J. Plasma Physics (2007), *vol.* 73, *part* 4, *pp.* 599–611. © 2006 Cambridge University Press
doi:10.1017/S0022377806004879 First published online 16 August 2006 Printed in the United Kingdom

Covariant form for the collision integral

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(Received 6 June 2006; accepted 28 June 2006)

Abstract. The collision integral that describes the evolution of a distribution of particles in a plasma due to Coulomb interactions between themselves or with other particles is generalized to include relativistic effects and the current–current interaction (in addition to the charge–charge interaction). This is achieved through a covariant version of a conventional derivation based on correlation functions for fluctuations in the plasma. The covariant theory is used to distinguish between longitudinal (charge–charge) and transverse (current–current) interactions. For highly relativistic particles, the current–current contribution is half the charge–charge contribution when Debye screening is unimportant, and is unaffected by Debye screening. It is shown that the classical theory is reproduced by a quantum electrodynamics calculation for electron–electron (Møller) scattering in the limit of small momentum transfer.

1. Introduction

The collision integral is a term in the Boltzmann equation that describes the effect of collisions between particles on the evolution of the particle distribution function. There is extensive literature on the form of the collision integral for Coulomb interactions in a plasma, first derived by Landau (e.g. Lifshitz and Pitaevskii 1981, Sec. 41), and detailed developments of the theory by Rosenbluth et al. (1957) and Trubnikov (1958) involving a Fokker–Planck approach. The generalization to the relativistic case was made by Beliaev and Budker (1956). A different approach is based on the statistical theory of plasmas; for example, Balescu (1963), Klimontovich (1967), Lifshitz and Pitaevskii (1981, Sec. 51). In this approach the random motions of particles in the plasma are assumed to generate fluctuations in the electric field, and these fluctuations are described by a correlation function. These fluctuations affect the distribution functions of the particles, and the collision integral is calculated by averaging over the fluctuating terms in the Vlasov equation to find the effect on the averaged distribution function. This approach is somewhat counterintuitive in that the theory of fluctuations in a 'collisionless' plasma is used to derive the theory for 'collisions' (specifically Coulomb interactions) between particles (e.g. Lifshitz and Pitaevskii 1981, Sec. 51).

Two generalizations of the existing theory of the collision integral are discussed in this paper. The first generalization relates to the neglect of current–current interactions. The emphasis on Coulomb interactions in the literature cited and in more recent literature (e.g. Braams and Karney 1987; Baral and Mohanty 2000, 2002; Saveiev and Nanbu 2002) implies that only charge–charge interactions are

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taken into account. The current–current interaction is negligible for non-relativistic particles, because it is smaller than the charge–charge interaction by a factor of same order as a relativistic correction. However, this neglect is not justified for relativistic particles, for which the current–current interaction is of the same order as the charge–charge interaction. This was confirmed to be the case in a closely related context (collisions in a highly relativistic degenerate plasma) by Heiselberg and Pethick (1993), who found the contribution of the current–current interaction to be equal to half that of the charge–charge interaction when Debye screening is unimportant and to be the dominant contribution when the charge–charge interaction is suppressed by Debye screening. The neglect of the current–current interaction in the theory of fluctuations is implicit in the emphasis on the longitudinal component of the electric field: the current–current interaction is mediated by the transverse component of the electromagnetic field. Thus the first generalization is to include the current–current interaction by generalizing the theory of fluctuations to include both the longitudinal and transverse components. The second generalization concerns the calculation of the probability for a 'collision' between two particles in a plasma. On the one hand, the form of the collision integral may be derived by postulating the probability for a binary interaction and using simple arguments to derive the relevant kinetic equation (e.g. Lifshitz and Pitaevskii 1981, Sec. 41). On the other hand, the only known rigorous classical method for calculating the collision integral is through the theory of fluctuations. The resulting form for the collision integral may be used to infer the form of the postulated probability indirectly. There is no standard classical derivation that leads to this scattering probability directly. However, a standard derivation in quantum electrodynamics (QED) can be modified to derive this probability directly. A standard QED derivation of the Feynman amplitude for Møller (electron–electron) scattering involves the photon propagator, in which the effect of plasma dispersion is usually neglected, but it is trivial to replace the photon propagator in vacuo by the photon propagator in the plasma (that is, including the plasma response tensor). This leads directly to a QED form for the scattering probability, which is shown here to reproduce the classically-derived probability in the relevant limit of small momentum transfer.

The first of these generalizations is carried out using a classical covariant version of the Vlasov theory and of the theory of fluctuations in a plasma. The advantage of the covariant formalism in this context is that it allows one to include all components of the electromagnetic field in a derivation that mimics, step-by-step, the conventional derivations where only the longitudinal field is included. To be more specific, the longitudinal field may be described by the (Fourier transformed) electrostatic potential $\Phi(k)$, and in the covariant theory this becomes the time component of the 4-potential, $A^\mu(k) = [\Phi(k), \mathbf{A}(k)]$, where $\mathbf{A}(k)$ is the vector potential; step-by-step derivations involving $\Phi(k)$ are simply rewritten in 4-tensor notation to generalize them to include all components of the electromagnetic field. As indicated, the second generalization is straightforward in QED, and the relevant question concerns the conditions under which the probability calculated directly using QED reduces to the probability inferred by the indirect classical calculation.

In Sec. 2 a general form of the Landau-type derivation of the form of the collision integral is written down and generalized to a covariant formalism. In Sec. 3, a convention derivation of the collision integral using the theory of fluctuations is generalized to a covariant form, and used to identify the most general form for the classical scattering probability. In Sec. 4, the plasma is assumed isotropic, and

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the separation into longitudinal and transverse components is made explicitly; the probability is evaluated for both non-relativistic and highly relativistic particles. In Sec. 5, the generalization of the QED probability for Møller scattering is shown to reproduce the classically derived probability for sufficiently small momentum transfers.

Except where quantities are introduced and defined, natural units ($\hbar = c = 1$) are used in this paper.

2. Covariant form for the collision integral

In this section, a standard method for deriving the form of the collision integral is generalized to a covariant formalism.

2.1. Form of the collision integral

Let $f(t, \mathbf{x}, \mathbf{p})$ be the distribution of particles (charge q, mass m) in six-dimensional phase space, with differential element $d^3\mathbf{x}d^3\mathbf{p}/(2\pi)^3$, and let $\bar{f}(\mathbf{p})$ be the distribution averaged over fluctuations. Let $w(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k})$ be the probability that particles with initial 3-momenta $\mathbf{p}_1$, $\mathbf{p}_2$ are scattered with momentum transfer $\mathbf{k}$, such that the final momenta are $\mathbf{p}_1' = \mathbf{p}_1 + \mathbf{k}$, $\mathbf{p}_2' = \mathbf{p}_2 - \mathbf{k}$. Detailed balance requires that the probability be the same from the inverse scattering. The net rate of scattering is then determined by

$$w(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k})[\bar{f}(\mathbf{p}_1)\bar{f}(\mathbf{p}_2) - \bar{f}(\mathbf{p}_1 + \mathbf{k})\bar{f}(\mathbf{p}_2 - \mathbf{k})]$$

for momentum transfers in the range $d^3\mathbf{k}/(2\pi)^3$. The rate of change of $\bar{f}(\mathbf{p}_1)$ is found by considering the difference between the net gains from scatterings $\mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2 \leftrightarrow \mathbf{p}_1, \mathbf{p}_2 - \mathbf{k}$ and the net losses from $\mathbf{p}_1, \mathbf{p}_2 \leftrightarrow \mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2 - \mathbf{k}$. On expanding in $\mathbf{k}$, the first-order terms cancel and the second-order terms give the collision integral

$$\frac{d\bar{f}(\mathbf{p}_1)}{dt} = \frac{\partial}{\partial \mathbf{p}_1}\left\{\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{p}_2}{(2\pi)^3}\,\mathbf{k}\mathbf{k}\,w(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k})\left[\bar{f}(\mathbf{p}_2)\frac{d\bar{f}(\mathbf{p}_1)}{d\mathbf{p}_1} - \bar{f}(\mathbf{p}_1)\frac{d\bar{f}(\mathbf{p}_2)}{d\mathbf{p}_2}\right]\right\}, \tag{2.1}$$

where a dyadic notation is used to minimize confusion with the 4-tensor notation used below. It is implicit in (2.1) that only one species of particle is involved: the generalization to collisions between different species is trivial, involving only notational changes, and is ignored here to keep the notation as simple as practicable.

2.2. Covariant collision integral

A covariant form of (2.1) is obtained by repeating the calculation using the covariant notation, defined in Appendix A. Specifically, $F(x, p)$ is the distribution function in eight-dimensional phase space constructed from the 4-vectors $x^\mu = [t, \mathbf{x}]$, $p^\mu = [\varepsilon, \mathbf{p}]$, with $\varepsilon = (m^2 + |\mathbf{p}|^2)^{1/2}$ in natural units. Let $W(p_1, p_2, k)$ be the probability for such a collision with 4-momentum transfer k, such that the 4-momenta of the particles change to $p_1' = p_1 + k$, $p_2' = p_2 - k$. Repeating the analysis leading to (2.1)

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gives the covariant form of the collision integral

$$\frac{dF(p_1)}{d\tau} = \frac{\partial}{\partial p_1^\mu}\left\{\int \frac{d^4k}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}k^\mu k^\nu W(p_1,p_2,k)\left[F(p_2)\frac{\partial F(p_1)}{\partial p_1^\nu} - F(p_1)\frac{\partial F(p_2)}{\partial p_2^\nu}\right]\right\}, \tag{2.2}$$

where the derivative with respect to proper time, τ, may be interpreted as $u_1^\alpha\partial_\alpha$, where $u^\mu = p^\mu/m$ is the 4-velocity and $\partial_\mu = \partial/\partial x^\mu$ is the 4-derivative.

The probability $W(p_1,p_2,k)$ in (2.2) is related to the probability $w(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k})$ by translating between one notation and the other, as shown in Appendix A.

2.3. Two-scale separation of the Vlasov equation

An important point in the derivation of the collision integral through the theory of fluctuations is that it is based on the collisionless form of the Vlasov equation. Specifically, the two-particle distribution function is assumed to be the outer product of two single-particle distribution functions, and no collisional term is included in the Vlasov equation. The covariant form for the collisionless Vlasov equation is

$$\left(u^\alpha\partial_\alpha + qF^{\alpha\beta}(x)u_\beta\frac{\partial}{\partial p^\alpha}\right)F(x,p) = 0. \tag{2.3}$$

where $qF^{\alpha\beta}(x)u_\beta$ is the electromagnetic 4-force: the Maxwell 4-tensor expressed in terms of the 4-potential, $A^\mu(x)$, by $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$. Let $F(x,p) = \bar{F}(p) + \delta F(x,p)$ denote a separation into variations on two scales: a fast–short scale involving fluctuations and a slow–long scale on which the average distribution evolves secularly. On the slow–long variation, writing $u^\alpha\partial_\alpha\bar{F}(p) \to d\bar{F}(p)/d\tau$, the evolution is described by

$$\frac{d\bar{F}(p)}{d\tau} = -\left\langle qF^{\alpha\beta}(x)u_\beta\frac{\partial}{\partial p^\alpha}\delta F(x,p)\right\rangle, \tag{2.4}$$

where angular brackets denote an ensemble average over the fluctuations. On the fast–short scale one linearizes in the fluctuating quantities to find

$$u^\alpha\partial_\alpha\delta F(x,p) + qF^{\alpha\beta}(x)u_\beta\frac{\partial\bar{F}(p)}{\partial p^\alpha} = 0. \tag{2.5}$$

One Fourier transforms on the fast–short scale:

$$qF^{\alpha\beta}(x)u_\beta = -i\int\frac{d^4k}{(2\pi)^4}\,e^{ikx}\,ku\,G^{\alpha\beta}(k,u)\,A_\beta(k), \quad G^{\mu\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu}{ku}. \tag{2.6}$$

Then (2.5) gives

$$ku\,\delta F(k,p) + q\,ku\,G^{\alpha\beta}(k,u)\,A_\beta(k)\frac{\partial}{\partial p^\alpha}\bar{F}(p) = 0. \tag{2.7}$$

The solution of (2.7) consists of two parts,

$$\delta F(k,p) = \delta F_{\rm ud}(k,p) - q\,G^{\alpha\beta}(k,u)\,A_\beta(k)\frac{\partial}{\partial p^\alpha}\bar{F}(p), \tag{2.8}$$

with the part $\delta F_{\rm ud}(k,p)$ non-zero only for $ku = 0$. The part $\delta F_{\rm ud}(k,p)$ averages to zero, but has a non-zero autocorrelation function that described purely statistical

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fluctuations in the particle distribution; the subscript 'ud' is for 'undressed' to denote that these fluctuations are the same as for uncharged particles.

2.4. Fluctuations in the current

We are particularly interested in the fluctuations in the 4-current, which are determined by

$$J^\mu(k) = q \int \frac{d^4p}{(2\pi)^4}\, u^\mu \delta F(k,p). \tag{2.9}$$

Ignoring the term $\delta F_{\rm ud}(k,p)$ in (2.8), the remaining term in (2.9) gives $J^\mu(k) = \Pi^{\mu\nu}(k)A_\nu(k)$, where

$$\Pi^{\mu\nu}(k) = q^2 \int \frac{d^4p}{(2\pi)^4} u^\mu G^{\alpha\nu}(k,u) \frac{\partial \bar{F}(p)}{\partial p^\alpha} \tag{2.10}$$

is the linear response 4-tensor. (Only one species of particle is retained explicitly, and the response tensor implicitly includes a sum over all species.) Let the other contribution to the current, from the term $\delta F_{\rm ud}(k,p)$ in (2.8), be denoted $J^\mu_{\rm ud}(k)$. This term is regarded as a source term. Inserting these two terms in Maxwell's equations gives

$$\Lambda^{\mu\nu}(k)A_\nu(k) = -\mu_0 J^\mu_{\rm ud}(k), \quad \Lambda^{\mu\nu}(k) = k^2 g^{\mu\nu} - k^\mu k^\nu + \mu_0 \Pi^{\mu\nu}(k). \tag{2.11}$$

The solution of the equation

$$\Lambda^\mu{}_\alpha(k) D^{\alpha\nu}(k) = \mu_0 \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \tag{2.12}$$

defines the photon propagator (or Green's function) for the medium. The photon propagator is defined only to within a gauge transformation, which involved adding arbitrary terms proportional to k^μ and k^ν. The solution of (2.11) is

$$A^\mu(k) = -D^\mu{}_\nu(k) J^\nu_{\rm ud}(k), \tag{2.13}$$

which is determined only to within an arbitrary gauge transformation, implicit in the choice of gauge for $D^{\mu\nu}(k)$.

3. Correlation functions

In this section, correlation functions are defined and the correlations that lead to the collision integral are evaluated. The correlations are related to fluctuations in the distribution function, which are 'purely statistical' in the sense that the fluctuations in the number of N random events with a mean $\langle N \rangle$ has a variance $\langle (\delta N)^2 \rangle = \langle N \rangle$. The implied correlation in the distribution function is well known (e.g. Klimontovich 1967), and is expressed here in terms of the covariant formalism.

3.1. Definition

In a steady state the level of fluctuations in any quantity may be described in terms of its autocorrelation function, and correlations between fluctuations in two quantities may be described in terms of their cross-correlation function. Let the correlation function of two quantities, $Q_1(x)$ and $Q_2(x)$, be

$$\langle Q_1(x) Q_2(x') \rangle = \langle Q_1 Q_2 \rangle (x - x'), \tag{3.1}$$

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where the average, denoted by angular brackets, may be over space-time (specifically over $\frac{1}{2}(x+x')$), over the initial conditions, or over an ensemble. The average of the Fourier transform of the quantities is related to the Fourier transform of their correlation function by

$$\langle Q_1(k)Q_2^*(k')\rangle = \langle Q_1Q_2^*\rangle(k)(2\pi)^4\delta^4(k-k'). \tag{3.2}$$

3.2. Correlation function for the distribution function

The autocorrelation function for the fluctuations $\delta F_{\rm ud}(k,p)$ follows by rewriting the known correlation function for $f(t,\mathbf{x},\mathbf{p})$ (for example, Equation (5.16) of Klimontovich (1967), and Equation (51.14) of Lifshitz and Pitaevskii (1981)), in the present notation. This gives

$$\langle \delta F_{\rm ud}(k,p)\delta F_{\rm ud}(k',p')\rangle = 2\pi\delta(ku)(2\pi)^4\delta^4(k+k')(2\pi)^4\delta^4(p-p')\bar{F}(p), \tag{3.3}$$

which describes fluctuations that are purely statistical. The correlation function for the Fourier transform of the current, $J^\mu_{\rm ud}(k)$, is related to the Fourier transform of the correlation function, $\langle J\,J^*\rangle^{\mu\nu}_{\rm ud}(k)$, by

$$\langle J^\mu_{\rm ud}(k)\,J^{*\nu}_{\rm ud}(k')\rangle = (2\pi)^4\delta^4(k-k')\langle J\,J^*\rangle^{\mu\nu}_{\rm ud}(k), \tag{3.4}$$

and (2.9), (2.11) and (3.4) imply

$$\langle J\,J^*\rangle^{\mu\nu}(k) = q^2\int\frac{d^4p}{(2\pi)^4}\,u^\mu u^\nu\,2\pi\delta(ku)\,\bar{F}(p). \tag{3.5}$$

The correlation function for the 4-potential is determined by (2.13), (3.5):

$$\langle AA^*\rangle^{\mu\nu}(k) = D^{\mu\rho}(k)D^{*\nu\sigma}(k)q^2\int\frac{d^4p'}{(2\pi)^4}\,u'_\rho u'_\sigma\,2\pi\delta(ku')\bar{F}(p'). \tag{3.6}$$

One other correlation function needed below follows similarly from (2.13) and (3.3):

$$\langle A_\beta(k)\,\delta F_{\rm ud}(k',p)\rangle = -(2\pi)^4\delta^4(k+k')q2\pi\delta(ku)D_{\beta\gamma}(k)u^\gamma\bar{F}(p). \tag{3.7}$$

3.3. Kinetic equation due to fluctuations

The secular changes are described by (2.4), and the foregoing correlation functions are needed to evaluate the right-hand side. First note that the derivative with respect to p^α may be moved to the left, using $\partial F^{\alpha\beta}(x)u_\beta/\partial p^\alpha = 0$, so that the correlation function that needs to be evaluated is

$$\langle F^{\alpha\beta}(x)u_\beta\,\delta F(x,p)\rangle = -i\int\frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\,e^{i(k+k')x}ku\,G^{\alpha\beta}(k,u)\langle A_\beta(k)\delta F(k',p)\rangle. \tag{3.8}$$

There are two contributions to this correlation function, arising from the two terms on the right-hand side of (2.8).

The contribution from the final term in (2.8)–(3.8) is evaluated using the correlation function (3.6). The resulting expression is superficially imaginary, and only a real part contributes to the kinetic equation (2.4). A contribution is found by imposing the causal condition. The relevant contribution is from the factor $G^{\alpha\beta}(k,u)$ in (2.8), which is evaluated at $k'\to -k$ in (3.8). Hence, the Landau prescription $1/k'u\to 1/(k'u+i0)$ becomes $1/ku\to 1/(ku-i0)$, implying an imaginary part $1/ku\to i\pi\delta(ku)$. On relabeling $p\to p_1$, $p'\to p_2$, this contribution to the collision integral is of the form of the first term inside the square brackets

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in (2.2). This allows one to infer the form of the probability from the theory of fluctuations:

$$W(p_1, p_2, k) = 2\pi^2 q^4 \delta(ku_1)\delta(ku_2)\, |D_{\alpha\beta}(k)\, u_1^\alpha u_2^\beta|^2. \tag{3.9}$$

with $p_1 = mu_1$, $p_2 = mu_2$.

In evaluating the other contribution to (3.8), from the term $\delta F_{\rm ud}(k, p)$ in (2.8), one uses the correlation function (3.7). In this case, the relevant real part is from the antihermitian part of the photon propagator in (3.7). This antihermitian part is from the antihermitian part of the response tensor,

$$D^{\Lambda\mu\nu}(k) = -D^{*\mu\alpha}(k)\Pi^\Lambda_{\alpha\beta}(k)D^{\beta\nu}(k), \tag{3.10}$$

which follows from (2.12) with the antihermitian part of $\Lambda^{\mu\nu}(k)$ arising from the antihermitian part of $\Pi^{\mu\nu}(k)$, which is given by imposing the Landau prescription on the resonant denominator ($1/ku \to 1/(ku+i0)$) in (2.9). The resulting contribution, after relabeling the momenta appropriately, gives the second term in square brackets in (2.2), with $W(p_1, p_2, k)$ given by (3.9).

This completes the formal generalization of the derivation of the collision integral to the covariant form (2.2). This derivation leads to the identification (3.9) of the scattering probability, $W(p_1, p_2, k)$. Despite the relatively simple form of this probability, there is no well-known direct classical derivation of it. In Sec. 5, the classical limit of the QED calculation (for Møller scattering) is shown to reproduce it in the limit of small momentum transfer.

4. Separation into longitudinal and transverse parts

In this section, the plasma is assumed isotropic and the response is separated into longitudinal and transverse parts. The limits of non-relativistic and highly relativistic particles are discussed using this separation.

4.1. Projection operators

The separation into longitudinal and transverse parts may be made in a covariant manner by introducing projection operators, $L^{\mu\nu}(k, \tilde{u})$ and $T^{\mu\nu}(k, \tilde{u})$, respectively, which apply in an arbitrary frame in which the rest frame of the plasma is moving with 4-velocity $\tilde{u}$. These tensors are defined such that in the rest frame, $\tilde{u}^\mu = [1, \mathbf{0}]$, their mixed-tensor components reduce to $\mathbf{kk}/|\mathbf{k}|^2$, $\mathbf{1}-\mathbf{kk}/|\mathbf{k}|^2$, respectively, and that they satisfy the charge-continuity and gauge-invariance conditions, $k_\mu L^{\mu\nu}(k, \tilde{u}) = 0$, $k_\nu L^{\mu\nu}(k, \tilde{u}) = 0$, and similarly for $T^{\mu\nu}(k, \tilde{u})$. Explicit forms for these tensors are given in Appendix B. In an isotropic medium, one has

$$\Pi^{\mu\nu}(k) = \Pi^L(k)\, L^{\mu\nu}(k, \tilde{u}) + \Pi^T(k)\, T^{\mu\nu}(k, \tilde{u}), \tag{4.1}$$

$$\Pi^L(k) = \frac{k^4}{(k\tilde{u})^4}\, L_{\mu\nu}(k, \tilde{u})\, \Pi^{\mu\nu}(k), \quad \Pi^T(k) = \frac{1}{2} T_{\mu\nu}(k, \tilde{u})\, \Pi^{\mu\nu}(k). \tag{4.2}$$

The separation of $D^{\mu\nu}(k)$ into its longitudinal parts is performed in the same way. The explicit forms for these parts are

$$D^L(k) = \frac{(k\tilde{u})^4}{k^4} \frac{\mu_0}{(k\tilde{u})^2 + \mu_0 \Pi^L(k)}, \quad D^T(k) = \frac{\mu_0}{k^2 + \mu_0 \Pi^T(k)}. \tag{4.3}$$

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4.2. Scattering probability in an isotropic plasma

For an isotropic plasma, the factor that appears squared in the scattering probability (3.9) reduces to

$$D_{\alpha\beta}(k)\,u_1^\alpha u_2^\beta = D^L(k)\frac{k^4(u_1\tilde{u})(u_2\tilde{u})}{(k\tilde{u})^2[k^2-(k\tilde{u})^2]} + D^T(k)\left[u_1u_2 - \frac{k^2(u_1\tilde{u})(u_2\tilde{u})}{k^2-(k\tilde{u})^2}\right], \quad (4.4)$$

where $ku_1 = 0$, $ku_2 = 0$ are used.

In the rest frame of the plasma, (4.4) with (4.3) gives

$$D_{\alpha\beta}(k)\,u_1^\alpha u_2^\beta = -\frac{\mu_0\gamma_1\gamma_2}{|\mathbf{k}|^2}\left[\frac{\omega^2}{\omega^2+\mu_0\Pi^L(k)} - \frac{\mathbf{k}\times\mathbf{v}_1\cdot\mathbf{k}\times\mathbf{v}_2}{\omega^2-|\mathbf{k}|^2+\mu_0\Pi^T(k)}\right]. \quad (4.5)$$

Then the probability (3.9) becomes

$$W(p_1,p_2,k) = 2\pi^2\mu_0^2q^4\gamma_1\gamma_2\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}_1)\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}_2)$$
$$\times\,\frac{1}{|\mathbf{k}|^4}\left|\frac{\omega^2}{\omega^2+\mu_0\Pi^L(k)} + \frac{\mathbf{k}\times\mathbf{v}_1\cdot\mathbf{k}\times\mathbf{v}_2}{\omega^2-|\mathbf{k}|^2+\mu_0\Pi^T(k)}\right|^2, \quad (4.6)$$

where the numerator in the final term arises from $\omega^2 - |\mathbf{k}|^2\mathbf{v}_1\cdot\mathbf{v}_2$ by using the δ-functions to write $\omega^2 = \mathbf{k}\cdot\mathbf{v}_1\,\mathbf{k}\cdot\mathbf{v}_2$.

The scattering of one particle by another can be interpreted as a transfer of 4-momentum, k. In an isotropic plasma one may identify this momentum transfer as being through either a virtual longitudinal wave or a virtual transverse wave, corresponding to the terms in (4.6) involving $\Pi^L(k)$ and $\Pi^T(k)$, respectively. These correspond to charge–charge and current–current interactions, respectively.

4.3. Collisions of non-relativistic particles

In conventional derivations of the collision integral for non-relativistic particles, the simplifying assumption is made that only virtual longitudinal waves are important. This is well justified for non-relativistic particles because the transverse term in (4.5) is smaller than the longitudinal term by a factor of order $v^2 \ll 1$, where $v = |\mathbf{v}_1 - \mathbf{v}_2|$ is the relative speed of the particles.

A further simplification in the non-relativistic case is to approximate the longitudinal response by $\mu_0\Pi^L(k) \approx \omega^2/|\mathbf{k}|^2\lambda_{\rm D}^2$, where $\lambda_{\rm D} = V/\omega_p$ is the Debye length, and $T = mV^2$ is the thermal energy. This approximation applies for $\omega^2/|\mathbf{k}|^2V^2 \ll 1$. With this approximation the longitudinal contribution in (4.6) becomes $\omega^2/[\omega^2 + \mu_0\Pi^L(k)] \to |\mathbf{k}|^2\lambda_{\rm D}^2/[1+|\mathbf{k}|^2\lambda_{\rm D}^2]$. A further simplification is to replace this term by unity, and to cut off the resulting logarithmically divergent integral for $|\mathbf{k}| < 1/\lambda_{\rm D}$. The logarithm is identified as the Coulomb logarithm, so that the collision integral is proportional to the Coulomb logarithm whenever this approximation made. This is the case in all of the treatments of the collision integral discussed above, including those where relativistic effects are included. However, the neglect of the transverse term is not well justified in the relativistic case.

4.4. Collisions of highly relativistic particles

In discussing the highly relativistic limit it is helpful to separate $\mathbf{v}_1$, $\mathbf{v}_2$ into components parallel ($\parallel$) and perpendicular ($\perp$) to $\mathbf{k}$. Then one has $\omega = |\mathbf{k}|v_{1\parallel} = |\mathbf{k}|v_{2\parallel}$, and $\mathbf{k}\times\mathbf{v}_1\cdot\mathbf{k}\times\mathbf{v}_2 = |\mathbf{k}|^2v_{1\perp}\,v_{2\perp}\cos\phi$, where ϕ is an azimuthal angle (Heiselberg and Pethick 1993). In the isotropic case, only the probability averaged over angles contributes and $\cos^2\phi$ in the square of the current–current term is then replaced

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by 1/2. The interference between the charge–charge and current–current terms is proportional to $\cos\phi$, and averages to zero. The relative contribution of the charge–charge and current–current terms can be estimated by inserting appropriate approximations for $\Pi^{L,T}(k)$ into (4.6) and comparing the magnitudes of the two terms. The response functions for a relativistic thermal plasma have long been known (e.g. Prentice 1967 Godfrey et al. 1975), but are relatively cumbersome in general. In the limit of small $\omega/|\mathbf{k}|$ of relevance here, they have relatively simple approximations:

$$\mu_0\Pi^L(k) \approx \frac{\omega^2}{|\mathbf{k}|^2\lambda_{\mathrm{D}}^2}, \qquad \mu_0\Pi^T(k) \approx -i\frac{\pi}{2}\frac{\omega a(\rho)}{|\mathbf{k}|}, \quad a(\rho) = \frac{\rho+1}{\rho^2}\frac{e^{-\rho}}{K_2(\rho)}, \tag{4.7}$$

with $\rho = m/T = 1/V^2$, and where the Debye length is defined in the same way as in a non-relativistic plasma: $\lambda_{\mathrm{D}}^{-2} = \omega_p^2\rho$. The parameter $a(\rho)$ involves the Macdonald function $K_2(\rho)$, which is approximately $(\pi/2\rho)^{1/2}e^{-\rho}$ in the non-relativistic case, $\rho \gg 1$, and approximately $2/\rho^2$ in the extreme relativistic case, $\rho \ll 1$, giving $a(\rho) \approx 1/2$. Note that the dominant term in the transverse part is the imaginary part associated with Landau damping. One has $|\Pi^T(k)| \ll |\Pi^L(k)|$ in this limit.

Further simplification occurs for highly relativistic particles, when one has $v_{1\perp}^2 = v_{2\perp}^2 = 1-\omega^2/|\mathbf{k}|^2$. For $\omega^2 \ll |\mathbf{k}|^2$, the ratio of the charge–charge and current–current terms in (4.6) is then determined by the ratio of the terms

$$\left|\frac{\omega^2}{\omega^2-\mu_0\Pi^L(k)}\right|^2 \approx \frac{1}{|1+1/|\mathbf{k}|^2\lambda_{\mathrm{D}}^2|^2},$$

$$\left|\frac{\mathbf{k}\times\mathbf{v}\cdot\mathbf{k}\times\mathbf{v}'}{\omega^2-|\mathbf{k}|^2-\mu_0\Pi^T(k)}\right|^2 \approx \frac{1}{2}\frac{1}{1+(\pi^2/4)a^2(\rho)\omega^2/\lambda_{\mathrm{D}}^4|\mathbf{k}|^6}. \tag{4.8}$$

The ratio of the contributions of the current–current and charge–charge contributions is determined by two parameters, $|\mathbf{k}|^2\lambda_{\mathrm{D}}^2|$ and $(\pi^2/4)a^2(\rho)\omega^2/|\mathbf{k}|^2$. In a highly relativistic plasma, with $a(\rho) \approx 1/2$, the second parameter is of order $\omega^2/|\mathbf{k}|^2$, which is necessarily less than unity. It follows that the contributions of the current–current term is half that of the charge–charge term for $|\mathbf{k}|^2\lambda_{\mathrm{D}}^2| \gg 1$, and that it is the dominant term for $|\mathbf{k}|^2\lambda_{\mathrm{D}}^2| \ll 1$. The charge–charge contribution is suppressed by Debye screening for $|\mathbf{k}|^2\lambda_{\mathrm{D}}^2| \ll 1$, but the current term is not (Heiselberg and Pethick 1993). It follows that the error in neglecting the current–current interaction can be corrected by multiplying by $(3/2)^2$ for momentum transfers $|\mathbf{k}| > 1/\lambda_{\mathrm{D}}$. However, for smaller momentum transfers, $|\mathbf{k}| \ll 1/\lambda_{\mathrm{D}}$, when the charge–charge interaction is screened out, the current–current interaction is unaffected by Debye screening and implies a form of scattering that is not usually considered. The current–current contribution, of the same form as the charge–charge contribution, with an extra factor of one, and with a larger logarithmic factor, due to the cutoff being extended to $|\mathbf{k}| \ll 1/\lambda_{\mathrm{D}}$.

5. Møller scattering in a plasma

In this section it is shown that the probability of Møller scattering (and of Bhabha scattering), calculated using QED and modified by the presence of the plasmas, reproduces the probability calculated in Sec. 3 using the theory of fluctuations.

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5.1. Transition amplitude for Møller scattering

When a standard procedure for calculating processes in QED (e.g. Berestetskii et al. 1971, Sec. 82) is applied to the transition from an initial state in which two electrons have momenta, $\mathbf{p}_1, \mathbf{p}_2$ to a final state in which the two electrons have momenta $\mathbf{p}'_1, \mathbf{p}'_2$, the transition rate is of the form

$$w_{\mathrm{i}\to\mathrm{f}} = V\,(2\pi)^4\,\delta^4(p_1+p_2-p'_1-p'_2)\frac{|M_{\mathrm{fi}}|^2}{8\varepsilon_1\varepsilon_2\varepsilon'_1\varepsilon'_2V^4}\,\frac{Vd^3\mathbf{p}'_1}{(2\pi)^3}\,\frac{Vd^3\mathbf{p}'_2}{(2\pi)^3}, \tag{5.1}$$

where V is the normalization volume, the differentials arise form density of final states factors, and M_{fi} is the Feynman amplitude. The relevant Feynman amplitude for Møller scattering iM_{fi} contains two terms:

$$iM_{\mathrm{fi}} = ie^2\big[\overline{u}_{s'_2}(\mathbf{p}'_2)\gamma^\mu u_{s_2}(\mathbf{p}_2)D_{\mu\nu}(p_1-p'_1)\overline{u}_{s'_1}(\mathbf{p}'_1)\gamma^\nu u_{s_1}(\mathbf{p}_1)$$

$$-\,\overline{u}_{s'_1}(\mathbf{p}'_1)\gamma^\mu u_{s_2}(\mathbf{p}_2)D_{\mu\nu}(p_1-p'_2)\overline{u}_{s'_2}(\mathbf{p}'_2)\gamma^\nu u_{s_1}(\mathbf{p}_1)\big]. \tag{5.2}$$

The first term describes the direct interaction, and the second term describes the exchange interaction. The parameters s_1, s_2, s'_1, s'_2 describe the spins of the electrons; for unpolarized electrons one averages over the initial spins and sums over the final spins. (The amplitude for Bhabha scattering is closely analogous, but with no exchange term.)

The inclusion of the medium involves interpreting the photon propagator as that in the medium, rather than that in vacuo in conventional QED treatments of Møller scattering.

5.2. Limit of small momentum transfer

Interest here is in the limit in which the momentum transfer is small, specifically, $k = p'_1 - p_1 = p_2 - p'_2$ much smaller than p_1, p_2. In the direct term in (5.2), the argument of the photon propagator is k, and in the exchange term it is $p_1 - p'_2 \approx p_1 - p_2$, so that the two terms are approximately in the ratio $1/k^2$ to $1/(p_1-p_2)^2$, which is large for small momentum transfers. Hence, in this limit the exchange term can be neglected.

When only the direct term is retained, it is relatively simple to evaluate the modulus squared of M_{fi} averaged over the initial spins and summed over the final spins. This gives

$$\overline{|M_{\mathrm{fi}}|^2} = 4e^4 D_{\mu\nu}(p_1-p'_1)D^*_{\alpha\beta}(p_1-p'_1)F^{\mu\alpha}(p'_2,p_2)F^{\nu\beta}(p'_1,p_1), \tag{5.3}$$

with $F^{\mu\nu}(p,p')$ equal to the trace of $\frac{1}{4}(\not{p}+m)\gamma^\mu(\not{p}'+m)\gamma^\nu$, giving

$$F^{\mu\nu}(p,p') = p^\mu p'^\nu + p'^\mu p^\nu + g^{\mu\nu}(m^2 - pp'). \tag{5.4}$$

Setting $p'_1 = p_1 + k$, $p'_2 = p_2 - k$ in (5.3), expanding in k and retaining only the leading terms gives

$$D^*_{\mu\nu}(k)D_{\alpha\beta}(k)F^{\mu\alpha}(p'_2,p_2)F^{\nu\beta}(p'_1,p_1) \approx 4|p^\mu_2 p^\nu_1 D_{\mu\nu}(k)|^2. \tag{5.5}$$

The factor (5.5) is the same as the factor involving the photon propagator on the right-hand side of the classically-derived probability (3.9), indicating that (3.9) does indeed correspond to the small-k limit of Møller scattering.

5.3. Probability for Møller scattering

In order to compare the QED result with the classical result derived in Sec. 3, the QED result needs to be written so that the momentum transfer, k, appears

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explicitly. To make the k-dependence explicit, one may rewrite (5.1) using the identity

$$(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2') = \int \frac{d^4k}{(2\pi)^4}\,(2\pi)^4\delta^4(p_1'-p_1-k)\,(2\pi)^4\delta^4(p_2'-p_2+k). \quad (5.6)$$

The differentials $d^3\mathbf{p}_1'/(2\pi)^3$, $d^3\mathbf{p}_2'/(2\pi)^3$ in (5.1) may be canceled with the three-dimensional parts of the δ-functions in (5.6). The probability for Møller scattering, $w(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k})$, is defined by writing the transition rate in the form

$$w_{\mathrm{i}\to\mathrm{f}} = \frac{w(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k})}{V}\frac{d^3\mathbf{k}}{(2\pi)^3}, \quad (5.7)$$

with $\mathbf{p}_1' = \mathbf{p}_1+\mathbf{k}$, $\mathbf{p}_2' = \mathbf{p}_2-\mathbf{k}$ now implicit. Thus, $w(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k})$ is the probability for the scattering of electrons with momentum transfer, $\mathbf{k}$, in the range $d^3\mathbf{k}/(2\pi)^3$. such that the initial 4-momenta p_1, p_2, become the final 4-momenta $p_1' = [\varepsilon_1',\mathbf{p}_1+\mathbf{k}]$, $p_2' = [\varepsilon_2',\mathbf{p}_2-\mathbf{k}]$. The probability is identified as

$$w(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}) = \int \frac{d\omega}{2\pi}\,2\pi\delta(\omega+\varepsilon_1'-\varepsilon_1)2\pi\delta(\omega-\varepsilon_2+\varepsilon_2')\,\frac{|M_{\mathrm{fi}}|^2}{8\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}, \quad (5.8)$$

where M_{fi} is given by (5.2), with $p_1' = p_1+k$, $p_2' = p_2-k$.

In the limit of small k, using (5.5), the probability (5.8) reduces to

$$\begin{aligned} w(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}) &= \int \frac{d\omega}{2\pi}\,2\pi\delta(\omega+\varepsilon_1-\varepsilon_1')2\pi\delta(\omega-\varepsilon_2'+\varepsilon_2)\frac{e^4|p_2^\mu p_1^\nu D_{\mu\nu}(k)|^2}{2\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'} \\ &\approx \int \frac{d\omega}{2\pi}\,2\pi\delta(\omega-\mathbf{k}\cdot\mathbf{v}_1)2\pi\delta(\omega-\mathbf{k}\cdot\mathbf{v}_2)\frac{e^4|p_2^\mu p_1^\nu D_{\mu\nu}(k)|^2}{2(\varepsilon_1\varepsilon_2)^2}, \end{aligned} \quad (5.9)$$

where, in the small-k limit, one has $\varepsilon_1' \approx \varepsilon_1+\mathbf{k}\cdot\mathbf{v}_1$, $\varepsilon_2' \approx \varepsilon_2-\mathbf{k}\cdot\mathbf{v}_2$.

5.4. Comparison with the classical result

The classical result (3.9) for the probability $W(p_1,p_2,k)$ translates to the probability $w(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k})$ using the identity (A 2) derived in Appendix A:

$$w(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}) = \int \frac{d\omega}{2\pi}\frac{W(p_1,p_2,k)}{\gamma_1\gamma_2} = \int \frac{d\omega}{2\pi}\,\frac{2\pi^2e^4}{m^2}\,\delta(kp_1)\delta(kp_2)\frac{|p_1^\mu p_2^\nu D_{\mu\nu}(k)|^2}{\gamma_1\gamma_2}. \quad (5.10)$$

By inspection (5.8) and (5.10) are the same, apart from minor differences in notation. This establishes that the classically derived probability corresponds to Møller scattering (and also of Bhabha scattering) in the limit of small momentum transfer. Conversely, the relativistic quantum counterpart of the classically derived probability is the probability (5.8) with (5.3).

6. Conclusions

The use of a covariant formalism allows one to generalize the derivation of the collision integral to include both relativistic effects and the effects of the current–current interaction in addition to the charge–charge interaction. The current–current interaction, which is unimportant for non-relativistic particles, is equal to half the charge–charge contribution for highly relativistic particles for $|\mathbf{k}|\lambda_{\mathrm{D}} \gg 1$, and continues to apply for $|\mathbf{k}|\lambda_{\mathrm{D}} \ll 1$ when Debye screening makes the charge–charge interaction negligible. This implies that there is a regime of distant (current–current

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mediated) encounters between relativistic particles that is unjustifiably neglected by the usual assumption that interactions at distances $\gg \lambda_{\rm D}$ are negligible.

An interesting formal point is that there is no standard classical procedure for calculating the probability of scattering one particle by another due to their interaction via the self-consistent electromagnetic field in a plasma. The probability can be identified only indirectly through the collision integral calculated from the theory of fluctuations (e.g. Lifshitz and Pitaevskii 1981, Sec. 51). Here it is shown that the classically derived probability is reproduced as a limit of the QED treatment of electron–electron (Møller) scattering or of electron–positron (Bhabha) scattering. The important, but trivial, generalization of the QED calculation is to replace the photon propagator, which is the Green's function for the electromagnetic field in vacuo, by the photon propagator in a medium. The classical limit corresponds to the limit of small momentum transfer, k, and is reproduced by retaining only the leading term in an expansion in k.

Is there a classical method by which the scattering probability can be calculated directly? Formally, the answer is presumably positive: one should be able to translate the steps in a QED calculation into classical counterparts. However, the answer appears to be negative in practice: no such classical counterpart of QED exists for processes that involve more than one particle simultaneously.

Acknowledgement

I thank Qinghuan Luo for helpful comments on the manuscript.

Appendix A. Covariant formalism

The formalism used here assumes natural units ($\hbar = c = 1$), 4-vectors $x^\mu = [t, \mathbf{x}]$, $p^\mu = [\varepsilon, \mathbf{p}]$, $u^\mu = [\gamma, \gamma\mathbf{v}]$, $k^\mu = [\omega, \mathbf{k}]$ and a metric tensor $g^{\mu\nu}$ that is diagonal $1, -1, -1, -1$, so that one has $x_\mu = [t, -\mathbf{x}]$, etc. Invariants, such as $k^\mu u_\mu = \gamma(\omega - \mathbf{k}\cdot\mathbf{v})$ are written ku, and invariants such as such as $k^\mu k_\mu = \omega^2 - |\mathbf{k}|^2$ are written k^2. The 3-tensor counterparts of second-rank 4-tensors follow from the 3-tensor components of, say, $J^\mu(k) = \Pi^{\mu\nu}(k)A_\nu(k)$ being written $J^i(k) = \Pi^i{}_j(k)A_j(k)$, where Latin indices denote space components, so that the 3-tensor is identified as $\Pi^i{}_j(k)$, which is numerically equal to $-\Pi^{ij}(k)$. To minimize confusion with signs, 3-tensor relations are written in dyadic form, specifically, the mixed tensor components of $L^{\mu\nu}(k\tilde{u}), T^{\mu\nu}(k\tilde{u})$ in the rest frame $\tilde{u}^\mu = [1, \mathbf{0}]$ have dyadic forms $\mathbf{kk}/|\mathbf{k}|^2$, $\mathbf{1} - \mathbf{kk}/|\mathbf{k}|^2$, that is, $k^i k_j$, $g^i{}_j$ translate into $-\mathbf{kk}$, $\mathbf{1}$, respectively.

The use of the distribution function, $F(x, p)$, in eight-dimensional phase space, although relatively unfamiliar, has a long history, summarized, for example, by Dewar (1977), cf. also the recent discussion by Lehmann (2006). $F(x, p)$ is related to the distribution function, $f(t, \mathbf{x}, \mathbf{p})$, in six-dimensional phase space, by the infinitesimal contributions to the proper number density being given by $[d^4p/(2\pi)^4]F(x, p)$ and by $[d^3\mathbf{p}/(2\pi)^3]f(t, \mathbf{x}, \mathbf{p})/\gamma$. The specific relation is

$$F(x, p) = 4\pi m\delta(p^2 - m^2)f(t, \mathbf{x}, \mathbf{p}). \tag{A 1}$$

The probabilities $W(p_1, p_2, k)$ and $w(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k})$ are defined such that the rate of transitions from the initial states p_1, p_2 for momentum transfer in an infinitesimal range is given by $[d^4k/(2\pi)^4][d^4p_1/(2\pi)^4][d^4p_2/(2\pi)^4]\, W(p_1, p_2, k)F(p_1)\, F(p_2)$ and $[d^3\mathbf{k}/(2\pi)^3]\ [d^3\mathbf{p}_1/(2\pi)^3]\ [d^3\mathbf{p}_2/(2\pi)^3]\ w(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k})\ f(\mathbf{p}_1)\ f(\mathbf{p}_2)$, respectively.

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Using (A 1) this implies the relation between them:

$$\int \frac{d\omega}{2\pi}\, W(p_1, p_2, k) = \gamma_1\gamma_2\, w(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}). \tag{A 2}$$

Appendix B. Longitudinal and transverse projection tensors

The separation into longitudinal and transverse parts may be carried out in an arbitrary frame in which the rest frame of the plasma (in which it is assumed isotropic) has 4-velocity $\tilde{u}$ using the following tensors and identities:

$$a^{\mu\nu}(k,u) = G^{\mu\alpha}(k,u)G_{\alpha}{}^{\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu}{ku} - \frac{k^\nu u^\mu}{ku} + \frac{k^2 u^\mu u^\nu}{(ku)^2}, \tag{A 1}$$

$$L^{\mu\nu}(k,u) = \frac{k^2}{k^2-(ku)^2}\left[a^{\mu\nu}(k,u) - \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\right], \tag{A 2}$$

$$T^{\mu\nu}(k,u) = \frac{1}{k^2-(ku)^2}\left[-(ku)^2\, a^{\mu\nu}(k,u) + k^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\right], \tag{A 3}$$

$$L^{\mu\sigma}(k,u)L_{\sigma}{}^{\nu}(k,u) = \frac{k^2}{(ku)^2}L^{\mu\nu}(k,u), \quad T^{\mu\sigma}(k,u)T_{\sigma}{}^{\nu}(k,u) = T^{\mu\nu}(k,u), \tag{A 4}$$

$$L^{\mu\sigma}(k,u)T_{\sigma}{}^{\nu}(k,u) = 0, \quad L^{\mu}{}_{\mu}(k,u) = \frac{k^2}{(ku)^2}, \quad T^{\mu}{}_{\mu}(k,u) = 2. \tag{A 5}$$

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