} \frac{\partial S}{\partial X_\nu} > 0 \quad \text{on } K. \tag{1·16}$$

Here, $\langle\, , \rangle$ denotes the inner product on $\tilde{M}$, summation convention is used, and $\nabla^\mu S = \tilde{g}^{\mu\nu}\, \partial S / \partial X_\nu$. Given such a space-like function, one can in effect introduce an auxiliary Riemannian metric on K:

(1·17) Lemma. *Let S be space-like on a compact set $K \subset \tilde{M}$ and put, in any local coordinate system,*

$$\sigma^{\mu\nu} = \nabla^\mu S\, \nabla^\nu S = \tfrac{1}{2} \langle \nabla S, \nabla S \rangle\, \tilde{g}^{\mu\nu}. \tag{1·17}$$

Also, for $v \in C^\infty(K)$ and $k = 1, 2, \ldots$, put

$$\overset{(k)}{\sigma}(v) = \sigma^{\mu_1\nu_1} \ldots \sigma^{\mu_k\nu_k} \nabla_{\mu_1} \ldots \nabla_{\mu_k} v\, \nabla_{\nu_1} \ldots \nabla_{\nu_k} v. \tag{1·18}$$

Then $\overset{(k)}{\sigma}(v)$ is, at all points of K, a positive definite quadratic form in the k-th order covariant derivatives of v.

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Let $d\tilde{m}$ be the invariant volume element on M, $d\tilde{m} = |\tilde{g}|^{\frac{1}{2}} dX$ in local coordinates; also, put $\overset{(0)}{\sigma}(v) = v^2$. The energy estimates in question are then as follows.

(1·19) LEMMA. *Let $K \subset \overline{\mathbb{R}}^+ \times S^{n-1} \times \mathbb{R}$ be a compact set, with non-empty interior, and assume that there exists a space-like C^∞ function S on K. Let $\Omega \subset K$ be a connected open set with piece-wise smooth boundary $\partial\Omega$ such that*

$$\begin{cases} \partial\Omega = \Sigma_1 \cup \Sigma_2 & \text{where } \Sigma_2 \text{ is space-like and} \\ \nabla S \text{ points out of } \Omega \text{ at every point of } \Sigma_2. \end{cases} \tag{1·20}$$

Then there are constants C_k, $k = 1, 2, \ldots$, such that

$$\int_\Omega \sum_{j=0}^{k} \overset{(j)}{\sigma}(v)\, d\tilde{m} \leqslant C_K \left(\int_\Omega \sum_{j=0}^{k-1} \overset{(j)}{\sigma}(Pv)\, d\tilde{m} + \int_{\Sigma_1} \phi_k(v) \right), \tag{1·21}$$

for all $v \in C^\infty(\overline{\Omega})$, where the $\phi_k(v)$ are densities on Σ_1 which are quadratic in the covariant derivatives of v of order $\leqslant k$.

Remark. A surface Σ is space-like if every point $q \in \Sigma$ has a neighbourhood U in which Σ is given by a space-like defining function. (This agrees with the usual definition. For a piece-wise smooth surface, there is an obvious variant, in terms of supporting planes, at points at which Σ is not smooth.) If U is such a neighbourhood of a point $q \in \Sigma_2$ and the defining function F is chosen so that $F \leqslant 0$ in $\overline{\Omega} \cap U$, then the condition (1·20) becomes $\langle \nabla S, \nabla F \rangle > 0$ on $\Sigma_2 \cap U$.

We now turn to the proof of Theorem 1·6. To apply Lemma 1·19, one needs a space-like function on some neighbourhood of $\rho = 0$. This is provided by the following lemma.

(1·22) LEMMA. *The function*

$$S = \tau - r_0^2 \rho \tag{1·23}$$

is space-like with respect to $d\bar{s}^2$ on $[0, r_0^{-1}] \times S^{n-1} \times \mathbb{R}$, if r_0 is sufficiently large.

Proof. In terms of $x = r\theta$ and t the pull-back of S is $S^* = t - r - r_0^2 r^{-1}$, and the principal symbol of P is

$$p(.\,, \operatorname{grad} S) = r^2 \left(\left(\frac{\partial S^*}{\partial t} \right)^2 - \sum_{i,j=1}^{n} h^{ij} \frac{\partial S^*}{\partial x_i} \frac{\partial S^*}{\partial x_j} \right).$$

It follows from (1·4) that, for certain functions $\epsilon^{ij}(\rho, \theta) \in C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1})$,

$$h^{ij} = \delta_{ij} + r^2 \epsilon^{ij} \left(\frac{1}{r}, \theta \right).$$

So

$$\begin{aligned} p(.\,, \operatorname{grad} S) &= r^2 \left(1 - \left(1 - \frac{r_0^2}{r^2} \right)^2 - r^{-2} \left(1 - \frac{r_0^2}{r^2} \right)^2 \sum_{i,j=1}^{n} \epsilon^{ij} \theta_i \theta_j \right) \\ &= 2r_1^2 - \frac{r_0^4}{r^2} - \left(1 - \frac{r_0^2}{r^2} \right)^2 \sum_{i,j=1}^{n} \epsilon^{ij} \theta_i \theta_j. \end{aligned}$$

Reverting to (ρ, θ), we can write this as

$$p(.\,, \operatorname{grad} S) = r_0^2 + r_0^2 z - z^2 \sum_{i,j=1}^{n} \epsilon^{ij}(\rho, \theta)\, \theta_i \theta_j,$$

where

$$z = 1 - r_0^2 \rho^2;$$

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note that $0 \leqslant z \leqslant 1$ when $0 \leqslant \rho \leqslant r_0^{-1}$. Put

$$\mu(r_0) = \sup\left\{\sum_{i,j=1}^{n} \epsilon^{ij}(\rho,\theta)\,\theta_i\theta_j\colon \quad 0 \leqslant \rho \leqslant r_0^{-1}, \quad \theta \in S^{n-1}\right\}.$$

Then

$$p(.,\operatorname{grad} S) \geqslant r_0^2 + r_0^2 z - \mu(r_0)\,z^2 \geqslant \min\left(r_0^2, 2r_0^2 - \mu(r_0)\right) \quad \text{if} \quad 0 \leqslant z \leqslant 1.$$

As $\mu(r_0)$ is a non-decreasing function of r_0, one has $2r_0^2 > \mu(r_0)$ when r_0 is sufficiently large, and so the Lemma is proved.

(1·24) *Remark.* The proof also shows that the function

$$T = \tau + r_0^2\rho + \frac{2}{\rho}\left(= t + r + \frac{r_0^2}{r}\right)$$

is space-like on $(0, r_0^{-1}] \times S^{n-1} \times \mathbb{R}$.

Proof of Theorem 1.6. By hypothesis, the expanding wave u satisfies (1·5). By (1·5), one therefore has

$$(1\cdot25) \qquad v \in C^\infty((0,a^{-1}) \times S^{n-1} \times \mathbb{R}), \quad Pv = 0, \quad v = 0 \quad \text{if} \quad \tau + \frac{1}{\rho} < t_0.$$

One may assume that the r_0 of Lemma 1·22 is greater than a, and we shall do so from now on. Observe first that it follows from (1·25) that

$$(1\cdot26) \qquad v = 0 \quad \text{if} \quad 0 < \rho \leqslant r_0^{-1}, \quad S = \tau - r_0^2\rho < \tau_0 = t_0 - 2r_0.$$

Indeed, this is equivalent to the assertion that $u = 0$ for $r \geqslant r_0$ and

$$0 \leqslant t < t_0 + r^{-1}(r - r_0)^2,$$

that is to say for $S^* < t_0 - 2r_0$, which is a consequence on the compact dependence domain property of $\Box$ and of Lemma 1·22, as S^* is space-like with respect to ds^2. It can also be deduced from Lemma 1·19. It is easy to check that if, $T_0 > \tau_0 + 4r_0$, then the set

$$\Omega = \left\{(\rho,\theta,\tau)\colon \tau + \frac{1}{\rho} > t_0,\ S < \tau_0,\ T < T_0\right\}$$

is a relatively compact subset of $(0, r_0^{-1}] \times S^{n-1} \times \mathbb{R}$. Take $K = \bar{\Omega}$ in Lemma 1·19, and define S by (1·23). It is easy to check, in view of Lemma 1·22 and Remark 1·24, that those portions of $\partial\Omega$ on which either $S = \tau_0$ or $T = T_0$ can be taken as Σ_2. Then v vanishes to all orders on Σ_1, where of course $t = \tau + \rho^{-1} = t_0$, and, as also $Pv = 0$, it follows from (1·21) with $k = 1$ that $v = 0$ on Ω. As T_0 can be arbitrarily large, (1·26) follows.

To prove the Theorem, chose some $\tau_1 > \tau_0$ and take the compact set K of Lemma 1·19 to be

$$K = \{(\rho,\theta,\tau)\colon 0 \leqslant \rho \leqslant r_0^{-1},\ \theta \in S^{n-1},\ \tau_0 \leqslant S = \tau - r_0^2\rho \leqslant \tau_1\},$$

so that $S = \tau - r_0^2\rho$ is space-like on K. It is then easily shown that, if $\lambda > \tau_1 + 2r_0$, the set

$$\Omega_\lambda = \left\{(\rho,\theta,\tau)\colon 0 < \rho < r_0^{-1},\ \theta \in S^{n-1},\ \tau_0 < S < \tau_1,\ t = \tau + \frac{1}{\rho} < \lambda\right\}$$

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is a connected open subset of K. Now any surface $S =$ constant is space-like if $0 \leqslant \rho \leqslant r_0^{-1}$, and so is the surface $t = \tau + \rho^{-1} = \lambda$. It is then immediate from the Remark following the enunciation of Lemma 1·19 that (1·20) holds if one sets

$$\Sigma_2 = (\bar{\Omega}_\lambda \cap \{S = \tau_1\}) \cup (\bar{\Omega}_\lambda \cap \{t = \lambda\}).$$

The estimates (1·21) therefore hold with $\Omega = \Omega_\lambda$. In the second member one has $Pv = 0$, and, also, v vanishes to all orders on $S = \tau_0$, by (1·26). So one is only left with integrals over $\{\rho = r_0^{-1}\}$ and these are clearly independent of λ for fixed τ_1. The crucial point is now that $\bar{\Omega}_\lambda \to K$ as $\lambda \to \infty$; essentially, this is so because $\rho = 0$ is a characteristic of P. One can thus let $\lambda \to \infty$ to obtain energy estimates 'up to the horizon',

$$\sum_{j=0}^{k} \int_K \overset{(j)}{\sigma}(v)\, d\tilde{m} \leqslant B_k, \quad k = 1, 2, \ldots \tag{1·27}$$

where the B_k depend only on the restrictions of the (covariant) derivatives of v to $\{r_0^{-1}\} \times S^{n-1} \times [\tau_0 + r_0, \tau_1 + r_0]$.

Now the first members of (1·27) are locally equivalent to Sobolev norms. Let $\Omega \subset S^{n-1}$ be a compact coordinate neighbourhood and denote the coordinate system by h: $\Omega \to \mathbb{R}^{n-1}$. Then h^{-1} gives a parametrization $h\Omega \ni X \mapsto \theta(X) \in \Omega$. It follows from the Sobolev inequalities and (1·27) that, for every $k \geqslant 0$, $l \geqslant 0$ and every $(n-1)$-multi-index α, there is a constant $A_{k,l,\alpha}$ such that

$$\left| \left(\frac{\partial}{\partial \rho}\right)^k \left(\frac{\partial}{\partial \tau}\right)^l \left(\frac{\partial}{\partial X}\right)^\alpha v(\rho, \theta(X), \tau) \right| \leqslant A_{k,l,\alpha} \quad \text{if} \quad 0 \leqslant \rho \leqslant \tau_0^{-1},\ X \in h\Omega,\ \tau_0 \leqslant \tau \leqslant \tau_1.$$

It is an elementary consequence of this, with k replaced by $k+1$, that

$$(\partial/\partial\rho)^k (\partial/\partial\tau)^l (\partial/\partial X)^\alpha v(\rho, \theta(X), \tau)$$

tends to a limit as $\rho \to 0$, uniformly in (X, τ) on $h\Omega \times (\tau_0, \tau_1)$. As $v = 0$ for $\tau \leqslant \tau_0$, S^{n-1} is compact, and τ_1 is arbitrary, this clearly implies the Theorem, and we are done.

Put

$$v_k(\theta, \tau) = \lim_{\rho \to 0} \frac{\partial^k v}{\partial \rho^k}, \quad k = 0, 1, \ldots. \tag{1·28}$$

Then Taylor's theorem for v gives the following description of the asymptotic behaviour of u:

(1·29) THEOREM. *Let $u(x,t)$ be an expanding wave. Then there exist functions $v_k(\theta, \tau) \in C^\infty(S^{n-1} \times \mathbb{R})$, $k = 0, 1, \ldots$, such that u has the asymptotic expansion*

$$u(r\theta, r+\tau) \sim \sum_{k=0}^{\infty} r^{-\frac{1}{2}(n-1)-k} v_k(\theta, \tau), \tag{1·30}$$

valid in the topology of $C^\infty(S^{n-1} \times \mathbb{R})$. Similar expansions for the derivatives of u can be derived from (1·30) *by term-by-term differentiation.*

(1·31) *Remark.* It has already been mentioned that the surface $\rho = 0$ is a characteristic of P. This can be inferred from (1·14) which gives, on the normal bundle of $\rho = 0$,

$$d\tilde{s}^2\big|_{\rho = d\rho = 0} = -|d\theta|^2.$$

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As this is of corank 2, $\rho = 0$ is a null surface of $d\tilde{s}^2$, hence a characteristic of $\tilde{\Box}$ and so also of P. It follows that the f_k for $k \geqslant 1$ can be derived from f_0 by solving a certain set of transport equations. The function f_0 will be called the *radiation field* of the expanding wave u; by (1·30),

$$v_0(\theta, \tau) = \lim_{r \to \infty} r^{(n-1)/2} u(r\theta, r+\tau). \tag{1·32}$$

There are of course analogous results for 'contracting' waves. Suppose that $u \in C^\infty(\{|x| > a\} \times \mathbb{R})$ satisfies

$$\Box u = 0 \quad \text{if} \quad |x| > a, \quad u = 0 \quad \text{if} \quad |x| > a, \quad t > t_0. \tag{1·33}$$

Then $u(x, -t)$ is an expanding wave, and so the asymptotic behaviour of

$$r^{(n-1)/2} u(r\theta, -r+\tau), \quad (\theta, \tau) \in S^{n-1} \times \mathbb{R}, \tag{1·34}$$

as $r \to \infty$ can be deduced from Theorem 1·29.

Theorem 1·29 also holds for expanding-wave solutions of the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + a(x)\, u = 0, \tag{1·35}$$

where $a(x)$ and the $b^i(x)$ are in $C^\infty(\mathbb{R}^n)$, provided that, for $x \neq 0$,

$$a(x) = |x|^{-2} \alpha\left(\frac{1}{|x|}, \frac{x}{|x|}\right), \quad b^i(x) = |x|^{-2} \beta^i\left(\frac{1}{|x|}, \frac{x}{|x|}\right), \quad i = 1, \ldots, n, \tag{1·36}$$

where $\alpha(\rho, \theta)$ and the $\beta^i(\rho, \theta)$ are restrictions to $\mathbb{R}^+ \times S^{n-1}$ of functions of class $C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1})$. A straightforward computation shows that Lemma 1·9 still holds, and the remainder of the proof goes through as before.

The hypothesis (1·4) can be weakened. Suppose that, for $x \neq 0$,

$$h_{ij} = \delta_{ij} + |x|^{-1} \epsilon_{ij}(|x|^{-1}, x/|x|), \tag{1·37}$$

where the $\epsilon_{ij}(\rho, \theta)$ are again restrictions to $\mathbb{R}^+ \times S^{n-1}$ of functions of class $C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1})$, and that

$$\sum_{i,j=1}^{n} \epsilon_{ij}(0, \theta)\, \theta_i \theta_j = 2\epsilon_0 = \text{constant}. \tag{1·38}$$

(This is a condition on the asymptotic behaviour of the coefficient of dr^2 when the Lorentzian metric ds^2 is written in terms of polar coordinates (r, θ) and t.) From this one has

$$\sum_{i,j=1}^{n} \epsilon_{ij}(\rho, \theta)\, \theta_i \theta_j = 2\epsilon_0 + \rho\epsilon_1(\rho, \theta), \quad \epsilon_1 \in C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1}). \tag{1·39}$$

Put

$$\rho = r^{-1}, \quad \tau = t - r - \epsilon_0 \log r,$$

and define $d\tilde{s}^2$, as before, by (1·11). Then

$$d\tilde{s}^2 = \rho^2 d\tau^2 - 2(1 + \epsilon_0 \rho)\, d\rho\, d\tau + (\epsilon_0^2 - \epsilon_1)\, d\rho - |d\theta|^2 + 2 \sum_{i,j=1}^{n} \epsilon_{ij} \theta_i d\theta_j d\rho - \rho \sum_{i,j=1}^{n} \epsilon_{ij} d\theta_i d\theta_j.$$

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This extends by continuity to a C^∞ Lorentzian metric on $\overline{\mathbb{R}}^+ \times S^{n-1} \times \mathbb{R}$. One then gets an analogue of Lemma 1·9 if one defines v by

$$V(\rho, \theta, \tau) = r^{(n-1)/2} u(r\theta, r + \epsilon_0 \log r + \tau), \quad (\theta, \tau) \in S^{n-1} \times \mathbb{R}. \tag{1·40}$$

Theorem 1·6 is proved as before, and Theorem 1·29 holds with the obvious modification, for expanding-wave solutions of (1·35), subject to (1·36).

The condition (1·39) holds automatically if $n = 1$, or if the metric has radial symmetry. Suppose for example that

$$ds^2 = dt^2 - c^{-2} |dx|^2,$$

where $c = g(|x|)$, $g(r)$ being the restriction of a positive even C^∞ function on $\mathbb{R}$ to $\overline{\mathbb{R}}^+$. To satisfy (1·37) one must require that, for $r > 0$, $g(r) = \gamma(1/r)$, where $\gamma \in C^\infty(\overline{\mathbb{R}}^+)$ and $\gamma(0) = 1$. (All that is really needed here is $\gamma(0) > 0$, of course.) Instead of (1.40), one can then work with

$$v(\rho, \theta, \tau) = r^{(n-1)/2} u(r\theta, \psi(r) + \tau), \quad \psi(r) = \int_0^r \frac{ds}{g(s)},$$

so that the radiation field of an expanding wave is obtained by going to infinity along the bicharacteristics of the characteristics $t - \psi(r) =$ constant.

Section 2.

We shall now consider the initial value problem

$$\Box u = 0 \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R}, \quad u = f_0(x), \quad \frac{\partial u}{\partial t} = f_1(x) \quad \text{when} \quad t = 0, \tag{2·1}$$

where $\Box$ is the d'Alembertian of a metric (1·1) which satisfies the hypothesis (1·4). It is not difficult to extend the argument to perturbed wave equations for which (1·37) and (1·38) hold.

It will be assumed for the time being that the data $f = (f_0, f_1)$ are in $C_0^\infty(\mathbb{R}^n)$. It is well known that the initial-value problem (2·1) then has a unique solution

$$u(x, t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}).$$

Moreover, $x \mapsto u(x, t)$ has compact support for all $t \in \mathbb{R}$. This follows in particular from the first part of the Proof of Theorem 1·6 applied to $u(x, t)$ for $t > 0$ and to $u(x, -t)$ for $t < 0$. But it is usually proved by integrating the identity

$$h^{\frac{1}{2}} \frac{\partial u}{\partial t} \Box u = \frac{1}{2} \frac{\partial}{\partial t} \left(\left(\sum_{i,j=1}^n h^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \left(\frac{\partial u}{\partial t} \right)^2 \right) h^{\frac{1}{2}} \right) - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(h^{\frac{1}{2}} h^{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} \right), \tag{2·2}$$

over a suitable domain. It then follows from the same identity that u has an important property: it satisfies the *energy identity*

$$\frac{1}{2} \int \left(\sum_{i,j=1}^n h^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \left(\frac{\partial u}{\partial t} \right)^2 \right) h^{\frac{1}{2}} dx = E = \text{constant}, \tag{2·3}$$

where

$$E = \frac{1}{2} \int \left(\sum_{i,j=1}^n h^{ij} \frac{\partial f_0}{\partial x_i} \frac{\partial f_0}{\partial x_j} + f_1^2 \right) h^{\frac{1}{2}} dx. \tag{2·4}$$

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Furthermore, the function $u(x,t)\,H(t)$, where H is the Heaviside function, restricted to $\{|x| > a\} \times \mathbb{R}$, is an expanding wave if a is large enough. By Theorem 1·29, $uH(t)$ has a radiation field,

$$\lim_{r\to\infty} (r^{(n-1)/2}\, u(r\theta, r+\tau), \quad (\theta,\tau) \in S^{n-1} \times \mathbb{R}.$$

This will here be called the (forward) radiation field of u.

It will presently become clear that, for an initial-value problem, the radiation field of $u_t = \partial u/\partial t$ is more important than the radiation field of u itself. As this is determined by the initial data f, we shall denote it by $\mathscr{R}f$ or by $\tilde{f}(\theta,\tau)$:

$$\mathscr{R}f \equiv \tilde{f}(\theta,\tau) = \lim_{r\to\infty} (r^{(n-1)/2}\, u_t(r\theta, r+\tau)). \tag{2·5}$$

Note that $\tilde{f} \in C^\infty(S^{n-1} \times \mathbb{R})$ by Theorem 1·6. We shall now show that $\tilde{f} \in L_2(S^{n-1} \times \mathbb{R})$ and that $\|\tilde{f}\|_{L_2(S^{n-1}\times\mathbb{R})} \leqslant E^{\frac{1}{2}}$. This will follow from

(2·6) LEMMA. *Let $u(x,t)$ be the solution of* (2·1), *where it is assumed that*

$$f \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n).$$

Let λ be a real number and $t > \lambda$, and put

$$E'_{\lambda,\,t} = \frac{1}{2}\int_{|X|>t-\lambda} \left(\sum_{i,j=1}^{n} h^{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + u_t^2 \right) h^{\frac{1}{2}}\, dx. \tag{2·7}$$

Then

$$\lim_{t\to\infty} E'_{\lambda,t} = \int_{-\infty}^{\lambda} d\tau \int_{S^{n-1}} \tilde{f}^2\, d\omega(\theta)), \tag{2·8}$$

where $d\omega(\theta)$ is the surface measure on S^{n-1}.

Proof. By Theorem 1·6,

$$u(x,t) = |x|^{(1-n)/2}\, v\left(\frac{1}{|x|}, \frac{x}{|x|}, t-|x|\right), \tag{2·9}$$

where (in the present case) $v(\rho,\theta,\tau) \in C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1} \times \mathbb{R})$. The proof of Theorem 1·6 also shows that, if $r_0 > \sup\{|x| : x \in \operatorname{supp} f_0 \cup \operatorname{supp} f_1\}$ is sufficiently large, then

$$v(\rho,\theta,\tau) = 0 \quad \text{if} \quad 0 \leqslant \rho \leqslant r_0^{-1}, \quad \tau \leqslant -2r_0. \tag{2·10}$$

As $\tau = t - |x| < \lambda$ in the integral defining $E'_{\lambda,t}$, we can compute this integral by substituting (2·9). A convenient way of handling the dependence of v on $\theta \in S^{n-1}$ is to extend $\theta \mapsto v$ as a homogeneous function of degree zero of an auxiliary variable ζ, so that

$$v(\rho,\theta,\tau) = \tilde{v}(\rho,\zeta,\tau)|_{\zeta=\theta}, \tag{2·11}$$

where $\tilde{v}(\rho,\zeta,\tau) \in C^\infty(\overline{\mathbb{R}}^+ \times (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R})$, and $|\theta| = 1$. As before, we put

$$r = |x| = \rho^{-1}, \quad x/|x| = \theta, \quad t-|x| = t-r = \tau.$$

Then one has, from (2·9) and (2·11),

$$\begin{cases} r^{(n-1)/2}\dfrac{\partial u}{\partial t} = \dfrac{\partial v}{\partial \tau}, \\ r^{(n-1)/2}\dfrac{\partial u}{\partial x_i} = -\theta_i\left(\dfrac{\partial v}{\partial \tau} + \frac{1}{2}(n-1)\,\rho v + \rho^2 \dfrac{\partial v}{\partial \rho}\right) \\ \qquad\qquad + \rho \sum_{j=1}^{n} (\delta_{ij} - \theta_i\theta_j) \left.\dfrac{\partial \tilde{v}}{\partial \xi_j}\right|_{\zeta=\theta}, \quad i = 1,\ldots,n. \end{cases} \tag{2·12}$$

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Also,

$$\begin{cases} h^{\frac{1}{2}}\,dx = \bar{h}^{\frac{1}{2}} r^{n-1}\,dr\,d\omega(\theta), \\ \bar{h} = \det(\delta_{ij} + \rho^2 \epsilon_{ij}) = 1 + \rho^2 \bar{\epsilon}(\rho, \theta), \quad \bar{\epsilon} \in C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1}). \end{cases} \tag{2·13}$$

As t is fixed and $r = t - \tau$, one can use (θ, τ) as variables of integration in (2·7); this gives

$$E'_{\lambda,t} = \int_{-\infty}^{\lambda} d\tau \int_{\rho^{n-1}} F|_{\rho=(t-\tau)^{-1}}\, d\omega(\theta), \tag{2·14}$$

where

$$\begin{cases} F = \frac{1}{2} r^{n-1} \bar{h}^{\frac{1}{2}} \left(\sum_{i,j=1}^{n} h^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \left(\frac{\partial u}{\partial t} \right)^2 \right) \\ \quad = \frac{1}{2} r^{n-1} \bar{h}^{\frac{1}{2}} \left(\left(\frac{\partial u}{\partial t} \right)^2 + |\,\mathrm{grad}\, u|^2 + \frac{1}{r^2} \sum_{i,j=1} \epsilon^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right), \end{cases} \tag{2·15}$$

by (1·4). From (2·12) one has, first,

$$\begin{aligned} F_0 &= \tfrac{1}{2} r^{n-1} \bar{h}^{\frac{1}{2}} \left(\left(\frac{\partial u}{\partial t} \right)^2 + |\mathrm{grad}\, u|^2 \right) \\ &= \tfrac{1}{2} \bar{h}^{\frac{1}{2}} \left(\left(\frac{\partial v}{\partial \tau} \right)^2 + \left(\frac{\partial v}{\partial \tau} + \tfrac{1}{2}(n-1)\rho v + \rho^2 \frac{\partial v}{\partial \rho} \right)^2 + \rho^2 \sum_{i=1}^{n} \left(\frac{\partial \tilde{v}}{\partial \xi_i} \right)^2 \Big|_{\zeta=\theta} \right). \end{aligned}$$

For $t > \lambda + r_0$ one has $\rho = (t-\tau)^{-1} \leqslant (t-\lambda)^{-1} \leqslant r_0^{-1}$ in the integral. By (2·10) one can therefore make $t \to \infty$ under the integral sign, as the convergence is uniform. In view of (2·13) and as $\partial v/\partial \tau|_{\rho=0} = \tilde{f}(\theta, \tau)$, this gives

$$\lim_{t\to\infty} \int_{-\infty}^{\lambda} d\tau \int_{S^{n-1}} F_0\,|_{\rho=(t-\tau)^{-1}}\, d\omega(\theta) = \int_{-\infty}^{\lambda} d\tau \int_{S^{n-1}} \tilde{f}^2\, d\omega(\theta).$$

It is also evident from (2·12) and (2·15) that

$$F - F_0 = \tfrac{1}{2} \bar{h}^{\frac{1}{2}} r^{n-3} \sum_{i,j=1}^{n} \epsilon^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = \rho^2 G(\rho, \theta, \tau),$$

where G is quadratic in v and its first-order derivatives, so that $G \in C^\infty(\overline{\mathbb{R}}^+ \times S^{n-1} \times \mathbb{R})$, and $G = 0$ for $\tau < -2r_0$ when $t > \lambda + r_0$. Hence

$$\lim_{t\to\infty} \int_{-\infty}^{\lambda} d\tau \int_{S^{n-1}} (F - F_0)\,|_{\rho=(t-\tau)^{-1}}\, d\omega(\theta) = 0$$

and so, by (2·14), the Lemma is proved.

As an immediate consequence of this Lemma and the energy identity, we have

(2·16) PROPOSITION. *Let $u(x,t)$ be the solution of (2·1) with C_0^∞ initial data f. Then*

$$E = \|\tilde{f}\|^2_{L_2(S^{k-1}\times\mathbb{R})} + \lim_{\lambda\to\infty} \lim_{t\to\infty} E''_{\lambda t}, \tag{2·17}$$

where, with $t > \lambda$,

$$E''_{\lambda,t} = \frac{1}{2} \int_{|X|<t-\lambda} \left(\sum_{i,j=1}^{n} h^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \left(\frac{\partial u}{\partial \tau} \right)^2 \right) h^{\frac{1}{2}}\, dx. \tag{2·18}$$

Thus $\tilde{f} \in L_2(S^{n-1} \times \mathbb{R})$ and

$$\|\tilde{f}\|_{L_2(S^{k-1}\times\mathbb{R})} \leqslant E^{\frac{1}{2}}. \tag{2·19}$$

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Proof. One can write (2·3) as

$$E'_{\lambda,t} + E''_{\lambda,t} = E = \text{constant}.$$

By Lemma 2.6 this gives, as $t \to \infty$,

$$E = \int_{-\infty}^{\lambda} d\tau \int \tilde{f}^2 \, d\omega + \lim_{t\to\infty} E''_{\lambda,t};$$

the existence of $\lim_{t\to\infty} E''_{\lambda,t}$ follows from that of $\lim_{t\to\infty} E'_{\lambda,t}$. Letting $\lambda \to \infty$ one obtains (2·17), which implies (2·19) as, clearly, $E''_{\lambda,t} \geqslant 0$.

(2·20) *Remark.* The radiation field of an expanding wave depends on the choice of origin in $\mathbb{R}^{n+1}$. It is easy to see that this does not affect the identity (2·17). Let u be an expanding wave defined on $\{|x| > a\} \times \mathbb{R}$ and let (x_0, t_0) be a point of $\mathbb{R}^{n+1}$. Clearly, u is also an expanding wave with respect to the translated coordinates $x' = x - x_0$, $t' = t - t_0$, on $\{|x'| > a + x_0\} \times \mathbb{R}$; its radiation field is

$$\begin{aligned} \tilde{f}'(\theta, \tau) &= \lim_{r\to\infty} (r^{(n-1)/2} u(x_0 + r\theta, r + t_0 + \tau)) \\ &= \lim_{r\to\infty} \left(\frac{r}{|x_0 + r\theta|}\right)^{(n-1)/2} \nu\left(\frac{1}{|x_0 + r\theta|}, \frac{x_0 + r\theta}{|x_0 + r\theta|}, t_0 + r + \tau - |x_0 + r\theta|\right), \end{aligned}$$

whence, by Theorem 1·6,

$$\tilde{f}'(\theta, \tau) = \tilde{f}(\theta, t_0 + \tau - \langle x_0, \theta \rangle). \tag{2·21}$$

So (2·17) is unchanged when $\tilde{f}$ is replaced by $\tilde{f}'$.

Following Lax and Phillips, one can now introduce the *space H_E of initial data with finite energy*. This is the completion of the space of (pairs of) C_0^∞ initial data f with respect to the norm

$$\|f\|_{H_E} = \left(\frac{1}{2} \int \left(\sum_{i,j=1}^{n} h^{ij} \frac{\partial f_0}{\partial x_i} \frac{\partial f_0}{\partial x_j} + f_1^2 \right) h^{\frac{1}{2}} \, dx \right)^{\frac{1}{2}}, \tag{2·22}$$

which is just the square root of the second member of (2·4). This completion is a Hilbert space H_E if $n \geqslant 3$. (If $n = 1$ or $n = 2$ one has to factor out pairs $(f_0, 0)$ with f_0 a constant function; we assume for simplicity that $n \geqslant 3$.) Clearly, H_E consists of all f such that $f_1 \in L_2(\mathbb{R}^n)$ and the distributions $\partial f/\partial x_i$, $i = 1, \ldots, n$ are in $L_2(\mathbb{R}^n)$. One should really write H_E^h to indicate the dependence on the metric; to simplify the notation we shall not do so. Different metrics satisfying (1·4) give energy spaces that are equivalent as Banach spaces, but have different Hilbert structures.

It is immediate from the energy identity (2·3) that, if $f^{(j)}$ are C_0^∞ data converging in H_E to $f \in H_E$, and $u^{(j)}$ is the solution of $\Box u^{(j)} = 0$ with initial data $f^{(j)}$, then the vectors

$$U(t) f^{(j)} = (u^{(j)}(\,.\,, t), u_t^{(j)}(\,.\,, t)) \tag{2·23}$$

converge in H_E for each $t \in \mathbb{R}$; we denote the limit by $U(t)f$. Its first component $(U(t)f)_0 = u(x, t)$ solves the initial-value problem as a distribution valued function $u\colon \mathbb{R} \to \mathscr{D}'(\mathbb{R}^n)$ which is in fact C^∞ by partial hypoellipticity. But one has more.

(2·24) PROPOSITION. *The solution operators $\{U(t)\colon t \in \mathbb{R}\}$ are a strongly continuous group of unitary operators $H_E \to H_E$.*

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Proof. Virtually by definition, $U(t)$ is an isometry, and the group property $U(t+s) = U(t)\,U(s)$ follows from the uniqueness of the solution of the initial-value problem, as $\Box$ is invariant under translation in t. To prove that $U(t)$, for $t \neq 0$ (of course $U(0)$ is the identity) is unitary, we fix a $T \neq 0$ and an $f \in H_E$. Let $f^{(j)}, j = 1, \ldots,$ be a sequence of C_0^∞ data converging to f in H_E, and for each j let $u^{(j)}(x,t)$ be the solution of the initial value problem

$$\Box u^{(j)} = 0, \quad u^{(j)}(x,T) = f_0^{(j)}, \quad u_t^{(j)}(x,T) = f_1^{(j)}.$$

Setting $g^{(j)} = (u^{(j)}(x,\theta), u_t^{(j)}(x,0))$, one has $U(T)\,g^{(j)} = f^{(j)}$; by the isometry of U the $g^{(j)}$ converge to some $g \in H_E$ in energy norm, and $U(T)\,g = f$. Hence $U(t)$ is surjective, hence unitary.

To prove that $\{U(t)\}$ is a strongly continuous group, it is clearly sufficient to show that

$$\|U(t)f - f\|_{H_E} \to 0 \quad \text{as} \quad t \to 0, \tag{2·25}$$

for all $f \in H_E$. Suppose first that f is C_0^∞, let u be the solution of (2·1), so that $u = (U(t)f)_0$, and set $v = U(t)f - f$. Then

$$\frac{\partial v}{\partial t} = Av + Af, \quad A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}. \tag{2·26}$$

A simple manipulation gives

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_{i,j=1}^{n} h^{ij}\frac{\partial v_0}{\partial x_i}\frac{\partial v_0}{\partial x_j} + v_1^2\right)h^{\frac{1}{2}}\Big) = h^{\frac{1}{2}}\left(v_n \Delta f_0 + \sum_{i,j=1}^{n} h^{ij}\frac{\partial f_1}{\partial x_i}\frac{\partial v_0}{\partial x_j}\right) + \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(h^{\frac{1}{2}}h^{ij}v_1\frac{\partial v_0}{\partial x_j}\right).$$

As $v \in C_0^\infty \times C_0^\infty$ and $v_{t=0} = 0$, integration of this over $\mathbb{R}^n(0,t)$ gives

$$\|v(t)\|_{H_E}^2 = 2\int_0^t (v(s), Af)_{H_E}\,ds \leqslant 2\|Af\|_{H_E}\int_0^t \|v(s)\|_{H_E}\,ds.$$

A routine argument shows that this implies

$$\|U(t)f - f\|_{H_E} = \|V(t)\|_{H_E} \leqslant t\|Af\|_{H_E}.$$

Now let $f \in H_E$ and let $f^{(j)}, j = 1, 2, \ldots,$ be C_0 data converging to f in H_E. Then

$$\begin{aligned}\|U(t)f - f\|_{H_E} &\leqslant \|U(t)f - U(t)f^{(j)}\|_{H_E} + \|f^{(j)} - f\|_{H_E} + \|U(t)f^{(j)} - f^{(j)}\|_{H_E} \\ &\leqslant 2\|f - f^{(j)}\|_{H_E} + t\|Af^{(j)}\|_{H_E},\end{aligned}$$

by the estimate just obtained and the unitarity of U. This easily gives (2.25), and so we are done.

(2·27) *Remark.* We have deduced Proposition 2·24 from the theory of hyperbolic equations. Conversely, one can give an operator theoretic proof. Defining A as in (2·26) as an unbounded operator on H_E with domain $\{f \in H_E : Af \in H_E\}$, one can follow the proof of ((6), 137, theorem 1·2) to conclude that A is skew self-adjoint and so generates a unitary group by Stone's theorem, using the obvious complexification of H_E. Regularity can then be proved *a posteriori*. Alternatively, one can use the Friedrichs extension of $-\Delta$ and the spectral theorem, as in (8), section 4.

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As an immediate consequence of Proposition 2·16 we now have:

(2·28) THEOREM. *The map $\mathscr{R}$ defined by (2·5) for C_0^∞ initial data extends by continuity to a map $\mathscr{R}$: $H_E \to L_2(S^{n-1}\times\mathbb{R})$ such that*

$$\|\mathscr{R}f\|_{L_2(S^{n-1}\times\mathbb{R})} \leqslant \|f\|_{H_E} \tag{2·29}$$

and

$$\mathscr{R}\circ U(s)f(\theta,\tau) = \mathscr{R}f(\theta, s+\tau). \tag{2·30}$$

Proof. As the inequality (2·19) of the Proposition is just (2·29) in another notation, it is immediate that $\mathscr{R}$ extends by continuity to a map $H_E \to L_2(S^{n-1}\times\mathbb{R})$ that satisfies (2·29). As to (2·30), we only have to prove this for C_0^∞ data, as these are dense in H_E. With $(U(t)f)_1 = u(x,t)$ one then has, by (2·5),

$$\mathscr{R}\circ U(s)f = \lim_{r\to\infty}(r^{(n-1)/2}u_t(r\theta, s+r+\tau)) = \mathscr{R}f(\theta, s+\tau),$$

which is (2.30).

Remark. It is clear that $\mathscr{R}$ gives a 'twisted' translation representation of the group $\{U(t)\}$. If one sets $\check{\mathscr{R}}f(\theta,\tau) = \mathscr{R}f(\theta,-\tau)$, then

$$\check{\mathscr{R}}\circ U(s) = T_s\check{\mathscr{R}}, \tag{2·31}$$

where T_s denotes translation by s in the τ variable. So $\check{\mathscr{R}}f$ is a translation representer of f in the sense of Lax and Phillips. But Theorem 2·28 leaves open the possibility that $\mathscr{R}$ is not an isometry. We shall return to this important question presently.

The definition of the map $\mathscr{R}$ which has been given is rather indirect, and it is clearly of interest to have a more explicit characterization of $\mathscr{R}f$ as an asymptotic limit.

(2·32) PROPOSITION. *Let $f\in H_E$ and put $\mathscr{R}f = \tilde{f}(\theta,\tau)$. Let $\phi(\theta,\tau)\in C_0^\infty(S^{n-1}\times\mathbb{R})$ and put (for $x\neq 0$)*

$$\psi(x,t) = |x|^{(1-n)/2}\phi(x/|x|, t-|x|). \tag{2·33}$$

Then, with $u_t = (U(t)f)_1$,

$$\lim_{t\to\infty}\int u_t(x,t)\,\psi(x,t)\,dx = \int \tilde{f}(\theta,\tau)\,\phi(\theta,\tau)\,d\omega(\theta)\,d\tau. \tag{2·34}$$

Proof. Note first that

$$\|\psi(\,.\,,t)\|^2_{L_2(\mathbb{R}^n)} = \int_0^\infty\int(\phi(\theta,t-r))^2\,dr\,d\omega(\theta) \leqslant \|\phi\|^2_{L_2(S^{k-1}\times\mathbb{R})}. \tag{2·35}$$

Let $f^{(j)}$ be a sequence of C_0^∞ data converging to f in H_E, and put, as usual,

$$U(t)f^{(j)} = (u^{(j)}(x,t), u_t^{(j)}(x,t)).$$

Then $H(t)\,u^{(j)}(x,t)$ is an expanding wave on $\{|x| > a_j\}\times\mathbb{R}$ for some a_j, and one can appeal to Theorem 1·6. As $\phi\in C_0^\infty(S^{n-1}\times\mathbb{R})$, there are numbers τ_1 and $\tau_2 > \tau_1$ such that $\phi = 0$ if $\tau < \tau_1$ or $\tau > \tau_2$. So for $t > \tau_2$ one has

$$\begin{aligned}\int u_t^{(j)}(x,t)\,\psi(x,t)\,dx &= \int_0^\infty\int_{S^{n-1}} u_t^{(j)}(r\theta,t)\,\phi(\theta,t-r)\,r^{(n-1)/2}\,dr\,dw(\theta)\\ &= \int_{\tau_1}^{\tau_2}\int_{S^{n-1}} v_\tau^{(j)}\left(\frac{1}{t-\tau},\theta,\tau\right)\phi(\theta,\tau)\,d\tau\,d\omega(\theta),\end{aligned}$$

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where $v^{(j)}$ is related to $u^{(j)}$ as v is related to u in Theorem 1·6. So, by uniform convergence,

$$\lim_{t\to\infty}\int u_t^{(j)}(x,t)\,\psi(x,t)\,dx = \int \tilde{f}^{(j)}\phi\,d\omega\,d\tau, \tag{2·36}$$

as $v_\tau^{(j)}(0,\theta,\tau) = \tilde{f}^{(j)}(\theta,\tau)$.

Now

$$\begin{aligned}\left|\int u_t\psi\,dx - \int \tilde{f}\phi\,d\omega\,d\tau\right| &\leqslant \left|\int u_t^{(j)}\psi\,dx - \int f^{(j)}\phi\,d\omega\,d\tau\right| \\ &\quad + \left|\int (u_t - u_t^{(j)})\,\psi\,dx\right| + \left|\int (\tilde{f}^{(j)} - \tilde{f})\,\phi\,d\omega\,d\tau\right|.\end{aligned} \tag{2·37}$$

By Theorem 2·29 and Schwarz's inequality,

$$\left|\int (\tilde{f}^{(j)} - \tilde{f})\,\phi\,d\omega\,d\tau\right| \leqslant \|f^{(j)} - f\|_{H_E}\|\phi\|_{L_2(S^{n-1}\times\mathbb{R})}.$$

Again,

$$\left|\int (u_t - u_t^{(j)})\,\psi\,dx\right| \leqslant \|u_t - u_t^{(j)}\|_{L_2(\mathbb{R}^n)}\|\psi\|_{L_2(\mathbb{R}^n)},$$

whence, by (2·35) and as $u_t - u_t^{(j)} = (U(t)f - U(t)f^{(j)})$,

$$\left|\int (u_t - u_t^{(j)})\,\psi\,dx\right| \leqslant \sqrt{2}\,\|f - f^{(j)}\|_{H_E}\|\phi\|_{L_2(S^{n-1}\times\mathbb{R})}.$$

It therefore follows from (2·36) and (2·37) that

$$\limsup_{t\to\infty}\left|\int u_t\psi\,dx - \int f\phi\,d\omega\,d\tau\right| \leqslant (1+\sqrt{2})\,\|f - f^{(j)}\|_{H_E}\|\phi\|_{L_2(S^{n-1}\times\mathbb{R})},$$

and, as $\|f - f^{(j)}\|_{H_E} \to 0$ when $j \to \infty$, the Proposition is proved.

For the standard wave equation

$$\Box_0 u \equiv \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = 0, \tag{2·38}$$

the hypothesis (1·4) is satisfied trivially as all the ϵ_{tj} are identically zero. So Theorems 1·6 and 2·28 hold. In this case it is easy to prove that $\mathscr{R}\colon H_E \to L_2(S^{n-1}\times\mathbb{R})$ is an isometry.

(2·39) PROPOSITION. *For the standard wave equation,*

$$\|\mathscr{R}f\|_{L_2(S^{n-1}\times\mathbb{R})} = \|f\|_{H_E}.$$

Proof. It is sufficient to prove this for C_0^∞ data, as these are dense in H_E. If n is odd, then the wave equation satisfies the sharp Huygens principle, as the support of the forward fundamental solution (such that $\Box_0 E = \delta$, $\operatorname{supp} E \subset \{t \geqslant 0\}$) is the forward characteristic cone $\{(x,t)\colon t = |x|\}$. It follows that if

$$a > \sup\{|x|\colon x \in \operatorname{supp} f_0 \cup \operatorname{supp} f_1\} \tag{2·40}$$

then $u(x,t) = (U(t)f)_1$ vanishes for $t > |x| + a$ (as well as for $t < |x| - a$). So $E''_{\lambda,t}$, which is defined by (2·18), is zero for $\lambda > a$, and the assertion follows at once from (2·17).

If n is even, then the solution of the initial-value problem is (see for example (3)) (for $t > 0$)

$$u(x,t) = C_n\int_{\Sigma_t}(t^2 - |x-x'|^2)^{(1-n)/2} f_1(x')\,dx' + C_n\frac{\partial}{\partial t}\int_{\Sigma_t}(t^2 - |x-x'|^2)^{(1-n)/2} f_0(x')\,dx', \tag{2·41}$$

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where $C_n = \frac{1}{2}\pi^{(n-1)/2}\Gamma((3-n)/2)$, $\Sigma_t = \{x' : |x-x'| < t\}$, and the integrals are in fact distributions (finite parts of divergent integrals). However, if a satisfies (2·40), $t > \lambda > a$ and $|x| = r < t-\lambda$, then, for $x' \in \operatorname{supp} f_0 \cup \operatorname{supp} f_1$,

$$t^2 - |x-x'|^2 \geqslant t^2 - (r+a)^2 \geqslant (\lambda - a)(t+r+a) > 0. \tag{2·42}$$

The integrals on the right-hand side of (2·41) are then ordinary integrals. Consider the first one, which, by (2·40) and (2·42), can be written as an integral over $\mathbb{R}^n$,

$$u'(x,t) = C_n \int (t^2 - |x-x'|^2)^{(1-n)/2} f_1(x')\, dx'.$$

Here, one can differentiate under the integral sign, to obtain

$$\frac{\partial u'}{\partial t} = -(n-1) C_n t \int (t^2 - |x-x'|^2)^{-(n+1)/2} f_1(x)\, dx'.$$

So, by (2·42)

$$\left|\frac{\partial u'}{\partial t}\right| \leqslant (n-1) C_n \|f_1\|_{L_1(\mathbb{R}^n)} t (t^2 - (r+a)^2)^{-(n+1)/2}.$$

Similarly,

$$\left|\frac{\partial u'}{\partial x_i}\right| \leqslant (n-1) C_n \|f_1\|_{L_1(\mathbb{R}^n)} (r+a)(t^2 - (r+a)^2)^{-(n+1)/2}.$$

Hence

$$\int_{|X|<t-\lambda} \left(\left(\frac{\partial u'}{\partial t}\right)^2 + |\operatorname{grad} u'|^2 \right) dx \leqslant C \int_0^{t-\lambda} \frac{(t^2 + n(r+a)^2)\, r^{n-1}\, dr}{(t^2 - (r+a)^2)^{n+1}}.$$

But

$$\frac{t^2 r^{n-1}}{(t+r+a)^{n+1}} \leqslant 1, \quad \frac{(r+a)^2 r^{n-1}}{(t+r+a)^{n+1}} \leqslant 1,$$

so that

$$\int_0^{t-\lambda} \frac{(t^2 + n(r+a)^2)\, r^{n-1}}{(t^2 - (r+a)^2)^{n+1}}\, dr \leqslant \int_0^{t-\lambda} \frac{dr}{(t-r-a)^{n+1}} = \frac{1}{n}\left(\frac{1}{(\lambda-a)^n} - \frac{1}{(t-a)^n}\right),$$

whence, with another constant C,

$$\int_{|x|<t-\lambda} \left(\left(\frac{\partial u'}{\partial t}\right)^2 + |\operatorname{grad} u'|^2 \right) dx \leqslant C\lambda^{-n} \quad \text{if} \quad t > \lambda \geqslant 2a. \tag{2·43}$$

The other term in the second member of (2·41),

$$u''(x,t) = C_n \frac{\partial}{\partial t} \int (t^2 - |x-x'|^2)^{(1-n)/2} f_0(x')\, dx',$$

can be treated in the same way. One obtains the estimate

$$\int_{|x|<t-\lambda} \left(\left(\frac{\partial u''}{\partial t}\right)^2 + |\operatorname{grad} u''|^2 \right) dx \leqslant C'\lambda^{-n-1} \quad \text{if} \quad t > \lambda \geqslant 2a. \tag{2·44}$$

Now it is clear from (2·18) and (2·41) that

$$E''_{\lambda,t} \leqslant \int_{|x|<t-\lambda} \left(\left(\frac{\partial u'}{\partial t}\right)^2 + |\operatorname{grad} u'|^2 \right) dx + \int_{|x|<t-\lambda} \left(\frac{\partial u''}{\partial t} + |\operatorname{grad} u''|^2 \right) dx,$$

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and so (2·43), (2·44) give

$$\lim_{\lambda\to\infty}\lim_{t\to\infty} E''_{\lambda,t} = 0.$$

The assertion therefore follows from (2·17), and the Proposition is proved.

(2·45) *Remark.* A connected open set $X \subset \mathbb{R}^n$ is called an *exterior domain* if its complement is compact. Let X be such a domain, with smooth boundary ∂X. The energy space $H_E(X)$ is now the completion of $C_0^\infty(X) \times C_0^\infty(X)$ with respect to the norm

$$\|f\|_{H_E(X)} = \left(\frac{1}{2}\int\left(\sum_{i,j=1}^{n} h^{ij}\frac{\partial f_0}{\partial x_i}\frac{\partial f_0}{\partial x_j} + f_1^2\right)h^{\frac{1}{2}}\,dx\right)^{\frac{1}{2}}. \tag{2·46}$$

Instead of the initial-value problem (2·1) one can consider the initial-boundary-value problem

$$\begin{cases} \Box u = 0 \quad \text{on} \quad X\times\mathbb{R}, \quad u_{\partial X\times\mathbb{R}}| = 0, \\ u|_{t=0} = f_0(x), \quad u_t|_{t=0} = f_1(x), \quad f = (f_0,f_1)\in C_0^\infty(X)\times C_0^\infty(X). \end{cases} \tag{2·47}$$

This has a smooth solution which again satisfies an energy identity, and to which Theorem 1·6 can be applied, so that a radiation field of u_t exists. It is clear that the proof of Theorem 2·28 goes through as before. This is also the case when the Dirichlet condition is replaced by the Neumann condition $\partial u/\partial n|_{\partial X\times\mathbb{R}} = 0$.

Section 3

For perturbed wave equations, the radiation field map $\mathscr{R}$ is not necessarily an isometry, but only satisfies the inequality (2·29),

$$\|\mathscr{R}f\|_{L_2(S^{n-1}\times\mathbb{R})} \leqslant \|f\|_{H_E}. \tag{3·1}$$

We shall show that there are always initial data for which one has equality in (3·1).

The hypersurfaces

$$S(x,t) = t - r - \frac{r_0^2}{r} = \text{constant}, \quad r = |x| \geqslant r_0, \tag{3·2}$$

are space-like if r_0 is sufficiently large, by Lemma 1·22. We choose such an r_0 and keep it fixed throughout. Note that the equation $S(x,t) = \tau_1$ can be put into the form

$$r = s(t) = \tfrac{1}{2}(t-\tau_1) + (\tfrac{1}{4}(t-\tau_1)^2 - r_0^2)^{\frac{1}{2}}, \quad t \geqslant \tau_1 + 2r_0. \tag{3·3}$$

A solution of $\Box u = 0$ will be called a *receding wave* if there is a $\tau_1 \in \mathbb{R}$ and a $T_1 > \tau_1 + 2r_0$ such that

$$u = 0 \quad \text{if} \quad t > T_1 \quad \text{and} \quad |x| < s(t). \tag{3·4}$$

The boundary of the support of u will in fact be a union of bicharacteristic curves, and locally a characteristic; the characterization (3·4) of receding waves ensures this, but is easier to handle. For the standard wave equation one can easily show that (3·4) can be replaced by

$$u = 0 \quad \text{if} \quad t > r + \tau_1. \tag{3·4′}$$

Initial data (not necessarily with finite energy) that give rise to a receding wave will be called *eventually outgoing*. For the standard wave equation, it is clear from (3·4′) that this agrees with the terminology of Lax and Phillips(6).

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Receding waves can be constructed by 'starting at infinity'.

(3·5) LEMMA. *Let $\psi(\theta,\tau)\in C^\infty(S^{n-1}\times\mathbb{R})$ and suppose that there is a τ_1 such that $\psi=0$ for $\tau>\tau_1$. Then there is a unique receding wave $u(x,t)\in C^\infty(\mathbb{R}^{n+1})$ satisfying* (3·4), *with radiation field ψ,*

$$\lim_{r\to\infty}\,(r^{(n-1)/2}u(r\theta,r+\tau))=\psi(\theta,\tau).$$

Moreover, if one puts

$$v(\rho,\theta,\tau)=r^{(n-1)/2}u(r\theta,r+\tau),\quad\text{where}\quad\rho=r^{-1},\tag{3·6}$$

then $v\in C^\infty([0,\infty)\times S^{n-1}\times\mathbb{R})$.

Proof. Recall that if $\Box u=0$ and v is defined by (3·6) for $\rho>0$ then $Pv=0$, where P is given by (1·13) in terms of the conformal metric $d\tilde{s}^2=ds^2/r^2$, and has coefficients that are functions of (ρ,θ) which are C^∞ up to $\rho=0$. Conversely, if $Pv=0$, then (3·6) defines a solution of $\Box u=0$. In view of this, we first set up an auxiliary boundary value problem, as follows:

$$\begin{cases} P\tilde{v}=0 \quad\text{on}\quad \Omega=\{(\rho,\theta,\tau)\colon 0<\rho\leqslant r_0^{-1},\theta\in S^{n-1},\tau-r_0^2\rho<\tau_1\},\\ \tilde{v}=\psi \quad\text{on}\quad \Sigma_0=\{0\}\times S^{n-1}\times(-\infty,\tau_1],\\ \tilde{v}=\partial\tilde{v}/\partial\tau=0 \quad\text{on}\quad \Sigma_1=\{(\rho,\theta,\tau)\colon 0\leqslant\rho\leqslant r_0^{-1},\theta\in S^{n-1},\tau-r_0^2\rho=\tau_1\}.\end{cases}\tag{3·7}$$

We begin by solving this at the formal series level. A straightforward but somewhat tedious computation shows that, for any $\tilde{v}$ that is smooth up to $\rho=0$, one has, for $k=0,1,\ldots,$

$$\left(\frac{\partial}{\partial\rho}\right)^{k+1}P\tilde{v}\Big|_{\rho=0}=-2\frac{\partial}{\partial\tau}\left(\frac{\partial^{k+1}\tilde{v}}{\partial\rho^{k+1}}\Big|_{\rho=0}\right)+\sum_{j=0}^{k}Q_{kj}\left(\frac{\partial^j v}{\partial\rho^j}\Big|_{\rho=0}\right),\tag{3·8}$$

where the Q_{kj} are differential operators with smooth coefficients (depending only on θ) on $S^{n-1}\times\mathbb{R}$. For given $\tilde{v}_0=\tilde{v}|_{\rho=0}=\psi$, these equations are a set of transport equations; they can be solved by putting

$$\tilde{v}_0=4,\quad \tilde{v}_{k+1}=-\frac{1}{2}\sum_{j=0}^{k}\int_\tau^\infty Q_{kj}\tilde{v}_j(\,.\,,\tau')\,d\tau',\quad k=0,1,\ldots.\tag{3·9}$$

Note that all the $\tilde{v}_k$ are C^∞ and vanish for $\tau>\tau_1$. The formal power series

$$\sum_{k=0}^{\infty}\tilde{v}_k(\theta,\tau)\,\rho^k$$

then satisfies (3·7). By the Borel theorem, one can replace it by a function $\tilde{V}(\rho,\theta,\tau)$ such that

$$\begin{cases}\tilde{V}\in C^\infty((0,\infty)\times S^{n-1}\times\mathbb{R}),\quad \tilde{V}=0\quad\text{for}\quad\tau>\tau_1,\\ \partial^k\tilde{v}/\partial\rho^k|_{\rho=0}=\tilde{v}_k,\quad k=0,1,\ldots,\quad P\tilde{v}\text{ vanishes to all orders as }\quad\rho\downarrow 0.\end{cases}\tag{3·10}$$

Now $\Sigma_0\cup\Sigma_1$ clearly has an open neighbourhood Ω_0 in Ω with the following property: if $(\rho_0,\theta_0,\tau_0)\in\Omega_0$, then the future dependence domain of (ρ_0,θ_0,τ_0) meets Ω in a relatively compact set. (Every point of $\Sigma_0\cup\Sigma_1$ has such a neighbourhood, and the global statement follows by a covering argument.) It follows that the boundary value problem

$$P\tilde{W}=-P\tilde{V}\quad\text{in }\Omega_0,\quad \tilde{W}=0\quad\text{on }\Sigma_0\cup\Sigma_1$$

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has a unique solution in Ω_0 that is smooth up to $\Sigma_0 \cup \Sigma_1$ and vanishes there to all orders. The function

$$\tilde{v} = \tilde{V} + \tilde{W}$$

then satisfies (3·7) in Ω_0, by (3·10); uniqueness is clear, by an energy estimate.

Clearly, Σ_1 corresponds to the hypersurface (3·3) under the map $(x,t) \mapsto (\rho, \theta, \tau)$. So the hypersurface $\Sigma_{T_1}: t = \tau + \rho^{-1} = T_1$ meets Σ_1 if $T_1 > \tau_1 + 2r_0$; as one has energy estimates for $\tilde{W}$ (or for $\tilde{v}$) in any set bounded by $\Sigma_0, \Sigma_1, \Sigma_{T_1}$ and an auxiliary hypersurface $\tau - r_0^2\rho = \tau_0 < \tau_1$, $\Sigma_T \cap \Omega$ must be Ω_0. Hence $\tilde{v}$ is well defined in

$$\{(\rho, \theta, \tau): 0 \leqslant \rho \leqslant 1/(T-\tau),\ \theta \in S^{n-1},\ \tau - r_0^2\rho \leqslant \tau_1\}.$$

Under the map $(\rho, \theta, \tau) \mapsto (x,t)$, the restriction of $\tilde{v}$ to this set pulls back to a C^∞ solution $\tilde{u}$ of $\Box\tilde{u} = 0$ defined on

$$\{(x,t): t \geqslant \tau_1, |x| \geqslant s(t)\},$$

which vanishes to all orders on $|x| = s(t)$, $t > T_1$. (In fact, it vanishes for $t > T_1$ when $|x| - s(t)$ is sufficiently small.) We now put

$$(3\cdot11) \qquad \begin{cases} u = \tilde{u} & \text{if} \quad |x| > s(t), \quad t > T_1, \\ = 0 & \text{if} \quad |x| \leqslant s(t), \quad t > T_2, \end{cases}$$

to obtain a C^∞ solution of $\Box u = 0$ in $\mathbb{R}^n \times (T_1, \infty)$. This can be continued backwards, for all t, by solving a Cauchy problem, taking (u, u_t) as data on any hyperplane $t = T > T_1$. Clearly, u has all the required properties, and the Lemma is proved.

One can now establish the existence of receding waves with finite energy.

(3·12) LEMMA. *Let*

$$(3\cdot13) \qquad \Phi = \{\phi \in C^\infty(S^{n-1} \times \mathbb{R}) \cap L_2(S^{n-1} \times \mathbb{R}): \textit{there is a } \tau_1 \textit{ such that } \phi = 0 \textit{ for } \tau > \tau_1\}.$$

For $\phi \in \Phi$, *put*

$$(3\cdot14) \qquad \psi(\theta, \tau) = -\int_\tau^\infty \phi(\theta, \tau')\, d\tau',$$

so that $\psi = 0$ *for* $\tau > \tau_1$, *and* $\partial\psi/\partial\tau = \phi$. *Let* $u(x,t)$ *be the receding wave of Lemma* (3·5) *with radiation field* ψ, *and put* $f = (u(x,0), u_t(x,0))$. *Then*

$$(3\cdot15) \qquad \|\mathscr{R}f\|_{L_2(S^{n-1}\times\mathbb{R})} = \|f\|_{H_E}.$$

Proof. By (2·2), one has

$$(3\cdot16) \qquad \frac{1}{2}\frac{\partial}{\partial t}\left(h^{\frac{1}{2}}\left(\sum_{i,j=1}^n h^{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j} + \left(\frac{\partial u}{\partial t}\right)^2\right)\right) - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(h^{\frac{1}{2}}h^{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\right) = 0.$$

Let ρ and T be positive numbers, $\rho > r_0$, $T > T_1$. We shall integrate (3·16) over the (open) set $\Omega_{\rho,T}$ bounded by $t = 0$, $t = T$ and a hypersurface Γ_ρ given by

$$t = r + \frac{\rho}{r} - 2\rho, \quad r \geqslant \rho.$$

This is a bounded set; in fact

$$(3\cdot17) \qquad \Omega_{\rho,T} = \{(x,t): 0 < t < \tau,\ |x| < \sigma(t) = \rho + \tfrac{1}{2}t + (\tfrac{1}{4}t^2 + \rho t)^{\frac{1}{2}}\}.$$

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With the notation

$$E_{a,b} = \frac{1}{2}\int_{|X|<a}\left(\sum_{i,j=1}^{n} h^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \left(\frac{\partial u}{\partial t}\right)^2\right)\Big|_{t=0} h^{\frac{1}{2}}\,dx \tag{3·18}$$

one then obtains, by the divergence theorem,

$$E_{\sigma,T} = E_{\rho,0} + E' \tag{3·18}$$

where now $\sigma = \sigma(T)$, and E' is an integral over $\Gamma_\rho \cap \bar{\Omega}_{\rho,T}$. As $\rho > r_0$, it follows from Lemma 1·22 that Γ_ρ is space-like. Hence the integrand of E' is a positive definite quadratic form in $\mathrm{grad}_{(x,t)} u$. (This well-known fact can be proved by a simple manipulation, or an appeal to Lemma A 2.). Hence

$$E_{\rho,0} \leqslant E_{\sigma,T}. \tag{3·19}$$

In the integral $E_{\sigma,T}$ one has, by (3·4) and (3·17),

$$\tfrac{1}{2}(T-\tau_1) + (\tfrac{1}{4}(T-\tau_1)^2 - r_0^2)^{\frac{1}{2}} < r < \rho + \tfrac{1}{2}T + (\rho T + \tfrac{1}{4}T^2)^{\frac{1}{2}},$$

whence

$$-2\rho + O\left(\frac{1}{T}\right) < T - r < \tau_1 + O\left(\frac{1}{T}\right)$$

as $T \to \infty$. By Lemma 3·5, one can thus replace u by v in the integrand, and compute $\lim_{T\to\infty} E_{\sigma,T}$ exactly like $\lim_{t\to\infty} E'_{\lambda,t}$ in the proof of Lemma 2·6. By (3·14) one then obtains (as $\phi = 0$ for $\tau > \tau_1$)

$$E_{\rho,0} \leqslant \lim_{T\to\infty} E_{\sigma,T} = \int_{-2\rho}^{\infty} d\tau \int_{S^{n-1}} \phi^2\, d\omega(\theta).$$

Letting $\rho \to 0$ gives

$$\|f\|^2_{H_E} \leqslant \|\phi\|^2_{L_2(S^{n-1}\times\mathbb{R})}, \tag{3·20}$$

with f as in the Lemma. Hence $f \in H_E$; also it is clear from Proposition 2·32 and Lemma 3·5 that $\mathscr{R}f = \phi$. So the assertion (3·15) follows from (3·20) and (3·1), and the Lemma is proved.

Let us denote the subset of H_E on which $\mathscr{R}$ is an isometry by H_E^1,

$$H_E^1 = \{f \in H_E : \|\mathscr{R}f\|_{L_2(S^{n-1}\times\mathbb{R})} = \|f\|_{H_E}\}. \tag{3·21}$$

(3·22) LEMMA. *If $f \in H_E$ and $g \in H_E^1$, then*

$$(f,g)_{H_E} = (\mathscr{R}f, \mathscr{R}g)_{L_2(S^{n-1}\times\mathbb{R})}. \tag{3·23}$$

Proof. If $\lambda \in \mathbb{R}$ then $f + \lambda g \in H_E$, so

$$\|f+\lambda g\|^2_{H_E} \geqslant \|\mathscr{R}f + \lambda\mathscr{R}g\|^2_{L_2(S^{n-1}\times\mathbb{R})}$$

or, as $g \in H_E^1$ by hypothesis,

$$\|f\|^2_{H_E} - \|\mathscr{R}f\|^2_{L_2(S^{n-1}\times\mathbb{R})} + 2\lambda((f,g)_{H_E} - (\mathscr{R}f, \mathscr{R}g)_{L_2(S^{n-1}\times\mathbb{R})}) \geqslant 0.$$

As this must hold for all $\lambda \in \mathbb{R}$, the coefficient of 2λ must vanish, whence (3·23).

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In particular, if both f and g are in H_E^1, then it is clear from (3·21) and (3·23) that $f+g \in H_E^1$, as then

$$\|\mathscr{R}f+\mathscr{R}g\|^2_{L_2(S^{n-1}\times\mathbb{R})} = \|f\|^2_{H_E} + 2(f,g)_{H_E} + \|g\|^2_{H_E}$$
$$= \|f+g\|^2_{H_E}.$$

It is therefore obvious that H_E^1 is a subspace, and it is immediate from the definition that H_E^1 is closed. So we have proved

(3·24) PROPOSITION. *H_E^1 is a closed subspace of H_E.*

It is now easy to establish the principal result of this section.

(3·25) THEOREM. (i) *The restriction of $\mathscr{R}$ to H_E^1 is a surjective isometry*

$$H_E^1 \to L_2(S^{n-1}\times\mathbb{R}).$$

(ii) *H_E is the direct orthogonal sum of H_E^1 and the nullspace of $\mathscr{R}$,*

$$H_E^0 = \{f \in H_E : \mathscr{R}f = 0\}. \tag{3·26}$$

Proof. $\mathscr{R}H_E^1$ is a closed subspace of $L_2(S^{n-1}\times\mathbb{R})$, as the map $\mathscr{R}$ is an isometry when restricted to H_E^1. By Lemma 3·12, the set Φ defined by (3·13) is a subset of $\mathscr{R}H_E^1$. But Φ is clearly dense in $L_2(S^{n-1}\times\mathbb{R})$, and so (i) follows.

To prove (ii) we note first that, if $f \in H_E^0$ and $g \in H_E^1$, then $(f,g)_{H_E} = 0$ by (3·23) and (3·26). Hence

$$H_E^0 \subset (H_E^1)^\perp.$$

On the other hand, if $f \in (H_E^1)^\perp$, then it follows from (3·23) that $\mathscr{R}f$ is orthogonal to $\mathscr{R}H_E^1$; by (i), which we have just proved, this implies that $\mathscr{R}f = 0$, so $f \in H_E^0$. Hence also

$$(H_E^1)^\perp \in H_E^0.$$

So $(H_E^1)^\perp = H_E^0$, and (ii) is proved.

(3·27) *Remark.* By Lemma 3·12, there is a map $\mathscr{I}: \Phi \to H_E$ such that $\mathscr{R}\circ\mathscr{I}$ is the identity on Φ. (This can be extended by continuity to all of $L_2(S^{n-1}\times\mathbb{R})$ and is in fact the projection $H_E \to H_E^1$.) Furthermore, Lemma 3·12 asserts that $F = \mathscr{I}\Phi \subset H_E^1$. Repeating the proof of part (ii) of the theorem, one sees that the orthogonal complement of the closure $\bar{F}$ of F in H_E is also H_E^0. Hence $\bar{F} = H_E^1$.

If $f \in F$ then $U(t)f$ is a receding wave, by construction, so the data f are eventually outgoing. In fact, all eventually outgoing data are in H_E^1. To prove this, we first extend the identity (2·17), which has been established only for C_0^∞ data, to H_E.

Put, if $f \in H_E$ and $a > 0$,

$$\|f\|^0_{H_E} = \left(\frac{1}{2}\int_{|X|<a}\left(\sum_{i,j=1}^n h^{ij}\frac{\partial f_0}{\partial x_i}\frac{\partial f_0}{\partial x_j} + f_1^2\right)h^{\frac{1}{2}}\,dx\right)^{\frac{1}{2}}. \tag{3·28}$$

As this is a semi-norm on H_E, one has

$$|\|f\|^a_{H_E} - \|g\|^a_{H_E}| \leqslant \|f-g\|^a_{H_E}. \tag{3·29}$$

(3·30) LEMMA. *If $f \in H_E$, $\lambda \in \mathbb{R}$ and $t > \lambda$, then*

$$\lim_{t\to\infty}\|U(t)f\|^{t-\lambda}_{H_E} = \left(\|f\|^2_{H_E} - \int_{-\infty}^{\lambda} d\tau \int (\mathscr{R}f)^2\,d\omega(\theta)\right)^{\frac{1}{2}}. \tag{3·31}$$

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Proof. Let $f^{(j)}$, $j = 1, 2, \ldots$ be a sequence of C_0^∞ data converging to f in H_E as $j \to \infty$. It follows from (3·29) that, for any positive integers j and k,

$$\begin{aligned}|\|U(t)f^{(j)}\|_{H_E}^{t-\lambda} - \|U(t)f^{(k)}\|_{H_E}^{t-\lambda}| &\leqslant \|U(t)f^{(j)} - U(t)^{(k)}\|_{H_E}^{t-\lambda} \\ &\leqslant \|U(t)f^{(j)} - U(t)f^{(k)}\|_{H_E} = \|f^{(j)} - f^{(k)}\|_{H_E}.\end{aligned}$$

Hence $\|U(t)f^{(j)}\|_{H_E}^{\lambda-t}$ converge as $j \to \infty$, and

$$\lim_{j\to\infty} \|U(t)f^{(j)}\|_{H_E}^{\lambda-t} = \|U(t)f\|_{H_E}^{\lambda-t}. \tag{3·32}$$

Also

$$|\|U(t)f\|_{H_E}^{t-\lambda} - \|U(t)f^{(j)}\|_{H_E}^{t-\lambda}| \leqslant \|f - f^{(j)}\|_{H_E}. \tag{3·33}$$

Now by Lemma 2·6,

$$\lim_{t\to\infty} \|U(t)f^{(j)}\|_{H_E}^{t-\lambda} = \left(\|f^{(j)}\|_{H_E}^2 - \int_{-\infty}^{\lambda} d\tau \int (\mathscr{R}f^{(j)})^2 d\omega(\theta)\right)^{\frac{1}{2}}$$

whence

$$\lim_{j\to\infty} \lim_{t\to\infty} \|U(t)f^{(j)}\|_{H_E}^{t-\lambda} = \left(\|f\|_{H_E}^2 - \int_{-\infty}^{\lambda} d\tau \int (\mathscr{R}f)^2 d\omega(\theta)\right)^{\frac{1}{2}}.$$

The order of the limits can be reversed, as it is clear from (3·33) that convergence in (3·32) is uniform in t, and so (3·31) follows.

(3·34) PROPOSITION. *H_E^1 is the closure of the subspace of H_E consisting of eventually outgoing data.*

Proof. By definition, data $f \in H_E$ are eventually outgoing if $u(x,t) = (U(t)f)_0$ is a receding wave, so that (3·4) holds: there is a τ_1 and a $T_1 > \tau_1 + 2r_0$ such that

$$u = 0 \quad \text{if} \quad t > T_1 \quad \text{and} \quad |x| < s(t) = \tfrac{1}{2}(t-\tau_1) + (\tfrac{1}{4}(t-\tau_1)^2 - r^2)^{\frac{1}{2}}.$$

(Here r_0 is such that the hypersurfaces $|x(t)| - s(t)$, $t > \tau_1 + 2r_0$, are space-like for any τ_1.) Now if $\lambda > \tau_1$, then

$$s(t) - (t-\lambda) = (\tfrac{1}{4}(t-\tau_1)^2 - r_0^2)^{\frac{1}{2}} - \tfrac{1}{2}(t-\tau_1) + (\lambda - \tau_1)$$

is positive if

$$\tfrac{1}{4}(t-\tau_1)^2 - r_0^2 > \tfrac{1}{4}(t-\tau_1)^2 + (\lambda-\tau_1)^2 - (\lambda-\tau_1)(t-\tau_1),$$

that is to say if

$$t > \lambda + \frac{r_0^2}{\lambda - \tau_1}.$$

Then

$$\|U(t)\|_{H_E}^{t-\lambda} = 0 \quad \text{if} \quad t > \max(T_1, \lambda + (\lambda-\tau_1)^{-1} r_0^2),$$

and it follows from (3·31) that $f \in H_E^1$. So all eventually outgoing data are in H_E^1. But, by Remark 3·27, H_E^1 contains a dense subset of eventually outgoing data, and so the Proposition is proved.

(3·34) *Remark.* It also follows from Lemma 3·30 that, for data f in H_E^1, all energy is eventually outgoing, in the sense that

$$\lim_{\lambda\to\infty} \lim_{t\to\infty} \|U(t)f\|_{H_E}^{t-\lambda} = 0.$$

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Clearly, this implies the local energy decay property for such data,

$$\lim_{t\to\infty}\|U(t)f\|_{H_E^a}=0 \quad \text{if} \quad f\in H_E^1 \quad \text{and} \quad a\in\mathbb{R}^+. \tag{3·35}$$

So far, we have considered what may be called the forward radiation fields of solutions of $\Box u = 0$. It is a simple matter to extend the theory to 'backward' radiation fields, by virtue of the following lemma:

(3·36) LEMMA. *Define* $\sigma: H_E \to H_E$ *by*

$$\sigma(f_0,f_1)=(f_0,-f_1). \tag{3·37}$$

Then

$$U(-t)f=\sigma U(t)\sigma f. \tag{3·38}$$

Proof. It is clearly sufficient to prove this for C_0^∞ data, as these are dense in H_E. If one splits such data f up as

$$f=f^{(0)}+f^{(1)}, \quad f^{(1)}=(f_0,0), \quad f^{(1)}=(0,f_1),$$

then $$u(x,t)=u^{(0)}(x,t)+u^{(1)}(x,t), \quad u^{(j)}=(U(t)f^{(i)})_0, \quad j=0 \text{ or } 1.$$

By the uniqueness of the solution of an initial-value problem, $u^{(0)}$ is even in t and $u^{(1)}$ is odd in t. Hence

$$u(r,-t)=u^{(0)}(x,t)-u^{(1)}(x,t),$$
$$-u_t(x,-t)=u_t^{(0)}(r,t)-u_t^{(1)}(r,t),$$

which is (3·38), written out in full.

For C_0^∞ data we can now set, with $u(x,t)=(U(t)f)_0$ as usual,

$$\overline{\mathscr{R}}f(\theta,\tau)=\lim_{r\to\infty}r^{(n-1)/2}u_t(r\theta,-r-\tau). \tag{3·39}$$

The existence of this limit follows from Theorem 1·6, with $-\tau$ instead of τ, applied to $u(x,-t)$. By (3·38) one then has

$$\overline{\mathscr{R}}f(0,\tau)=\mathscr{R}\sigma f(\theta,\tau). \tag{3·40}$$

So all properties of $\mathscr{R}$ can be transferred, *mutatis mutandis*, to $\overline{\mathscr{R}}$.

It is clear that the case where $\mathscr{R}$ is an isometry is of particular interest. By combining Theorem 3·25 and Proportion 3·34, one obtains:

(3·41) PROPOSITION. *The following are equivalent:*

(i) $\mathscr{R}: H_E \to (S^{n-1}\times\mathbb{R})$ *is a bijective isometry;*

(ii) *the kernel* H_E^0 *of* $\mathscr{R}$ *is trivial;*

(iii) *eventually outgoing data are dense in* H_E*;*

(iv) $\overline{\mathscr{R}}: H_E \to L_2(S^{n-1}\times\mathscr{R})$ *is a bijective isometry.*

Proof. $\mathscr{R}$ is an isometry if and only if $H_E = H_E^1$. By Theorem 3·25, this is equivalent to $H_E^0 = 0$, and $\mathscr{R}$ is then bijective. Proposition 3·34 shows that (iii) implies $H_E^1 = H_E$, and that (i) implies (iii). Finally, σ is a bijection, and so it follows from (3·40) that the kernel of $\overline{\mathscr{R}}$ is trivial if and only if $H_E^0 = 0$; the equivalence of (iv) and (i) is then also immediate, by the time-reversed version of Theorem 3·25.

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The map
$$f \mapsto \check{\mathscr{R}} f(\theta, \tau) = \mathscr{R} f(0, -\tau)$$
is also a bijection $H_E \to L_2(S^{n-1} \times \mathbb{R})$ if one of the equivalent conditions of the Proposition obtains. By (2·31) it is a translation representation of the unitary group $\{(U(t)\}$. As $\overline{\mathscr{R}}$ has been defined by (3·39), it is thus clear from (3·40) that $\overline{\mathscr{R}}$ is also a translation representation of $\{U(t)\}$. In this case one can therefore define a Lax–Phillips scattering operator
$$S\colon L_2(S^{n-1} \times \mathbb{R}) \to L_2(S^{n-1} \times \mathbb{R})$$
by setting
$$S\phi = \check{\mathscr{R}} \circ \overline{\mathscr{R}}^{-1} \phi = \check{\mathscr{R}} \circ \sigma \circ \mathscr{R}^{-1} \phi, \quad \phi \in L_2(S^{n-1} \times \mathbb{R}). \tag{3·42}$$
The (partial) Fourier transform of $S\phi$ then has a distribution kernel which is the scattering matrix for the perturbed-wave equations discussed in this paper.

As before (see Remark 2·45), the arguments in this section remain valid for the initial-boundary-value problem (2·47). This includes the scattering problem for the standard wave equation considered by Lax and Phillips. Condition (iii) of Proposition 3.41 is closely related to the outgoing subspace condition, and is proved for n odd in (6) and for even n in (7). In each case, it is deduced from local energy decay. For perturbed wave equations such that the perturbation has compact support (so that there is an $a > 0$ such that $h_{ij}(x) = \delta_{ij}$, $i, j = 1, \ldots, n$, if $|x| > a$), the local energy decay property follows in effect from the work of Iwasaki (5); the requisite unique continuation property can be established by following the proof of ((10), 280, lemma 2). So in this case $\mathscr{R}$ is an isometry. It is an open question whether there are in fact any perturbed wave equations satisfying (1·4) for which $\mathscr{R}$ is not an isometry, that is to say for which $H_E^0 \neq 0$; it may well be that this is not the case.

Section 4

We have constructed maps $\mathscr{R}\colon H_E \to L_2(S^{n-1} \times \mathbb{R})$ and $\mathscr{I}\colon L_2(S^{n-1} \times \mathbb{R}) \to H_E$ such that $\mathscr{R} \circ \mathscr{I}$ is the identity map on $L_2(S^{n-1} \times \mathscr{R})$ and $\mathscr{I} \circ \mathscr{R}$ is the projection $H_E \to H_E^1$. (For the definition of $\mathscr{I}$, see Remark 3·27.) We shall now show that each of these maps is generated by a distribution kernel, and then use Lemma 3·22 to express both of these kernels in terms of the radiation field of the forward fundamental solution of the differential operator $\Box$. For simplicity, we shall work on $\mathbb{R}^n \times \mathbb{R}$, but the arguments go through without change for Dirichlet or Neumann problems on exterior domains.

Let M_0 denote the Riemannian manifold which results when $\mathbb{R}^n$ is equipped with the metric
$$\sum_{i,j=1}^{n} h_{ij}(x)\, dx_i dx_j,$$
which is again assumed to satisfy (1.4). The space of distributions $\mathscr{D}'(M_0)$ can be defined as the dual of the space of C_0^∞ densities on M_0. However, the invariant volume element $dm(x) = h^{\frac{1}{2}}(x)\, dx$, where dx is Lebesgue measure, is a canonical density on M_0, so that one can treat $\mathscr{D}'(M_0)$ as the dual of $C_0^\infty(M_0)$, with the understanding that any $\phi \in C_0^\infty(M_0)$ is to be identified with the density $\phi\, dm$. Similarly, we shall view $\mathscr{D}'(S^{n-1} \times \mathbb{R})$ as the dual of $C_0^\infty(S^{n-1} \times \mathbb{R})$, by identifying $\psi \in C_0^\infty(S^{n-1} \times \mathbb{R})$ with the density $\psi\, d\omega(\theta)\, d\tau$.

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Consider the initial-value problem

$$\Box u = 0 \quad \text{on} \quad M_0 \times \mathbb{R}, \quad u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \phi(x) \in C_0^\infty(M_0). \tag{4·1}$$

By Theorem 1·6, the solution $u(x,t)$ of this problem has a smooth forward radiation field

$$\phi^\#(\theta,\tau) = \lim_{r\to\infty} (r^{(n-1)/2} u(r\theta, r+\tau)) \in C^\infty(S^{n-1}\times\mathbb{R}). \tag{4·2}$$

We first show that the map $\phi \mapsto \phi^\#$ is generated by a distribution kernel. This will follow from

(4·3) LEMMA. *The map $\phi \mapsto \phi^\#$ is a continuous map $C_0^\infty(M_0) \to C_0^\infty(S^{n-1}\times\mathbb{R})$.*

Proof. The topology of $C^\infty(S^{n-1}\times\mathbb{R})$ can be generated by the semi-norms

$$\sup\left\{\left|\left(\frac{\partial}{\partial X}\right)^\alpha \left(\frac{\partial}{\partial\tau}\right)^k \psi(\theta,\tau)\right| : \theta(X)\in\Omega, \tau\in J\right\}, \tag{4·4}$$

where $\Omega \subset S^{n-1}$ is a compact coordinate neighbourhood with local coordinates $X \in \mathbb{R}^{n-1}$, $J \subset \mathbb{R}$ a bounded interval, α an $(n-1)$-multi-index, and k a non-negative integer. To prove the lemma it is sufficient to show that (4·4), with $\psi = \phi^\#$, can be estimated in terms of the usual semi-norms $\sup|(\partial/\partial x)^\beta \phi|$ when $\phi \in C_0^\infty(K)$, where K is a compact subset of $\mathbb{R}^n$.

It is clear from Lemma 1·22 that, if $\operatorname{supp}\phi \subset K$, then there is a $\tau_0 = \tau_0(K)$ such that $\phi^\#(\theta,\tau) = 0$ if $\tau < \tau_0$. So one can replace $J = (\tau_1,\tau_2)$, say, by $J_0 = (\tau_0,\tau_2)$, which is non-empty only if $\tau_2 > \tau_0$. The proof of Theorem 1·6 shows that a semi-norm (4·4) of $\phi^\#$ can then be estimated in terms of the L_∞ norms of

$$V(\rho,\theta,\tau) = r^{(n-1)/2} u(r\theta, r+\tau), \quad r = \rho^{-1},$$

and its derivatives up to some (finite) order, restricted to

$$\{\rho,\theta,\tau) = \{r_0^{-1}\} \times S^{n-1} \times (\tau_0, \tau_2 + r_0).$$

These can, in turn, be expressed in terms of u and its derivatives. But these can be estimated in terms of $C_0^\infty(K)$ semi-norms of ϕ by standard energy estimates and the Sobolev inequalities, and so the Lemma follows.

It follows from Lemma 4·3 that the map $\phi \mapsto \phi^\#$ is generated by a Schwartz kernel $G(\theta,\tau,x) \in \mathscr{D}'(S^{n-1}\times\mathbb{R}\times M_0)$, so that

$$\int \phi^\#(\theta,\tau)\,\psi(\theta,\tau)\,d\omega(\theta)\,d\tau = \langle G, \psi\otimes\phi\rangle, \quad \phi\in C_0^\infty(M_0), \ \psi\in C_0^\infty(S^{n-1}\times\mathbb{R}). \tag{4·5}$$

Now if $f = (f_0, f_1) \in C_0^\infty(M_0)\times C_0^\infty(M_0)$ then it is clear from the definition of $\phi^\#$ and Duhamel's principle that

$$\mathscr{R}f = \frac{\partial^2 f_0^\#}{\partial\tau^2} + \frac{\partial f_1^\#}{\partial\tau}.$$

By (4·5) we therefore have

(4·6) PROPOSITION. *Let $G \in \mathscr{D}'(S^{n-1}\times\mathbb{R}\times M_0)$ be the Schwartz kernel of the map $\phi\mapsto\phi^\#$ and let $f\in C_0^\infty(M_0)\times C_0^\infty(M_0)$. Then*

$$\mathscr{R}f = G_{\tau\tau} f_0 + G_\tau f_1 \tag{4·7}$$

where $G_\tau = (\partial/\partial\tau)\,G$ and $G_{\tau\tau} = (\partial/\partial\tau)^2 G$.

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Let us now consider the map $\mathscr{I}$. If $\psi \in C_0(S^{n-1} \times \mathbb{R})$ then $\psi \in \Phi$ where Φ is defined by (3·13). By Lemma 3·12 there is then a unique receding wave $u(x,t)$ whose time derivative $u_t(x,t)$ has radiation field ψ, and $\mathscr{I}\psi$ is by definition $f = (u(x,0),\ u_t(x,0))$. Moreover, (3·15) can be written as

$$\|\psi\|_{L_2(S^{n-1}\times\mathbb{R})} = \|\mathscr{I}\psi\|_{H_E}.$$

This implies that $\mathscr{I}$ is a continuous map $C_0^\infty(S^{n-1} \times \mathbb{R}) \to \mathscr{D}'(M_0) \times \mathscr{D}'(M_0)$ and so is generated by a distribution kernel which of course consists of a pair of distributions $(H_0, H_1) \in \mathscr{D}'(M_0 \times S^{n-1} \times \mathbb{R}) \times \mathscr{D}'(M_0 \times S^{n-1} \times \mathbb{R})$. As the range of $\mathscr{I}$ is C^∞, it follows from the closed-graph theorem that $\mathscr{I}$ is in fact a continuous map

$$C_0^\infty(S^{n-1} \times \mathbb{R}) \to C^\infty(M_0) \times C^\infty(M_0).$$

(This can also be proved directly by means of energy estimates and Sobolev inequalities.) We have therefore proved

(4·8) Lemma. *The map $\mathscr{I}$, restricted to $C_0^\infty(S^{n-1} \times \mathbb{R})$, is generated by a distribution kernel $H = (H_0, H_1) \in \mathscr{D}'(M_0 \times S^{n-1} \times \mathbb{R}) \times \mathscr{D}'(M_0 \times S^{n-1} \times \mathbb{R})$. This kernel is regular to the right, that is to say $\mathscr{S}$ is a continuous map $C_0^\infty(S^{n-1} \times \mathbb{R}) \to C^\infty(M_0) \times C^\infty(M_0)$.*

One can now use Lemma 3·22 to express H in terms of the kernel G of the map $\phi \mapsto \phi^\#$, or rather in terms of the adjoint map ${}^tG\colon C_0^\infty(S^{n-1} \times \mathbb{R}) \to \mathscr{D}'(M_0)$ which is defined by

$$\langle {}^tG\psi, \phi\rangle = \langle G, \psi \otimes \phi\rangle, \quad \psi \in C_0^\infty(S^{n-1} \times \mathbb{R}), \quad \phi \in C_0^\infty(M_0). \tag{4·9}$$

(4·10) Proposition. *With G as in Proposition 4·6, one has*

$$\mathscr{I}\psi = (-{}^tG\psi, {}^tG_\tau\psi), \quad \psi \in C_0^\infty(S^{n-1} \times \mathbb{R}). \tag{4.11}$$

Proof. Let us write $\mathscr{I}\psi = f$. By Lemma 3·12, one has $f \in H_E^1$, so also $U(t)f \in H_E^1$ for all t. Put

$$U(t)f = U(t)\mathscr{I}\psi = (u(x,t),\ u_t(x,t)). \tag{4·12}$$

By Lemma 3·22, one then has

$$(U(t)f, (0,\phi))_{H_E} = (\mathscr{R}U(t)f, \mathscr{R}(0,\phi))_{L_2(S^{n-1}\times\mathbb{R})};$$

for all $\phi \in C_0^\infty(M_0)$. As

$$\mathscr{R}U(t)f = \mathscr{R}U(t)\mathscr{I}\psi = \psi(\theta, \tau + t), \quad \mathscr{R}(0,\phi) = \phi_\tau^\#(\theta,\tau) \quad \text{where} \quad \phi_\tau^\# = \partial\phi^\#/\partial\tau,$$

one concludes that

$$\begin{aligned}\int u_t(x,t)\,\phi(x)\,dm(x) &= \int \psi(\theta,\tau+t)\,\phi_\tau^\#(\theta,\tau)\,d\omega(\theta)\,d\tau \\ &= -\int \psi_\tau(\theta,\tau+t)\,\phi^\#(\theta,\tau)\,d\omega/(\theta)\,d\tau.\end{aligned} \tag{4·13}$$

As u is a receding wave, and ψ has compact support, one can integrate this over (t,∞) and interchange the order of integration by Fubini's theorem to obtain

$$\int u(x,t)\,\phi(x)\,dm(x) = -\int \phi^\#(\theta,\tau)\,\psi(\theta,\tau+t)\,d\omega(\theta)\,d\tau.$$

By (4·5), this can be written as

$$\langle u(.,t), \phi(.)\rangle = -\langle G, T_{-t}\psi \times \phi\rangle$$

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where $T_{-t}\psi(\theta,\tau) = \psi(\theta,\tau+t)$. So, by (4·9),

$$u(.,t) = -{}^tGT_{-t}\psi, \tag{4·14}$$

whence (or directly from (4·13))

$$u_t(.,t) = {}^tG_\tau T_{-t}\psi. \tag{4·15}$$

As $\mathscr{I}\psi = (u(.,0), u_t(.,0))$, the equations (4·14) and (4·15) reduce to (4·11) when $t = 0$, and the Proposition is proved.

It follows from ${}^tG\psi = -(\mathscr{I}\psi)_0$ and Lemma 4·8 that $\psi \mapsto {}^tG\psi$ is a map

$$C_0^\infty(S^{n-1}\times\mathbb{R}) \to C^\infty(M_0).$$

In view of Lemma 4·3, G is therefore a regular distribution kernel, and we have

(4·16) COROLLARY. *The map $\phi \mapsto \phi^\# = G\phi$ extends by continuity to a map*

$$\mathscr{E}'(M_0) \to \mathscr{D}'(S^{n-1}\times\mathbb{R}).$$

One can thus define the radiation field of the distribution $u\in\mathscr{D}'(M)$, where $M = M_0\times\mathbb{R}$, which is the solution of the generalized initial-value problem

$$\Box u = f(x)\otimes\delta(t),\ \operatorname{supp} u \subset M_0\times\{t\geqslant 0\}, \tag{4·17}$$

where $\delta(t)$ is Dirac measure on $\mathbb{R}$, provided that $f\in\mathscr{E}'(M_0)$. Let δ_y be the Dirac distribution on M_0,

$$\langle\delta_y(x), \phi(x)\rangle = \phi(y), \quad \phi\in C_0^\infty(M_0). \tag{4·18}$$

Note that this is the map generated by the distribution kernel $\delta(x,y)\in\mathscr{D}'(M_0\times M_0)$ that is defined by

$$\langle\delta,\chi\rangle = \int\chi(x,x)\,dm(x), \quad \chi\in C_0^\infty(M_0\times M_0).$$

Let us denote the solution of (4·17) by E_y when $f = \delta_y$, so that

$$\Box_{(x,t)}E_y(x,t) = \delta_y(x)\otimes\delta(t), \quad \operatorname{supp} E_y \subset M_0\times\{t\geqslant 0\}. \tag{4·19}$$

This distribution E_y is just the forward fundamental solution of the differential operator $\Box$ with 'pole' $(y,0)$. It is well known that, corresponding to E_y, one has a distribution kernel $E(x,t,y)\in\mathscr{D}'(M\times M_0)$ such that

$$\langle E, \chi\otimes\phi\rangle = \int\langle E_y(x,t), \chi(x,t)\rangle\,\phi(y)\,dm(y), \quad \chi\in C_0^\infty(M), \quad \phi\in C_0^\infty(M_0),$$

and that the solution of the initial-value problem (4·1) is just $u = E\phi$. So it is now clear that G is in effect the radiation field of the forward fundamental solution E_y.

When one of the equivalent conditions of Proposition 3·14 holds, so that $\mathscr{R}$ is an isometry, then $\mathscr{I}\circ\mathscr{R}: H_E \to H_E$ is the identity. As the map $\mathscr{I}$ extends to a map $L_2(S^{n-1}\times\mathscr{R}) \to H_E$, the equation (4·11) then gives the inverse of $\mathscr{R}$, and (4·14), (4·15) give $U(t)f$ in terms of $\psi = \mathscr{R}f$. In particular, if one takes $f = (0,\phi)$ one has

(4·20) PROPOSITION. *Assume that $\mathscr{R}$ is an isometry and put $G_\tau = K$. For $\phi\in L_2(M_0)$, put $\psi = K\phi$. Then $\phi = {}^tK\psi$.*

For the standard wave equation, this is equivalent to the Radon transform inversion formula.

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Appendix

This appendix is an outline of the proof of the energy estimates used in Section 1. To simplify the notation, we consider an N-dimensional C^∞ manifold M equipped with a Lorentzian metric ds^2 whose signature is $2-N$. We first recall some standard facts and notations.

In local coordinates $x = (x_1, \ldots, x_N)$ we put

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(x)\, dx_i dx_j,$$

and set $\det g_{ij} = g$ so that the invariant measure on M is $dm = |g|^{\frac{1}{2}} dx$. Recall that the metric gives, at every point $p \in M$, a canonical isomorphism $TM_p \to T^*M_p$, which together with its inverse is the classical 'raising and lowering' of (tensor) suffixes. One can thus in effect identify TM_p and T^*M_p, and denote the inner products on TM_p and T^*M_p, as well as the pairing of TM and T^*M_p, by $\langle , \rangle$. In particular, if $S\colon M \to \mathbb{R}$ is a C^1 function, then ∇S will stand both for the gradient of S and for the corresponding vector field whose components in local coordinates are

$$\nabla^i S = \sum_{j=1}^{N} g^{ij}(x) \frac{\partial S}{\partial x_j}, \quad i = 1, \ldots, N,$$

where $(g^{ij}) = (g_{ij})^{-1}$, and

$$\langle \nabla S, \nabla S \rangle = \sum_{i,j=1}^{N} g^{ij} \frac{\partial S}{\partial x_i} \frac{\partial S}{\partial x_j}.$$

The Laplace–Beltrami operator of the metric is called the d'Alembertian of M; it is

$$\Box v = \sum_{i=1}^{N} \nabla^i \nabla_i v = |g|^{-\frac{1}{2}} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(|g|^{\frac{1}{2}} g^{ij} \frac{\partial v}{\partial x_j} \right).$$

Let p be a point of M. A non-zero vector $\xi \in TM_p$ is called time-like if $\langle \xi, \xi \rangle > 0$, space-like if $\langle \xi, \xi \rangle < 0$, and null if $\langle \xi, \xi \rangle = 0$. The (open) set of time-like vectors at p has two connected components, and two vectors ξ, η are in the same component (have the same 'time orientation') if and only if $\langle \xi, \eta \rangle > 0$. This extends in an obvious way to null vectors, and (by the isomorphism) to covectors.

A C^1 hypersurface Σ is called space-like if $T\Sigma$ contains only space-like vectors. It is easy to show that this will be so if and only if the conormal is time-like. So if S is (locally) a defining function for Σ, the condition for Σ to be space-like is

$$\langle \nabla S, \nabla S \rangle > 0 \quad \text{on } \Sigma. \tag{A 1}$$

A null hypersurface is one for which $\langle \nabla S, \nabla S \rangle = 0$; such a hypersurface is a characteristic of the d'Alembertian, or more generally of the differential operator

$$Pv = \Box v + \langle a, \nabla v \rangle + bv,$$

where $b \in C^\infty(M)$ and a is a C^∞ vector field on M. We shall derive energy estimates for P, which imply the estimates (1·21) of Lemma 1·19. We begin with a simple but useful lemma.

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(A 2) LEMMA. *Let p be a point of M, and let ξ and η be time-like vectors at p with the same time orientation. Put, for any $w \in T^*M_p$,*

$$\sigma(\xi, \eta; w) = \langle \xi, w\rangle \langle \eta, w\rangle - \tfrac{1}{2}\langle \xi, \eta\rangle \langle w, w\rangle. \tag{A 3}$$

Then $w \mapsto \sigma(\xi, \eta; w)$ is a positive definite quadratic form on T^*M_p. If η is allowed to be a null vector, then $w \mapsto \sigma$ is positive semidefinite.

Proof. Put

$$\zeta = \langle \xi, \eta\rangle w - \langle \xi, w\rangle \eta$$

where η is viewed as a covector. Then $\langle \xi, \zeta\rangle = 0$. Hence ζ is either space-like, or $\zeta = 0$. (There is a coordinate chart at p such that

$$(g_{ij})_p = \operatorname{diag}(-1, \ldots, -1, 1) \quad \text{and} \quad \xi = (0, \ldots, 0, 1).$$

So $\langle \xi, \zeta\rangle = 0$ gives $\zeta_N = 0$, whence $\langle , \zeta\rangle = -\zeta_1^2 - \ldots - \zeta_{N-1}^2$.) Now

$$\langle \xi, \eta\rangle \sigma(\xi, \eta; w) = \langle \eta, \eta\rangle (\langle \xi, w\rangle)^2 - \langle \zeta, \zeta\rangle,$$

so $\sigma \geqslant 0$ as $\langle \eta, \eta\rangle \geqslant 0$ and $\langle \zeta, \zeta\rangle \leqslant 0$. Hence $w \mapsto \sigma$ is positive semi-definite if ξ is time-like, η is time-like or null, and $\langle \xi, \eta\rangle > 0$. If η is time-like and $\sigma(\xi, \eta; w) = 0$ for some $w \in T^*M_p$, then $\langle \zeta, \zeta\rangle = 0$ and $\langle \xi, w\rangle = 0$. Thus $\zeta = 0$, and the definition shows that therefore $w = 0$, and so σ is positive definite.

(A 4) *Remark.* In local coordinates,

$$\begin{cases} \sigma(\xi, \eta; w) = \sigma^{ij}(\xi, \eta) w_i w_j, \\ \sigma^{ij}(\xi, \eta) = \tfrac{1}{2}(\xi^i \eta^j + \xi^j \eta^i - \langle \xi, \eta\rangle g^{ij}). \end{cases} \tag{A 5}$$

So the Lemma shows that, if ξ and η are time-like and $\langle \xi, \eta\rangle > 0$, then the symmetric matrix $(\sigma^{ij}(\xi, \eta))$ is positive definite. In particular, this holds if $\eta = \xi$. Now one can find a chart at p in which both $(\sigma^{ij}(\xi, \xi))$ and $(\sigma^{ij}(\xi, \eta))$ are diagonal. It follows that, if one sets

$${}^k\sigma^{I,J}(\xi, \eta) = \sigma^{i_1 j_1}(\xi, \xi) \ldots \sigma^{i_{k-1}, j_{k-1}}(\xi, \xi)\, \sigma^{i_K j_K}(\xi, \eta), \tag{A 6}$$

where $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$, and w is a tensor of type $(0, k)$, then

$$w \mapsto {}^k\sigma(\xi, \eta; w) = \sum_{I,J} {}^k\sigma^{I,J}(\xi, \eta) w_I w_J \tag{A 7}$$

is positive definite, provided that ξ and η are time-like and have the same time orientation. This proves Lemma 1·17.

We shall in fact use Lemma A 2 when ξ and η are C_0^∞ vector fields. Assume that this is so, and let $v \in C^\infty(M)$. Then

$$\sigma(\xi, \eta; \nabla v) = E(v)(\xi, \eta), \tag{A 8}$$

where $E(v)$ is a symmetric tensor field, given in local coordinates by

$$E_{ij}(v) = \nabla_i v \nabla_j v - \tfrac{1}{2}\langle \nabla v, \nabla v\rangle g_{ij}, \quad i, j = 1, \ldots, N. \tag{A 9}$$

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Here ∇_i denotes covariant differentiation but, as v is a scalar, one just has $\nabla_i v = \partial v/\partial x_i$ in local coordinates. Another covariant differentiation, and contraction over one suffix, give

$$\sum_{j=1}^{N} \nabla_j E_i^j = \sum_{j=1}^{N} \nabla_i(\nabla_j v\, \nabla^j v - \tfrac{1}{2}\langle \nabla v, \nabla v\rangle \delta_i^j)$$
$$= (\nabla_i v)\, \Box v + \sum_{j=1}^{N} (\nabla_j \nabla_i v - \nabla_i \nabla_j v)\, \nabla^j v.$$

But $\nabla_i \nabla_j v = \nabla_j \nabla_i v$, and hence

$$\sum_{j=1}^{N} \nabla_j E_i^j = (\nabla_i v)\, \Box v, \quad i = 1, \ldots, N. \tag{A 10}$$

Restricting M to an open subset if necessary, we make the following hypothesis:

there exists an $S \in C^\infty(M)$ such that $\langle \nabla S, \nabla S\rangle > 0$ on M. (A 11)

Clearly, this implies that $dS \neq 0$ on M, so that the sets $\{p\colon S(p) = \text{constant}\}$ are hypersurfaces, necessarily space-like; for this reason, a function satisfying (A 11) is called space-like in this paper.

We can now establish the basic energy estimate for the differential operator P defined above.

(A 12) LEMMA. *Let $K \subset M$ be a compact set with non-empty interior, and let $\Omega \subset M$ be an open set with piece-wise smooth boundary $\partial\Omega$ satisfying the following condition: $\partial\Omega = \Sigma_1 \cup \Sigma_2$, where Σ_2 is space-like or null, and ∇S points out of Ω at all points of Σ_2. Then there is a constant $C = C(K)$ such that, for all $v \in C^2(\bar{\Omega})$,*

$$\int_\Omega (\sigma(\nabla S, \nabla S;\, \nabla v) + v^2)\, dm \leqslant C\left(\int_\Omega (Pv)^2\, dm + \int_{\Sigma_1} \phi(v, \nabla v)\right), \tag{A 13}$$

where ϕ is a density on Σ_1 that is quadratic in $(v, \nabla v)$.

Proof. Let t be a positive real number. It follows from (A 10) that

$$t\, e^{-ts}\left(\sum_{i,j=1}^{N} E_{ij} \nabla^i S \nabla^j S + \langle \nabla S, \nabla S\rangle v^2\right)$$
$$+ \sum_{j=1}^{N} \nabla_j \left(e^{-ts}\left(\sum_{i=1}^{N} E_i^j \nabla^i S + v^2 \nabla^i S\right)\right) = e^{-ts}(\langle \nabla S, \nabla v\rangle Pv + Q), \tag{A 14}$$

where Q is a quadratic form in $(v, \nabla v)$. We shall integrate this over Ω and apply the divergence theorem to the second term on the right-hand side. A convenient form of this theorem is as follows. Suppose first that Ω' is a bounded open set with a smooth boundary given by a defining function F such that $F = 0$ on $\partial\Omega'$ and $F < 0$ at points of Ω' near $\partial\Omega'$, and that X is a C^1 vector field on $\bar{\Omega}'$. Then

$$\int_{\Omega'} \sum_{i=1}^{N} \nabla_i X^i\, dm = \int_{\partial\Omega'} \langle \nabla F, X\rangle\, m_F. \tag{A 15}$$

Here m_F is an $(N-1)$-form such that $dF \wedge m_F = dm$, and $\partial\Omega'$ is oriented by $m_F > 0$; the restriction of m_F is unique, and the second member of (A 15) is independent of the choice of defining function F. In general, the second member of (A 15) is a sum of terms

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of this type, one for each smooth portion of $\partial\Omega$; for simplicity we shall retain the notation (A 15). The integral of the second term in the first member of (A 14) thus gives two contributions,

$$\sum_{k=1}^{2}\int_{\Sigma_k} e^{-ts}\Big(\sum_{i,j=1}^{N} E_{ij}\nabla^i S\nabla^j F_k + \langle\nabla S, \nabla F_k\rangle v^2\Big)\, m_{F_k} = \sum_{k=1}^{2}\int_{\Sigma_k} e^{-ts}(\sigma(\nabla S, \nabla F_k;\, \nabla v) + \langle\nabla S, \nabla F_k\rangle v^2)\, m_{F_k},$$

by (A 8). By Lemma A 2, (A 11), and the hypothesis on Σ_2, the integral over Σ_2 is non-negative. So, using (A 8) again, we obtain, from (A 14),

$$t\int_\Omega \mathrm{e}^{-ts}(\sigma(\nabla S, \nabla S;\, \nabla v) + \langle\nabla S, \nabla S\rangle v^2)\, dm \leqslant \int_\Omega e^{-ts}(\langle\nabla S, \nabla v\rangle Pv + \phi)\, dm - \int_{\Sigma_1} e^{-ts}(\sigma(\nabla S, \nabla F_2;\, \nabla v) + \langle\nabla S, \nabla F_1\rangle v^2)\, m_{F_1}.$$

But $$\langle\nabla S, \nabla v\rangle Pv + Q \leqslant \tfrac{1}{2}(Pv)^2 + \tfrac{1}{2}(\langle\nabla S, \nabla v\rangle)^2 + Q = \tfrac{1}{2}(Pv)^2 + Q_1(v, \nabla v),$$

where Q_1 is a quadratic form in $(v, \nabla v)$. By Lemma A 2 and (A 11),

$$\nabla v \mapsto \sigma(\nabla S, \nabla S;\, \nabla v)$$

is a positive definite quadratic form; as K is compact, there is thus a constant $C' = C'(K)$ such that, on K,

$$Q_1 \leqslant C'(\sigma(\nabla S, \nabla S;\, \nabla v) + \langle\nabla S, \nabla s\rangle v^2).$$

Taking $t > C'$ we see now that

$$\int_\Omega e^{-ts}(\sigma(\nabla S, \nabla S;\, \nabla v) + \langle\nabla S, \nabla S\rangle v^2)\, dm \leqslant \frac{1}{2(t-c')}\Big(\int_\Omega (Pr)^2\, dm - 2\int_{\Sigma_1} e^{-ts}(\sigma(\nabla S, DF_1;\, \nabla v) + (\langle\nabla S, \nabla F_1\rangle v^2)\, m_{F_1}\Big),$$

and this clearly implies the lemma.

To establish similar estimates for derivatives of higher order, we put, for multi-indices I, J of length $k > 1$,

$${}^kE_{I,J}(v) = (\nabla_{i_1}\dots\nabla_{i_k}v)(\nabla_{j_1}\dots\nabla_{j_k}v) - \tfrac{1}{2}g_{i_1,j_1}\sum_{l=1}^{N}\nabla_l\nabla_{i_2}\dots\nabla_{i_k}v\nabla^l\nabla_{j_2}\dots\nabla_{j_k}v.$$

A manipulation which utilizes the commutation relations for covariant derivatives then gives

$$\sum_{j_1=1}^{N}\nabla^{j_1}({}^kE_{I,J}) = (\nabla_{i_1}\dots\nabla_{i_k}v)\nabla_{j_2}\dots\nabla_{j_k}\Box v + \dots,$$

where the terms omitted are quadratic in the covariant derivatives of v of order $\leqslant k$. On the other hand,

$$\Sigma^k E_{I,J}(v)\,\xi^{i_1}\dots\xi^{i_k}\xi^{j_1}\dots\xi^{j_{k-1}}\eta^{i_k} = {}^k\sigma(\xi, \eta;\, \nabla v)$$

where ${}^k\sigma$ is as in (A 6), (A 7), and is positive definite (respectively, positive semi-definite) if ξ and η are time-like (respectively, ξ time-like and η null) and have the same time

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orientation. Proceeding as in the proof of Lemma A 12, one can then obtain integral estimates from which the following is easily derived by induction:

(A 16) THEOREM. *Let K and Ω be as in Lemma* A 12, *and let ${}^k\sigma$ be defined by* (A 7) *for $k > 1$, ${}^1\sigma = \sigma$ and ${}^0\sigma = v^2$. Then there are positive constants $C_k = C_k(K)$, $k = 1, 2, \ldots$, such that, for all $v \in C^\infty(\bar{\Omega})$,*

$$\sum_{j=0}^{k} \int_\Omega {}^j\sigma(\nabla S, \nabla S; v)\, dm \leqslant C_k \left(\sum_{j=0}^{k-1} \int_\Omega {}^j\sigma(\nabla S, \nabla S, Pv)\, dm + \int_{\Sigma_1} \phi_k \right),$$

where the ϕ_k are densities on Σ_1 which are quadratic in v and its covariant derivatives of order $\leqslant k$.

This becomes Lemma 1·19 when $P = \tilde{\Box} + \gamma$.

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