Ergod. Th. & Dynam. Sys. (2017), **37**, 664–672 doi:10.1017/etds.2015.53
© *Cambridge University Press, 2015*

Normality in Pisot numeration systems

ADRIAN-MARIA SCHEERER
TU Graz, Austria
(e-mail: scheerer@math.tugraz.at)

(*Received* 25 *February* 2015 *and accepted in revised form* 14 *May* 2015)

Abstract. Copeland and Erdös [Note on normal numbers. *Bull. Amer. Math. Soc. (N.S.)* **52** (1946), 857–860] showed that the concatenation of primes when written in base 10 yields a real number that is normal to base 10. We generalize this result to Pisot number bases in which all integers have finite expansion.

1. *Introduction*

Let x be a real number and $b \geq 2$ a positive integer. Then x has a b-adic representation of the form

$$x = \lfloor x \rfloor + \sum_{i=1}^{\infty} \epsilon_i b^{-i},$$

where $\epsilon_i \in \{0, 1, \ldots, b-1\}$ are the digits of x and $\lfloor x \rfloor$ is the integer part of x, the biggest integer less than or equal to x. We call x *normal to base b* if any block $d = d_1 d_2 \cdots d_k$ of $k \geq 1$ digits occurs with the expected frequency in the b-adic representation of x. This means that

$$\lim_{n\to\infty} \frac{1}{n} N_d(x, n) = \frac{1}{b^k},$$

where $N_d(x, n)$ counts the occurrences of the block d within the first n digits of x. A real number x is called *absolutely normal* if it is normal to every base $b \geq 2$.

The terminology of a normal number can be extended to the context when the underlying base is no longer an integer. Rényi [**21**] introduced and Parry [**20**] studied numeration systems with respect to real bases $\beta > 1$. Each real number x has a representation of the form

$$x = \sum_{i=L}^{-\infty} \epsilon_i \beta^i,$$

with digits $\epsilon_i \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$. One way to produce the digits is the so-called *greedy algorithm* using the transformation $T_\beta : x \mapsto \beta x \pmod 1$ on the unit interval. In a natural way, T_β corresponds to the shift operator on the set W^∞ of right-infinite sequences over $\{0, 1, \ldots, \lceil \beta \rceil - 1\}$ and each x corresponds to its sequence of digits. A sequence ω in

https://doi.org/10.1017/etds.2015.53 Published online by Cambridge University Press

W^∞ is called *μ-normal* for a given shift-invariant measure μ on W^∞ if all possible finite patterns of digits occur in ω with asymptotic frequency given by μ. Consequently, the real number x is called *μ-normal* if its sequence of digits is μ-normal. Details will be made clear in the next section.

From a modern approach, using Birkhoff's point-wise ergodic theorem, it is immediate that almost all numbers are normal to a fixed† base b. The map $T_b : x \mapsto bx \pmod 1$ on the unit interval is the underlying ergodic transformation which preserves Lebesgue measure. Knowing this, the existence of normal numbers to base b is in a certain sense not very surprising. However, in this context, there are two observations that make the study of normal numbers interesting.

First: The explicit construction of normal numbers. The study of normal numbers dates back to Borel [**4**], who in 1909 showed that Lebesgue-almost all numbers are absolutely normal. However, the first explicit example of a normal number is due to Champernowne [**5**] in 1933. He showed that the concatenation of integers, when written in base 10,

$$0, 1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\ldots,$$

is normal to base 10. Copeland and Erdös [**6**] showed that

$$0, 2\,3\,5\,7\,11\,13\,17\,19\ldots,$$

i.e. the concatenation of primes in base 10, is normal to base 10. Besicovitch [**2**] showed that the decimal formed by concatenation of the squares in base 10 is normal to base 10. This construction was extended to general integer-valued polynomials by Davenport and Erdös [**7**] and by Schiffer [**22**] and Nakai and Shiokawa [**17**, **18**] to more general polynomial settings. Nakai and Shiokawa [**19**] also evaluated polynomials at primes, and Madritsch [**13**] showed that numbers generated by pseudo-polynomial sequences along the primes are normal. Further constructions of normal numbers in the spirit of Copeland and Erdös and Erdös and Davenport include [**15**, **16**].

These constructions, most notably the one due to Champernowne, have subsequently been generalized to other number systems. We mention the works by Ito and Shiokawa [**12**], who generalized the Champernowne construction to real bases $\beta > 1$, and by Madritsch and Mance [**14**], who modified the construction to produce normal sequences in general symbolic dynamical systems. Bertrand-Mathis and Volkmann [**1**] gave a generalized Copeland–Erdös construction to symbolic dynamical systems.

In this paper we prove a polynomial Copeland–Erdös construction to bases which are not integers. We use results from the work of Bertrand-Mathis and Volkmann [**1**], which in turn extended the original work of Copeland and Erdös [**6**].

Second: Not *all* numbers are normal (almost all cannot be improved to all; hence, T_b is not uniquely ergodic), so the ergodic theorem is strict. This can be seen in context of the following 'test' for ergodicity.

THEOREM 1.1. (See [**3**, Theorem 1.4]) *Let $\mathcal{F}_0$ be a field‡ generating $\mathcal{F}$. If T respects§ every A in $\mathcal{F}_0$, then T is ergodic.*

† Borel's result also follows, but for the moment we want to restrict the discussion to a single base.
‡ The underlying sigma-field $\mathcal{F}$ is in our case the Borel sigma-algebra which is generated by b-adic intervals.
§ Almost every orbit under T visits A with the expected asymptotic frequency.

https://doi.org/10.1017/etds.2015.53 Published online by Cambridge University Press

Hence, normal numbers, or rather the existence of non-normal numbers, can be used to test the underlying transformation for unique ergodicity. Although the Lebesgue measure is not the only T_b-invariant probability measure on the unit interval, it is the only one that maximizes entropy (see the following section). Accepting this for the moment as a definition of uniqueness, it is possible to study normality in greater generality with respect to a given transformation that can be different from T_b.

2. *Preliminaries*

In the following we present a condensed introduction to β-expansions, Pisot numbers and symbolic dynamical systems—the context in which we want to state our result.

Let $\beta > 1$ be a fixed real number. A *β-expansion* of a non-negative real number x is a representation of x as a sum of integer powers of β of the form

$$x = \sum_{i=L}^{-\infty} \epsilon_i \beta^i, \tag{2.1}$$

where the digits $\epsilon_i \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ are obtained by the following *greedy algorithm*. Let $L \in \mathbb{Z}$ be such that $\beta^L \leq x < \beta^{L+1}$ and put $\epsilon_L = \lfloor x/\beta^L \rfloor$ and $r_L = \{x/\beta^L\}$. For $L \geq i > -\infty$, define recursively $\epsilon_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \{\beta r_{i+1}\}$. β-expansions have been introduced and studied by Rényi [**21**] and Parry [**20**].

Let T_β be the *β-transformation* $T_\beta : [0, 1) \to [0, 1)$, $x \mapsto \{\beta x\}$. The digits in the β-expansion (2.1) are given by $\epsilon_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$. Rényi [**21**] showed that there is a unique normalized measure μ_β on $[0, 1)$ that is invariant under T_β and equivalent to the Lebesgue measure. This measure also maximizes the entropy of the corresponding symbolic dynamical system and we use it to define normal numbers in base β; see below.

A *Pisot number* β is a real algebraic integer $\beta > 1$ such that all its conjugates have absolute value less than 1. For a Pisot number of degree d, we denote its conjugates by β_i, $i = 2, \ldots, d$, and the corresponding conjugations by σ_i.

We work with Pisot numbers such that every positive integer has finite β-expansion. A criterion for when this is the case can for example be found in [**8**].

In our notation concerning symbolic dynamical systems, we follow Bertrand-Mathis and Volkmann [**1**].

Let A be a finite alphabet. A^* is the set of all finite (possibly empty) words over A and $A^{\mathbb{N}}$ is the set of all (right-)infinite words over A. We call a subset L of A^* a language. L^* denotes the set of all finite concatenations of words from L. Let $W(L^*)$ be the set of all non-empty factors of words in L^*. For a finite word ω, we let $\|\omega\|$ be its length. A language L is said to be connecting† of order $j \geq 0$ if for any two words $a, b \in W(L^*)$ there is a word $u = u(a, b) \in W(L^*)$ of length j such that $aub \in W(L^*)$. For each $a, b \in W(L^*)$, we choose one $u = u(a, b)$ and introduce the notation $a \oplus b := aub$. In the applications we have in mind, this intermediary word will simply consist of zeros. We write $W(L^*) = \bigcup_{n \geq 1} L_n$, where L_n is the subset of words in $W(L^*)$ of length n. Also, denote by L'_n the subset of $W(L^*)$ of words of length less or equal to n. For a language L, we denote by $W^\infty = W^\infty(L)$ the set of all infinite words generated by L, i.e. the set of all $\omega = a_1 a_2 \cdots$ such that $a_i a_{i+1} \cdots a_k \in L$ for all $1 \leq i < k < \infty$.

† A symbolic dynamical system having a connecting language is also said to have the *specification property*.

https://doi.org/10.1017/etds.2015.53 Published online by Cambridge University Press

We introduce the discrete topology on the alphabet A and the corresponding product topology on the set of sequences $A^{\mathbb{N}}$. With each language L, we associate the symbolic dynamical system

$$S_L = (W^\infty, \mathcal{B}, T, I),$$

where $W^\infty = W^\infty(L)$; $\mathcal{B}$ is the σ-algebra generated by all cylinder sets of $A^{\mathbb{N}}$, i.e. sets of the form

$$c(\omega) = \{a_1 a_2 \cdots \in A^{\mathbb{N}} \mid a_1 a_2 \cdots a_n = \omega\}$$

for some word $\omega \in A^*$ of length n. T is the shift operator and I is the set of all T-invariant probability measures μ on $\mathcal{B}$. We will write $\mu(\omega)$ instead of $\mu(c(\omega))$ for a finite word ω.

With each symbolic dynamical system S_L, we associate the entropy

$$h(W^\infty) = \sup_{\mu \in I} h(\mu),$$

where $h(\mu)$ denotes the entropy of the measure μ (cf. [**3**, Ch. 2]). In the context of the symbolic dynamical system generated by a β-expansion, the measure μ_β is precisely the (unique) measure with maximum entropy equal to $\log \beta$ (cf. [**11**] and the work by Bertrand-Mathis and Volkmann [**1**]). In the following, we will work with this maximal measure.

For an infinite word $\omega = \omega_1 \omega_2 \cdots \in W^\infty$ and a block $d = d_1 d_2 \cdots d_k \in L$, we denote by $N_d(\omega, n)$ the number of occurrences of d within the first n letters of ω. If the word ω is finite, we denote by $N_d(\omega)$ the occurrences of d in it. An infinite word $\omega \in W^\infty$ is called *μ-normal* or a *μ-normal sequence* if for all $d \in W(L^*)$

$$\lim_{n \to \infty} \frac{1}{n} N_d(\omega, n) = \mu(d).$$

We note that a non-negative real number x is *μ_β-normal* if and only if the sequence of digits in its β-expansion is a μ_β-normal sequence.

In this terminology, the main result of Bertrand-Mathis and Volkmann [**1**] is the following.

THEOREM 2.1. *Let L be a connecting language and $a_1, a_2, \ldots$ a sequence of different elements of $W(L^*)$, $\|a_1\| \le \|a_1\| \le \cdots$, satisfying the generalized Copeland–Erdös condition:*

$$\textit{for all } \epsilon > 0 \textit{ there exists } n_0(\epsilon) \textit{ for all } n \ge n_0 \; \sharp\{a_\nu \mid \|a_\nu\| \le n\} > |L'_n|^{1-\epsilon}.$$

Then the infinite word $a = a_1 a_2 \cdots \in W^\infty$ is normal.

The symbolic dynamical system *generated by the β-shift* (or *corresponding to the β-expansion*) arises naturally when viewing the β-expansions of real numbers $x \in [0, 1)$ as infinite words over the alphabet $\{0, 1, \ldots, \lceil \beta \rceil\}$. W^∞ is the set of these right-infinite sequences, $\mathcal{B}$ the σ-algebra generated by all cylinder sets, T the shift operator (it corresponds to the β-transformation T_β) and I the set of all T-invariant probability measures on $\mathcal{B}$. We work with the unique entropy-maximizing measure μ in I. It corresponds to μ_β on $[0, 1)$ in the sense that for a finite word ω, the measure of the cylinder set $\mu(\omega)$ is the same as the measure $\mu_\beta(\tilde{\omega})$ of the set $\tilde{\omega}$ of all real numbers in $[0, 1)$ whose β-expansion starts with ω. We speak of these two concepts interchangeably.

https://doi.org/10.1017/etds.2015.53 Published online by Cambridge University Press

For a given Pisot number β, denote by $(n)_\beta$ the word over $\{0, 1, \ldots, \lceil\beta\rceil - 1\}$ that corresponds to the β-expansion of the positive integer n. We prove the following polynomial generalization of [**6**].

THEOREM 2.2. *Let β be a Pisot number such that all integers have finite β-expansion and let the measure μ_β be as before. Let f be a polynomial of degree g that maps positive integers to positive integers. Then*

$$(f(2))_\beta \oplus (f(3))_\beta \oplus (f(5))_\beta \oplus (f(7))_\beta \oplus (f(11))_\beta \oplus \cdots$$

is a μ_β-normal sequence.

In the context of the dynamical system generated by the β-shift, the entropy is $\log\beta$. Hence, by [**1**, Lemma 2], we have bounds on the number of words of length n in $W(L^*)$. For n sufficiently large we have $\beta^n \ll |L_n| \ll \beta^n$, where the implied constants do not depend on n. Therefore,

$$|L'_n| = \sum_{\nu=1}^{n} |L_\nu| \ll \beta^n \tag{2.2}$$

for all large n.

3. *General case*

First we need upper and lower bounds of the length of the β-expansion of integers. Under the assumption of β being a Pisot number such that all integers have finite β-expansions we can in fact show that the lengths of these expansions are asymptotically of logarithmic order of magnitude.

Note that if $n = \sum_{i=L(n)}^{-R(n)} \epsilon_i \beta^i$, we call $\sum_{i=0}^{L(n)} \epsilon_i \beta^i$ its integer part and $\sum_{i=-1}^{-R(n)} \epsilon_i \beta^i$ its fractional part. In the following we will think of n as fixed and omit writing the dependency on it in the lengths L and R.

LEMMA 3.1. *Let β be a Pisot number of degree d such that all natural numbers have finite β-expansion. For the length $R(n)$ of the fractional part of n upper and lower bounds of the following form hold, for sufficiently large n:*

$$\delta \log n \le R(n) \le \delta' \log n,$$

where δ is a positive constant (specified in the proof) and the difference $\delta' - \delta > 0$ can be chosen to be arbitrarily small.

Proof. We have $n/\beta^{L+1} \in [0, 1)$. Following an argument of Proposition 3.5 in Frougny and Steiner [**9**], for a certain number k the number $T_\beta^k(n/\beta^{L+1})$ is an element of the finite set

$$Y = \left\{ y \in \mathbb{Z}[\beta] \cap [0, 1) \mid |\sigma_j(y)| < 1 + \frac{\lfloor\beta\rfloor}{1 - |\sigma_j(\beta)|} \text{ for } 2 \le j \le d \right\}.$$

To see this, let the β-expansion of n for the moment be $\epsilon_1\epsilon_2 \cdots$ and put $z := n/\beta^{L+1}$. Then, for all $k \ge 0$,

$$T_\beta^k(z) = \beta T_\beta^{k-1}(z) - \epsilon_k = \cdots = \beta^k z - \sum_{l=1}^{k} \epsilon_l \beta^{k-l}.$$

https://doi.org/10.1017/etds.2015.53 Published online by Cambridge University Press

Hence, for all $k \geq 0$ and $2 \leq j \leq d$,

$$|\sigma_j(T_\beta^k(z))| = \left|\sigma_j(\beta)^k \sigma_j(z) - \sum_{l=1}^{k} \epsilon_l \sigma_j(\beta)^{k-l}\right| < |\sigma_j(\beta)|^k |\sigma_j(z)| + \frac{\lfloor \beta \rfloor}{1 - |\sigma_j(\beta)|}.$$

Let k be equal to

$$\max_{2\leq j\leq d} \left\lceil -\frac{\log |\sigma_j(n/\beta^{L+1})|}{\log |\sigma_j(\beta)|} \right\rceil = L + 1 + \max_{2\leq j\leq d} \left\lceil \frac{\log n}{\log |\sigma_j(\beta)^{-1}|} \right\rceil,$$

which we write as

$$k = L + 1 + \delta \log n + O(1) \quad \text{where } \delta := \max_{2\leq j\leq d} \frac{1}{\log |\sigma_j(\beta)^{-1}|}.$$

Note that the $O(1)$ constant coming from the ceiling lies in $[0, 1)$. For this choice of k, $T_\beta^k(n/\beta^{L+1}) \in Y$.

Let W be the maximum length of the β-expansions of the elements in Y. We therefore obtain asymptotic bounds of R,

$$\delta \log n \leq R \leq W + 1 + \delta \log n \leq \delta' \log n,$$

where $\delta' > \delta$ can be chosen arbitrarily close. □

In the course of the proof of Theorem 2.2 we deal with a problem caused by a constant coming from the length of the words we want to patch together. This issue can be circumvented by dividing the words into smaller subwords and glueing them back together afterwards. The following lemma ensures normality of the resulting word when patched back together.

LEMMA 3.2. *Let $v = v_1 v_2 v_3 \cdots$ and $w = w_1 w_2 w_3 \cdots$ be μ-normal words such that $\|v_i\| = \|w_i\|$, $\|v_i\| \to \infty$ and assume that the quantities $\|v_{N+1}\|/(\|v_1\| + \cdots + \|v_N\|)$ and $N/(\|v_1\| + \cdots + \|v_N\|)$ tend to zero as N tends to infinity. Then the word*

$$u = v_1 w_1 v_2 w_2 v_3 w_3 \cdots$$

is μ-normal.

Proof. We work with a fixed finite string $d = d_1 \cdots d_k$ of length k. Since v is μ-normal, we have

$$N_d(v, n) = \sum_{i=1}^{N} N_d(v_i) + O(N) + O(\|v_{N+1}\|) \longrightarrow \mu(d)n$$

as $n \to \infty$, where N is chosen such that $\|v_1\| + \cdots + \|v_N\| \leq n < \|v_1\| + \cdots + \|v_{N+1}\|$. The $O(N)$ contribution comes from possible occurrences in between two words v_i and v_{i+1}. Hence, by the assumptions,

$$\sum_{i=1}^{N} N_d(v_i) \longrightarrow \mu(d)n$$

https://doi.org/10.1017/etds.2015.53 Published online by Cambridge University Press

as $n \to \infty$. Then, by arguing similarly,

$$\begin{aligned} N_d(u, n) &= \sum_{i=1}^{N} N_d(v_i) + \sum_{i=1}^{N} N_d(w_i) + O(N) + O(\|w_{N+1}\|) \\ &= \mu(d)\frac{n}{2} + \mu(d)\frac{n}{2} + o(n). \end{aligned}$$

Here, N is chosen such that $\|v_1\| + \|w_1\| + \cdots + \|v_N\| + \|w_N\| \le n < \|v_1\| + \|w_1\| + \cdots + \|v_{N+1}\| + \|w_{N+1}\|$. This shows the normality of u. □

To verify the conditions of Lemma 3.2 in our application, we need some basic number theoretic input.

LEMMA 3.3. *Denote by p_N the Nth prime number. We have for $N \to \infty$*

$$(1) \quad \frac{\log p_{N+1}}{\log p_1 + \cdots + \log p_N} \longrightarrow 0 \quad \textit{and} \quad (2) \quad \frac{N}{\log p_1 + \cdots + \log p_N} \longrightarrow 0.$$

Proof. This is a consequence of the prime number theorem. We have

$$\sum_{i=1}^{p_N} \log p_i = \theta(p_N) \sim p_N \sim N \log N.$$ □

With these preliminaries, we can prove our theorem.

Proof of Theorem 2.2. The polynomial $f(n) = a_g n^g + \cdots + a_1 n + a_0$ behaves asymptotically like $a_g n^g$. Hence, for any $\epsilon > 0$ and any n large enough,

$$f(n) \le (1+\epsilon) a_g n^g \quad \text{and} \quad f(n) \ge (1-\epsilon) a_g n^g.$$

Thus, as a consequence of Lemma 3.1, we have the upper bounds

$$\begin{aligned} R(f(n)) &\le R((1+\epsilon) a_g n^g) \\ &\le \delta'(\log(1+\epsilon) + \log a_g + g \log n) \\ &\le C' \log n \end{aligned}$$

for some constant $C' > \delta' g$ arbitrarily close and n large enough. Similarly, we obtain lower bounds of the form

$$R(f(n)) \ge C \log n,$$

where $C < \delta g$ is a positive constant and can be chosen arbitrarily close if n is assumed to be large enough.

A direct consequence is an asymptotic upper bound for the total length of $f(n)$ when written in base β:

$$\begin{aligned} \|(f(n))_\beta\| &= L(f(n)) + 1 + R(f(n)) \\ &\le L((1+\epsilon) a_g n^g) + 1 + R((1+\epsilon) a_g n^g) \\ &\le \frac{g \log n + \log a_g}{\log \beta} + O(1) + C' \log n \\ &\le C \frac{\log n}{\log \beta} \end{aligned}$$

for some (other) constant C. Note that C depends only on f and β.

https://doi.org/10.1017/etds.2015.53 Published online by Cambridge University Press

However, applying Theorem 2.1 directly does not work as we do not have control of the size of the constant C. This can be avoided by choosing an integer $m \geq 0$ such that $C/2^m \leq 1$ and dividing the words $(f(n))_\beta$ into 2^m words of (almost) equal length. Then we can apply Theorem 2.1 to show normality of the concatenations of those shorter words. They can subsequently be patched back together applying Lemma 3.2 multiple times. Note that the lower bounds for R enable us to use Lemma 3.2.

For a prime number p, we have $\|(f(p))_\beta\| \leq C(\log p/\log \beta)$, so $\|(f(p))_\beta\| \leq N$ is implied by $C(\log p/\log \beta) \leq N$. This is equivalent to

$$p \leq \beta^{N/C}.$$

Thus, counting primes below $\beta^{N/C}$,

$$\pi(\beta^{N/C}) \sim \frac{\beta^{N/C}}{N/C \log \beta} \geq \frac{C}{\log \beta}\beta^{N(C^{-1}-\epsilon)}$$

for any $\epsilon < 1$ arbitrarily close and N large enough. Here, we see why we require $C \leq 1$, namely so that the condition of Theorem 2.1,

$$\pi(\beta^{N/C}) \geq (\beta^N)^{1-\epsilon}$$

for any $\epsilon' > 0$, is implied by $\beta^{N(C^{-1}-\epsilon)}$ being eventually greater than $(\beta^N)^{1-\epsilon}$. Inserting the intermediary word to obtain admissibility in base β does not destroy the normality of the sequence, since we are inserting a word of constant length. □

4. *Final remarks*

Let $\phi = (1+\sqrt{5})/2$ be the golden ratio, i.e. the dominating root of the polynomial $x^2 - x - 1$. All positive integers have finite ϕ-expansion (see for example [**8**, Theorem 2]). Considering this special case is interesting insofar that we can provide an *exact* formula on the length of the fractional part. Let n be a positive integer and denote by $L+1$ and R the lengths of its integer and fractional parts when written in base ϕ. From the greedy algorithm, we already know that

$$\phi^L \leq n < \phi^{L+1} \Rightarrow L = \left\lfloor \frac{\log n}{\log \phi} \right\rfloor.$$

In [**10**], it is proved that the fractional part R of n satisfies $R = L$ or $R = L+1$, depending on whether L is even or odd. However, even with such an exact formula it is not possible to establish the normality to base ϕ of the word

$$(2)_\phi 0 (3)_\phi 0 (5)_\phi 0 \cdots$$

by directly applying the generalized Copeland–Erdös criterion of Theorem 2.1. Moreover, the method employed in [**10**] does not seem to yield an exact formula for the length of the fractional part when the underlying base is a Pisot number of degree greater than two. It seems to be an interesting open problem to obtain such a formula.

https://doi.org/10.1017/etds.2015.53 Published online by Cambridge University Press

Acknowledgements. The author would like to thank Robert Tichy and Manfred Madritsch for their support, supervision and discussions about the subject. Special thanks go to Tomáš Vávra who showed me [**10**] and to Wolfgang Steiner who very patiently explained to me the proof of Proposition 3.5 in [**9**]. Part of this research was carried out at the Université de Lorraine in Nancy and at the Max-Planck Institute for Mathematics in Bonn and the author would like to thank these institutions for their hospitality. The author also would like to thank the anonymous referee for his/her useful comments. The author was supported by the Austrian Science Fund (FWF): I 1751-N26; W1230, Doctoral Program 'Discrete Mathematics'; and SFB F 5510-N26.

References

[**1**] A. Bertrand-Mathis and B. Volkmann. On (ϵ, k)-normal words in connecting dynamical systems. *Monatsh. Math.* **107**(4) (1989), 267–279.
[**2**] A. S. Besicovitch. The asymptotic distribution of the numerals in the decimal representation of the squares of the natural numbers. *Math. Z.* **39**(1) (1935), 146–156.
[**3**] P. Billingsley. *Ergodic Theory and Information*. John Wiley, New York, London, Sydney, 1965.
[**4**] É. Borel. Les probabilités dénombrables et leurs applications arithmétiques. *Rend. Circ. Mat. Palermo* (2) **27**(1) (1909), 247–271.
[**5**] D. G. Champernowne. The construction of decimals normal in the scale of ten. *J. Lond. Math. Soc.* **s1–8**(4) (1933), 254–260.
[**6**] A. H. Copeland and P. Erdös. Note on normal numbers. *Bull. Amer. Math. Soc. (N.S.)* **52** (1946), 857–860.
[**7**] H. Davenport and P. Erdös. Note on normal decimals. *Canad. J. Math.* **4** (1952), 58–63.
[**8**] C. Frougny and B. Solomyak. Finite beta-expansions. *Ergod. Th. & Dynam. Sys.* **12**(4) (1992), 713–723.
[**9**] C. Frougny and W. Steiner. Minimal weight expansions in Pisot bases. *J. Math. Cryptol.* **2**(4) (2008), 365–392.
[**10**] P. J. Grabner and H. Prodinger. Additive irreducibles in α-expansions. *Publ. Math. Debrecen* **80**(3–4) (2012), 405–415.
[**11**] F. Hofbauer. β-shifts have unique maximal measure. *Monatsh. Math.* **85**(3) (1978), 189–198.
[**12**] S. Ito and I. Shiokawa. A construction of β-normal sequences. *J. Math. Soc. Japan* **27** (1975), 20–23.
[**13**] M. G. Madritsch. Construction of normal numbers via pseudo-polynomial prime sequences. *Acta Arith.* **166**(1) (2014), 81–100.
[**14**] M. G. Madritsch and B. Mance. Construction of μ-normal numbers. *Preprint*, 2012, available at http://arxiv.org/abs/1206.4950.
[**15**] M. G. Madritsch and R. F. Tichy. Construction of normal numbers via generalized prime power sequences. *J. Integer Seq.* **16**(2) (2013), Article 13.2.12.
[**16**] M. G. Madritsch, J. M. Thuswaldner and R. F. Tichy. Normality of numbers generated by the values of entire functions. *J. Number Theory* **128**(5) (2008), 1127–1145.
[**17**] Y. Nakai and I. Shiokawa. A class of normal numbers. *Jpn. J. Math. (N.S.)* **16**(1) (1990), 17–29.
[**18**] Y. Nakai and I. Shiokawa. Discrepancy estimates for a class of normal numbers. *Acta Arith.* **62**(3) (1992), 271–284.
[**19**] Y. Nakai and I. Shiokawa. Normality of numbers generated by the values of polynomials at primes. *Acta Arith.* **81**(4) (1997), 345–356.
[**20**] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416.
[**21**] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* **8** (1957), 477–493.
[**22**] J. Schiffer. Discrepancy of normal numbers. *Acta Arith.* **47**(2) (1986), 175–186.

https://doi.org/10.1017/etds.2015.53 Published online by Cambridge University Press